MATRICES OF INTEGERS ORDERING DERIVATIVES*

BY

JOSEPH MILLER THOMAS

1. **Introduction.** Riquier† in his treatise on partial differential equations has employed matrices of integers, which he calls cotes, to establish order relations among the derivatives of the unknown functions. The matrix effecting a given ordering of the derivatives is not uniquely determined. Certain simple transformations which preserve order relations have been employed by Riquier and Janet. The object of the present paper is to study systematically the matrices in question with special attention to equivalence. The principal result is a method of reducing any matrix to a canonical form which characterizes all matrices establishing the same order relations as the given one.

Some of the transformations are applicable only to a restricted class of matrices. The totality of transformations described has the property that any transformation preserving order can be expressed as a product of them.

Except when the contrary is expressly stated, the results obtained are valid whatever the first cotes of the independent variables may be.

It is expected to follow this paper with another which will treat the existence of a matrix establishing given order relations.

2. **Definitions and completeness of ordering.** Consider a rectangular array of integers, the term integer including zero. Let there be $n+r$ rows, the first n rows corresponding to independent variables x and the last r to unknown functions u. The number in the qth column will be called the *qth cote* of the corresponding variable. We shall use the ordinary matrix notation for the cotes: c_q^p will be the qth cote of the pth independent variable and γ_q^α the qth cote of the αth unknown.

The qth cote of the derivative

$$D_i u_\alpha = \frac{\partial^{i_1+i_2+\cdots+i_n} u_\alpha}{\partial x_1{}^{i_1} \partial x_2{}^{i_2} \cdots \partial x_n{}^{i_n}}$$

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† C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Paris, 1910, p. 195. For an exposition of the application of cotes to the proof of existence theorems one may consult the following papers also:

M. Janet, *Sur les systèmes d'équations aux dérivées partielles*, Journal de Mathématiques Pures et Appliquées, (8), vol. 3 (1920), p. 65.

J. M. Thomas, *Riquier's existence theorems*, Annals of Mathematics, (2), vol. 30 (1929), p. 285.

is defined as

$$C_q = \sum_{p=1}^{n} c_q^p i_p + \gamma_q^\alpha . \tag{2.1}$$

By definition also the derivative $D_i u_\alpha$ precedes or follows $D_j u_\beta$, whose cotes we denote by C_q', according as the first non-zero difference of the set

$$C_1 - C_1', C_2 - C_2', \cdots \tag{2.2}$$

is negative or positive.

If all the differences (2.2) are zero, the given matrix will not establish an order relation between the two derivatives. If the matrix is augmented by columns of arbitrarily chosen integers, the order relations established by the original matrix are not disturbed because the additional cotes will only play a rôle when the original cotes give no answer. Moreover, if the new columns are properly chosen, additional order relations are established by the augmented matrix.

In particular, if the last column is made $0, 0, \cdots, 0, 1, \cdots, r$, any two u's whose relative order is not established by the cotes before the last will have the order relation of their subscripts. Consequently, the augmented matrix completely orders the unknowns, and the equations

$$\gamma_q^\alpha - \gamma_q^\beta = 0 \qquad (q = 1, 2, \cdots, s), \tag{2.3}$$

where s represents the total number of cotes, imply $\alpha=\beta$, that is, if the differences (2.2) formed for two unknowns are all zero, the unknowns are the same.

Likewise if the matrix is further augmented by the n columns

$$\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{array},$$

and if all the differences (2.2) for the derivatives $D_i u_\alpha$ and $D_j u_\beta$ are zero, the vanishing of the last n differences gives $i_p = j_p$. The vanishing of the other differences then shows that (2.3) hold, that is, u_α and u_β are the same. Consequently the vanishing of the differences (2.2) implies that the derivatives are identical.

THEOREM 1. *For any matrix of integers there exists an augmented matrix whose ordering is consistent with that of the original and is complete.*

In the future, unless the contrary is expressly stipulated, we shall deal only with matrices whose ordering is complete so that equality of all the corresponding cotes of two derivatives implies identity of the derivatives.

3. **Simplest transformations preserving order.** Certain transformations which can be performed upon the elements of a matrix without disturbing the order relations are rather obvious. It is clear that the qth cotes of all the unknowns can be increased by the same integer μ_q without altering the differences (2.2), that is, an admissible transformation is

$$\bar{\gamma}_q^\alpha = \gamma_q^\alpha + \mu_q \qquad (\alpha = 1, 2, \cdots, r). \tag{3.1}$$

Clearly a second is

$$\bar{c}_q^p = \lambda c_q^p, \quad \bar{\gamma}_q^\alpha = \lambda \gamma_q^\alpha, \quad \lambda > 0 \qquad (p = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, r). \tag{3.2}$$

In these formulas, λ may be fractional provided its denominator is a divisor of all the elements of the qth column. Thus the highest common factor of the elements of any column can be removed.

If we put

$$\bar{c}_q^p = c_q^p + \sum \lambda_q^\sigma c_\sigma^p, \quad \bar{\gamma}_q^\alpha = \gamma_q^\alpha + \sum \lambda_q^\sigma \gamma_\sigma^\alpha,$$

we have

$$\overline{C}_q = C_q + \sum \lambda_q^\sigma C_\sigma,$$

whence for the differences of the sequence (2.2) the transformation

$$\overline{C}_q - \overline{C}_q' = C_q - C_q' + \sum \lambda_q^\sigma (C_\sigma - C_\sigma').$$

Since the qth cote plays a rôle in determining order (i.e. the vanishing or sign of $C_q - C_q'$ is of significance) only when the first $q-1$ cotes are equal, if we fix the range of the index σ as follows:

$$\bar{c}_q^p = c_q^p + \sum_{\sigma=1}^{q-1} \lambda_q^\sigma c_\sigma^p, \quad \bar{\gamma}_q^\alpha = \gamma_q^\alpha + \sum_{\sigma=1}^{q-1} \lambda_q^\sigma \gamma_\sigma^\alpha, \tag{3.3}$$

we have

$$\overline{C}_q - \overline{C}_q' = C_q - C_q'$$

whenever either $\overline{C}_q - \overline{C}_q'$ or $C_q - C_q'$ has significance. Hence (3.3) is a transformation preserving order.

