

A PROOF OF THE GENERALIZED SECOND-LIMIT THEOREM IN THE THEORY OF PROBABILITY*

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Introduction. A function $F(x)$, defined for all real x , will be called a “law of probability,” if the following conditions are satisfied:

- (i) $F(x)$ is monotone non-decreasing in $(-\infty, \infty)$ and continuous to the left,
- (ii) $F(-\infty) = 0, F(\infty) = 1$. †

A particular case is represented by $dF(x) = f(x)dx$, where $f(x)$, summable and ≥ 0 , is the “probability density” or “law of distribution” for x .

The expression $\int_{-\infty}^{\infty} x^s dF(x)$ is called the “ s th moment” of the distribution, s taking values $0, 1, 2, \dots$.

The Second Limit-Theorem, which was the starting point of this paper, can be stated, with A. Markoff, ‡ as follows:

If a sequence of laws of probability $F_k(x)$ ($k = 1, 2, \dots$) is such that they admit moments of all orders, and if

$$\lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} x^s dF_k(x) = \pi^{-1/2} \int_{-\infty}^{\infty} x^s e^{-x^2} dx \quad (s = 0, 1, \dots),$$

then, for all x ,

$$\lim_{k \rightarrow \infty} \int_{-\infty}^x dF_k(x) = \pi^{-1/2} \int_{-\infty}^x e^{-x^2} dx.$$

Markoff’s proof is rather complicated, being based on the distribution of roots and other properties of Hermite polynomials, also on the so-called Tchebycheff inequalities in the theory of algebraic continued fractions. He points out that the theorem still holds if we replace the law of probability $\pi^{-1/2} \int_{-\infty}^x e^{-x^2} dx$ by a more general one: $\int_{-\infty}^x f(x) dx$ (in which case, however, his considerations need many supplements). §

* Presented to the Society, April 18, 1930; received by the editors August 22, 1930.

† In fact, if X is a fortuitous variable (finite, not necessarily bounded), and if $F(x)$ is the probability that $X < x$, then $F(x)$ will satisfy these conditions, provided we assume that the principle of total probabilities still holds for a countable infinity of inconsistent events.

‡ A. Markoff, *Theory of Probability*, 4th edition (1924, in Russian), p. 522.

§ Cf. J. Chokhate, *Sur la convergence des quadratures mécaniques dans un intervalle infini . . .*, Comptes Rendus, vol. 186 (1928), pp. 344–346.

The same theorem has recently attracted the attention of many investigators: R. von Mises,* G. Pólya,† Paul Lévy,‡ Cantelli,§ Jacob|| and others.

The object of this paper is (a) to establish a general limit-theorem, removing many restrictions imposed otherwise on the functions involved and their moments, so that the above statement dealing with the law of probability $\pi^{-1/2} \int_{-\infty}^x e^{-x^2} dx$ (we shall call it hereafter the "classical case") is therein included as a very special case; (b) to give an elementary proof, which does not use either characteristic functions or algebraic continued fractions, being based on a well known Montel-Helly theorem concerning sequences of monotonic functions.

A brief account will first be given of the "moments-problem" to which the theorem in question is closely related.

1. The moments-problem. *Given a certain interval (a, b), finite or infinite, and an infinite sequence of real constants c₀, c₁, . . . , find a function ψ(x), non-decreasing in (a, b), ¶ such that*

$$\int_a^b x^s d\psi(x) = c_s \quad (s = 0, 1, \dots).$$

We call this the moments-problem corresponding to the data {c_s}. We may assume, without loss of generality, ψ(a) = 0.

It is known that if (a, b) be finite, then the moments-problem cannot have more than one solution (if it has any),** if we generally agree to consider as

* R. von Mises, *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 4 (1919), pp. 1-97.

† G. Pólya, *Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung . . .*, Mathematische Zeitschrift, vol. 8 (1920), pp. 171-181.

‡ Paul Lévy, *Calcul des Probabilités*, Paris, 1925, Chapter IV.

§ F. P. Cantelli, *Una nuova dimostrazione del secondo teorema-limite . . .*, Rendiconti di Palermo, vol. 52 (1928), pp. 151-174.

|| Jacob, *De l'application des intégrales généralisées de Fourier au calcul des probabilités*, Comptes Rendus, vol. 188 (1929), pp. 541-43, 754-56.

