ON REVERSIBLE DYNAMICAL SYSTEMS*

BY

G. BALEY PRICE

INTRODUCTION

In recent years great advances have been made in the theory of dynamical systems. Nevertheless comparatively few specific systems have been treated by modern methods in such a way as to set forth the characteristic features of the entire system. Among the specific systems which have been treated, we may mention (1) the restricted problem of three bodies [Birkhoff 5],† which is an irreversible dynamical system and further complicated by the presence of singularities; (2) the determination of the geodesics on a surface of negative curvature [Morse 2, 3; Birkhoff 4, pp. 238–248]; (3) a simple type of reversible dynamical systems on surfaces of revolution [Price 1, 2]; (4) the billiard ball problem [Birkhoff 6; 4, pp. 169–179]. The purpose of the present paper is to study reversible dynamical systems, with the emphasis on those which have an oval of zero velocity.

It has been known for a long time that among the small oscillations [Whittaker 1, chap. VII] of a reversible dynamical system at a position of stable equilibrium there are two fundamental periodic orbits which join two points of the oval of zero velocity and are traced with a backward and forward motion. Part I of the present paper is devoted to proving that similar orbits exist in the actual system at least for sufficiently restricted values of the energy constant. These orbits appear to be one of the characteristic features of a reversible dynamical system with an oval of zero velocity. The property of reversible systems which distinguishes them from irreversible systems is that their trajectories in the manifold of states of motion are grouped in symmetric pairs [see §2].

Using methods developed by Poincaré [1, vol. I, chap. III] and elaborated by Painlevé [1], Horn [1] has shown that there exist certain periodic orbits in the actual system which reduce to the fundamental periodic orbits of the small oscillations system, but their exact nature was not determined. In the present treatment a parameter μ is introduced in such a way that $\mu=0$

---

* Part of a paper presented to the Society, February 25, 1933, under the title *A study of certain dynamical systems with applications to the generalized double pendulum*; received by the editors February 9, 1934.

† We shall refer in this manner to the bibliography at the end of this paper.

gives the limiting case of small oscillations. Then the equations of motion are integrated in terms of series in parameters by Poincaré's method, and two theorems on the analytic continuation of the fundamental periodic orbits of the limiting integrable system are proved. The second theorem applies only when the system is symmetric in the position of equilibrium. These theorems show not only that analytic continuation is possible in certain cases in which Horn's method fails, but also that the continued orbits always touch the oval of zero velocity.

Part II, using the results of Part I, is devoted to a detailed study of motion in the neighborhood of a position of stable equilibrium in the case of two degrees of freedom. The methods are those of analytic continuation [Birkhoff 4, pp. 139–143; Poincaré 1, vol. I, chap. III], surfaces of section, and surface transformations [Birkhoff 1, 3]. First, the manifold of states of motion is studied, and a convenient representation of it is given in 3-space. Then the limiting integrable system is treated in detail. Next some general theorems on a common type of surface of section are given. At this point the presence of the oval of zero velocity introduces essential difficulties, since the surface of section is formed from a periodic orbit which joins two points of it. The existence of periodic orbits is established by applying the general theory of surface transformations. Poincaré's Last Geometric Theorem is useful here. When the system is symmetric in the position of equilibrium, the transformation on the surface of section can be factored in certain ways, and more specific results concerning periodic orbits are established.

The author wishes to acknowledge his indebtedness and express his thanks to Professor Birkhoff, who proposed a problem which led to this paper and made valuable suggestions concerning the method of treatment.

Part I. Analytic continuation of periodic orbits

1. Definitions and assumptions. Let (y_i) and (y_i'), $i=1, \cdots, n$, be the position and velocity coordinates respectively of a dynamical system with kinetic energy T and force function U. Throughout the paper a prime will denote a derivative with respect to the time. Let the (y_i) be principal coordinates [Whittaker 1, chap. VII], and assume that T and U have the following specific forms:

$$(1) \qquad T = \frac{1}{2} \sum_{i,j=1}^{n} [\delta_{ij} + T_{ij}(y_1, \cdots, y_n)] y_i' y_j' \qquad (\delta_{ii} = 1; \delta_{ij} = 0, i \neq j),$$

$$(2) \qquad U = \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2 y_i^2 + u(y_1, \cdots, y_n),$$

where

$$(3) \qquad \left| \delta_{ij} + T_{ij}(y_1, \cdots, y_n) \right| \neq 0,$$

and the λ_i are either real or pure imaginary with

$$(4) \qquad \lambda_i \neq 0 \qquad (i = 1, \cdots, n).$$

Here the $T_{ij}(y_i)$ are functions which vanish at (0), and $u(y_i)$ has no terms of degree lower than the third; we assume that these are entire functions of the indicated arguments.