If we like, we may combine (3.1), (3.2), (3.3) into the single transformation

$$\bar{c}_q^p = \sum_{\sigma=1}^{q} \lambda_q^\sigma c_\sigma^p, \quad \bar{\gamma}_q^\alpha = \sum_{\sigma=1}^{q} \lambda_q^\sigma \gamma_\sigma^\alpha + \mu_q, \quad \lambda_q^q > 0 \tag{3.4}$$

$$(p = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, r).$$

The λ's in these formulas are not necessarily integers. The only restriction is that the result of applying (3.4) be a matrix of integers. Thus the inverse of a transformation (3.4) in general has fractional coefficients, and the set of matrices to which it can be applied is restricted.

We may summarize the transformation (3.3) in

THEOREM 2. *A matrix of integers formed by increasing or diminishing the elements of a column in a given matrix by any equimultiples of the corresponding elements in any column which precedes it establishes the same order relations as the original.*

It is evident that by use of (3.1) the cotes of all the unknowns can be made positive. If all the first cotes of the independent variables are positive,* we can accomplish the same result for the whole matrix by using the subsequent transformation

$$\bar{c}_q^p = c_q^p + (a+1)c_1^p, \quad \bar{\gamma}_q^\alpha = \gamma_q^\alpha + (a+1)\gamma_1^\alpha, \tag{3.5}$$

where a is the numerical value of the numerically greatest negative cote. A particular result of this is

THEOREM 3. *If the first cote of every independent variable is unity, the cotes of both independent variables and unknowns can be made positive without altering the order.*†

Consider two derivatives D_i and D_j of the same unknown, the ordering being complete. From the fact that these derivatives are identical if the differences (2.2) are zero, we know that the system

$$\sum_{p=1}^{n} c_q^p (i_p - j_p) = 0 \qquad (q = 1, 2, \cdots, s) \tag{3.6}$$

has only the trivial solution $i_p - j_p = 0$. Hence

THEOREM 4. *If the ordering is complete, the matrix of the cotes of the independent variables is of rank equal to the number of variables n.*

* It will be a result of Theorem 12, which is to follow, that the signs of the first cotes of the independent variables cannot be changed without changing the ordering.

† This result is given by Janet, who uses (3.1) and (3.5) for $c_1^p = 1$ to obtain it. Thus Janet employs a special transformation (3.4).

Since, conversely, (3.6) has only the trivial solution if $\|c\|$ is of rank n, we have also

THEOREM 5. *If there is only one unknown, a necessary and sufficient condition for complete ordering is that the matrix of cotes of the independent variables be of rank equal to the number of variables n.*

If a column of the whole matrix is linearly dependent on those to the left of it, its elements can be made zero by a transformation (3.4). The column can then be suppressed, for the difference (2.2) corresponding to it is always zero. Hence we may assume that *the rank of the matrix is equal to the number of its columns*. The rank is an integer between n and $n+r$. That it may be as high as $n+r$ follows from the existence of a matrix of integers of rank $n+r$.

If the rank of the whole matrix is $n+r$, the ordering is complete because the $n+1$ additional columns adjoined in §2 to insure completeness, being linearly dependent on those already there, are superfluous.

THEOREM 6. *A sufficient condition for complete ordering is that the matrix of cotes be of rank $n+r$.*

By transformations (3.1) the row of cotes corresponding to any unknown can be made zero. Consequently, the condition in Theorem 6 is not necessary. Likewise we have

THEOREM 7. *The rank of a matrix of cotes is not invariant under transformations preserving order.*

That the transformations considered in this section are not the only ones admissible can be seen from the following example:

$$
(3.7)\qquad
\begin{array}{c|ccc}
x & 1 & 2 & 1 \\
y & 1 & 2 & 0 \\
\hline
u & 1 & 1 & 0 \\
v & 2 & 2 & 0
\end{array}
\quad\Bigg\|\quad
\begin{array}{ccc}
2 & 1 & 1 \\
2 & 1 & 0 \\
\hline
1 & 1 & 0 \\
2 & 2 & 0
\end{array}\,.
$$

It is easy to give a direct proof that the above two matrices bring about the same ordering. We shall not do this, but shall content ourselves with remarking that no transformation (3.4) will throw one of them into the other. This is obvious because their first columns are not proportional for any choice of μ_1 in (3.1). Later (§8) a proof that the two matrices (3.7) are equivalent will be given by means of another sort of transformation.

4. **Properties of forms with integral coefficients.** For convenience of reference we state in a form best adapted to our purpose several consequences of well known theories.

Consider the system

$$\sum_{p=1}^{n} c_t^p k_p = b_t \qquad (t = 1, 2, \cdots, q), \tag{4.1}$$

where the rank of $\|c\|$ is the same as the number of equations q. If every b is divisible by the product of all the determinant factors of $\|c\|$, then (4.1) has a solution in integers because any determinant in the augmented matrix but not in the original contains a column composed solely of b's and hence is divisible by any determinant factor of $\|c\|$.*

In particular, if certain of the b's are made positive, others negative and the rest zero, we have a result conveniently stated as

THEOREM 8. *A set of integers k can be found for which the distribution of signs and zero values is anything we wish among the q linearly independent forms*

$$\sum_{p=1}^{n} c_t^p k_p \qquad (t = 1, 2, \cdots, q). \tag{4.2}$$

By choosing the b's divisible by a sufficiently great power of the product of the determinant factors we see that the values assumed by the non-zero forms can be made numerically greater than any given number.

