ON DEFINITIONS OF BOUNDED VARIATION FOR FUNCTIONS OF TWO VARIABLES*

BY

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1. **Introduction.** Several definitions have been given of conditions under which a function of two or more independent variables shall be said to be of bounded variation. Of these definitions six are usually associated with the names of Vitali, Hardy, Arzelà, Pierpont, Fréchet, and Tonelli respectively. A seventh has been formulated by Hahn and attributed by him to Pierpont; it does not seem obvious to us that these two definitions are equivalent, and we shall give a proof of that fact.

The relations between these several definitions have thus far been very incompletely determined, and there would appear to have been misconceptions concerning them. In the present paper we propose to investigate these relations rather fully, confining our attention to functions of two independent variables.

We first (§2) give the seven definitions mentioned above and a list of the known relations among them. In §3 some properties of the classes of functions satisfying the several definitions are established. In § 4 we determine, for each pair of classes, whether one includes the other or they overlap. In §5 further relations are found concerning the extent of the common part of two or more classes. We next (§6) give a list of similar relations when only bounded functions are admitted to consideration; in §7 additional like relations are obtained when only continuous functions are admitted. We conclude (§8) with a list of the comparatively few relations that are not yet fully determined.

2. **Definitions.** The function $f(x, y)$ is assumed to be defined in a rectangle $R(a \leqq x \leqq b,\ c \leqq y \leqq d)$. By the term *net* we shall, unless otherwise specified, mean a set of parallels to the axes:

$$x = x_i\ (i = 0, 1, 2, \cdots, m), \quad a = x_0 < x_1 < \cdots < x_m = b;$$
$$y = y_j\ (j = 0, 1, 2, \cdots, n), \quad c = y_0 < y_1 < \cdots < y_n = d.$$

Each of the smaller rectangles into which R is divided by a net will be called a *cell*. We employ the notation

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$$\Delta_{11}f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j),$$
$$\Delta f(x_i, y_i) = f(x_{i+1}, y_{i+1}) - f(x_i, y_i).$$

The *total variation function*, $\phi(\bar{x})$ $[\psi(\bar{y})]$, is defined as the total variation of $f(\bar{x}, y)$ $[f(x, \bar{y})]$ considered as a function of $y[x]$ alone in the interval (c, d) $[(a, b)]$, or as $+\infty$ if $f(\bar{x}, y)$ $[f(x, \bar{y})]$ is of unbounded variation.

DEFINITION V (Vitali-Lebesgue-Fréchet-de la Vallée Poussin*). *The function $f(x, y)$ is said to be of bounded variation† if the sum*

$$\sum_{i=0,\, j=0}^{m-1,\, n-1} |\Delta_{11}f(x_i, y_j)|$$

is bounded for all nets.

DEFINITION F (Fréchet). *The function $f(x, y)$ is said to be of bounded variation if the sum*

$$\sum_{i=0,\, j=0}^{m-1,\, n-1} \epsilon_i\bar{\epsilon}_j\Delta_{11}f(x_i, y_j)$$

is bounded for all nets and for all possible choices of $\epsilon_i = \pm 1$ and $\bar{\epsilon}_j = \pm 1$.

DEFINITION H (Hardy-Krause). *The function $f(x, y)$ is said to be of bounded variation if it satisfies the condition of definition V and if in addition‡ $f(\bar{x}, y)$ is of bounded variation in y (i.e., $\phi(\bar{x})$ is finite) for at least one $\bar{x}$ and $f(x, \bar{y})$ is of bounded variation in x (i.e., $\psi(\bar{y})$ is finite) for at least one $\bar{y}$.*

DEFINITION A (Arzelà). *Let (x_i, y_i) $(i = 0, 1, 2, \cdots, m)$ be any set of points satisfying the conditions*

$$a = x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_m = b;$$
$$c = y_0 \leqq y_1 \leqq y_2 \leqq \cdots \leqq y_m = d.$$

Then $f(x, y)$ is said to be of bounded variation if the sum

$$\sum_{i=0}^{m-1} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

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* References to most of the authors mentioned here in connection with the various definitions are given by Hahn, *Theorie der Reellen Funktionen*, Berlin, 1921, pp. 539–547, or by Hobson, *Theory of Functions of a Real Variable*, 3d edition, vol. 1, Cambridge, 1927, pp. 343–347. We need supplement these only by Tonelli, *Sulla quadratura delle superficie*, Accademia dei Lincei, Rendiconti, (6), vol. 3 (1926), pp. 357–362.

† *In the rectangle R* is always to be understood.

‡ The definition H as originally formulated imposed the two latter conditions for *every* x and *every* y, respectively, but it was shown by W. H. Young that the three conditions were redundant and that the definition could be reduced to the form given here. See Hobson, loc. cit., p. 345.

DEFINITION P (Pierpont). *Let any square net be employed, which covers the whole plane and has its lines parallel to the respective axes. The side of each square may be denoted by D, and no line of the net need coincide with a side of the rectangle R. A finite number of the cells of the net will then contain points of R, and we may denote by ω_ν the oscillation of $f(x, y)$ in the νth of these cells, regarded as a closed region. The function $f(x, y)$ is said to be of bounded variation if the sum*

$$\sum_\nu D\omega_\nu$$

is bounded for all such nets in which D is less than some fixed constant.

DEFINITION P_H (Hahn's version of definition P). *Let any net be employed in which we have $m=n$ and $x_{i+1}-x_i=(b-a)/m$, $y_{i+1}-y_i=(d-c)/m$ $(i=0, 1, 2, \cdots, m-1)$. Then there are m^2 congruent rectangular cells and we may let ω_ν' stand for the oscillation of $f(x, y)$ in the νth cell, regarded as a closed region. The function $f(x, y)$ is said to be of bounded variation if the sum*

$$\sum_{\nu=1}^{m^2} \frac{\omega_\nu'}{m}$$

is bounded for all m.*

DEFINITION T (Tonelli). *The function $f(x, y)$ is said to be of bounded variation if the total variation function $\phi(\bar{x})$ is finite almost everywhere in (a, b), and its Lebesgue integral over (a, b) exists (finite), while a symmetric condition is satisfied by $\psi(\bar{y})$.*

It may be of interest to indicate briefly how a set of definitions seemingly so diverse came to be formulated. Definition V, perhaps the most natural analogue of that of bounded variation for a function of one variable, is sufficient to insure the existence of the Riemann-Stieltjes double integral $\int_a^b\int_c^d g(x, y)d_xd_yf(x, y)$ for every continuous function $g(x, y)$. The existence of this integral†, when $g(x, y)$ is the product of a continuous function of x and a continuous function of y, is also implied by condition F, which is weaker than V (see relation (3) below). Definition H singles out a class of functions which it is convenient to consider in the study of double Fourier series. Definition A, also a rather natural analogue of that of bounded variation for a function

* It may readily be proved that this is equivalent to assuming that there exists some infinite sequence of values of m, say $m_k(k=1, 2, 3, \cdots; m_{k-1}<m_k)$, with m_k/m_{k-1} bounded, for which this sum is bounded.

† In a certain restricted sense; see Fréchet, *Sur les fonctionnelles bilinéaires*, these Transactions, vol. 16 (1915), pp. 215–234, especially pp. 225–227. Several questions concerning these double integrals are considered in a forthcoming paper by Clarkson.

of one variable, expresses a condition necessary and sufficient that $f(x, y)$ be expressible as the difference of two bounded monotone functions.* Definition P, or P_H, is a natural extension to functions of two variables of the notion of *bounded fluctuation*, to use Hobson's terminology, which is equivalent to that of bounded variation for functions of one variable. Condition T is necessary and sufficient that the surface $z=f(x, y)$, where $f(x, y)$ is continuous, be of finite area in the sense of Lebesgue; this definition also is useful in connection with double Fourier series.

For simplicity we shall also use the letters V, F, H, A, P, P_H, and T to represent the classes of functions satisfying the respective definitions. The class of bounded functions will be denoted by B and the class of continuous functions by C; a product, such as $V \cdot T \cdot C$, will stand for the common part of the two or more classes named.†

The only relations that seem to be already known among the several definitions may be indicated as follows‡:

(1) $P_H > A > H$, (2) $A \cdot C > H \cdot C$, (3) $F > V > H$,

(4) $V \cdot C > H \cdot C$, (5) $T \cdot C > A \cdot C$, (6) $V \cdot T \cdot C = H \cdot C$.

3. **Some properties of functions belonging to these classes.**§ We first prove the following theorem.

THEOREM 1. *If $f(x, y)$ is in class H, the total variation function $\phi(\bar{x})$ $[\psi(\bar{y})]$ is of bounded variation in the interval (a, b) $[(c, d)]$.*||

Assume the contrary; then, given any $M>0$, there exists a set of numbers x_i $(i=0, 1, 2, \cdots, n)$ with

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* Monotone in the sense of Hobson, loc. cit., p. 343.

† From the definitions the following relations are easily seen: $V>V \cdot B$, $F>F \cdot B$, $T>T \cdot B$, $H=H \cdot B$, $A=A \cdot B$, $P=P \cdot B$, and $P_H=P_H \cdot B$.

‡ For a proof of the relation $A \geqq H$ see for example Hobson, loc. cit., pp. 345–346; the relation $A>H$ then follows from an example given by Küstermann, *Funktionen von beschränkter Schwankung in zwei reellen Veränderlichen*, Mathematische Annalen, vol. 77 (1916), pp. 474–481. Since Küstermann's example is continuous, it also gives us $A \cdot C>H \cdot C$. A proof of the relation $P_H>A$ is given by Hahn, loc. cit., pp. 546–547. From the definitions we clearly have $V \geqq H$, and the relations $V>H$ and $V \cdot C>H \cdot C$ may then be inferred from the example $f(x, y)=x \sin (1/x)(x \neq 0)$, $f(0, y)=0$. That F is $\geqq V$ is obvious from the definition; the definite inequality $F>V$ is established by Littlewood, *On bounded bilinear forms in an infinite number of variables*, Quarterly Journal of Mathematics, Oxford Series, vol. 1 (1930), pp. 164–174. The relations $T \cdot C>A \cdot C$ and $V \cdot T \cdot C=H \cdot C$ are stated by Tonelli, loc. cit.