The equations of motion in the Lagrangian form can be written down at once; because the determinant in (3) is not zero, these equations can be put in the form

$$(5) \qquad y_i'' = \lambda_i^2 y_i + \sum_{k,j=1}^{n} F_{kj}^{(i)}(y_1, \cdots, y_n) y_k' y_j' + G_i(y_1, \cdots, y_n),$$

where $i = 1, \cdots, n$. The integral of energy is

$$(6) \qquad T = U + h/2.$$

Since we assume T to be a positive definite quadratic form in the velocities, it follows from (6) that the motion takes place in the regions $U + h/2 \geqq 0$, bounded by the *oval of zero velocity* Z with the equation $U + h/2 = 0$. This system may be interpreted as a particle of unit mass moving on a surface with

$$(7) \qquad ds^2 = 2T(dt)^2$$

and (y_i) as the coordinates, and acted on by forces derived from U [Birkhoff 1, pp. 202 and 212–213; 4, pp. 23–25]. A curve on the *characteristic surface* (7) defined by a solution $[y_i(t)]$ of (5) will be termed an *orbit* of the particle, and the curve $[y_i(t); y_i'(t)]$ in the *manifold of states of motion* M with the equation (6) will be called a *trajectory* or *stream line*.

2. **Reversible dynamical systems.** Now M is a $(2n-1)$-manifold in the $2n$-space with coordinates $(y_i; y_i')$. We agree once and for all to exclude those values of h which lead to a double point on Z, i.e., we assume the equations

$$(8) \qquad U + h/2 = 0, \qquad \partial U/\partial y_i = 0 \qquad (i = 1, \cdots, n)$$

have no simultaneous solution. With this restriction, M is an analytic manifold, and there is no equilibrium solution $y_i = \text{constant}$.

The usual existence theorems of differential equations [Birkhoff 4, pp. 1–14] can be applied to the system (5), (6). The orbits are regular curves except at those points at which they touch Z, for only at these points can (y_i') vanish. If an orbit touches Z, the particle approaches and recedes from

it along one and the same curve, and spends only a finite length of time in the neighborhood of the point. On the other hand, the trajectories are the stream lines of a steady fluid motion in M. Since there are no double points on Z, $(y_i'; y_i'')$ cannot vanish; hence the trajectories are regular curves without exception.

Dynamical systems of the type introduced in §1 are known as reversible [Birkhoff 1, p. 205; 4, pp. 27–29]; their fundamental property is stated in the following theorem.

THEOREM 1. *An orbit of a reversible dynamical system may be traced in either direction; or again, the stream lines are paired, each stream line of a pair being the symmetric image of the other in the n-plane* $y_i' = 0$, $i = 1, \cdots, n$.

The proof follows from the fact that if $[y_i(t); y_i'(t)]$ is a solution of (5), then $[y_i(-t); -y_i'(-t)]$ is also a solution. The coordinates of symmetric points on the two trajectories are obtained by combining the coordinates of a point on the orbit with the two possible directions of the velocity vector. Each trajectory of a pair will be called the *symmetric trajectory of the first kind* of the other. If T^* is a trajectory, its symmetric image of the first kind may be represented by V_1T^*, where V_1 may be thought of as a transformation.

THEOREM 2. *A necessary and sufficient condition that* T^* *and* V_1T^* *be identical is that the orbit which corresponds to* T^* *pass through a point of* Z.

The proof of this theorem and the following one are left to the reader.

THEOREM 3. *A necessary and sufficient condition that an orbit which passes through a point of* Z *be periodic is that it pass through a second point of* Z, *distinct from the first.*

A periodic orbit which passes through a point of Z is thus a curve joining two points of Z, and the particle traces this curve with a backward and forward motion. The length of time required for the particle to pass from one of the points of Z to the other is the same in either direction.

In certain cases, T and U are symmetric in the origin of coordinates on the characteristic surface, i.e.,

$$T(-y_i; y_i') = T(y_i; y_i'), \qquad U(-y_i) = U(y_i). \tag{9}$$

THEOREM 4. *If* T *and* U *satisfy* (9), *the orbits are paired, each orbit of a pair being the symmetric image of the other in the origin; or again, the trajectories are paired, each trajectory of a pair being the symmetric image of the other in the origin of coordinates in* M.

The proof follows from the fact that if $[y_i(t); y_i'(t)]$ is a trajectory, then

$[-y_i(t);\ -y_i'(t)]$ is also a trajectory when (9) holds. Each trajectory of a pair will be called the *symmetric trajectory of the second kind* of the other. If T^* is a trajectory, then V_2T^* will denote the symmetric trajectory of the second kind. Thus when (9) holds, the trajectories are related by fours. The group associated with T^* is T^*, V_2T^*, V_1T^*, and $V_1V_2T^*$.