The above result for a single form will be applied to prove

THEOREM 9. *If the non-homogeneous equations*

$$\sum_{p=1}^{n} c_t^p k_p + \kappa_t = 0 \qquad (t = 1, 2, \cdots, q-1) \tag{4.3}$$

have a solution and if the matrices

$$\|c_t^p\|, \ \|c_t^p c_q^p\| \qquad (t = 1, 2, \cdots, q-1) \tag{4.4}$$

are of rank $\rho-1$, ρ, respectively, there exist integral values of the k's which satisfy (4.3) *and for which*

$$\sum_{p=1}^{n} c_q^p k_p + \kappa_q \tag{4.5}$$

has arbitrary sign.

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* An exposition of the theory of systems (4.1) with integral coefficients will be found in §12 of a paper by O. Veblen and P. Franklin, *On matrices whose elements are integers*, Annals of Mathematics, (2), vol. 23 (1921), p. 1.

The general solution in integers of (4.3) is any particular solution plus the general solution of the corresponding homogeneous system*

$$\sum_{p=1}^{n} c_t^p k_p = 0 \qquad (t = 1, 2, \cdots, q-1). \tag{4.6}$$

If on substitution of the general solution of (4.3) in (4.5) the variables cancel out, every solution of (4.6) will satisfy

$$\sum_{p=1}^{n} c_q^p k_p = 0. \tag{4.7}$$

But the general solution of (4.6) depends on $n-\rho+1$ parameters, whereas that of the system composed of (4.6) and (4.7) involves only $n-\rho$. Hence when the general solution of (4.3) is substituted in it, (4.5) still contains a variable form, which by Theorem 8 and the remark immediately following can be made to assume either a positive or a negative value exceeding $|\kappa_q|$ numerically. The desired result is therefore established.

The following is geometrically obvious:

THEOREM 10. *The expressions*

$$\sum_{p=1}^{n} c_q^p k_p + \kappa_q, \quad \sum_{p=1}^{n} c_q^p k_p + \bar{\kappa}_q \tag{4.8}$$

do not have contradictory signs if and only if the form

$$\sum_{p=1}^{n} c_q^p k_p \tag{4.9}$$

does not assume a value on the segment† $(-\kappa_q, -\bar{\kappa}_q)$.

Coupling the above with the theorem on existence of solutions in integers‡ we get

THEOREM 11. *The expressions* (4.8) *do not have opposite signs for integers* k *satisfying* (4.3) *if and only if the invariant factors of*

$$\left\| \begin{matrix} c_t^p & c_q^p \\ \kappa_t & \kappa \end{matrix} \right\| \text{ and } \left\| c_t^p \; c_q^p \right\| \qquad (p = 1, 2, \cdots, n;\ t = 1, 2, \cdots, q-1) \tag{4.10}$$

are the same for no value of κ *on the segment* $(\kappa_q, \bar{\kappa}_q)$.

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* O. Veblen and P. Franklin, loc. cit.

† By the term "segment (a, b)" we mean the set of integers x satisfying $a<x<b$ or $b<x<a$, according as $a<b$ or $b<a$.

‡ O. Veblen and P. Franklin, loc. cit., p. 10.

5. Equivalence in the case of a single unknown. Two matrices of cotes will be called *equivalent* if they establish exactly the same order relations among the derivatives.

Suppose two matrices $\|c\|$ and $\|\bar{c}\|$ ordering the derivatives of a single unknown are equivalent. When the derivatives D_i, D_j are compared, the differences (2.2) computed in the two systems of cotes are

$$\sum_{p=1}^{n} c_q^p k_p \qquad (q = 1, 2, \cdots, n), \tag{5.1}$$

$$\sum_{p=1}^{n} \bar{c}_q^p k_p \qquad (q = 1, 2, \cdots, n), \tag{5.2}$$

where

$$k_p = i_p - j_p. \tag{5.3}$$

A necessary condition for equivalence is that the first non-vanishing expression (5.1) have the same sign as the first non-vanishing expression (5.2), whatever integral values be given to the k's.

If the matrix

$$\|c_1^p \ \bar{c}_1^p\| \qquad (p = 1, 2, \cdots, n)$$

is of rank two, by Theorem 8 the first pair of forms (5.1) and (5.2) can be made to have opposite signs for integral values of the k's. Hence

$$\bar{c}_1^p = \lambda_1{}^1 c_1{}^p.$$

To proceed by induction we assume

$$\bar{c}_t^p = \sum_{\sigma=1}^{t} \lambda_t{}^\sigma c_\sigma^p \qquad (p = 1, 2, \cdots, n; t = 1, 2, \cdots, q-1). \tag{5.4}$$

If the matrix

$$\| c_1^p \cdots c_q^p \ \bar{c}_q^p \| \tag{5.5}$$

is of rank $q+1$, Theorem 8 shows the existence of integral values of the k's for which the first $q-1$ of forms (5.1) vanish and for which the qth forms (5.1) and (5.2) have opposite signs. Because of (5.4) the first $q-1$ of forms (5.2) also vanish for these k's. Thus the equivalence of the matrices is contradicted and the rank of (5.5) must be q. Therefore formulas (5.4) hold also for $t=q$, and the induction is complete.

Now for values of the k's making the first $q-1$ forms (5.1) zero, the qth form (5.2) reduces to

$$\lambda_q{}^q \sum_{p=1}^{n} c_q^p k_p.$$

Since its sign must not be opposite to that of the qth form (5.1), we conclude $\lambda_q^q \geqq 0$. The fact that the ordering is complete, that is, the rank of $\|\bar{c}\|$ is n, excludes the value zero; for if λ_q^q were zero, any k's making the first $q-1$ forms (5.2) zero would also make the qth zero.