¶ The case of ψ(x) having but a finite number of points of increase in (a, b) is trivial.

** A simple proof is the following. The existence of two solutions ψ₁(x), ψ₂(x) implies

$$\int_a^b x^s dF(x) = 0 \quad (s = 0, 1, \dots; F(x) = \psi_1 - \psi_2; F(a) = 0),$$

$$F(b) = F(a) \quad (\text{for } s = 0), \int_a^b x^l F(x) dx = 0 \quad (l = 0, 1, \dots; \text{integration by parts}).$$

The latter relations lead to the required conclusion: F(x) = 0 at all points of continuity in (a, b), by the following reasoning due to Stieltjes (*Correspondance d'Hermite et Stieltjes*, Paris, 1905, pp. 337-338). If such a point z exists (a < z < b), where F(z) ≠ 0, then 1 - (x - z)²/M > 0 (a ≤ x ≤ b), for a sufficiently large M; hence, it is easily seen that

$$I = \int_a^b F(x) [1 - (x - z)^2/M]^n dx,$$

identical two solutions $\psi_1(x)$, $\psi_2(x)$ which coincide at all points of continuity.* We express this property by saying that the moments-problem for a finite interval is "determined."

On the other hand, the moments-problem for an infinite interval may be "indeterminate," i.e., it may admit infinitely many solutions. In fact, in the formula

$$\int_0^{\infty} y^{a-1} e^{-by} dy = \frac{\Gamma(a)}{b^a}$$

take

$$b = k + di \quad (k > 0), \quad a = (n + 1)/\lambda, (2n + 1)/\lambda \quad (n = 0, 1, \dots),$$

$$\frac{d}{k} = \tan \lambda\pi, \quad \tan \frac{\mu\pi}{2} \quad (\lambda, \mu \text{ defined below}), \quad y = x^\lambda,$$

and we get functions having all moments = 0:†

$$\int_0^{\infty} x^n e^{-\kappa x^\lambda} \sin(\kappa x^\lambda \tan \lambda\pi) dx = 0 \quad (\kappa > 0, 0 < \lambda < \frac{1}{2}),$$

$$\int_{-\infty}^{\infty} x^n e^{-\kappa x^\mu} \cos\left(\kappa x^\mu \tan \frac{\mu\pi}{2}\right) dx = 0$$

$$\left(\kappa > 0, 0 < \mu = \frac{2s}{2s + 1} < 1, s \text{ a positive integer}\right).$$

Hence we get infinitely many non-decreasing functions

$$(1) \quad \int_0^x e^{-\kappa x^\lambda} [1 + h \sin(\kappa x^\lambda \tan \lambda\pi)] dx,$$

$$\int_{-\infty}^x e^{-\kappa x^\mu} \left[1 + h \cos\left(\kappa x^\mu \tan \frac{\mu\pi}{2}\right)\right] dx \quad (-1 \leq h \leq 1),$$

for n very large, is different from zero, which is impossible, I being a linear combination of the moments of $F(x)$, all of which vanish. (We notice that such M does not exist for (a, b) infinite.) Moreover, if $\psi_i(x)$ is continuous to the left, then $\psi_1(x) = \psi_2(x)$ everywhere in (a, b) , since $\psi_i(x-0) = \lim \psi_i(X)$, where $X(<x)$ converges to x , being always a point of continuity of $\psi_i(x)$, $i=1, 2$.

* Also at the points a, b , if (a, b) be finite; this, however, necessarily follows from the relations

$$\psi_{1,2}(a) = 0, \quad \int_a^b d\psi_1(x) = \int_a^b d\psi_2(x) = c_0.$$

† These have been given by Adamoff (*Proof of a theorem of Stieltjes*, Proceedings of the Kazan Mathematical Society (1911, in Russian)) and by Stekloff (*Application de la théorie de fermeture . . .*, Mémoires de l'Académie des Sciences, Petrograd, vol. 33 (1915)), but the original statement is due to Stieltjes (loc. cit., p. 230).