§ Only properties of the total variation functions $\phi(\bar{x})$ and $\psi(\bar{y})$ are considered here; other properties will be examined in a forthcoming paper.

|| This property is not enjoyed by all functions of class A; indeed it is easily seen (compare example (C) below) that $f(x, y)$ may be in A and yet ϕ and ψ be everywhere discontinuous. It is clear that if $f(x, y)$ is in V, $\phi[\psi]$ is either everywhere infinite or of bounded variation.

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and such that

$$\sum_{i=1}^{n} | \phi(x_i) - \phi(x_{i-1}) | > M.$$

Consider any two successive points (x_{i-1}, c) and (x_i, c). From the definition of $\phi(\bar{x})$, there exists a set of points p_j on the line $x = x_i$, and their projections p_j' on the line $x = x_{i-1}$, such that we have

$$\left| \sum | f(p_j) - f(p_{j-1}) | - \sum | f(p_j') - f(p_{j-1}') | \right| \geqq | \phi(x_i) - \phi(x_{i-1}) | /2.$$

Hence for the net N composed of the boundary lines of the rectangle, the lines $x = x_{i-1}$ and $x = x_i$, and the horizontal lines through the points p_j, the V-sum is

$$V_N(f) = \sum_{i=0,j=0}^{m-1,n-1} | \Delta_{11} f(x_i, y_j) | \geqq | \phi(x_i) - \phi(x_{i-1}) | /2.$$

By a repetition of this process for each interval (x_{i-1}, x_i) we may prove the existence of a net N' for which the V-sum $V_{N'}(f)$ is $\geqq M/2$, thus contradicting the hypothesis that $f(x, y)$ is of class H.

Before proving Theorem 2 we demonstrate the following lemma, due essentially to Borel.†

LEMMA 1. *Let E be a bounded set of positive interior measure and let the sequence of functions $\{f_n(x)\}$ defined on E converge to the limit function $f(x)$ at each point of E. Then, if ϵ is any positive number and if $E_n(\epsilon)$ denotes the subset of E where*

$$f(x) - f_n(x) | > \epsilon,$$

we have

$$\lim_{n \to \infty} m_i E_n(\epsilon) = 0.$$

Since $m_i E_n(\epsilon)$ is the least upper bound of the measures of the measurable subsets of $E_n(\epsilon)$, it suffices to show that if $\{E_n'(\epsilon)\}$ is any sequence of measurable sets contained respectively in $\{E_n(\epsilon)\}$, then $\lim_{n\to\infty} mE_n'(\epsilon)$ is 0. This will be true if E^*, the complete limit of $\{E_n'(\epsilon)\}$, is of measure zero; but E^* is a null set, since the sequence $\{f_n(x)\}$ cannot converge at any point of E^*.

† See Borel, *Leçons sur les Fonctions de Variables Réelles*, Paris, 1905, p. 37, where essentially this lemma is indicated but not proved.

THEOREM 2. *If a function $f(x, y)$ is in class P_H, and E is the set of points $\bar{x}[\bar{y}]$ in the interval (a, b) $[(c, d)]$ for which $\phi(\bar{x})$ $[\psi(\bar{y})]$ is infinite, then m_iE is zero.*†

In particular, if E is measurable (as it would be if for example $f(x, y)$ were continuous; cf. Theorem 4), it is of measure zero.

To prove Theorem 2, assume $f(x, y)$ is in P_H and that E, the set of points $\bar{x}$ for which $\phi(\bar{x})$ is infinite, is of positive interior measure. On E define the sequence of functions $f_n(x)$ as follows. For a fixed n, let R be divided by a net N into n^2 congruent rectangles, and at the point $\bar{x}$ of E let

$$g_n(\bar{x}) = \sum_{i=1}^{n}[\text{oscillation of } f(\bar{x}, y) \text{ in the interval } y_{i-1} \leqq y \leqq y_i].$$

Let $f_n(\bar{x}) = 1/g_n(\bar{x})$. Then $\lim_{n\to\infty} f_n(x)$ is 0 at each point of E, and hence at each point of E', some arbitrarily selected measurable subset of E of positive measure.

Let $\epsilon > 0$ be given. By Lemma 1 there exists an r such that $m_iE_r(\epsilon)$ is $<\epsilon$, where $E_r(\epsilon)$ is the subset of E' on which $|f_r(x)|$ is $>\epsilon$. Consider the net N which defines $f_r(x)$. Let λ be the number of columns of N in which points of the set $E' - E_r(\epsilon)$ occur. We have

$$\text{(a)} \qquad m_e[E' - E_r(\epsilon)] \leqq \lambda(b - a)/r.$$

From the relations

$$m_e[E' - E_r(\epsilon)] = mE' - m_iE_r(\epsilon) \quad \text{and} \quad m_iE_r(\epsilon) < \epsilon$$

we have

$$m_e[E' - E_r(\epsilon)] > mE' - \epsilon,$$

and hence by (a)

$$\lambda > r(mE' - \epsilon)/(b - a).$$

But in each of the λ columns of N which contain points of the set $E' - E_r(\epsilon)$ the sum of the oscillations of $f(x, y)$ in the several cells is at least $1/\epsilon$. Hence for the net N we have

$$\frac{1}{r}\sum_{\nu=1}^{r^2}\omega_\nu' \geqq \lambda/(\epsilon r) > (mE' - \epsilon)/[\epsilon(b - a)].$$

Since mE' is >0, this last quantity increases indefinitely with $1/\epsilon$, while if $f(x, y)$ is in P_H the sum on the left must be bounded.

† If $f(x, y)$ is in A, $\phi[\psi]$ is clearly bounded.

The part of the theorem concerning $\psi(\bar{y})$ may of course be demonstrated in the same manner.

We may note here that *the set E of points $\bar{x}$ for which $\phi(\bar{x})$ is infinite may nevertheless be everywhere dense in the interval* (a, b), as the following example shows. Define the function $f(x, y)$ on the unit square $I(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$ as follows. Let the rational points of the segment $0 \leqq x \leqq 1$ be enumerated and designated by $x_1, x_2, x_3, \cdots, x_n, \cdots$. On each line $x = x_j$ define $f(x, y)$ as 1 for y irrational and $> 1 - 1/2^j$, and zero otherwise. When x is irrational let $f(x, y)$ be zero for all y. For convenience we denote by S_j the segment $1 - 1/2^j \leqq y \leqq 1$ of the line $x = x_j$.

Clearly $f(x, y)$ is of unbounded variation in y for each fixed rational x, and these points are everywhere dense in the interval $(0, 1)$. But $f(x, y)$ is in P_H. For consider any square net of n^2 cells on I. In all cells of such a net except for those which contain more than one point of some segment S_j, the oscillation is zero; in the remainder the oscillation is 1. Let M be the number of the latter. Then M is at most equal to $M_1 + M_2 + M_3 + \cdots + M_p + n$, where M_j is the number of cells containing more than one point of S_j, and p is the largest integer for which $1/2^p$ exceeds $1/n$. But M_j is less than $2 + n/2^{j-1}$; hence we have

$$\sum_{\nu=1}^{n^2} \omega'_\nu = M/n < 5$$

and $f(x, y)$ is in P_H.

As a preliminary to the proof of our third theorem we shall first establish another lemma.

Let A denote any set of k real numbers,

$$A: \qquad a_1, a_2, a_3, \cdots, a_k,$$

and let $\theta = \sum_{i=1}^{k} |a_i|$. With this set we may associate 2^k sums of the form

$$\pm a_1 \pm a_2 \pm a_3 \pm \cdots \pm a_k.$$

These sums occur in 2^{k-1} pairs, of opposite sign, $\pm S_j$ $(j = 1, 2, 3, \cdots, 2^{k-1})$, the subscripts being assigned arbitrarily. Let S_j $(j = 1, 2, 3, \cdots, 2^{k-1})$ be that one of the jth pair which is positive, or zero if each sum in the pair vanishes. Denote by $\sum A$ the sum $\sum_{j=1}^{2^{k-1}} S_j$.

LEMMA 2. *We have* $\sum A \geqq M_k \theta$, *where*

$$M_k = \begin{cases} (k-1)! \Big/ \left[\left(\dfrac{k-1}{2}\right)!\right]^2 & \text{for } k \text{ odd}, \\ k!/(2[(k/2)!]^2) & \text{for } k \text{ even}. \end{cases}$$

Since we shall make use of this result only for k odd, and a similar proof can be given for k even, we confine ourselves to the

Proof for k odd. Without loss of generality we may assume the a_i to be non-negative, since both θ and $\sum A$ are invariant under the change of sign of any a_i.

In the particular case in which all the a_i are equal we have $\sum A = M_k\theta$. For let $[S_j]_h$ $(h=0, 1, 2, \cdots, (k-1)/2)$ denote in this case the set of expressions of the form $\pm\theta/k\pm\theta/k\pm\theta/k\pm \cdots \pm\theta/k$ in which exactly h minus signs occur. Then each S_j in $[S_j]_h$ has the value $(k-2h)\theta/k$. In $[S_j]_h$ there will be exactly $\binom{k}{h}$ sums S_j. Hence, adding, we obtain $\sum A = M_k\theta$.

We wish to show that *in every case* $\sum A \geqq M_k\theta$. Let A be any set and let a' and a'' be any two elements of A. Let S_j' $(j=1, 2, 3, \cdots, n)$ be the 2^{k-3} sums obtained from the set composed of the remaining elements of A. Then the 2^{k-1} sums S_j may be written in an array of four columns thus:

$$\begin{array}{llll} S_1' + a' + a'', & |S_1' - a' - a''|, & |S_1' + a' - a''|, & |S_1' - a' + a''|, \\ S_2' + a' + a'', & |S_2' - a' - a''|, & |S_2' + a' - a''|, & |S_2' - a' + a''|, \\ \cdots & \cdots & \cdots & \cdots \\ S_n' + a' + a'', & |S_n' - a' - a''|, & |S_n' + a' - a''|, & |S_n' - a' + a''|. \end{array}$$

If we denote by C_1, C_2, C_3, and C_4 the sums of the respective columns, we have $\sum A = C_1+C_2+C_3+C_4$. By comparison with the sum obtained when absolute value signs are omitted from the third and fourth columns, we have at once

$$\sum A \geqq C_1 + C_2 + 2\sum_{j=1}^{n} S_j'.$$

But if in the set A we replace a' and a'' each by $(a'+a'')/2$ to form the set A', we see that $\sum A'$ is precisely the right-hand member of this inequality. Therefore, *if in a set A any two elements are each replaced by their arithmetic mean, $\sum A$ is not increased.*

Now assume the existence of a set A of k elements with $\sum_{i=1}^{k} a_i = \theta$ (and hence with arithmetic mean θ/k), and with $\sum A = M_k\theta - \delta$, where δ is some positive number. Let ξ be the absolute value of the greatest deviation from θ/k of any one a_i. There are a finite number of the a_i, then, whose deviation in absolute value exceeds $\xi/2$. Select one such and pair with it some element whose algebraic deviation is of the opposite sign. Replace each of these by half their sum. By repeating this operation we form the set A_1, with the same arithmetic mean θ/k, for which the greatest deviation from the mean does not exceed $\xi/2$, while $\sum A_1$ is $\leqq \sum A$. This process may be repeated as many times as we may desire, to yield a set A_p whose elements deviate from the mean by as little as we wish. But as $\sum A$ is evidently a continuous function of the elements of A, for sufficiently large p we must have both

$$\left| \sum A_p - M_k\theta \right| < \delta \text{ and } \sum A_p \leqq \sum A = M_k\theta - \delta.$$

From this contradiction follows Lemma 2 for k odd.