THEOREM 5. *If T and U satisfy* (9), *a necessary and sufficient condition that an orbit which passes through the origin be periodic is that it pass through the origin a second time.*

The proof follows from the fact that if an orbit passes through the origin, it is its own symmetric image in the origin. Hence, the complete orbit can be obtained by reflecting in the origin the part between any two successive passages through the origin. If an orbit passes through the origin at time $t=t_0$ and closes at time $t=t_1$, then it passes through the origin at time $t=(t_0+t_1)/2$ also.

As a consequence of Theorems 1 and 5, we have the following theorem, of special importance later.

THEOREM 6. *If an orbit passes through the origin at time $t=0$ and touches Z at time $t=t^*$, it is a periodic orbit with period $4t^*$ and joins two points of Z which are symmetric in the origin.*

3. **The limiting integrable system.** The equations of motion and the integral of energy are given by (5) and (6). We now replace the energy constant h by μ^2 and make the change of variables

$$y_i = \mu x_i \qquad (i = 1, \cdots, n). \tag{10}$$

The equations of motion and the integral of energy in the new variables are

$$\begin{aligned} x_i'' &= \lambda_i^2 x_i + \mu \sum_{k,j=1}^{n} F_{kj}^{(i)}(\mu x_1, \cdots, \mu x_n) x_k' x_j' + \frac{1}{\mu} G_i(\mu x_1, \cdots, \mu x_n), \\ \sum_{i=1}^{n} x_i'^2 &+ \sum_{k,j=1}^{n} T_{kj}(\mu x_1, \cdots, \mu x_n) x_k' x_j' \\ &= \sum_{i=1}^{n} \lambda_i^2 x_i^2 + \frac{1}{\mu^2} u(\mu x_1, \cdots, \mu x_n) + 1, \end{aligned} \tag{11}$$

where $i=1, \cdots, n$, and T_{kj}, G_i, and u are entire functions whose series expansions have no terms of degree lower than the first, second, and third respectively. Now for every value of $\mu \neq 0$ these equations represent the actual system (5) and (6) for the value $h=\mu^2$ of the energy constant. On the other hand, μ may be considered as a parameter in the equations of the

system. The system is analytic in the parameter for all values of the parameter, including $\mu=0$.

For $\mu=0$ the system (11) is integrable, being

$$
\begin{aligned}
x_i'' &= \lambda_i^2 x_i \qquad (i = 1, \cdots, n), \\
\sum_{i=1}^{n} x_i'^2 &= \sum_{i=1}^{n} \lambda_i^2 x_i^2 + 1.
\end{aligned} \tag{12}
$$

This system is a limiting case of the actual system (5) and (6) obtained by reducing the energy to zero and at the same time altering the units of length according to (10). The solution of (12) for which $(x_i;\ x_i')$ reduces at time $t=0$ to $(\alpha_i;\ \beta_i)$ is

$$
\begin{aligned}
x_i &= (\alpha_i/2)[\exp(\lambda_i t) + \exp(-\lambda_i t)] + (\beta_i/(2\lambda_i))[\exp(\lambda_i t) - \exp(-\lambda_i t)], \\
x_i' &= (\alpha_i\lambda_i/2)[\exp(\lambda_i t) - \exp(-\lambda_i t)] + (\beta_i/2)[\exp(\lambda_i t) + \exp(-\lambda_i t)],
\end{aligned} \tag{13}
$$

where $i=1, \cdots, n$, and

$$
\sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} \lambda_i^2 \alpha_i^2 + 1. \tag{14}
$$

We now suppose that k of the λ_i, $0<k\leqq n$, are pure imaginary. We can suppose the notation is so chosen that they are $\lambda_1, \cdots, \lambda_k$. Then among the solutions (13) there are k of special importance.

THEOREM 7. *The limiting integrable system* (12) *has the k fundamental periodic trajectories*

$$
\begin{aligned}
x_i &= 0, \\
x_i' &= 0 \qquad (i\neq j;\ i = 1, \cdots, n), \\
x_j &= (1/|\lambda_j|)\sin(|\lambda_j| t + \theta_j), \\
x_j' &= \cos(|\lambda_j| t + \theta_j) \qquad (j = 1, \cdots, k).
\end{aligned} \tag{15}
$$

4. Solutions of the equations of motion in terms of series in parameters. Our ultimate aim is to show that analytic continuation of the fundamental periodic trajectories of the limiting integrable system is possible. With this end in view, we shall obtain the solution of (11) in terms of series in certain parameters, following a method developed by Poincaré [1, vol. I, pp. 58–63; Moulton 1, chap. III].