solutions of the same moments-problem for $(0, \infty)$ and $(-\infty, \infty)$ respectively. Either of the following conditions ensures the determined character of the moments-problem for an infinite interval:

$$(2) \quad \sum_{n=1}^{\infty} c_{2n}^{-1/(2n)} \text{ diverges}^* \quad \left(c_n = \int_{-\infty}^{\infty} x^n dF(x) \right);$$

$$(3) \quad dF(x) = p(x)dx \quad (p(x) \geq 0 \text{ on } (a, b))$$

with $p(x) < M|x|^{\alpha-1}e^{-\kappa|x|^\lambda}$ for $|x| \geq x_0$, sufficiently large (M, α, κ are positive constants); $\lambda \geq \frac{1}{2}$ for $(a, b) = (0, \infty)$, $\dagger \lambda \geq 1$ for $(a, b) = (-\infty, \infty)$. \ddagger

On the other hand, the moments-problem is indeterminate if $d\psi(x) = p(x)dx$, and for $|x|$ sufficiently large (see (1))

$$p(x) > e^{-\kappa|x|^\lambda} \quad (\kappa > 0) \text{ with } \lambda < \frac{1}{2} \text{ for } (0, \infty), \lambda < 1 \text{ for } (-\infty, \infty).$$

2. A generalized statement of the second limit-theorem. *Given a sequence of laws of probability $F_n(x)$ ($n=1, 2, \dots$), with the following properties: (i) the moments $m_n^{(r)} = \int_{-\infty}^{\infty} x^r dF_n(x)$ of all orders $r=0, 1, \dots$ exist for $n=1, 2, \dots$, or at least from a certain rank n on (possibly depending on r); (ii) the quantities $m_1^{(r)}, m_2^{(r)}, \dots$, for any $r=0, 1, \dots$, lie, when they exist, between two fixed limits independent of n (but possibly dependent on r). Then a subsequence $\{C_i(x) \equiv F_{n_i}(x)\}$ ($i=1, 2, \dots; n_1 < n_2 < \dots; n_i \rightarrow \infty$) can be extracted such that $(\alpha) \lim_{i \rightarrow \infty} \int_{-\infty}^{\infty} x^r dC_i(x)$ exists ($=m_r$), \S for $r=0, 1, \dots$; (β) the subsequence $\{C_i(x)\}$ converges for any x to one fixed law of probability $\psi(x)$, save, perhaps, at its points of discontinuity; $(\gamma) \int_{-\infty}^{\infty} x^r d\psi(x)$ exists and $=m_r$ ($r=0, 1, \dots$). \parallel*

The proof will be arranged in several steps.

3. Existence of m_r ($r=1, 2, \dots$). We apply here the classical "diagonal method." The hypothesis of the uniform boundedness of $\{m_n^{(r)}\}$ for all $r=1, 2, \dots$, enables us to extract from the sequence $\{m_n^{(1)}\}$ a subsequence $\{m_{p_i}^{(1)}\}$ converging to a finite limit m_1 . The sequence $\{m_{p_i}^{(2)}\}$ gives rise to a subsequence $\{m_{q_i}^{(2)}\}$ converging to a finite limit m_2 , and so on. We thus get a sequence $\{m_{p_1}^{(r)}, m_{q_2}^{(r)}, \dots\}$ converging to a finite limit m_r , for any

* T. Carleman, *Sur les équations intégrales singulières . . .*, Uppsala, 1923, p. 219.

\dagger Stieltjes, *Recherches sur les fractions continues*, Oeuvres, vol. II, pp. 402-559, where (3) is given for $(0, \infty)$ only. The corresponding condition for $(-\infty, \infty)$ follows directly.

\ddagger It can be easily shown that, if $x_0 > 1$ and if, in $(1, x_0)$, $p(x)$ is bounded in the sense of Lebesgue, i.e., disregarding a set of zero measure, then (3) is included in (2). Furthermore, if $\psi_1(x), \psi_2(x)$ are solutions of a determined moments-problem for the interval (a, ∞) ($a=0, -\infty$), we can arrange so as to have $\psi_1(x) = \psi_2(x)$ everywhere in (a, ∞) , since $\psi_1(x) = \psi_2(x)$ at all points of continuity, it being permissible, as above, to take $\psi_i(x)$ continuous to the left ($i=1, 2$).

\S m_0 exists and $=1$, by definition of law of probability.