We may now prove

THEOREM 3. *If $f(x, y)$ is in class F, the total variation function $\phi(\bar{x})$ $[\psi(\bar{y})]$ is either everywhere infinite, or is bounded and integrable in the sense of Riemann* over the interval (a, b) $[(c, d)]$.*

Let $f(x, y)$ be in class F, and for some x_0 $(a \leqq x_0 \leqq b)$ let $\phi(x_0) = M_1$, a finite number. Consider the function $f'(x, y) = f(x, y) - f(b, y)$. Clearly $f'(x, y)$ is also in class F.

Let x_1 $(a \leqq x_1 \leqq b)$ be any number distinct from x_0, and let (x_1, y_i) $(i = 0, 1, 2, \cdots, n)$ be any set of $n+1$ points on the line $x = x_1$ with

$$c = y_0 < y_1 < y_2 < \cdots < y_n = d.$$

We have

$$f(x, y) = f(x_0, y) + f'(x, y) - f'(x_0, y),$$

whence

$$\begin{aligned}
&\sum_{i=1}^{n} \left| f(x_1, y_i) - f(x_1, y_{i-1}) \right| \\
&\quad = \sum_{i=1}^{n} \left| f(x_0, y_i) + f'(x_1, y_i) - f'(x_0, y_i) - f(x_0, y_{i-1}) - f'(x_1, y_{i-1}) \right. \\
&\qquad \left. + f'(x_0, y_{i-1}) \right| \\
&\quad \leqq \sum_{i=1}^{n} \left| f(x_0, y_i) - f(x_0, y_{i-1}) \right| + \sum_{i=1}^{n} \left| \Delta_{11} f'(x_0, y_{i-1}) \right| .
\end{aligned}$$

Since $f'(x, y)$ is in F, there exists a number M_2 such that we have

$$\sum_{i=0,\, j=0}^{m-1,\, n-1} \epsilon_i \bar{\epsilon} \Delta_{11} f'(x_i, y_j) < M_2$$

for any net. But $\sum_{i=1}^{n} \left| \Delta_{11} f'(x_0, y_{i-1}) \right|$ is the sum of the absolute values of the differences $\Delta_{11} f'(x_i, y_j)$ in one column of cells of the net composed of the

* It is easily seen from the proof that discontinuities of $\phi[\psi]$ can occur only at a denumerable set of points; indeed, for any $\epsilon > 0$, the number of points at which $\phi[\psi]$ has a saltus $> \epsilon$ is finite. It is appropriate to remark also that example (E) below shows that $\phi[\psi]$ may be bounded but not of bounded variation.

four vertical lines $x=a$, $x=x_0$, $x=x_1$, $x=b$ and the $n+1$ horizontal lines $y=y_i$ $(i=0, 1, 2, \cdots, n)$. Hence, a fortiori, it is less than M_2 and we have

$$\sum_{i=1}^{n} | f(x_1, y_i) - f(x_1, y_{i-1}) | < M_1 + M_2;$$

thus $\phi(\bar{x})$ is bounded by the latter number.

Now assume that $\phi(\bar{x})$ is bounded but not integrable in the Riemann sense. Then E, the set of points in the interval (a, b) at which $\phi(\bar{x})$ is discontinuous, must be of positive exterior measure. Let E_n be the subset of E such that at each point of E_n the saltus of $\phi(\bar{x})$ exceeds $2/n$. Then E is $\sum_{n=1}^{\infty} E_n$, and so for some fixed k we must have $m_e E_k > 0$. Let x_1 be any point of E_k and let Δ_1 be an interval of length not exceeding $\frac{1}{2} m_e E_k$, with center x_1. Within Δ_1 there must be a point x_1' such that $|\phi(x_1) - \phi(x_1')| > 1/k$. We may assume without loss of generality that $\phi(x_1)$ is $> \phi(x_1')$. Let m and M be any constants satisfying the inequalities

$$\phi(x_1') < m < M < \phi(x_1), \quad M - m > 1/k.$$

Then there exists a set of points on the line $x = x_1$,

$$p_0(x_1, c), p_1, p_2, \cdots, p_r(x_1, d),$$

and their horizontal projections q_i $(i=0, 1, 2, \cdots, r)$ on the line $x=x_1'$, such that we have

$$\sum_{i=1}^{r} | f(p_i) - f(p_{i-1}) | > M, \quad \sum_{i=1}^{r} | f(q_i) - f(q_{i-1}) | < m,$$

and hence

$$\text{(b)} \qquad \sum_{i=1}^{r} | f(p_i) - f(p_{i-1}) - f(q_i) + f(q_{i-1}) | > M - m > 1/k.$$

Now consider the net N_1 on R consisting of the four vertical lines $x=a$, $x=x_1$, $x=x_1'$, $x=b$, and the $r+1$ horizontal lines through the points p_i $(i=0, 1, 2, \cdots, r)$. From (b) it is seen that the sum of the absolute values of the terms $\Delta_{11} f(x_i, y_j)$ associated with the single column of cells which stands on the interval (x_1, x_1') exceeds $1/k$.

Since the length of Δ_1 is $\leqq m_e E_k/2$, there is a point x_2 of E_k exterior to Δ_1. Surround x_2 with an interval Δ_2 of which it is the center, of length not exceeding $m_e E_k/4$ and small enough so that it does not overlap Δ_1. Proceeding as before, we prove the existence of a second net N_2 of which one column of cells possesses the property that the sum of the absolute values of the terms $\Delta_{11} f(x_i, y_j)$ associated with it exceeds $1/k$. It is clear that the net composed of all the lines in both N_1 and N_2 has two distinct columns of cells each possessing this property.

This process may be repeated indefinitely, and so we see that there exists a $\theta>0$ such that, given any integer k, there exists a net on R in at least k columns of which the sum of the absolute values of the differences $\Delta_{11}f(x_i, y_j)$ in the several cells of the column exceeds θ. We proceed to show that under these conditions the sum

$$F_N(f) = \sum_{i=0,\, j=0}^{m-1,\, n-1} \epsilon_i \bar{\epsilon}_j \Delta_{11}f(x_i, y_j)$$

may be made arbitrarily large by proper choice of the net N and the ϵ_i's and $\bar{\epsilon}_j$'s.

Let k (taken odd for convenience) be given, and let N be a net, of n rows and m columns of cells, such that in at least k columns the above condition is satisfied. Consider the matrix $\|a_{ij}\|$ for which $a_{ij}=\Delta_{11}\, f(x_{i-1}, y_{j-1})$ and in which all the a_{ij} but those arising from the k columns noted above are suppressed. This matrix has, then, n rows and k columns; renumbering the columns consecutively, we have

$$\|a_{ij}\| = \left\| \begin{matrix} a_{1n} & a_{2n} & \cdots & a_{kn} \\ \cdot & \cdot & \cdot & \cdot \\ a_{12} & a_{22} & \cdots & a_{k2} \\ a_{11} & a_{21} & \cdots & a_{k1} \end{matrix} \right\|,$$

with $\sum_{j=1}^{n} |a_{ij}|>\theta$ $(i=1, 2, 3, \cdots, k)$.

If it can be shown that the sum

$$F' = \sum_{i=1,\, j=1}^{k,\, n} \delta_i\bar{\delta}_j a_{ij} \qquad (|\delta_i| = |\bar{\delta}_j| = 1)$$

for some choice of the δ_i's and $\bar{\delta}_j$'s is arbitrarily large with k, the proof will be complete, since $\max F_N(f)$ is $\geqq \max F'$.

The δ_i's may be chosen in 2^{k-1} essentially distinct ways. Let S_{jp} represent the absolute value of the sum yielded by the jth row of $\|a_{ij}\|$ with the pth such choice, and let

$$F_p' = \sum_{j=1}^{n} S_{jp} \qquad (p = 1, 2, 3, \cdots, 2^{k-1}).$$

Then each F_p' is a particular value of F' corresponding to some choice of the δ_i's and $\bar{\delta}_j$'s. We may write

$$\sum_{p=1}^{2^{k-1}} F_p' = \sum_{j=1}^{n} \left[\sum_{p=1}^{2^{k-1}} S_{jp} \right],$$

and by Lemma 2 we have

$$\sum_{p=1}^{2^{k-1}} S_{jp} \geqq M_k\theta_j \ (j = 1, 2, 3, \cdots, n), \text{ where } \theta_j = \sum_{i=1}^{k} |\, a_{ij}\, |,$$

whence

$$\sum_{p=1}^{2^{k-1}} F_p' \geqq \sum_{j=1}^{n} M_k\theta_j \geqq M_k k\theta = \frac{k!}{\left[\left(\frac{k-1}{2}\right)!\right]^2}\,\theta,$$

since k is odd. *It follows that at least one F' exceeds*

$$\frac{k!}{\left[\left(\frac{k-1}{2}\right)!\right]^2\cdot 2^{k-1}}\,\theta.$$

By Stirling's formula we see that this quantity increases without limit with k; therefore the sum $F_N(f)$ can have no bound, contrary to the hypothesis that $f(x, y)$ is in class F. Thus Theorem 3 is proved.