Set

$$
x_i = x_i, \qquad x_i' = y_i \qquad (i = 1, \cdots, n). \tag{16}
$$

The system (11) thus takes the form

$$
(17)\qquad
\begin{aligned}
&\frac{dx_i}{dt} = y_i,\\
&\frac{dy_i}{dt} = \lambda_i^2 x_i + \mu Y_i(x_1, \cdots, x_n; y_1, \cdots, y_n; \mu),\\
&\sum_{i=1}^{n} y_i^2 + \sum_{i,j=1}^{n} T_{ij}(\mu x_1, \cdots, \mu x_n) y_i y_j\\
&\qquad = \sum_{i=1}^{n} \lambda_i^2 x_i^2 + \frac{1}{\mu^2} u(\mu x_1, \cdots, \mu x_n) + 1,
\end{aligned}
$$

where $i=1, \cdots, n$, and the Y_i are entire functions of the indicated arguments. Now set

$$
(18)\qquad t = t^*(t_0 + \tau)/t_0.
$$

Here t^* is the new independent variable, τ is a parameter, and t_0 is a constant whose value will be specified later. Now transform from the variables $(x_i; y_i)$ to new variables $(p_i; q_i)$ by means of

$$
(19)\qquad
\begin{aligned}
x_i &= p_i + \gamma_i,\\
y_i &= q_i + \delta_i \qquad\qquad (i \neq j,\ i = 1, \cdots, n),\\
x_j &= p_j + \gamma_j + \frac{1}{|\lambda_j|} \sin\left(\frac{t_0+\tau}{t_0} |\lambda_j| t^* + \theta_j\right),\\
y_j &= q_j + \delta_j + \qquad \cos\left(\frac{t_0+\tau}{t_0} |\lambda_j| t^* + \theta_j\right).
\end{aligned}
$$

Here $(\gamma_i; \delta_i)$ are to be considered as parameters in the transformation. The equations of motion in the new variables are

$$
(20)\qquad
\begin{aligned}
\frac{dp_i}{dt^*} &= \frac{t_0+\tau}{t_0}(q_i + \delta_i),\\
\frac{dq_i}{dt^*} &= \frac{t_0+\tau}{t_0}\left[\lambda_i^2(p_i + \gamma_i) + \mu Q_i(p_i; q_i; \gamma_i; \delta_i; \mu; \tau; t^*)\right],
\end{aligned}
$$

where $i=1, \cdots, n$. The right hand members of these equations can be expanded as power series in the $2n$ variables $(p_i; q_i)$ and the $(2n+2)$ parameters $(\gamma_i; \delta_i; \mu; \tau)$ with coefficients which are analytic functions of t^*. On carrying through the details of Poincaré's method and transforming back to the original variables $(x_i; y_i)$, we find the following solution of (17):

$$
\begin{aligned}
x_i &= (1/2)[\exp(\lambda_i t^*) + \exp(-\lambda_i t^*)]\gamma_i \\
&\quad + (1/(2\lambda_i))[\exp(\lambda_i t^*) - \exp(-\lambda_i t^*)]\delta_i + \cdots, \\
y_i &= (\lambda_i/2)[\exp(\lambda_i t^*) - \exp(-\lambda_i t^*)]\gamma_i \\
&\quad + (1/2)[\exp(\lambda_i t^*) + \exp(-\lambda_i t^*)]\delta_i + \cdots, \qquad (21) \\
x_j &= \frac{1}{|\lambda_j|}\sin\left[\frac{t_0+\tau}{t_0}|\lambda_j|\, t^* + \theta_j\right] + (1/2)[\exp(\lambda_j t^*) + \exp(-\lambda_j t^*)]\gamma_j \\
&\quad + (1/(2\lambda_j))[\exp(\lambda_j t^*) - \exp(-\lambda_j t^*)]\delta_j + \cdots, \\
y_j &= \cos\left[\frac{t_0+\tau}{t_0}|\lambda_j|\, t^* + \theta_j\right] + (\lambda_j/2)[\exp(\lambda_j t^*) - \exp(-\lambda_j t^*)]\gamma_j \\
&\quad + (1/2)[\exp(\lambda_j t^*) + \exp(-\lambda_j t^*)]\delta_j + \cdots,
\end{aligned}
$$

where $i \neq j$, $i = 1, \cdots, n$. This solution has the following properties:

(I) The series in (21) are series in the $(2n+2)$ parameters $(\gamma_i;\ \delta_i;\ \mu;\ \tau)$. Except for the terms in μ, the series are written out completely up to terms of the second degree. The coefficients are real analytic functions of the real variable t^*. From (18) we see that the values of $(x_i;\ y_i)$ at time $t = t_0 + \tau$ are obtained from (21) by setting $t^* = t_0$.