\parallel It follows that there is necessarily at least one solution of the moments-problem corresponding to the data $\{m_r\}$.

$r=1, 2, \dots$. The corresponding laws of probability $\gamma_1(x) \equiv F_{p_1}(x)$, $\gamma_2(x) \equiv F_{q_2}(x), \dots$ clearly satisfy (α) . The reasoning still holds if none of the $F_n(x)$ has all of its moments finite.

4. Existence of a limiting law of probability $\psi(x)$. This follows by applying to $\{F_n(x)\}$ the Montel-Helly* theorem on monotonic functions. We state it in a slightly generalized form:

If a family $\{f(x)\}$ of functions, non-decreasing on $(-\infty, \infty)$, is uniformly bounded in any finite interval (i.e. $|f(x_0)| < A(x_0)$ at any finite point x_0 , $A(x_0)$ being the same for all $f(x)$), then from any infinite sequence of this family we can extract a subsequence which converges, for any x , to a non-decreasing function. Moreover, the convergence is uniform in any interval, where the limit-function is continuous.

The theorem holds, with proper modifications, for families of functions of bounded variation.

In order to prove (β) , it suffices to apply this theorem to the sequence $\{\gamma_i(x)\}$, since $0 \leq \gamma_i(x) \leq 1$ for $-\infty \leq x \leq \infty$. We extract then from it a sequence $\{C_p(x)\}$ converging everywhere to a non-decreasing function $\phi(x)$, and we take $\psi(x) \equiv \phi(x-0)$.

The following remarks are important:

(i) The limit-function $\psi(x)$ of the $\{C_p(x)\}$ varies *effectively* from 0 to 1, i.e.

$$\psi(-\infty) = 0, \quad \psi(\infty) = \int_{-\infty}^{\infty} d\psi(x) = 1. \dagger$$

(ii) For any $r=0, 1, \dots$, the convergence of

$$\lim_{a \rightarrow -\infty, b \rightarrow \infty} \int_a^b x^r dF_n(x) \text{ to } \int_{-\infty}^{\infty} x^r dF_n(x)$$

is uniform with respect to n , at least from a certain rank n on.

(iii) $\lim_{x \rightarrow \infty} x^s(1 - F_n(x)) = 0$, $\lim_{x \rightarrow -\infty} |x|^s F_n(x) = 0$ ($n = 1, 2, \dots$; $s > 0$ arbitrary).

(i) follows from the inequalities

$$(4) \quad \int_b^{\infty} x^r dF_n(x) \leq \frac{m_n^{(2r+2)}}{b^{r+2}} \quad (b > 1); \quad \int_{-\infty}^a x^r dF_n(x) \leq \frac{m_n^{(2r+2)}}{|a|^{r+2}} \quad (a < -1),$$

* (a) P. Montel, *Sur les suites infinies des fonctions*, Annales de l'École Normale Supérieure, 1907; (b) E. Helly, *Über lineare Funktionaloperationen*, Sitzungsberichte der Akademie der Wissenschaften, Wien, vol. 121 (1912), pp. 265-297.

† This is by no means obvious. Take, for example, $F_n(x) = 0$ ($x < -n$), $= \frac{1}{2}(-n \leq x \leq n)$, $= 1$ ($x > n$). Here $\lim_{n \rightarrow \infty} F_n(x) = \psi(x) \equiv \frac{1}{2}$ for $-\infty \leq x \leq \infty$.

which, applied to $\{C_p(x)\}$, yield, for $r=0$ and $p \rightarrow \infty$,

$$(5) \quad (0 \leq) 1 - \psi(b) \leq \frac{1 + m_2}{b^2}, \quad 0 \leq \psi(a) \leq \frac{1 + m_2}{a^2} \quad (b, -a > 1),$$

and this proves (i) by letting $b \rightarrow \infty, a \rightarrow -\infty$.

(ii) follows directly from (4), taking into account the uniform boundedness of $\{m_n^{(2r+2)}\}$ ($n=1, 2, \dots$).

In order to establish (iii), we write

$$1 - F_n(b) = \int_b^\infty dF_n(x) \leq \int_b^\infty \left(\frac{x}{b}\right)^{2r} dF_n(x); \quad b^s [1 - F_n(b)] \leq \frac{m_n^{(2r)}}{b^{2r-s}}$$

($b > 0, 0 < s < 2r$),

and a similar expression for $|a|^s F_n(a)$ ($a < -1$).