THEOREM 4. *If $f(x, y)$ is in class C, then $\phi(\bar{x})$ and $\psi(\bar{y})$ are lower semi-continuous functions.**

Let the interval (c, d) be divided into 2^n equal parts by the numbers

$$y_0 = c,\ y_1,\ y_2, \cdots,\ y_{2^n} = d$$

and set

$$\phi_n(\bar{x}) = \sum_{i=1}^{2^n} |\, f(\bar{x}, y_i) - f(\bar{x}, y_{i-1})\, |\, .$$

Since $f(\bar{x}, y)$ is continuous in y, we have $\lim_{n\to\infty} \phi_n(\bar{x}) = \phi(\bar{x})$, and since $f(\bar{x}, y_i)$ $(i=0, 1, 2, \cdots, n)$ is continuous in $\bar{x}$, $\phi_n(\bar{x})$ is a continuous function of $\bar{x}$. Moreover the sequence $\{\phi_n(\bar{x})\}$ is non-decreasing; hence $\phi(\bar{x})$ is lower semi-continuous.† Similarly $\psi(\bar{y})$ is of like character.

4. **Relations between pairs of classes.** We shall establish the following:

(7) $P = P_H$, (8) $T > H$, (9) $V \ngtreqless P,\ P \ngtr V$,

(10) $A \ngtreqless V, V \ngtr A$, (11) $V \ngtreqless T,\ T \ngtr V$, (12) $A \ngtreqless T,\ T \ngtr A$,

(13) $P \ngtreqless T, T \ngtr P$, (14) $F > V$, (15) $F \ngtreqless A,\ A \ngtr F$,

(16) $F \ngtreqless P, P \ngtr F$, (17) $F \ngtreqless T, T \ngtr F$.

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* It is apparent from the proof that the hypothesis that $f(x, y)$ be continuous in each variable separately is sufficient here.

† See, for example, Hobson, loc. cit., 2d edition, vol. 2, 1926, p. 149.

Proof of (7). Let us first assume $f(x, y)$ to be in class P and consider *any* net of n^2 cells as in definition P_H. Without loss of generality we may suppose $d-c \geqq b-a$. Then there exists a net of square cells as used in definition P for which we have $D=(b-a)/n$ and whose vertical lines include $x=a$ and $x=b$. No square of the P net can overlap more than two cells of the P_H net; hence we have

$$\sum \omega_\nu' / n \leqq \frac{2}{b-a} \cdot \sum D\omega_\nu,$$

which is bounded. This establishes the relation $P \leqq P_H$.

Now assume $f(x, y)$ to be in P_H. Again let us suppose $d-c \geqq b-a$, and consider any square net N, as used in definition P, for which we have $D<(b-a)/2$. Let n be the largest integer satisfying the inequality $(b-a)/n \geqq D$; then we have

$$\frac{b-a}{n+1} < D \leqq \frac{b-a}{n} \quad \text{and} \quad \frac{b-a}{n} < 2\frac{b-a}{n+1} < 2D,$$

and therefore

$$\text{(c)} \qquad D \leqq (b-a)/n < 2D.$$

Consider now the net N' of n^2 cells as used in the P_H definition. From the second part of (c) it is seen that one cell of N' can overlap no more than three columns of cells of N. The height of one cell of N' is $(d-c)/n$, and if p is the smallest integer satisfying the inequality $p \geqq (d-c)/(b-a)$, we have

$$(d-c)/n \leqq p(b-a)/n < 2pD;$$

hence one cell of N' can overlap no more than $2p+1$ rows of cells of N. Thus we have

$$\sum D\omega_\nu \leqq 3(2p+1) \sum D\omega_\nu' \leqq 3(2p+1)(b-a) \sum \omega_\nu' / n,$$

which is bounded. This establishes the relation $P_H \leqq P$, and we conclude the identity of the two classes.

Proof of (8). It was shown in §3 that if $f(x, y)$ is of class H, then the total variation functions $\phi(\bar{x})$ and $\psi(\bar{y})$ are of bounded variation. From this follows at once $T \geqq H$. Then from the example*

$$\text{(A)} \qquad f(x, y) = \begin{Bmatrix} 0, & x < y \\ 1, & x \geqq y \end{Bmatrix} \text{ in I, the unit square,}$$

which is in T but not H, we infer (8).

* See Hahn, loc. cit., p. 547.

Proof of (9). The first part follows from example (A), which is in P but not V. Example

$$\text{(B)} \qquad f(x, y) = \begin{Bmatrix} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{Bmatrix} \text{ in I,}$$

which is in V but not P, establishes the second part.

Proof of (10). The first of these relations follows from the second of (9). By taking sets of points (x_i, y_i) along the perimeter of the rectangle R, one sees immediately that a function of class A must satisfy the two latter conditions of definition H. Since there exist functions which are in A but not H, and by the last remark these must fail to be in V, the second of relations (10) follows.

Proof of (11). The first of these relations is shown by example (A), which is in T but not V, while example (B) shows the second.

Proof of (12). Example (A) establishes the first relation. The second is a consequence of the following example.

(C) Let E be a non-measurable set in the interval $0 \leqq x \leqq 1$, and let E' be the set of points on the downward sloping diagonal of I whose projection on the x-axis is E. Define $f(x, y)$ as 1 at all points of E' and zero at all other points of I. Then clearly $f(x, y)$ is in A; but it is not in T, since $\phi(\bar{x})$ is not measurable.

Proof of (13). The first relation follows from example

$$\text{(D)} \qquad f(x, y) = \begin{Bmatrix} 1 \text{ for } x \text{ and } y \text{ both rational} \\ 0 \text{ otherwise} \end{Bmatrix} \text{ in I,}$$

which is in T but not P. The second follows from the second of (12).

Proof of (14). It has already been remarked that this relation, which is included in (3), has been established by Littlewood. His proof, however, depends upon the theory of bilinear forms in infinitely many variables; it may therefore be of interest to show how an example of a function which is in class F but not V can be constructed directly. Moreover, we can easily determine whether our example belongs to the classes P, A, and T; consequently it may be expected to be useful in proving other relations later.

We first make a preliminary observation. Consider a function $f(x, y)$ defined in R. For any net N let max $F_N(f)$ denote the maximum value which the sum

$$F_N(f) = \sum_{i=0,\,j=0}^{m-1,\,n-1} \epsilon_i \bar{\epsilon}_j \Delta_{11} f(x_i, y_j)$$

associated with N may be made to assume by a suitable choice of the ϵ_i's and $\bar{\epsilon}_j$'s. If an additional line, horizontal or vertical, be added to N to form the new net N', we have $\max F_N(f) \leqq \max F_{N'}(f)$. For, suppose a horizontal line be added. Then one row of cells of N is replaced by two new rows of cells of N'; and if the two $\bar{\epsilon}$'s associated with these rows in the sum $F_{N'}(f)$ be both assigned the same value as the $\bar{\epsilon}$ associated with the single replaced row in the sum $F_N(f)$, and all the remaining ϵ's and $\bar{\epsilon}$'s given identical values in the two sums, we have $F_N(f) = F_{N'}(f)$, from which the above observation follows.

By a "point-rectangle function" we shall mean a function $f(x, y)$ defined on R as follows: $f(x, y) = \pm 1$ (or some other constant) on each of a rectangular array of points p_{ij} in R, where the rows are equally spaced with each other and with the lines $y=c$, $y=d$, and the columns likewise, and p_{ij} is the point standing in the jth row and ith column of the array; $f(x, y) = 0$ at all other points of R. Let $\max F(f)$ denote the maximum value which the expression $F_N(f)$ can attain for all possible nets and choices of the ϵ's and $\bar{\epsilon}$'s, and $\max V(f)$ be the maximum value which the sum

$$V_N(f) = \sum_{i=0,\, j=0}^{m-1,\, n-1} \left| \Delta_{11} f(x_i, y_j) \right|$$

can attain with all possible nets N. We consider the problem of determining the value of $(\max F(f))/(\max V(f))$ for such a function.

Clearly $\max F(f)$ is attained by the use of a net consisting of one line through each row and column of points and one between every two rows and columns, together with the lines forming the boundary of R, since by our preliminary remark the net obtained by omitting any lines cannot yield a larger sum, and adding any line is extraneous as it merely introduces an additional row or column of cells each of which contributes zero to the sum. The position of the intermediate lines of the net is immaterial.

Let N, then, be such a net on R, and consider next the problem of choosing the ϵ's and $\bar{\epsilon}$'s so that $F_N(f)$ is a maximum.

Form the related matrix

$$\left\| a_{ij} \right\| = \left\| \begin{matrix} a_{1n} & a_{2n} & \cdots & a_{mn} \\ \cdot & \cdot & \cdot & \cdot \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{11} & a_{21} & \cdots & a_{m1} \end{matrix} \right\|$$

where $a_{ij} = f(p_{ij})$. Let

$$F' = \sum_{i=1,\, j=1}^{m,\, n} \delta_i \bar{\delta}_j a_{ij} \qquad (|\delta_i| = |\bar{\delta}_j| = 1).$$

Suppose the δ's and $\bar{\delta}$'s to be so chosen that $\max F'$ is attained. To a particular

$\bar{\delta}_k$ of the sum F' there correspond two consecutive $\bar{\ }$'s of the sum $F_N(f)$; namely, those attached to the two rows of cells of N whose top and bottom edges, respectively, pass through the kth row of points p. If $\bar{\delta}_k$ is positive, let these be assigned positive and negative values respectively, while if $\bar{\delta}_k$ is negative let their signs be fixed in the reverse order. Let all the ϵ's and $\bar{\epsilon}$'s be determined in this manner.