(II) If T^* be chosen arbitrarily, it is possible to find an ϵ such that the series converge absolutely and uniformly for $0 \leqq t^* \leqq T^*$, $|\gamma_i| \leqq \epsilon$, $|\delta_i| \leqq \epsilon$, $|\mu| \leqq \epsilon$, $|\tau| \leqq \epsilon$.

(III) The coefficients of all terms not explicitly written out in (21) vanish for $t^* = 0$. The solution (21) thus satisfies the initial conditions

$$
\begin{aligned}
\alpha_i &= \gamma_i, \\
\beta_i &= \delta_i \qquad\qquad (i \neq j,\ i = 1, \cdots, n), \\
\alpha_j &= (1/|\lambda_j|)\sin\theta_j + \gamma_j, \\
\beta_j &= \cos\theta_j + \delta_j.
\end{aligned} \qquad (22)
$$

(IV) For $\gamma_i = \delta_i = \mu = \tau = 0$, the trajectory (21) reduces to the fundamental periodic trajectory (15) of the limiting integrable system.

5. Analytic continuation of the fundamental periodic trajectories of the limiting integrable system. We shall now show that under certain conditions each of the k periodic trajectories (15) can be continued analytically for $\mu > 0$. We shall give the proof in the case $j = 1$; the proof in the other cases is similar.

From the solutions (21) we select for special consideration those for which $\theta_j = \pi/2$, $\delta_i = 0$, $i = 1, \cdots, n$. They are

$$
\begin{aligned}
x_1 &= (1/|\lambda_1|)\cos[(t_0+\tau)|\lambda_1|t^*/t_0] + (1/2)[\exp(\lambda_1 t^*) + \exp(-\lambda_1 t^*)]\gamma_1 \\
&\qquad + \cdots, \\
y_1 &= -\sin[(t_0+\tau)|\lambda_1|t^*/t_0] + (\lambda_1/2)[\exp(\lambda_1 t^*) - \exp(-\lambda_1 t^*)]\gamma_1 \\
&\qquad + \cdots, \\
x_i &= (1/2)[\exp(\lambda_i t^*) + \exp(-\lambda_i t^*)]\gamma_i \\
&\qquad + \cdots, \\
y_i &= (\lambda_i/2)[\exp(\lambda_i t^*) - \exp(-\lambda_i t^*)]\gamma_i \\
&\qquad + \cdots,
\end{aligned}
\tag{23}
$$

where $i=2, \cdots, n$. The orbits corresponding to (23) are characterized by the fact that at time $t^*=0$ they pass through a point of Z. Now when $\mu=0$, it is possible to determine the parameters γ_i and τ so that the orbit (23) passes through a second point of Z at time $t^*=t_0=\pi/|\lambda_1|$. We hereby define t_0 in (18). For $\mu=0$, we have only to take $\gamma_i=0$, $\tau=0$.

We now seek to determine $(\gamma_i;\tau)$ as functions of μ so that for every value of μ the orbit (23) passes through a second point of Z at time $t^*=t_0=\pi/|\lambda_1|$. A necessary and sufficient condition that an orbit pass through a point of Z is that all the velocities vanish simultaneously. We therefore have the following equations for determining $(\gamma_i;\tau)$ as functions of μ:

$$
\begin{aligned}
-\sin(|\lambda_1|\tau+\pi) - |\lambda_1|(\sin\pi)\gamma_1 + \cdots &= 0, \\
-|\lambda_i|\left(\sin\left|\frac{\lambda_i}{\lambda_1}\right|\pi\right)\gamma_i + \cdots &= 0, \\
\frac{\lambda_j}{2}\left[\exp\left(\frac{\lambda_j\pi}{|\lambda_1|}\right) - \exp\left(-\frac{\lambda_j\pi}{|\lambda_1|}\right)\right]\gamma_j + \cdots &= 0, \\
\lambda_1^2\left(\frac{1}{|\lambda_1|}+\gamma_1\right)^2 + \sum_{i=2}^{n}\lambda_i^2\gamma_i^2 + \frac{1}{\mu^2}u(\mu\gamma_1,\cdots,\mu\gamma_n) + 1 &= 0.
\end{aligned}
\tag{24}
$$

Here $i=2, \cdots, k$ and $j=k+1, \cdots, n$. The last equation states that the integral of energy is satisfied at time $t^*=0$. The equations (24) are power series in $(\gamma_i;\tau;\mu)$ with constant coefficients. For $\mu=0$ they have the solution $\gamma_i=\tau=0$. If the Jacobian with respect to $(\gamma_i;\tau)$ is not zero for $\gamma_i=\tau=0$, it is possible to solve (24) and obtain $(\gamma_i;\tau)$ as analytic functions of μ at least for μ small. Direct computation shows that this Jacobian does not vanish unless $|\lambda_i|/|\lambda_1|$, $i=2, \cdots, k$, is an integer. Since an orbit which passes through two distinct points of Z is periodic by Theorem 3, we have proved the following theorem [compare Birkhoff 4, pp. 139–143].