5. $\int_{-\infty}^\infty x^r d\psi(x)$ exists and $=m_r$ ($r=0, 1, \dots$). This statement being the fundamental part of the theorem, we give for it two proofs.

Proof I. We apply the two following theorems due respectively to Hamburger* and to Helly,† the proofs of both of which are very elementary.

HAMBURGER'S THEOREM. Suppose (i) $\int_{-\infty}^\infty [1/(z+x)]d\psi(x)$ converges for $z=iy$ with $y>0$, $\psi(x)$ denoting a function non-decreasing in $(-\infty, \infty)$; (ii) $F(z) \equiv \int_{-\infty}^\infty [1/(z+x)]d\psi(x)$ has, for $z=iy \rightarrow \infty$, an asymptotic expansion (in Poincaré's sense) $F(z) \sim \sum_{\nu=1}^\infty (-1)^\nu c_\nu / z^{\nu+1}$ (c_ν real). Then $\int_{-\infty}^\infty x^\nu d\psi(x)$ exists and $=c_\nu$ ($\nu=0, 1, \dots$).

HELLY'S THEOREM. Given a sequence $\{V_n(x)\}$ of functions of bounded variation on a finite interval (a, b) such that (i) the total variations of all $V_n(x)$ on (a, b) are uniformly bounded; (ii) $\lim_{n \rightarrow \infty} V_n(x) = v(x)$ ‡ exists for $a \leq x \leq b$, with the possible exception of a countable set of points not including a, b . Then $\lim_{n \rightarrow \infty} \int_a^b f(x) dV_n(x) = \int_a^b f(x) dv(x)$, for any continuous function $f(x)$.

Going back to the given sequence $\{F_n(x)\}$ and to the above function $\psi(x) = \lim_{p \rightarrow \infty} \{C_p(x)\}$, we notice, first, that $\int_{-\infty}^\infty f(x) d\psi(x), \int_{-\infty}^\infty f(x) dF_n(x)$ certainly exist if $f(x)$ is bounded on $(-\infty, \infty)$ and continuous on any finite interval. Furthermore, since, as we have seen, $\int_{-\infty}^\infty dC_p(x)$ converges uniformly (with respect to p), an easy application of Helly's theorem yields

* H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, I, II, III, *Mathematische Annalen*, vols. 81-82 (1920), pp. 235-319, 120-64, 168-87.

† Loc. cit.

‡ Necessarily of bounded variation, by virtue of the Montel-Helly theorem extended to this class of functions.

$$(6) \quad \lim_{p \rightarrow \infty} \int_{-\infty}^{\infty} f(x) dC_p(x) = \int_{-\infty}^{\infty} f(x) d\psi(x),$$

$$(7) \quad \lim_{p \rightarrow \infty} \int_{-\infty}^{\infty} \frac{dC_p(x)}{z+x} = \int_{-\infty}^{\infty} \frac{d\psi(x)}{z+x} \quad (z = iy; y > 0).$$

Consider now the expression

$$(8) \quad f_p(z) \equiv \int_{-\infty}^{\infty} \frac{dC_p(x)}{z+x} = \sum_{r=0}^{p-1} (-1)^r \frac{\alpha_p^{(r)}}{z^{r+1}} + \frac{(-1)^p}{z^{p+1}} I_{p,p} \quad (p \text{ arbitrary}),$$

$$I_{r,p} = \int_{-\infty}^{\infty} x^r \left(\frac{z}{z+x} \right) dC_p(x), \quad \alpha_p^{(r)} = \int_{-\infty}^{\infty} x^r dC_p(x).$$

Letting $p \rightarrow \infty$, using (7) and the property

$$\lim_{p \rightarrow \infty} \alpha_p^{(r)} = m_r \quad (r = 0, 1, \dots),$$

we get

$$(9) \quad F(z) \equiv \int_{-\infty}^{\infty} \frac{d\psi(x)}{z+x} = \sum_{r=0}^{p-1} \frac{(-1)^r m_r}{z^{r+1}} + \frac{(-1)^p}{z^{p+1}} \lim_{p \rightarrow \infty} I_{p,p}.$$