If both c and $\bar{c}$ are regarded as given in (5.4), the equations of that system corresponding to a fixed value of the index t form a system of n linear equations in the t unknowns

$$\lambda_t^1, \lambda_t^2, \cdots, \lambda_t^t.$$

The matrix of these equations consists of the first t columns of $\|c\|$ and consequently is of rank t. The λ's in question can therefore be determined from t of the equations by Cramer's rule as rational functions of the integers c, $\bar{c}$. The λ's are therefore all rational. We cannot conclude, however, that they are integers.

A necessary condition for equivalence is accordingly that formulas (3.4) hold, in so far as they apply to the independent variables. The condition was previously known to be sufficient. Hence

THEOREM 12. *When there is only one unknown, two matrices of cotes are equivalent if and only if their elements are related by the formulas*

$$\bar{c}_q^p = \sum_{\sigma=1}^{q} \lambda_q^\sigma c_\sigma^p, \quad \lambda_q^q > 0 \qquad (p, q = 1, 2, \cdots, n). \tag{5.6}$$

6. **Canonical form for one unknown.** Consider the matrix $\|c\|$ for a single unknown. At least one of the elements on the first column is different from zero. Select the highest non-zero element, say c_1^a. Multiply the elements of the second column by $|c_1^a|$. Replace the second column by itself plus or minus c_2^a times the first, the sign being chosen so that in the new equivalent matrix $c_2^a=0$. In the same way $c_3^a, \cdots, c_n^a$ can all be made zero. Suppose this done.

In the modified matrix there is a non-zero element on the second column. Suppose the highest one is c_2^b. As above, we make $c_3^b = \cdots = c_n^b = 0$. The zeros already obtained on the ath row persist under this operation.

The process is repeated until the nth column contains a single non-zero element, the $(n-1)$th at most two, and so on.

Finally, any positive factor common to all the elements of a column is removed.

The resulting matrix is called the *canonical form* of the original. Suppose we have two equivalent matrices in canonical form. By Theorem 12 the elements of their first columns must be related by the formulas $\bar{c}_1^p = \lambda_1^1 c_1^p$, where λ_1^1 is known to be rational. Suppose it is in its lowest terms. Its numerator will have to be a divisor of all the integers $\bar{c}_1^p$. Since these numbers are relatively prime, the numerator must be unity. In the same way the denominator is a factor of the relatively prime integers c_1^p and is consequently unity. Therefore $\lambda_1^1 = 1$ and the first columns are identical.

A consequence of this is that exactly the same row will have had $n-1$ of its elements reduced to zero in the two matrices. For the purposes of the present proof there is no loss of generality in supposing this common row is the first, a situation which can be realized by a change in notation. Hence we assume

$$c_a{}^1 = \bar{c}_a{}^1 = 0 \qquad (a = 2, 3, \cdots, n).$$

From the equivalence and Theorem 12 we have

$$\bar{c}_2{}^1 = \lambda_2{}^1 c_1{}^1 + \lambda_2{}^2 c_2{}^1,$$

whence by substitution and use of the fact that $c_1^1 \neq 0$ we get $\lambda_2^1 = 0$. By use of (5.6) we find

$$\bar{c}_2{}^p = \lambda_2{}^2 c_2{}^p,$$

and as before we conclude the identity of the columns. Since the first non-zero elements on the second columns occupy the same position in the two matrices, we may asume that $c_a^2 = \bar{c}_a^2 = 0$ for $a = 3, 4, \cdots, n$.

To complete the proof by induction, assume that

$$\bar{c}_t{}^p = c_t{}^p \qquad (p = 1, 2, \cdots, n;\ t = 1, 2, \cdots, q-1), \tag{6.1}$$

relations equivalent to

$$\lambda_t{}^t = 1,\ \lambda_t{}^p = 0 \qquad (t = 1, 2, \cdots, q-1;\ p < t), \tag{6.2}$$

and that

$$\bar{c}_a{}^t = c_a{}^t = 0 \qquad (t = 1, 2, \cdots, q-1;\ t < a). \tag{6.3}$$

In accordance with (6.1) we may assume that the highest non-zero element on the tth column is c_t^t. From (5.6) we have

$$\bar{c}_q^p = \sum_{\sigma=1}^{q} \lambda_q{}^\sigma c_\sigma^p. \tag{6.4}$$

Make in these relations $p = 1, 2, \cdots, q-1$ successively, and use (6.2), (6.3). There results

$$\overset{a}{\lambda}_q = 0, \qquad a < q.$$

These values introduced in (6.4) give

$$\bar{c}_q^p = \lambda_q^q c_q^p,$$

whence as before we conclude the identity of the two columns. Moreover, we may consistently assume that c_q^q is the highest non-zero element on the qth column, so that (6.3) hold for $t=q$. The induction is therefore complete.

THEOREM 13. *Two matrices of cotes ordering the derivatives of a single function are equivalent if and only if they are identical when reduced to canonical form.*

As an example, consider the case of three independent variables, choosing the first cotes all equal to unity, as is customary in applications.* The reduction to canonical form gives either

$$
\begin{array}{cccc}
x & 1 & 0 & 0 \\
y & 1 & p & 0 \\
z & 1 & q & \pm 1
\end{array}
\tag{6.5}
$$

or

$$
\begin{array}{cccc}
x & 1 & 0 & 0 \\
y & 1 & 0 & \pm 1 \\
z & 1 & \pm 1 & 0,
\end{array}
\tag{6.6}
$$

where p and q are relatively prime. The orderings as p and q take on all possible values are all distinct.

Of course (6.6) can be obtained from (6.5) by putting $p=\pm 1$, $q=0$ and *changing the order of the variables*, but (6.5) and (6.6) are not equivalent for any values of p, q.

7. **Canonical form for matrix ordering unknowns alone.** Suppose the unknowns arranged in a definite order. If we attribute to the first unknown the cote 0, to the second the cote 1, $\cdots$, to the rth the cote $r-1$, the unknowns are arranged in the given order by any matrix with its first column composed of the integers $0, 1, \cdots, r-1$ written in the appropriate order, and the other columns anything we wish. We shall call the single column whose formation is described above the *canonical form* for any matrix ordering the unknowns in the given manner.