THEOREM 8. *The jth fundamental periodic orbit of the limiting integrable system can be continued analytically for* $\mu>0$, *at least for* μ *small, if* λ_i/λ_j, $i\neq j$, $i=1, \cdots, k$, *is not an integer; each orbit of the continuation joins two points of* Z *and is periodic with period* $2(\pi/|\lambda_j|+\tau)$.

If the system has the symmetry specified by (9), another procedure is possible, which gives additional information and in certain cases additional results. Consider the trajectories (21) for which

$$\theta_j = 0, \qquad \gamma_i = 0 \qquad (i = 1, \cdots, n). \tag{25}$$

The characteristic property of the corresponding orbits is that they pass through the origin when $t^*=0$. By setting $\delta_i=\tau=0$ when $\mu=0$, we obtain an orbit which passes through a point of Z when $t^*=t_0=\pi/(2|\lambda_1|)$. We hereby define t_0 anew in (18). We propose to show that under certain conditions it is possible to determine $(\delta_i;\ \tau)$ as functions of μ so that the orbit determined by (21) and (25) has this property for all values of μ sufficiently small. The following equations determine these functions:

$$\begin{aligned}
\cos(\pi/2 + |\lambda_1|\tau) + (\cos \pi/2)\delta_1 + \cdots &= 0,\\
\left(\cos\left|\frac{\lambda_i}{\lambda_1}\right|\frac{\pi}{2}\right)\delta_i + \cdots &= 0,\\
(1/2)\left[\exp\left(\frac{\pi\lambda_j}{2|\lambda_1|}\right) + \exp\left(-\frac{\pi\lambda_j}{2|\lambda_1|}\right)\right]\delta_j + \cdots &= 0,\\
(1+\delta_1)^2 + \delta_2^2 + \cdots + \delta_n^2 - 1 &= 0.
\end{aligned} \tag{26}$$

Here $i=2, \cdots, k$, and $j=k+1, \cdots, n$. The first n equations express the condition that the velocities vanish for $t^*=t_0=\pi/(2|\lambda_1|)$, and the last equation states that the initial conditions satisfy the integral of energy. The equations (26) determine the analytic continuation of the first fundamental periodic trajectory (15); the equations for the others are similar.

Equations (26) have the solution $\delta_i=\tau=0$ when $\mu=0$. We can solve and get $(\delta_i;\ \tau)$ as analytic functions of μ if the Jacobian does not vanish when $\delta_i=\tau=0$. A direct computation shows that this Jacobian vanishes if and only if λ_i/λ_1, $i=2, \cdots, k$, is an *odd* integer. The orbits (21) for which (9), (25), and (26) hold pass through the origin for $t^*=0$ and touch Z when $t^*=\pi/(2|\lambda_1|)$; hence, by Theorem 6 they are periodic. We have thus proved the following theorem.

THEOREM 9. *If* (9) *holds, and if* λ_i/λ_j, $i\neq j$, $i=1, \cdots, k$, *is not an odd integer, the jth fundamental periodic orbit of the limiting integrable system can be continued analytically for* $\mu>0$; *each orbit of the continuation is a curve which*

passes through the origin, is symmetric in the origin, joins two points of Z, and is periodic with period $4(\pi/(2|\lambda_j|)+\tau)$.

If the system is symmetric in the origin, the periodic orbits whose existence is established by Theorem 8 are identical with those established by Theorem 9, because the continuation in each case is unique. However, since Theorem 9 fails only when λ_i/λ_j is an odd integer, we see that it proves that analytic continuation is possible in some cases when the first theorem fails.

COROLLARY 1. *At a maximum of the force function U in the case of two degrees of freedom at least one of the fundamental periodic orbits of the limiting integrable system can be continued analytically for $\mu>0$ unless $\lambda_1=\lambda_2$.*

This corollary follows from Theorem 8 and the fact that λ_1/λ_2 and λ_2/λ_1 are not both integers unless $\lambda_1=\lambda_2$. Important use will be made of this corollary in later work.

PART II. MOTION IN THE NEIGHBORHOOD OF A POSITION OF STABLE EQUILIBRIUM IN THE CASE OF TWO DEGREES OF FREEDOM

6. The manifold of states of motion. We continue the study of the dynamical systems of Part I, but we now restrict attention to motion in the neighborhood of a position of stable equilibrium in the case of two degrees of freedom. First we shall investigate the manifold of states of motion M.