Observing that $|z/(z+x)| \leq 1$ ($-\infty \leq x \leq \infty$, $z = iy$, $y \geq y_0 > 0$), we get

$$|I_{2s,p}| \leq \alpha_p^{(2s)};$$

$$|I_{2s-1,p}| \leq \left| \int_{-1}^1 x^{2s-1} dC_p(x) \right| + \int_1^{\infty} x^{2s} dC_p(x) + \int_{-\infty}^{-1} x^{2s} dC_p(x)$$

$$\leq 1 + \alpha_p^{(2s)},$$

$$\left| \lim_{p \rightarrow \infty} I_{2s-\sigma,p} \right| \leq \lim_{p \rightarrow \infty} (1 + \alpha_p^{(2s)}) = 1 + m_{2s} \quad (\sigma = 0, 1; s = 1, 2, \dots),$$

$$(10) \quad \lim_{z=iy \rightarrow \infty} \left| z^r \left[F(z) - \sum_{r=0}^{p-1} (-1)^r \frac{m_r}{z^{r+1}} \right] \right| = \lim_{z=iy \rightarrow \infty} \left| \frac{1}{z} \lim_{p \rightarrow \infty} I_{p,p} \right| = 0.$$

Formula (10), where ν is arbitrary, gives the asymptotic expansion of $F(z)$:

$$F(z) \sim \sum_{r=0}^{\infty} (-1)^r \frac{m_r}{z^{r+1}} \quad (z = iy \rightarrow \infty).$$

Hence, Hamburger's theorem is applicable and proves (γ).

Proof II. We restrict ourselves to the domain of real numbers, making use of the following extension of Helly's theorem to the infinite interval $(-\infty, \infty)$.

Given a sequence $\{v_n(x)\}$ defined on $(-\infty, \infty)$ such that (i) $v_n(x)$ is of bounded variation on any finite interval, (ii) all $v_n(x)$ and their total variations are uniformly bounded on any finite interval, (iii) $\lim_{n \rightarrow \infty} v_n(x) = v(x)$ exists for all x , with the possible exception of a countable set of points, (iv) $\int_a^b f(x) dv_n(x)$ converges uniformly (with respect to n) to $\int_a^b f(x) dv(x)$ ($a \rightarrow -\infty, b \rightarrow \infty$), if $f(x)$ is continuous everywhere (not necessarily uniformly). Then $\int_{-\infty}^{\infty} f(x) dv(x)$ exists and $= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) dv_n(x)$.

We notice, first, that $v(x)$ is of bounded variation (see footnote on page 538) on any finite interval, secondly (by virtue of Helly's theorem), that

$$\left| \int_b^{b'} f(x) dv(x) \right| = \left| \lim_{n \rightarrow \infty} \int_b^{b'} f(x) dv_n(x) \right| < \epsilon$$

($bb' > 0, n, |b|, |b'|$ sufficiently large; $\epsilon > 0$ arbitrarily small); hence, $\int_{-\infty}^{\infty} f(x) dv(x)$ exists. Furthermore,

$$\begin{aligned} \Delta_n = \left| \int_{-\infty}^{\infty} f dv - \int_{-\infty}^{\infty} f dv_n \right| &\leq \left| \int_{-\infty}^a f dv \right| + \left| \int_{-\infty}^a f dv_n \right| + \left| \int_b^{\infty} f dv \right| \\ &+ \left| \int_b^{\infty} f dv_n \right| + \left| \int_a^b f dv - \int_a^b f dv_n \right| \quad (a < 0, b > 0) \end{aligned}$$

can be made as small as we please by taking $-a, b$, and then n sufficiently large. Hence $\lim_{n \rightarrow \infty} \Delta_n = 0$.

Remark. *The new condition (iv) is essential.* The following example shows that if (iv) is not satisfied, the theorem may not hold: $f(x) = x, v_n(x) = 0 (x \leq 0), 1 - 1/2^n (0 < x \leq 4^n), 1 (x > 4^n)$. Here

$$\begin{aligned} \lim_{n \rightarrow \infty} v_n(x) &= v(x) = 0 (x \leq 0), 1 (x > 0); \\ \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} x dv_n(x) &= \lim_{n \rightarrow \infty} 4^n \cdot \frac{1}{2^n} = \infty \neq \int_{-\infty}^{\infty} x dv(x) = 0. * \end{aligned}$$

It suffices to apply the above theorem, with $f(x) = x^r (r = 0, 1, \dots)$, to the above sequence $\{C_p(x)\}$ which satisfies all four conditions stated, and (γ) is established.