If the matrix does not completely order all the unknowns, we may still give it a canonical form by assigning 0 cote to all the unknowns whose relative order is indeterminate but which precede all the rest, etc. We mean in this case by equivalence that the orderings are not only consistent but equally complete. Obviously we have

* Riquier, loc. cit., p. 207, footnote 2.

THEOREM 14. *Two matrices ordering unknowns alone are equivalent if and only if identical when reduced to canonical form.*

Consider now in a complete matrix a column in which the cotes of the independent variables are all zero. The corresponding difference (2.2) is the same for u_α, u_β as for $D_i u_\alpha$, $D_j u_\beta$, that is, only the relative order of the unknowns is of any consequence. Hence the column can be put in the canonical form for unknowns.

If several adjacent columns contain nothing but zeros in the places corresponding to the independent variables, they likewise can be replaced by a single column in the canonical form for a matrix of cotes alone. We may therefore replace a given matrix by an equivalent matrix in which no two adjacent columns have zeros in all places corresponding to the independent variables.

8. **Two additional transformations preserving order.** Let c be a positive integer. We write

$$\gamma_q^\alpha = c[\gamma_q^\alpha/c] + g_q^\alpha,$$

where [ ] denotes the "greatest integer in." The numbers g_q^α are non-negative and less than c. We introduce here the abbreviations

$$\kappa_q = \gamma_q^\alpha - \gamma_q^\beta, \quad \bar{\kappa}_q = \bar{\gamma}_q^\alpha - \bar{\gamma}_q^\beta, \tag{8.1}$$

which will be useful throughout the rest of the paper. For present purposes we in addition put

$$\bar{\gamma}_q^\alpha = c[\gamma_q^\alpha/c].$$

The difference $\kappa_q - \bar{\kappa}_q$, being equal to the difference of two non-negative numbers both less than c, is numerically less than c. Since $\bar{\kappa}_q$ is divisible by c, there is no multiple of c on the segment $(-\kappa_q, -\bar{\kappa}_q)$.

Suppose c is a factor of all the c_q^p on the qth column. Since any value assumed by the form (4.9) is divisible by c, Theorem 10 shows that the expressions (4.8) never have contradictory signs. Hence the qth cotes of the matrices

$$\left\| \begin{matrix} \cdots & \overset{p}{c_q} & \overset{p}{c_{q+1}} & \cdots \\ \cdots & \gamma_q^\alpha & \overset{\alpha}{\gamma_{q+1}} & \cdots \end{matrix} \right\| \tag{8.2}$$

and

$$\left\| \begin{matrix} \cdots & \overset{p}{c_q} & \overset{p}{c_q} & \overset{p}{c_{q+1}} & \cdots \\ \cdots & c[\gamma_q^\alpha/c] & c[\gamma_q^\alpha/c] + g_q^\alpha & \overset{\alpha}{\gamma_{q+1}} & \cdots \end{matrix} \right\| \tag{8.3}$$

are such that the expressions (4.8) have the same sign, if they are both different from zero.

If the first expression in (4.8) is zero, the number c must be a divisor of κ_q, since it is a divisor of c_q^p. It must therefore be a divisor of

$$\kappa_q - \bar{\kappa}_q = g_q^\alpha - g_q^\beta .$$

The difference on the right, being numerically less than c, must be zero. Hence $\kappa_q = \bar{\kappa}_q$, and the second expression (4.8) is also zero. The burden of the decision is thus thrown upon the $(q+1)$th and succeeding cotes in (8.2) and the $(q+2)$th and succeeding cotes in (8.3). As these series of cotes are identical, the decision they render is the same.

If the first expression in (4.8) is not zero and the second is zero, the decision is made by the qth column of (8.2) and the $(q+1)$th of (8.3). Since these columns are identical, the decision is the same.

Hence the two matrices are equivalent.

By two rather obvious transformations (8.3) becomes

$$\left\| \begin{matrix} \cdots & c_q^p/c & 0 & c_{q+1}^p & \cdots \\ \cdots & [\gamma_q^\alpha/c] & g_q^\alpha & \gamma_{q+1}^\alpha & \cdots \end{matrix} \right\| .$$

We accordingly have

THEOREM 15. *If all the qth cotes of the independent variables have a positive factor c in common, without disturbing order relations the qth column can be replaced by two columns the first of which has for elements c_q^p/c and the greatest integers in γ_q^α/c, and the second, zeros in the places corresponding to the independent variables and the non-negative remainders from the divisions γ_q^α/c in the others.**

Theorem 15 can be applied to show the equivalence of matrices (3.7). Reducing the first cotes of the independent variables in the second matrix by the above principle gives

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array} .$$

Multiply the first column by 2 and add to the second:

$$\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array} .$$

* Riquier, loc. cit., p. 202, gives essentially this result for the first column.

When the first cotes of the unknowns are increased by unity, the third column, being identical with the first, can be omitted. The first matrix of (3.7) results.

A converse to Theorem 15 can also be proved.

THEOREM 16. *If all the $(q+1)$th cotes of the independent variables are zero, by modifying the qth column appropriately the $(q+1)$th can be suppressed without disturbing order relations.*

Let the matrix be

$$\overline{M} = \left\| \begin{matrix} \cdots & \bar{c}_q^p & 0 & \cdots \\ \cdots & \bar{\gamma}_q^\alpha & \bar{\gamma}_{q+1}^\alpha & \cdots \end{matrix} \right\|.$$

It is only essential that the elements on the $(q+1)$th column be made non-negative. For definiteness, however, we suppose that column in canonical form (§7). Let c be a positive integer satisfying

$$\bar{\gamma}_{q+1}^\alpha < c \qquad (\alpha = 1, 2, \cdots, r).$$

Again we make the unessential restriction that c be the least integer satisfying the condition. Consider the matrix with the single column

$$M = \left\| \begin{matrix} \cdots & c\bar{c}_q^p & \cdots \\ \cdots & c\bar{\gamma}_q^\alpha + \bar{\gamma}_{q+1}^\alpha & \cdots \end{matrix} \right\|$$

in place of the two in $\overline{M}$. Theorem 15 shows that M is equivalent to $\overline{M}$ and the theorem is proved.