The equation of M is $T=U+h/2$. We restrict h henceforth to values for which the region of motion R about the origin on the characteristic surface is homeomorphic to a circular disc. The oval of zero velocity is a simple closed curve Z. There may be other regions of motion for the given value of h, but attention will be confined to the one R about the origin.

Suppose first that h is so restricted that U has only a single critical point in R, a maximum at the origin [see (2) and (4)]. Then the contour curves $U+h/2=c$ are simple closed curves surrounding the critical point. Then a homeomorphism between the points of R and the unit circle C: $u^2+v^2\leqq 1$ can be established as follows. Let the points on the curves $U+h/2=c$ correspond in a one-to-one and continuous manner with the points on the circle $u^2+v^2=(1-2c/h)$, each point (x_1^0, x_2^0) of R† corresponding to a point (u_0, v_0) of C. Then as c varies from $h/2$ to 0, the contour curve expands from the origin and sweeps through R; the corresponding circle expands from the origin and sweeps through C. Now a point of M is obtained by combining the coordinates (x_1^0, x_2^0) of a point of R with the coordinates of a point (y_1, y_2) on the ellipse $T(x_1^0, x_2^0; y_1, y_2)=U(x_1^0, x_2^0)+h/2$ [see (16) for the notation]. This ellipse is real and non-degenerate if (x_1^0, x_2^0) is an interior point of R; it is the

† For convenience, the variables y of §§1 and 2 have been replaced by x.

point ellipse $y_1=y_2=0$ when (x_1^0, x_2^0) is on Z. Let the points on this ellipse correspond to the points on the circle $\xi^2+\eta^2=1-(u_0^2+v_0^2)$, the points on corresponding rays through the origins corresponding. This circle degenerates to a point when and only when the ellipse degenerates to a point. We thus establish a one-to-one and continuous correspondence between the points of M and the unit 3-sphere S_3: $u^2+v^2+\xi^2+\eta^2=1$ in 4-space.

It is possible to give a representation of M in 3-space. Put R into correspondence with C in the way explained above. Then put the points of the ellipse $T(x_1^0, x_2^0; y_1, y_2)=U(x_1^0, x_2^0)+h/2$ into one-to-one and continuous correspondence with the points of the line segment $-[1-(u_0^2+v_0^2)]^{1/2} \leqq w \leqq [1-(u_0^2+v_0^2)]^{1/2}$, the two end points being considered identical, by means of

$$(27) \qquad (\arctan y_2/y_1)[1-(u_0^2+v_0^2)]^{1/2}=\pi w, \qquad -\pi \leqq \arctan y_2/y_1 \leqq \pi.$$

By definition, w shall be zero when y_1 and y_2 vanish simultaneously. We have thus put M into one-to-one and continuous correspondence with the points of the unit sphere S_2: $u^2+v^2+w^2 \leqq 1$, the points (u, v, w) and $(u, v, -w)$ of the bounding sphere being considered identical.

Consider the general case now. Assume that R is homeomorphic to C with no restriction on the number of critical points of U. Then by the method just explained, we can put M into correspondence with S_2 with the stated convention about points of the bounding sphere. But S_2 can be put into one-to-one and continuous correspondence with S_3, the unit 3-sphere in 4-space. We have proved this, because in the first case we put both into correspondence with M. We have thus proved the following theorem.

THEOREM 10. *If R is homeomorphic to a circular disc, then M is homeomorphic to S_3, and also to S_2 with the points (u, v, w) and $(u, v, -w)$ of the bounding sphere considered identical.*

It is obvious how these results are to be extended to dynamical systems with n degrees of freedom.

We have also the following important theorem concerning the steady fluid motion in M.

THEOREM 11. *The steady flow in M possesses an invariant volume integral.*

This result may be established most easily by transforming to Hamiltonian coordinates [Birkhoff 4, p. 212]. The result is well known, and the details are omitted [see also Birkhoff 1, pp. 211–212; Poincaré 1, vol. III, chaps. XXII–XXIII].

7. **The limiting integrable system.** A detailed study of the limiting integrable system will be made now. By setting $n=k=2$ in §3, we find that the equations of motion are

$$\frac{dx_i}{dt} = y_i, \qquad \frac{dy_i}{dt} = \lambda_i^2 x_i \qquad (i = 1, 2) \tag{28}$$

and that the integral of energy is

$$\sum_{i=1}^{2} (y_i^2 + |\lambda_i|^2 x_i^2) = 1. \tag{29}$$

The general solution of (28) and (29) is

$$\begin{aligned} x_i &= \frac{a_i}{|\lambda_i|} \sin(|\lambda_i| t + \theta_i), \qquad a_1^2 + a_2^2 = 1, \\ y_i &= a_i \cos(|\lambda_i| t + \theta_i) \qquad (i = 1, 2). \end{aligned} \tag{30}$$

The region of motion R^0 is the interior of the ellipse $|\lambda_1|^2 x_1^2 + |\lambda_2|^2 x_2^2 = 1$, whose boundary is the oval of zero velocity Z^0. The axes of this ellipse lie along the x_1- and x_2-axes and are respectively the first and second fundamental periodic orbits O_1^0 and O_2^0. The manifold of states of motion M^0 is the ellipsoid (29).