6. Special case. A direct corollary is the following

THEOREM. *If $\lim_{n \rightarrow \infty} m_{n^r}$ exists ($= m_r$) for $r = 0, 1, \dots$, then at least one fixed law of probability, say $F(x)$, exists such that m_r is its r th moment ($r = 0, 1, \dots$), and a subsequence $\{C_i(x) \equiv F_{n_i}(x)\}$ can be extracted from the given*

* $v_n(x), v(x)$ have each a single saltus $= 1/2^n, 1$ at $x = 4^n, 0$ respectively.

sequence $\{F_n(x)\}$ of laws of probability such that $\lim_{i \rightarrow \infty} C_i(x) = F(x)$ for any x . If, in addition, the $\{m_r\}$ are such that the corresponding moments-problem is determined, then the sequence $\{F_n(x)\}$ itself converges, for $n \rightarrow \infty$, to $F(x)$ at any point of continuity of $F(x)$.

We need a proof for the last part only. Assume that a point x_0 of continuity of $F(x)$ exists such that $\{F_n(x_0)\}$ does not converge to $F(x_0)$. Hence, a subsequence $\{C_k(x_0) \equiv F_{n_k}(x_0)\}$ can be extracted such that $C_k(x_0)$ converges, for $k \rightarrow \infty$, to a certain number $A \neq F(x_0)$. On the other hand, we have seen that the sequence $\{C_k(x)\}$ gives rise to a subsequence $\{d_i(x)\}$ which, for any x , converges, as $i \rightarrow \infty$, to a function $d(x)$, having the same moments $m_r (r=0, 1, \dots)$ as $F(x)$, and therefore, since, by hypothesis, the moments-problem corresponding to $\{m_r\}$ is determined,

$$(11) \quad \lim_{i \rightarrow \infty} d_i(x_0) = d(x_0) = F(x_0)$$

($F(x)$ being continuous at $x = x_0$), which is impossible, $\{d_i(x_0)\}$ being a subsequence of $\{C_k(x_0)\}$ which converges, but not to $d(x_0)$. We have seen also ((i), p. 537) that $F(-\infty) = 0, F(\infty) = 1$.

The condition that the moments-problem corresponding to $\{m_r\}$ be determined is not only sufficient for the validity of the limiting relation

$$\lim_{n \rightarrow \infty} F_n(x) = F(x)$$

(at any point of continuity of $F(x)$),* but it is also necessary. For if $F(x)$ and $\phi(x)$ be two distinct solutions of the moments-problem in question, then $F_n(x)$ should converge simultaneously to $F(x)$ and $\phi(x)$ at all points of continuity, while at least one such point x_0 exists where $F(x_0) \neq \phi(x_0)$.†

7. The classical case: $m_r = \pi^{-1/2} \int_{-\infty}^{\infty} x^r e^{-x^2} dx (r=0, 1, \dots)$. Here

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} x^r dF_n(x) = \pi^{-1/2} \int_{-\infty}^{\infty} x^r e^{-x^2} dx \quad (r = 0, 1, 2, \dots)$$

implies

$$\lim_{n \rightarrow \infty} \int_{-\infty}^x dF_n(x) = \pi^{-1/2} \int_{-\infty}^x e^{-x^2} dx \quad (x \text{ arbitrary}).$$

* Even everywhere in $(-\infty, \infty)$, $F(x)$ being a law of probability, hence continuous to the left (see Introduction).

† The conditions imposed by different writers on the quantities $\{m_r\}$ are such as to ensure the determined character of the corresponding moments-problem. In fact, one sees readily that the conditions of R. von Mises, Pólya, and Cantelli (loc. cit.)

$$m_{2n} \leq C \left(\frac{n}{c^2 e} \right)^n \quad (C, c = \text{const.}), \lim_{n \rightarrow \infty} m_{2n}^{1/(2n)} / n \text{ is finite, } m_{2n} \frac{[\psi(2n)]^{2n}}{2n!} < 1 \quad (n > n_1; \psi(n)_{n \rightarrow \infty} \rightarrow \infty)$$

are but special cases of Carleman's condition (2).