This choice will make $F_N(f)$ assume its maximum. For it will be seen that if a particular term $\delta_i \cdot \bar{\delta}_j a_{ij}$ has the value $+1$, then the four cells of N which have the point p_{ij} in common will together contribute $+4$ to the sum $F_N(f)$, while if this term has the value -1, these cells will contribute -4; so by this choice we have $F_N(f) = 4$ max F'. Suppose now that by some other choice of ϵ's and $\bar{\epsilon}$'s we should have $F_N(f) > 4$ max F'. If for some k we have $\bar{\epsilon}_{2k+1} = \bar{\epsilon}_{2k}$, the two rows of cells of N to which these $\bar{\epsilon}$'s are attached contribute zero to the sum $F_N(f)$; and such contribution as these rows make when $\bar{\epsilon}_{2k+1} = +1$ and $\bar{\epsilon}_{2k} = -1$ is minus that which they make when the values of the $\bar{\epsilon}$'s are interchanged. Hence if by *any* choice it be possible to make $F_N(f) > 4$ max F', there must exist *such a choice in which the ϵ's and $\bar{\epsilon}$'s occur by pairs with different signs*. But in this case we may, by reversing the process above, choose the δ's and $\bar{\delta}$'s in the sum F', and it will be seen that with this choice we have $4F' = F_N(f) > 4$ max F', which is a contradiction. Hence max $F_N(f) = 4$ max F', and since, as previously remarked, we may attain the maximum of $F(f)$ by using the net N, we have

$$\max F(f) = 4 \max F'.$$

If now we denote by max V' the sum

$$\sum_{i=1,\, j=1}^{m,n} | a_{ij} |,$$

we easily see that

$$\max V(f) = 4 \max V',$$

and hence

$$\frac{\max F(f)}{\max V(f)} = \frac{\max F'}{\max V'}.$$

We proceed to show that given any $\epsilon > 0$, there exists a matrix $\| a_{ij} \|$ with elements ± 1 for which $(\max F')/(\max V')$ is $< \epsilon$. This being so, we may then assert the existence of a "point-rectangle function" $f(x, y)$ for which $(\max F(f))/(\max V(f))$ is $< \epsilon$ for any preassigned $\epsilon > 0$.

Consider the matrix

$$\|a_{ij}\| = \left\| \begin{matrix} a_{1n} & a_{2n} & \cdots & a_{2^{n-1},n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{11} & a_{21} & \cdots & a_{2^{n-1},1} \end{matrix} \right\| \qquad (n \text{ odd}),$$

in which all the elements of the bottom row are $+1$, and the rest of the various columns consist of the 2^{n-1} possible ordered sets of $n-1$ elements each equal to $+1$ or -1. Consider max F' for this matrix. Let the $\bar{\delta}$'s be assigned in any arbitrary way; then in order to make F' as large as possible, choose the δ's so that the total sum contributed by each column shall be positive. If this be done, we see that

1 column of elements contributes n,

n columns of elements contribute $n-2$ each,

$\dfrac{n(n-1)}{2}$ columns of elements contribute $n-4$ each,

. . . .

$\dfrac{n(n-1)(n-2)\cdots\left(\dfrac{n+3}{2}\right)}{((n-1)/2)!}$ columns of elements contribute 1 each,

and hence

$$F' = n + n(n-2) + \frac{n(n-1)}{2}(n-4) + \cdots + \frac{n(n-1)(n-2)\cdots\left(\dfrac{n+3}{2}\right)}{((n-1)/2)!}.$$

Moreover this value is independent of the choice of the $\bar{\delta}$'s, so that max F' equals this expression. Clearly we have max $V' = n2^{n-1}$. Thus for matrices of this type, we have

$$\lim_{n\to\infty} \frac{\max F'}{\max V'} = 0,$$

since the expression for $(\max F')/(\max V')$ reduces to

$$\frac{(n-1)(n-2)\cdots\left(\dfrac{n+1}{2}\right)}{((n-1)/2)!\,2^{n-1}},$$

which by Stirling's formula is $O(1/n^{1/2})$, and so tends to zero with $1/n$.

We now construct example

(E), a function in class F but not V. Let I, the unit square, be divided into quarter squares, and let S_1 be the upper left-hand quarter square. Next divide the lower right-hand square into quarter squares, and let S_2 be that quarter which has a common vertex with S_1, etc. We obtain in this way an infinite sequence of square subdivisions of I converging toward the point $(1, 0)$. Now in S_1 let a "point-rectangle function" be defined for which we have $\max V(f)=1$ and $\max F(f)<1/2$; thus if the set of points p_{ij} contains $n_1 2^{n_1-1}$ points, $f(p_{ij})$ is $\pm 1/(4n_1 2^{n_1-1})$ for each i and j. Similarly, in each S_j, define a "point-rectangle function" for which $\max V(f)=1$, $\max F(f)<1/2^j$. At all remaining points of I let $f(x, y)$ be zero.

It is readily seen that this function is not in V. Consider any net N on I, and let S_k be the last S_j through which lines of this net pass. By adding a sufficient number of lines to insure the largest possible contribution from each $S_j (j \leqq k)$, which cannot decrease $\max F_N(f)$, we see that for the net N we have

$$\max F_N(f) < \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k}\cdot$$

Hence $f(x, y)$ is in class F, and relation (14) is established.

Proof of (15). The first part follows from example

$$\text{(F)} \qquad f(x, y) = \begin{cases} 1 \text{ on main diagonal (through } (0, 1), (1, 0)) \text{ of I}, \\ 0 \text{ elsewhere in I}, \end{cases}$$

which is in A but not in F. The second part may be deduced from example (B).

Proof of (16). Examples (F) and (B).

Proof of (17). Examples (F) and (B).

5. **Relations concerning the extent of the common part of two or more classes.** We first establish the following relations involving a single class on the one hand and the product of two classes on the other*:

(18)	$H = A\cdot V,$	(21)	$H < A\cdot T,$
(19)	$H = V\cdot T,$	(22)	$H < P\cdot T,$
(20)	$H = P\cdot V,$	(23)	$H < A\cdot F,$

* From this list are intentionally omitted all relations such as $P > P\cdot F$, in which the class on the left appears also on the right; the inequality is definite in the light of relations (9)–(13), (15)–(17). From this and all subsequent lists all relations involving "reducible" products (such as $P\cdot A$ which reduces to A by (1), and $P\cdot V$ which reduces to H by (20)) are also omitted.

(24) $H < P \cdot F$,

(25) $H < F \cdot T$,

(26) $T > A \cdot F$,

(27) $T > P \cdot F$,

(28) $A \not\gtreqless P \cdot T, P \cdot T \not> A$,

(29) $V \not\gtreqless A \cdot T, A \cdot T \not> V$,

(30) $V \not\gtreqless P \cdot T, P \cdot T \not> V$,

(31) $F \not\gtreqless A \cdot T, A \cdot T \not> F$,

(32) $F \not\gtreqless P \cdot T, P \cdot T \not> F$,

(33) $V \not\gtreqless A \cdot F, A \cdot F \not> V$,

(34) $V \not\gtreqless P \cdot F, P \cdot F \not> V$,

(35) $V \not\gtreqless F \cdot T, F \cdot T \not> V$,

(36) $A \not\gtreqless F \cdot T, F \cdot T \not> A$.

Proof of (18). By (1) and (3) we have $H \leqq A \cdot V$. But a function of class V satisfies the first condition of definition H, and a function which is in A satisfies the two latter conditions of H. Hence we have $A \cdot V \leqq H$, and (18) follows.

Proof of (19). From (3) and (8) follows $H \leqq V \cdot T$. But if $f(x, y)$ is in $V \cdot T$, it satisfies the first condition of definition H, and by definition T the total variation functions $\phi(\bar{x})$ and $\psi(\bar{y})$ must be finite almost everywhere; thus we have $V \cdot T \leqq H$, and hence (19).

Proof of (20). The relation $H \leqq P \cdot V$ follows from (1) and (3). But if $f(x, y)$ is in $P \cdot V$, by Theorem 2 the functions $\phi(\bar{x})$ and $\psi(\bar{y})$ are surely finite for at least one point in their respective intervals; and as the first condition of definition H is also satisfied, we have $P \cdot V \leqq H$, and hence (20).

Proof of (21). From (1) and (8) we obtain $H \leqq A \cdot T$. From example (F), which is in $A \cdot T$ but not H, (21) is inferred.

Proof of (22). The relation $H \leqq P \cdot T$ is a consequence of (1) and (8). Example (F) then establishes (22).

Proof of (23). By (1) and (3) we have $H \leqq A \cdot F$. Then consider example (E). That function was shown to be in class F; it is, moreover, in class A. For let (x_i, y_i) be any set of points as used in definition A. Then $f(x_i, y_i)$ vanishes at all these points excepting at most those which lie within one square S_j. For this set of points we have

$$\sum \left| \Delta f(x_i, y_i) \right| \leqq 2(2^{n_j - 1} + n_j - 1)\left[1/(n_j 2^{n_j - 1})\right],$$

where $n_j 2^{n_j-1}$ is the number of points in the array p_{ij} used to define the "point-rectangle function" in S_j. But as this expression is bounded, and indeed approaches zero with $1/n_j$, $f(x, y)$ is in A. Hence $f(x, y)$ is in $A \cdot F$, but since it is not in V it cannot be in H, from which fact (23) follows.

Proof of (24). This is implied by relation (23).

Proof of (25). By (3) and (8) we have $H \leqq F \cdot T$. Example (E) is clearly in class T, since $\phi(\bar{x})$ and $\psi(\bar{y})$ are zero except for a denumerable set of points; and since it is also in F but not in H, we infer (25).

Proof of (26). By Theorem 3, if $f(x, y)$ is in class $A\cdot F$, $\phi(\bar{x})$ and $\psi(\bar{y})$ must be bounded and integrable in the sense of Riemann, from which we have $T \geqq A\cdot F$. Then relation (26) follows from relation (12).

Proof of (27). By Theorems 2 and 3, if $f(x, y)$ is in class $P\cdot F$, $\phi(\bar{x})$ and $\psi(\bar{y})$ are bounded and integrable in the Riemann sense, whence follows the relation $T \geqq P\cdot F$. From relation (13), relation (27) then follows.

Proof of (28). The first part follows from example (A), which is in $P\cdot T$ but not A. The second part is implied by the second of the relations (12).

Proof of (29). The first of these relations follows from example (F); the second from the first of relations (10).

Proof of (30). Example (F); relation (11).

Proof of (31). Example (F); relation (15).

Proof of (32). Example (F); relation (16).

Proof of (33). The first relation is shown by example (E), which has been proved to be in F and A, but not in V. The second part is a consequence of example (B).

Proof of (34). Example (E); relation (9).

Proof of (35). Example (E); relation (11).

Proof of (36). The second part of this relation is a consequence of (12). To establish the first part we shall now exhibit a function which is in $F\cdot T$ but not A.