9. **Reduced form for the general matrix.** If the rank of the matrix

$$\|c_t^p\| \qquad (p = 1, 2, \cdots, n;\ t = 1, 2, \cdots, q-1)$$

is the same as that of the matrix with one additional column $t=q$, rational multipliers λ which satisfy

$$\sum_{\sigma=1}^{q} \lambda_q^\sigma c_\sigma^p = 0 \qquad (p = 1, 2, \cdots, n)$$

can be found, the last λ_q^q being different from zero. Because of the homogeneity of this relation, the λ's can always be rendered integral and λ_q^q positive. By a transformation (3.4) the elements in the qth column of $\|c\|$ can therefore be replaced by zeros. We suppose this done wherever possible. The matrix $\|c\|$ may then have r columns of zeros.

By transformations (3.4) and the transformations of Theorem 15 the square matrix obtained from $\|c\|$ by disregarding the columns of zeros can be put in the canonical form for a single unknown.

If equations (4.3) have no solution in integers whatever distinct values be given α, β in the definition of κ_t (8.1), the qth cotes of the unknowns play no rôle. They can be replaced by zeros. We assume this has been done whereever possible.

Let the columns whose elements in $\|c\|$ are all zero be treated as described at the end of §7 with the result that no two of them are consecutive and that each is in the canonical form described in §7. In addition, if for such a column, say the qth, there is no pair of distinct values α, β for which (4.3) have a solution and $\kappa_q \neq 0$ simultaneously, the qth cotes of the unknowns are useless and can be replaced by zeros. We assume this has been carried out wherever possible.

Finally any column containing only zeros is to be omitted.

The resulting matrix will be called a *reduced form* of the original.

Consider two equivalent matrices M and $\overline{M}$ in reduced form and such that the first $q-1$ columns of $\|c\|$ and $\|\bar{c}\|$ are identical.

If there is a non-zero element on the qth columns of both $\|c\|$ and $\|\bar{c}\|$, those columns, being corresponding columns in the canonical form of two equivalent matrices for a single unknown, are identical by Theorem 13.

Let $\bar{c}_q^p = 0$ for all values of p. For any k's satisfying (4.3) the first q differences (2.2) formed for $\overline{M}$ are

$$\bar{\kappa}_1 - \kappa_1, \cdots, \bar{\kappa}_{q-1} - \kappa_{q-1}, \bar{\kappa}_q.$$

The expression (4.5) must not have sign contradictory to the first of these which is not zero. The last is known to be different from zero for some choice of α, β. With α, β so chosen, the sign of (4.5) is fixed. The cotes c_q^p are therefore linearly dependent on $c_1^p, \cdots, c_{q-1}^p$: if they were not, Theorem 9 would say that (4.5) could be made to change sign for integral k's satisfying (4.3). Since the columns in $\|c\|$ linearly dependent on those preceding have been replaced by zeros, we have $c_q^p = 0$ for all values of p. The same argument also shows that $\bar{c}_q^p = 0$ if $c_q^p = 0$.

In any case, therefore, the qth columns of $\|c\|$ and $\|\bar{c}\|$ are identical, and by induction we have proved

THEOREM 17. *A necessary condition for the equivalence of two matrices of cotes is that the portions of them corresponding to the independent variables be identical when they are put in reduced form.*

10. **Transformations of the reduced form.** Let V be any transformation which converts a matrix M into an equivalent matrix $\overline{M}$:

$$(10.1) \qquad V(M) = \overline{M}.$$

Suppose

$$(10.2) \qquad U(M) = R, \quad S(\overline{M}) = \overline{R},$$

where R and $\overline{R}$ are in reduced form. By Theorem 17 R and $\overline{R}$ have their corresponding matrices $\|c\|$ identical. Hence there exists a transformation T which affects the cotes of the unknowns alone and which sends R into $\overline{R}$:

$$T(R) = \overline{R}.$$

By substitution from (10.2) we get

$$TU(M) = S(\overline{M}),$$

or

$$S^{-1}TU(M) = \overline{M}.$$

Comparison with (10.1) gives

$$V = S^{-1}TU.$$

Hence to complete the determination of all transformations preserving order it will suffice to consider the transformations of the cotes of the unknowns alone, the matrix being assumed in reduced form.

Let R and $\overline{R}$ be two equivalent matrices in reduced form. Consider any distinct pair α, β in the definition (8.1). Suppose that equations (4.3) have integral solutions, but that none of these makes (4.5) vanish. For such values of the k's the first q differences (2.2) formed for $\overline{R}$ can be written

$$(10.3) \qquad \bar{\kappa}_1 - \kappa_1, \bar{\kappa}_2 - \kappa_2, \cdots, \kappa_{q-1} - \bar{\kappa}_{q-1}, \sum_{p=1}^{n} c_q^p k_p + \bar{\kappa}_q.$$

The sign of (4.5) must not be opposite to that of the first non-vanishing difference in the sequence (10.3).

First case, $c_q^p \neq 0$ for some p. The quantities c_q^p are linearly independent of c^p, $\cdots$, c_{q-1}^p, and by Theorem 9 expression (4.5) can be made to change sign for values of the k's satisfying (4.3). Hence we have

$$(10.4) \qquad \bar{\kappa}_1 - \kappa_1 = \bar{\kappa}_2 - \kappa_2 = \cdots = \bar{\kappa}_{q-1} - \kappa_{q-1} = 0.$$

A further necessary condition on $\bar{\kappa}_q$ is furnished by Theorem 11.