In fact, the moments-problem corresponding to $\{m_r\}$ is determined (by virtue of (2) or (3)), and

$$F(x) = \pi^{-1/2} \int_{-\infty}^x e^{-x^2} dx$$

is continuous for all x .

We see that *the classical case is but a very special case of the general second limit-theorem.*

8. The determined character of the moments-problem in the classical case. The conditions (2, 3) ensuring the determined character of the moments-problem have been established by means of very profound, but also complicated, considerations (continued fractions, singular integral equations). It seems of interest to give an *elementary* proof involving a simple theorem of Pólya.*

We wish to prove the following

THEOREM. *Given a law of probability $\psi_1(x)$ such that*

$$(12) \quad m_{2r} = \int_{-\infty}^{\infty} x^{2r} d\psi_1(x) = \Gamma\left(\frac{2r+1}{\lambda}\right) / \Gamma\left(\frac{1}{\lambda}\right) = \frac{\lambda}{2\Gamma(1/\lambda)} \int_{-\infty}^{\infty} x^{2r} e^{-|x|^\lambda} dx,$$

$$m_{2r+1} = \int_{-\infty}^{\infty} x^{2r+1} d\psi_1(x) = 0 = \frac{\lambda}{2\Gamma(1/\lambda)} \int_{-\infty}^{\infty} x^{2r+1} e^{-|x|^\lambda} dx$$

($\lambda \geq 1$; $r = 0, 1, \dots$).

Then necessarily

$$\psi_1(x) = \frac{\lambda}{2\Gamma(1/\lambda)} \int_{-\infty}^x e^{-|z|^\lambda} dz$$

for any x . In other words, $\psi_1(x)$ is *uniquely determined* by (12).

Assume the existence of two such functions $\psi_1(x)$ and $\psi_2(x)$. Employing the reasoning of §1 (footnote on page 534) and using property (iii), page 537.

$$\lim_{x \rightarrow \infty} x^s (1 - \psi_i(x)) = \lim_{x \rightarrow -\infty} |x|^s \psi_i(x) = 0 \quad (i = 1, 2; s > 0 \text{ arbitrary}),$$

we conclude that

$$(13) \quad \int_{-\infty}^{\infty} x^l F(x) dx = 0 \quad (l = 0, 1, \dots; F = \psi_1 - \psi_2 = (1 - \psi_2) - (1 - \psi_1)).$$

* G. Pólya, *Über den Gaussischen Fehlergesetz*, *Astronomische Nachrichten*, vol. 208 (1919), No. 4981, pp. 185-192.

On the other hand, a reasoning similar to that of §4 leads, making use of (12) and the asymptotic expression for the Γ function, to

$$(14) \quad |F(x_0)| < \frac{2m_{2n}}{x_0^{2n}} \quad (n \text{ very large, } x_0 \text{ arbitrary}),$$

$$|F(x_0)| < C e^{(-1/2)|x_0|^\lambda} \quad \left(\left(\frac{2n}{\lambda} \right)^{2/\lambda+1/(2n)} < x_0^2 < \left(\frac{4n}{\lambda} \right)^{2/\lambda} \right),$$

$C = \text{const. independent of } n \text{ and } x_0.$

Therefore,

$$\int_{-\infty}^{\infty} |F(x)| e^{c|x|} dx \quad (0 < c < \frac{1}{2})$$

exists. But this is precisely the condition imposed by Pólya,* which leads to the required conclusion: $F(x) \equiv 0$ at all points of continuity.†

* Loc. cit. p. 187. For the classical case ($\lambda=2$) cf. M. H. Stone, *Developments on Hermite polynomials*, Annals of Mathematics, vol. 29 (1927), pp. 1-13.

† After the present paper had been prepared for publication, we came across an interesting article by A. Wintner: *Über den Konvergenzsatz der mathematischen Statistik*, Mathematische Zeitschrift, vol. 28 (1928), pp. 470-480, some of the results of which are similar to those obtained above.