As a preliminary step we define a matrix $\|a_{ij}\|$ in the following manner.* Let

$$a_{i1} = a_{1j} = 0 \qquad (i, j = 1, 2, 3, \cdots, n+1),$$

and determine the remaining elements by assigning the values of

$$\Delta_{ij} = a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1} + a_{ij} \quad (i, j = 1, 2, 3, \cdots, n)$$

to satisfy the conditions

$$|\Delta_{ij}| = 1 \qquad (i, j = 1, 2, 3, \cdots, n),$$

$$\sum_{j=1}^{n}\Delta_{ij}\Delta_{i'j} = 0 \text{ for } i' \neq i, \qquad \sum_{i=1}^{n}\Delta_{ij}\Delta_{ij'} = 0 \text{ for } j' \neq j.$$

It is known that there exist such orthogonal matrices $\|\Delta_{ij}\|$ for an infinite sequence of values of n. For such a matrix the sum

* We gratefully acknowledge our indebtedness to the late Dr. R. E. A. C. Paley for the construction of this matrix.

$$F = \sum_{i,j=1}^{n} \epsilon_i \bar{\epsilon}_j \Delta_{ij} = \sum_{i=1}^{n} \epsilon_i u_i \qquad (|\epsilon_i| = |\bar{\epsilon}_j| = 1),$$

where

$$u_i = \sum_{j=1}^{n} \Delta'_{ij}, \quad \Delta'_{ij} = \bar{\epsilon}_j \Delta_{ij},$$

is $O(n^{3/2})$, since the matrix $\|\Delta'_{ij}\|$ is also orthogonal, and by Schwarz's inequality we have $F \leqq (\sum \epsilon_i^2 \sum u_i^2)^{1/2} = n^{3/2}$.

Let $d_{ij} = a_{i,j+1} - a_{ij}$; then we have

$$d_{Ij} = \sum_{i=1}^{I} \Delta_{ij},$$

and the sum

$$\sum_{j=1}^{n} | d_{Ij} | = \sum_{j=1}^{n} \bar{\bar{\epsilon}}_j d_{Ij},$$

where

$$\bar{\bar{\epsilon}}_j = \operatorname{sgn} d_{Ij},$$

is also readily seen to be $O(n^{3/2})$ by a second application of Schwarz's inequality. This sum represents the "total variation" in the Ith row $(I = 1, 2, 3, \cdots, n+1)$ of $\|a_{ij}\|$. It is evident from symmetry that the total variation in the Jth column $(J = 1, 2, 3, \cdots, n+1)$ of $\|a_{ij}\|$ is also $O(n^{3/2})$.

Now for each j consider

$$\max_I | d_{Ij} | \qquad (I = 1, 2, 3, \cdots, n+1);$$

let I_j be the least value of I for which this maximum is assumed. If the numbers I_j do not increase monotonically with j, the columns of $\|\Delta_{ij}\|$ may be re-arranged so that they do; this will clearly leave undisturbed the properties of the matrix $\|\Delta_{ij}\|$ described above. Numbering the rows of both matrices $\|a_{ij}\|$ and $\|\Delta_{ij}\|$ from bottom to top we observe that the sum

$$\sum_{j=1}^{n} | d_{I_j, j} |$$

is part of what may be thought of as an Arzelà-sum for the matrix $\|a_{ij}\|$. But this sum, and therefore the maximum Arzelà-sum for the matrix, will not be $O(n^{3/2})$ if the matrix $\|\Delta_{ij}\|$ be defined thus*:

* The orthogonality of this matrix is shown by Paley, *On orthogonal matrices*, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 12 (1933), pp. 311–320. That $|d_{I_j,j}|$ is not $O(n^{1/2})$ is indicated by Paley, *Note on a paper of Kolmogoroff and Menchoff*, forthcoming in the Mathematische Zeitschrift,

$$\Delta_{ij} = \begin{cases} \chi(i-j) & \text{for } i \neq j,\ i \neq 0,\ j \neq 0; \\ +1 & \text{for } i = j; \\ -1 & \text{for } i = 0 \text{ or } j = 0 \text{ but } i \neq j; \end{cases}$$

where $\chi(m)$ is a real primitive Dirichlet's character to the prime modulus p, with $p = n-1 \equiv 3 \bmod 4$.

We may now construct example

(G), a function in $F \cdot T$ but not A. Let $\{S_k\}$ $(k=1, 2, 3, \cdots)$ be an infinite sequence of square subdivisions of the unit square I similar to that employed in example (E) but converging toward the point $(1, 1)$. By the above discussion there exists a matrix $\|a_{ij}^{(k)}\|$, of n_k rows and n_k columns, for which F and the total variation in each row and column is $<1/2^k$ while the maximum Arzelà-sum is >1. In S_k $(k=1, 2, 3, \cdots)$ let $p_{ij}^{(k)}$ be a square array of n_k^2 points, with rows and columns equally spaced. The points $p_{ij}^{(k)}$ then determine a set of square cells. In the cell whose vertices are $p_{ij}^{(k)}$, $p_{i,j+1}^{(k)}$, $p_{i+1,j}^{(k)}$ and $p_{i+1,j+1}^{(k)}$, including its boundary, let $f(x, y) = a_{ij}^{(k)}$ at each point except along the top and right-hand sides. At all other points of I let $f(x, y) = 0$. The function $f(x, y)$ is then in both F and T but is not in A.

We list the following additional relations and indicate briefly the proof of each*:

(37) $A \cdot T < P \cdot T$, (38) $A \cdot T \not\gtreqless F \cdot T, F \cdot T \not> A \cdot T$, (39) $A \cdot F < A \cdot T$,

(40) $A \cdot F < P \cdot T$, (41) $A \cdot F < F \cdot T$, (42) $P \cdot F < P \cdot T$,

(43) $A \cdot F = A \cdot F \cdot T$, (44) $P \cdot F = P \cdot F \cdot T$.

Proof of (37). Relations (1) and (28).
Proof of (38). Relations (36) and (31).
Proof of (39). Relations (26) and (31).
Proof of (40). Relations (1), (26), and (32).
Proof of (41). Relations (26) and (36).
Proof of (42). Relations (27) and (32).
Proof of (43). Relation (26).
Proof of (44). Relation (27).

6. **Relations between classes when only bounded functions are admitted to consideration.** Reasoning similar to that of the preceding sections readily shows that each of the forty-four relations given above remains valid if bounded functions alone are considered. We thus have the further results†:

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* Relations (43) and (44), together with (19), show that there are no irreducible products of three classes; hence such products need no further consideration.

† Numbers are used here and later to correspond with those of similar relations given above.

(3b) $V\cdot B > H$, (8b) $T\cdot B > H$,

(9b) $V\cdot B \not\gtrless P, P \not> V\cdot B$, (10b) $A \not\gtrless V\cdot B, V\cdot B \not> A$,

(11b) $V\cdot B \not\gtrless T\cdot B, T\cdot B \not> V\cdot B$, (12b) $A \not\gtrless T\cdot B, T\cdot B \not> A$,

and so on.

7. **Relations between classes when only continuous functions are admitted.** We establish the following set* of relations:

(1c) $T\cdot C > P\cdot C > A\cdot C > H\cdot C$, (9c) $P\cdot C \not\gtrless V\cdot C, V\cdot C \not> P\cdot C$,

(10c) $A\cdot C \not\gtrless V\cdot C, V\cdot C \not> A\cdot C$, (11c) $T\cdot C \not\gtrless V\cdot C, V\cdot C \not> T\cdot C$,

(14c) $F\cdot C > V\cdot C$, (15c) $A\cdot C \not\gtrless F\cdot C, F\cdot C \not> A\cdot C$,

(16c) $P\cdot C \not\gtrless F\cdot C, F\cdot C \not> P\cdot C$, (17c) $T\cdot C \not\gtrless F\cdot C, F\cdot C \not> T\cdot C$,

(23c) $H\cdot C < A\cdot F\cdot C$, (24c) $H\cdot C < P\cdot F\cdot C$,

(25c) $H\cdot C < F\cdot T\cdot C$, (33c) $V\cdot C \not\gtrless A\cdot F\cdot C, A\cdot F\cdot C \not> V\cdot C$,

(34c) $V\cdot C \not\gtrless P\cdot F\cdot C,\ P\cdot F\cdot C \not> V\cdot C$, (35c) $V\cdot C \not\gtrless F\cdot T\cdot C, F\cdot T\cdot C \not> V\cdot C$.

The proof of (1c) will be given in three parts.

Proof of $A\cdot C > H\cdot C$. This relation was established by Küstermann, loc. cit., who gave an example of a continuous function which is in class A but not H; a simpler example is given by Hahn, loc. cit. The following function, example

(H), will be found to exhibit the same property; moreover one may easily determine whether it is in classes F and T. Let $S_1, S_2, S_3, \cdots$ be an infinite sequence of square subdivisions of I converging toward the point (1, 0) as defined in example (E). In each S_j let $f(x, y)$ be defined by the surface of a regular square pyramid whose base is S_j and height $1/j$, and let $f(x, y)$ vanish over the rest of I. Then $f(x, y)$ is continuous; it is not in V and hence not in H. For if a net N be defined whose lines consist of the lines through the sides of the squares $S_j(j=1, 2, 3, \cdots, k)$ and lines horizontally and vertically through the centers of these squares, for this net we have

$$V_N(f) = 4\sum_{n=1}^{k} 1/n,$$

which may be arbitrarily large. But $f(x, y)$ is in class A. For if (x_i, y_i) be any set of points as used in the Arzelà definition, $f(x_i, y_i)$ vanishes except at such points as lie within one square S_j, and we have $\sum|\Delta f(x_i, y_i)| \leqq 2$.

* The correspondents of certain earlier relations do not appear here, since $T\cdot C$ includes both $P\cdot C$ and $A\cdot C$, whereas T includes neither P nor A. We omit also all relations such as $T\cdot C > T\cdot F\cdot C$, in which the class on the left (other than C) appears also on the right; in such relations the inequality is always definite by virtue of the relations (9c)–(11c) and (15c)–(17c).

This function is in class T, since $\phi(\bar{x})$ and $\psi(\bar{y})$ are continuous. It is not in class F; for if a net N as defined in the preceding paragraph be used, the ϵ_i's and $\bar{\epsilon}_j$'s can be chosen so that $F_N(f)=V_N(f)$, which is arbitrarily large.