If the value of $\bar{\kappa}_q$ is such that values of the k's satisfying (4.3) make the last expression (10.3) vanish, expression (4.5), which becomes $\kappa_q - \bar{\kappa}_q$ for the values in question, must not have opposite sign to

$$\sum_{p=1}^{n} c_{q+1}^{p} k_p + \bar{\kappa}_{q+1}. \tag{10.5}$$

Theorem 9 forces us to conclude $c_{q+1}^{p}=0$ for all values of p. If $\bar{\kappa}_{q+1}$ is also zero, then

$$\sum_{p=1}^{n} c_{q+2}^{p} k_p + \bar{\kappa}_{q+2} \tag{10.6}$$

must not have opposite sign to $\kappa_q - \bar{\kappa}_q$, which is not zero because (4.5) does not vanish. Since adjacent columns of $\|c\|$ cannot consist solely of zeros, $c_{q+2}^{p}\neq 0$ for some p. Theorem 9 shows that (10.6) can be made to change sign for values of k making the first $q+1$ differences (2.2) zero. It is therefore impossible for $\bar{\kappa}_{q+1}$ to be zero, and the sign of $\bar{\kappa}_{q+1}$ is the same as that of $\kappa_q - \bar{\kappa}_q$.

To summarize, the conditions are (10.4) and that stated in Theorem 11. The invariant factors of the two matrices (4.9) can become the same for $\kappa = \bar{\kappa}_q$ only if $c_{q+1}^{p}=0$ for all values of p, and in such a case there is the further condition that $\bar{\kappa}_{q+1}^{p}$ is not zero and has the same sign as $\kappa_q - \bar{\kappa}_q$. These conditions are also sufficient for equivalence, so far as comparison of derivatives of the unknowns u_α, u_β is concerned, because they assure that the first nonvanishing difference in the sequence (2.2) has the same sign for the two matrices. The quantities $\bar{\kappa}_{q+2}, \cdots$ are unrestricted, as is also κ_{q+1} except in the special case noted above

Second case, $c_q^p = 0$ *for all values of* p. In this case, $c_{q-1}^{p}\neq 0$ for some p. By Theorem 9 we can find integral k's satisfying the first $q-2$ of equations (4.3) and for which

$$\sum_{p=1}^{n} c_{q-1}^{p} k_p + \kappa_{q-1} \tag{10.7}$$

has arbitrary sign. Since (10.7) must not have opposite sign to the first nonzero expression in the sequence

$$\bar{\kappa}_1 - \kappa_1,\ \bar{\kappa}_2 - \kappa_2,\ \cdots,\ \bar{\kappa}_{q-2} - \kappa_{q-2},\ \sum_{p=1}^{n} c_{q-1}^{p} k_p + \bar{\kappa}_{q-1}, \tag{10.8}$$

we conclude as before that

$$\bar{\kappa}_1 - \kappa_1 = \bar{\kappa}_2 - \kappa_2 = \cdots = \bar{\kappa}_{q-2} - \kappa_{q-2} = 0, \tag{10.9}$$

and that the condition of Theorem 11 with q replaced by $q-1$ holds.

Moreover, by considering values of the k's satisfying all of (4.3) we find that $\bar{\kappa}_{q-1} - \kappa_{q-1}$ cannot have sign opposite to κ_q, which cannot be zero from the definition of q.

If there are integers k satisfying the first $q-2$ of equations (4.3) and making the last expression (10.8) vanish, the first of the following expressions

$$\bar{\kappa}_q, \quad \sum_{p=1}^{n} c^p_{q+1} k_p + \bar{\kappa}_{q+1}$$

which does not vanish for these values must not have sign opposite to the first of the following,

$$\kappa_{q-1} - \bar{\kappa}_{q-1}, \; \kappa_q,$$

which does not vanish. Theorem 9 and the facts that $\kappa_q \neq 0$ and $c^p_{q+1} \neq 0$ show that $\bar{\kappa}_q \neq 0$ in this case. Hence $\bar{\kappa}_q$ has the sign of $\kappa_{q-1} - \bar{\kappa}_{q-1}$, unless the latter expression vanishes. In the exceptional case the hypothesis under which we are working (i.e., that the last expression (10.8) vanishes) is surely fulfilled and $\bar{\kappa}_q$ has the same sign as κ_q.

To summarize, the conditions are (10.9) and Theorem 11 with q replaced by $q-1$. Further, if $\bar{\kappa}_{q-1} - \kappa_{q-1}$ is zero, $\bar{\kappa}_q$ has the sign of κ_q; the expression $\bar{\kappa}_{q-1} - \kappa_{q-1}$, if not zero, has the same sign as κ_q, and, whenever $\bar{\kappa}_q$ is of significance, opposite sign to $\bar{\kappa}_q$. These necessary conditions are readily seen to be sufficient as far as comparison of derivatives of u_α, u_β is concerned. The $\bar{\kappa}$'s not mentioned are subjected to no restriction.

Any transformation satisfying the restrictions given above for all pairs α, β will preserve order.

The number q defined above, being a function of α and β, might be written $q(\alpha, \beta)$. The same is true of κ.

As an example, consider the matrix

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}, \tag{10.10}$$

which is in reduced form. For it,

$$q(2, 3) = q(3, 1) = q(1, 2) = 4.$$

The matrix

$$
(10.11)\qquad
\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0\\
1 & 2 & 0 & 0 & 0\\
1 & 3 & 1 & 0 & 0\\
1 & 2 & 2 & 0 & 1\\
\hline
0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0\\
0 & 0 & 1 & 2 & 0
\end{array}
$$

is also in reduced form, and can be obtained from (10.10) by the process described under case two. For it, however,

$$q(2, 3) = 4, \quad q(3, 1) = q(1, 2) = 3.$$

This illustrates the fact that $q(\alpha, \beta)$ may decrease by unity under case two, and is therefore not an invariant.