Proof of $P\cdot C>A\cdot C$. We clearly have $P\cdot C\geqq A\cdot C$. To remove the possibility of equality consider example

(I), a function defined in I in precisely the same manner as example (H) except that the sequence of subsquares $\{S_j\}$ shall in this case converge toward the point (1, 1); i.e., example (I) is obtained from example (H) by changing the position of the x- and y-axes. As the function was in class P before the change, the new function is clearly in that class also; it is easily seen to be in class T but not in classes H, A, V, or F.

Proof of $T\cdot C>P\cdot C$. We first establish the relation $T\cdot C\geqq P\cdot C$.

Assume $f(x, y)$ to be of class $P\cdot C$ in R and suppose $\sum_{\nu=1}^{n^2}\omega_\nu'/n<M$. Let the sequence of functions $\{\phi_n(\bar{x})\}$ be defined in the interval $a\leqq\bar{x}\leqq b$ as follows: for a fixed n and $\bar{x}$,

$$\phi_n(\bar{x}) = \sum_{i=1}^{2^n}\left| f(\bar{x}, y_i) - f(\bar{x}, y_{i-1})\right|,$$

where $y_0=c$, $y_n=d$, and $y_i-y_{i-1}=(d-c)/2^n$ $(i=1, 2, 3, \cdots, 2^n)$. Then each $\phi_n(\bar{x})$ is continuous, and we have

$$\lim_{n\to\infty}\phi_n(\bar{x}) = \phi(\bar{x}); \tag{d}$$

moreover the sequence $\{\phi_n(\bar{x})\}$ is positive and non-decreasing. Hence* we have

$$\lim_{n\to\infty}\int\phi_n(x)\,dx = \int\phi(x)dx.$$

Now for any fixed n let the symmetrical net N of 2^{2n} congruent cells be considered, and let I_j $(j=1, 2, 3, \cdots, 2^n)$ be the jth of the 2^n equal parts into which N divides the interval $a\leqq x\leqq b$; then we have

$$\int\phi_n(x)dx = \sum_{j=1}^{2^n}\int_{I_j}\phi_n(x)dx.$$

Let B_j denote the least upper bound of $\phi_n(\bar{x})$ in I_j; then

$$\begin{aligned}\int\phi_n(x)dx &\leqq \sum_{j=1}^{2^n}\int_{I_j}B_j dx\\ &= [(b-a)/2^n]\sum_{j=1}^{2^n}B_j.\end{aligned}$$

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* See, for example, de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, vol. 1, Paris, 1914, p. 264, Theorem III.

But B_j is at most equal to the sum of the oscillations ω_ν' in the jth column of cells of N, whence

$$[(b-a)/2^n]\sum_{j=1}^{2^n} B_j \leqq (b-a)\sum_{\nu=1}^{2^{2n}}\omega_\nu'/2^n$$
$$< M(b-a).$$

Thus for all n, $\int\phi_n(x)dx$ is $<M(b-a)$ and consequently, by (d), $\phi(\bar{x})$ is summable. Since the same reasoning holds for $\psi(\bar{y})$, the relation $T\cdot C\geqq P\cdot C$ is proved.

We now construct example

(J), a function $f(x, y)$ in class $T\cdot C$ but not P, thus establishing the relation $T\cdot C>P\cdot C$. To this end we employ a result of Tonelli,* that if $f(x, y)$ is continuous and if the surface $z=f(x, y)$ is of finite area†, $f(x, y)$ is in class T.

Let N_j ($j=2, 3, 4, \cdots$) be the net which divides I, the unit square, into $2^{2(j-1)}$ equal subsquares Q_{ji} ($i=1, 2, 3, \cdots, 2^{2(j-1)}$). Thus N_{j+1} divides each subsquare Q_{ji} of N_j into four equal subsquares. We shall define the function $f(x, y)$ over I by a surface Z which will in turn be defined as the limit of a sequence $\{Z_j\}$ of polyhedral surfaces over I, Z_j corresponding to the net N_j.

Let Z_1 be a regular pyramid Δ_1 whose base is I and altitude 1. Its surface area may be denoted by $S/2$.

Let Z_2 be identical with Z_1 except over the squares of a set Q_2' concentric with the squares Q_2 of N_2. Let a second set of smaller concentric squares Q_2'' be chosen. The squares of Q_2' may be taken as small as desired, and, these having been chosen, the squares of Q_2'' may be selected as small as desired. Limitations on their size are presently to be imposed.

As a first limitation on Q_2' let the oscillation of Z_1 be less than $\frac{1}{2}$ in each square of Q_2'. Within Q_{2i}'' (where this is the square of Q_2'' interior to Q_{2i}', in turn interior to Q_{2i}) define Z_2 as a regular pyramid Δ_{2i} of altitude $\frac{1}{2}$ and with base in a horizontal plane. The plane of the base of Δ_{2i} may be so chosen that Δ_{2i} lies wholly between the two horizontal planes through the lowest and highest points of Z_1.

Figure 1 is intended to indicate a top elevation of the part of the surface Z_2 now being described. $ABCD$ is the space quadrilateral on Z_1 whose projection on the xy-plane is Q_{2i}; $A'B'C'D'$ is the space quadrilateral on Z_1 whose projection is Q_{2i}'; and $A''B''C''D''$ is the base of the pyramid Δ_{2i} whose projection is Q_{2i}''. Let $a, b, c,$ and d be the mid-points of the sides of $A''B''C''D''$. Then plane triangles may be interpolated between the space quadrilateral

* See Tonelli, loc. cit.

† In the sense of Lebesgue.

$A'B'C'D'$ and the base of Δ_{2i}; these triangles are $A'A''a$, $A'aB'$, $aB''B'$, etc. The plane triangles thus interpolated we use to define the part of Z_2 standing over the region between the two squares Q'_{2i} and Q''_{2i}; Z_2 so defined is continuous within Q'_{2i}, and hence throughout I. The position of the plane of the base of Δ_{2i} is further restricted merely by the condition that the oscillation of Z_2 in Q'_{2i} (which by the presence of the pyramid Δ_{2i} is not less than $\frac{1}{2}$) shall be $\frac{1}{2}$. Evidently, by decreasing the size of the squares Q'_{2i} and Q''_{2i}, we may make the

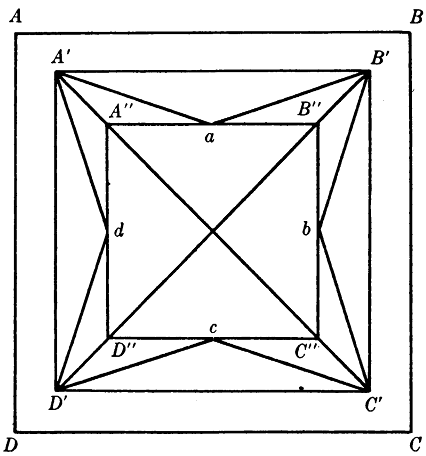

Fig. 1

surface area of Z_2 within Q'_{2i} as small as we wish; hence we may impose the final limitation upon the size of the squares, that the resulting area of Z_2 shall not exceed $S(\frac{1}{2}+\frac{1}{4})$. To provide for further subdivision we require that the lengths of the sides of the squares in Q_2, Q_2', and Q_2'' be relatively incommensurable.

Each succeeding surface Z_p is defined by means of the surface Z_{p-1} in a similar manner. Let Z_p be identical with Z_{p-1} except over the squares of a set Q_p' concentric with the squares Q_p of N_p. Let Q'_{pi} be chosen sufficiently small so that its perimeter does not intersect the perimeter of any previously chosen Q'_{ij} or Q''_{ij}. Let a second set of smaller concentric squares Q_p'' be chosen.

As the next limitation on Q_p' let the oscillation of Z_{p-1} be less than $1/p$ in each square of Q_p'. Within Q''_{pi} (where this is the square of Q_p'' interior to Q'_{pi}, in turn interior to Q_{pi}) define Z_p as a regular pyramid Δ_{pi} of altitude $1/p$ and with base in a horizontal plane. Q'_{pi} lies entirely within some smallest previously chosen Q'_{ij} (which may be I itself), Q'_{mn}. The plane of the base of Δ_{pi} may be so chosen that Δ_{pi} lies wholly between the two horizontal planes through the highest and lowest points of Z_{p-1} in Q'_{mn}.

Figure 2 is intended to indicate a top elevation of the part of the surface Z_p now being described. $ABCD$ is the space polygon on Z_{p-1} whose projection on the xy-plane is Q_{pi}; $A'p_1p_2 \cdots B' \cdots C' \cdots D' \cdots$ is the space polygon on Z_{p-1} whose projection is Q'_{pi}; and $A''B''C''D''$ is the base of the pyramid Δ_{pi} whose projection is Q''_{pi}. Let a, b, c, and d be the mid-points of the sides of $A''B''C''D''$. Then plane triangles may be interpolated between the space polygon $A'p_1p_2 \cdots B' \cdots C' \cdots D' \cdots$ and the base of Δ_{pi}; these triangles

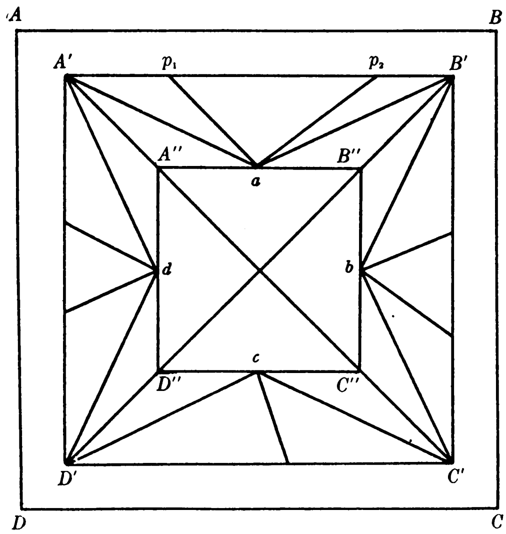

Fig. 2

are $A'A''a$, $A'ap_1$, p_1ap_2, etc. The plane triangles thus interpolated we use to define the part of Z_p standing over the region between the two squares Q'_p and Q''_{pi}; Z_p is then continuous within Q_{pi}', and hence throughout I. The position of the plane of the base of Δ_{pi} is further restricted merely by the condition that the oscillation of Z_p in Q'_{pi} (which by the presence of the pyramid Δ_{pi} is at least $1/p$) shall be $1/p$. The final limitation upon the size of the squares Q'_{pi} and Q''_{pi} is that the resulting surface area of Z_p shall not exceed $S(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \cdots +1/2^p)$. In order to show that this result may be effected, we need only prove that the total surface area of Z_p thus defined within Q'_{pi} may be made arbitrarily small by choosing the squares Q'_{pi} and Q''_{pi} sufficiently small.

Let any $\epsilon > 0$ be given. Then clearly there exists a δ_1 such that if the side of Q''_{pi} be taken less than δ_1, the surface area S' of the pyramid Δ_{pi} will be less than $\epsilon/2$. Now consider S'', the total surface area of the plane triangles between Q'_{pi} and Q''_{pi}. Of these triangles eight have the property that each has one side which coincides with half of a side of the base of Δ_{pi}; moreover the length of each of its other sides is bounded by $((d/2)^2+(1/p)^2)^{1/2}$, where d is the length of the diagonal of Q'_{pi}; whence we may assert that there exists a δ_2 such that if the side of Q''_{pi} be taken less than δ_2 in length, the surface area of these eight triangles will be less than $\epsilon/4$. There remain to be considered the rest of the triangles which contribute to S''. Each of these has one side whose length is bounded by $(\omega^2+l^2)^{1/2}$, where ω is the oscillation of Z_{p-1} in Q'_{pi} and l is the length of a side of Q'_{pi}; likewise each of its other sides is bounded by $((d/2)^2+(1/p)^2)^{1/2}$. Moreover the number of these triangles is limited, since the surface Z_{p-1} consists of a finite number of plane pieces. Now by taking Q'_{pi} sufficiently small we may make ω and l, and consequently one side of each of these triangles, as small as we please; hence there exists a δ_3 such that if the side of Q'_{pi} be taken less than δ_3, the combined areas of these remaining triangles will be less than $\epsilon/4$. If, then, we require that the side of Q'_{pi} be less than δ_3, and the side of Q''_{pi} be less than δ_1 and δ_2, the total area of the part of Z_p within Q'_{pi} will be less than ϵ.

Finally, to provide for further subdivision, we take the lengths of the sides of the squares in Q_p, Q'_p, and Q''_p relatively incommensurable.

Then if P is any point of I, and h_j denotes the height of Z_j over P, the sequence $\{h_j\}$ approaches a limit as j increases indefinitely. For if P does not lie within an infinite number of squares Q'_{ij}, all the h_j's are equal for sufficiently large j. If P does lie within an infinite sequence of such squares, we have $|h_j-h_p|<1/p$ for all $j>p$, and so the sequence $\{h_j\}$ converges. Let the surface Z be defined as the limit of the sequence $\{Z_j\}$.

Inasmuch as each Z_j is continuous, and the sequence $\{Z_j\}$ converges uniformly, the surface Z is continuous and defines a continuous function $f(x, y)$ over I. Moreover, as Z may be approximated arbitrarily closely by one of the sequence $\{Z_j\}$ of polyhedral surfaces, each of which is of area less than S, the area of Z does not exceed S; hence $f(x, y)$ is in class T. But for each net N_j we have

$$\sum_{\nu=1}^{n^2}\omega'_\nu/n \geqq [2^{2(j-1)}(1/j)]/2^{j-1} = 2^{j-1}/j,$$

and as the latter quantity increases indefinitely with j, the function $f(x, y)$ is not in class P.

Proof of (9c). Example (B); example (I).

Proof of (10c). Example (B); example (H).

Proof of (11c). Example (B); example (H).

Proof of (14c). Example (E) has already been given to exhibit a function which is in class F but not in class V. We now show how this example may be modified so as to be continuous without otherwise essentially altering its character.

Consider a "point-rectangle function" $f(x, y)$ such as is used in example (E), with $|f| = 1$ on the array of $n2^{n-1}$ points p_{ij} in R and with $(\max F(f)) / (\max V(f)) < \epsilon$. Surround each point p_{ij} by a square Q_{ij} with sides parallel to the axes and with p_{ij} as center; all these squares are taken equal in size and small enough so that they do not abut or overlap.

Let $f'(x, y)$ be defined on R as follows: within each Q_{ij} let $f'(x, y)$ be defined by the surface of a regular pyramid whose base is Q_{ij} and whose height is $f(p_{ij})$; let $f'(x, y) = 0$ at all other points of R. This function is continuous on R, and for it also we have

$$(\max F(f'))/(\max V(f')) < \epsilon. \tag{e}$$

For the following inequalities are easily seen to hold:

$$\max F(f') \geqq \max F(f), \quad \max V(f') \geqq \max V(f);$$

and we shall show that $\max F(f')$ does not exceed $\max F(f)$, whence (e) will follow.

Let N be any net on R. Construct a second net N' by adding lines to N as follows. Add the horizontal lines through the center and upper and lower sides of Q_{11}, and the corresponding vertical lines. If a horizontal line l of N passes through Q_{11}, add to N the horizontal line l' so that p_{11} is equidistant from l and l', and also the vertical lines l'' and l''' at the same distance from p_{11}. For each Q_{ij} add to N four lines bearing the same relation to it that l, l', l'' and l''' bear to Q_{11}. Let this construction be carried out for each horizontal and vertical line through Q_{11}, and then the process repeated for every other Q_{ij}.

The net N' thus defined is symmetric; each Q_{ij} is divided by N' in precisely the same way into t^2 rectangular subregions which are in general not square except for those along the diagonals, and for each of which, excepting those along the diagonals, the difference $\Delta_{11}f'(x_i,y_j)$ vanishes. For each of the squares along the diagonals the difference is in absolute value equal to one of the $t/2$ values $a, b, c, \cdots, k$, where $a+b+c+\cdots+k=1$. The values of t and of the numbers $a, b, c, \cdots, k$ depend upon the net N'. The situation for a particular Q_{ij} may be represented as in Figure 3, where the number within each cell is the value of the difference $\Delta_{11}f'$ for that cell. The figure represents a Q_{ij} when $f'(p_{ij}) = +1$ and $t/2 = 4$. It will be seen that the ϵ_i's and $\bar{\epsilon}_j$'s in the sum $F_{N'}(f')$ which attach to the cells yielding $\pm a$ may be chosen independently of those which attach to the cells yielding $\pm b$, etc., and that the maximum contribution obtainable from the cells yielding $\pm a$ is $a \cdot \max F(f)$; from those yielding $\pm b$, $b \cdot \max F(f)$; etc. Hence we have

$$\max F_{N'}(f') = a \cdot \max F(f) + b \cdot \max F(f) + \cdots + k \cdot \max F(f) = \max F(f).$$

But as $\max F_N(f')$ is $\leqq \max F_{N'}(f')$ (since N' was obtained from N by adding lines) and N was an arbitrary net, we conclude that $\max F(f') = \max F(f)$, which was to be proved.

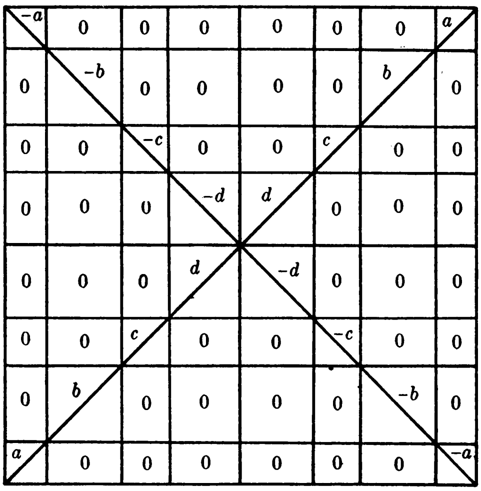

Fig. 3

(K) Since a "point-rectangle function" may be made continuous while retaining the same values for $\max F(f)$ and $\max V(f)$, we may clearly construct example (K) by modifying example (E) in this way, and so obtain a continuous function which is in class F but not V.

Proof of (15c). Example (B); example (H).

Proof of (16c). Example (B); relation (15c).

Proof of (17c). Example (B); relations (5) and (15c).

Proof of (23c). Example (K) is readily seen to be in class $A \cdot F \cdot C$, but not in V and hence not in H. Since we have $H \cdot C \leqq A \cdot F \cdot C$, (23c) follows.

Proof of (24c). Relation (23c).

Proof of (25c). Relation (23c).

Proof of (33c). Example (K); example (B).

Proof of (34c). Relation (33c); example (B).

Proof of (35c). Relation (33c); example (B).

8. **Open questions.** The following is a complete list of pairs of classes the

relations between which are not yet fully determined; in each case we give in parentheses a partial determination of the relation, with a reason therefor.

(45) $P, F\cdot T$ $(P \not\leqq F\cdot T$ by (13)),

(46) $A, P\cdot F$ $(A \not\leqq P\cdot F$ by (15)),

(47) $A\cdot F, P\cdot F$ $(A\cdot F \leqq P\cdot F$ by (1)),

(48) $A\cdot T, P\cdot F$ $(A\cdot T \not\leqq P\cdot F$ by (31)),

(49) $P\cdot F, F\cdot T$ $(P\cdot F \leqq F\cdot T$ by (27)),

(50) $P\cdot T, F\cdot T$ $(P\cdot T \not\leqq F\cdot T$ by (32)),

(36c) $A\cdot C, F\cdot T\cdot C$ $(A\cdot C \not\leqq F\cdot T\cdot C$ by (15c)),

(41c) $A\cdot F\cdot C, F\cdot T\cdot C$ $(A\cdot F\cdot C \leqq F\cdot T\cdot C$ by (1c)),

(45c) $P\cdot C, F\cdot T\cdot C$ $(P\cdot C \not\leqq F\cdot T\cdot C$ by (16c)),

(46c) $A\cdot C, P\cdot F\cdot C$ $(A\cdot C \not\leqq P\cdot F\cdot C$ by (15c)),

(47c) $A\cdot F\cdot C, P\cdot F\cdot C$ $(A\cdot F\cdot C \leqq P\cdot F\cdot C$ by (1)),

(49c) $P\cdot F\cdot C, F\cdot T\cdot C$ $(P\cdot F\cdot C \leqq F\cdot T\cdot C$ by (1c)).

The relations still to be determined present some interesting, but probably not simple, problems. We would hazard no conjecture concerning their nature except in the case of (36c) and (41c), the first of which is probably an overlapping relation and the second a definite inequality; that such is the case could be established by modifying example (G) so as to be continuous while preserving its other properties. We have no doubt that this modification is possible, but see no way to do it easily.

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