We may, of course, pass from (10.11) to (10.10) by the process of case one, and $q(3, 1)$, $q(1, 2)$ are seen to increase by unity.

We saw above, under case one, that $\bar{\kappa}_{q+1}$ can not be zero when all the expressions

$$\sum_{p=1}^{n} c_t^p k_p + \bar{\kappa}_t \qquad (t = 1, 2, \cdots, q)$$

vanish. Hence $q(\alpha, \beta)$ cannot increase by more than unity. It cannot decrease by more than unity, because if it did, it would increase by more than unity under the inverse transformation.

11. **Canonical form for the general matrix.** Consider a matrix in reduced form. By a transformation of the preceding section make the difference $\gamma_1^1 - \gamma_1^2$ numerically as small as possible. With this value fixed, make $\gamma_1^1 - \gamma_1^3$ numerically as small as possible. And so on, for each of the differences

$$\gamma_1^1 - \gamma_1^2, \gamma_1^1 - \gamma_1^3, \cdots, \gamma_1^1 - \gamma_1^r$$

in turn. Then treat the differences

$$\gamma_2^1 - \gamma_2^2, \gamma_2^1 - \gamma_2^3, \cdots, \gamma_2^1 - \gamma_2^r$$

successively in like manner. And so on.

The differences of the cotes of the unknowns are thus determined, provided we agree to choose the positive value when a difference can be made numerically least with either sign. We render the cotes themselves determinate by specifying that the algebraically least on each column be made zero by a transformation (3.1).

Any column containing only zeros is to be omitted.

The matrix finally obtained will be called the *canonical form* of the original. Since in reduction to canonical form we single out a matrix among all those equivalent to the given one by means of a certain minimum property, we evidently have

THEOREM 18. *Two matrices of cotes are equivalent if and only if identical when reduced to canonical form.*

When a matrix of cotes has been put in the above canonical form, the submatrix ordering the derivatives of any single unknown is of course in the canonical form of §6. Moreover, we can prove that any column whose elements in $\|c\|$ are all zero is in the canonical form of §7. To do this, suppose the ath column of the $\|c\|$ of a matrix in canonical form consists solely of zeros. Let that column be reduced by the process of §7. Suppose the differences

$$\gamma_a^1 - \gamma_a^2, \cdots, \gamma_a^1 - \gamma_a^b \tag{11.1}$$

are unchanged in the process, and that $\gamma_a^1 - \gamma_a^{b+1}$ is changed into $\bar{\gamma}_a^1 - \bar{\gamma}_a^{b+1}$. Now the reduction of §7 has the property of making all the differences which it changes numerically smaller. Hence

$$|\bar{\gamma}_a^1 - \bar{\gamma}_a^{b+1}| < |\gamma_a^1 - \gamma_a^{b+1}| .$$

In reducing the matrix as a whole to canonical form, however, when the differences (11.1) and those preceding them have been fixed at their final values, of all equivalent matrices we pick one for which $|\gamma_a^1 - \gamma_a^{b+1}|$ is least. Hence there is a contradiction, and all the differences of the ath cotes

$$\gamma_a^1 - \gamma_a^2, \cdots, \gamma_a^1 - \gamma_a^r$$

are the same in the two matrices. Since the algebraically least cote in both cases is zero, the ath cotes are identical, and the desired result is established.

As an example, consider the following reduced form in two unknowns and two independent variables:

$$\begin{array}{cccc} x & 1 & 0 & 0 \\ y & 1 & 1 & 0 \\ \hline u & a & b & 1 \\ v & 0 & 0 & 0 \end{array} . \tag{11.2}$$

There is a single distinct pair α, β. The differences (2.2) are

$$k_1 + k_2 + a, \; k_2 + b, \; 1 .$$

Hence $q=3$, and since $c_3^{\phi}=0$, we are under the second case of §10. κ_2 can be changed. Since the invariant factors of $\|c\|$ are both unity, the maximum change in κ_2 is unity, and the new $\bar{\kappa}_3$ is necessarily of significance. Consequently there are two possibilities: $\bar{\kappa}_2-\kappa_2$ has same sign as $+1$ and opposite sign to $\bar{\kappa}_3$; or $\bar{\kappa}_2-\kappa_2=0$ and $\bar{\kappa}_3$ has same sign as $+1$. If b is negative, the canonical form is therefore

$$\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline a & b+1 & 0 \\ 0 & 0 & 1 \end{array};$$

if b is positive, the canonical form is (11.2).

12. **Normal form.** If the transformation described in the discussion of Theorem 16 be applied to the canonical form, the columns containing nothing but zeros in $\|c\|$, other than the first, can be suppressed. The resulting matrix will be called the *normal form* of the original. The reason for making the process in Theorem 16 uniquely defined becomes apparent, and we have

THEOREM 19. *Two matrices of cotes are equivalent if and only if identical when reduced to normal form.*

It is to be noted that when the whole matrix is in normal form, the matrix $\|c\|$ has the same properties as for the canonical form *except that the cotes of the same column are not necessarily relatively prime.*

The normal form contains $n+1$ or n columns according as the first column of $\|c\|$ consists solely of zeros or not. Hence we have

THEOREM 20. *Any ordering effected by a matrix of cotes with the first cotes of the independent variables all equal to unity can be accomplished by a matrix with columns equal in number to the independent variables.*

Thus the normal form of

$$\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & r-1 & 0 & \cdots & 0 \end{array}$$

is

$$\begin{array}{cccc} r & 0 & \cdots & 0 \\ r & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r & 0 & \cdots & 1 \\ \hline 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r-1 & 0 & \cdots & 0 \end{array}.$$

Duke University,
Durham, N. C.