Consider also the representation of M^0 in S_2. Now R^0 is mapped on C by

$$u = |\lambda_1| x_1, \qquad v = |\lambda_2| x_2. \tag{31}$$

The representation in S_2 can be completed as explained in §6. A trajectory corresponds to a curve in S_2 which may be called a stream line or line of flow. Consider in particular the lines of flow which represent O_1^0 and O_2^0. From (27) and (31) we see that O_2^0 is represented by the ellipse

$$v^2 + 4w^2 = 1, \; u = 0. \tag{32}$$

The direction of flow is the same as that of the rotation which carries the positive w-axis into the positive v-axis. Similarly, O_1^0 is represented by a curve in $v=0$. It is composed of the diameter of S_2 which lies along the u-axis and the semi-circle $w=(1-u^2)^{1/2}$, $v=0$ [or $w=-(1-u^2)^{1/2}$, $v=0$]. The flow is such that its direction is positive along the u-axis.

THEOREM 12. *The surface SS^0: $x_1=0$, $y_1 \geqq 0$ is a surface of section for the limiting integrable system.*

From (29) we see that SS^0 is the semi-ellipsoid

$$y_1^2 + y_2^2 + |\lambda_2|^2 x_2^2 = 1, \; y_1 \geqq 0, \tag{33}$$

in the plane $x_1=0$. The boundary is given by $x_1=0$, $y_1=0$; it is the ellipse which bounds the semi-ellipsoid. The boundary of SS^0 is therefore the closed stream line which corresponds to O_2^0. Equations (30) show that any stream line crosses SS^0 when t has a value which satisfies $|\lambda_1|t+\theta_1=2m\pi$, m any integer; hence, every stream line crosses SS^0 an infinite number of times, and the interval of time between any two successive crossings is $2\pi/|\lambda_1|$.

Next we must show that the angle at which a trajectory crosses SS^0 is of the first order in the distance to the boundary. The direction components of the stream line are given by (28); the surface of section is defined as the intersection of the two 3-spaces $\sum_{i=1}^{2}(y_i^2+|\lambda_i|^2x_i^2)=1$, $x_1=0$ with $y_1\geqq 0$. By a straightforward calculation, using the formula developed in the next section for the angle of intersection of a curve and a 2-surface in 4-space, we find that if ψ is the angle at which a stream line crosses SS^0, then

$$\sin\psi = \frac{y_1}{(\lambda_2^4x_2^2+y_1^2+y_2^2)^{1/2}}. \tag{34}$$

Since SS^0 is the semi-ellipsoid (33), it is clear that y_1 may be taken as a measure of the distance of a point on it to the boundary. From (34) it then follows that ψ is of the first order in the distance to the boundary. The fact that every crossing of SS^0 by a stream line is in the same sense follows from $\sin\psi\geqq 0$ in (34), but it will be geometrically obvious when we consider the representation of SS^0 in S_2. Thus SS^0 satisfies all the requirements of Birkhoff's definition of a surface of section [Birkhoff 1, p. 268], and the proof is complete.

Now consider the representation of SS^0 in S_2. Since SS^0 lies in $x_1=0$, (31) shows that the corresponding surface in S_2 lies in $u=0$. Again, since $y_1\geqq 0$ on SS^0, it follows from (27) that $-[1-v^2]^{1/2}/2\leqq w\leqq[1-v^2]^{1/2}/2$; hence, SS^0 is represented by the ellipse E: $v^2+4w^2\leqq 1$. We have seen already that the boundary of E represents O_2^0.

An orbit on which $y_1>0$ corresponds to a stream line in S_2 on which u is increasing. When the orbit crosses the x_2-axis, the stream line in M^0 crosses SS^0, and the stream line in S_2 passes through E. On the other hand, if $y_1<0$ when the orbit crosses the x_2-axis, the stream line in M^0 does not cross SS^0, and the stream line in S_2 passes through $u=0$ on the exterior of E. It is thus possible to visualize the flow in M^0. We observe among other things that all stream lines cross the representation of SS^0 in S_2 in the same sense.

Let a transformation T be defined on SS^0 as follows: a stream line which crosses SS^0 at P has its next succeeding crossing at P' and its kth succeeding crossing at $P^{(k)}$. Then $P'=T(P)$ and $P^{(k)}=T^k(P)$. We proceed to study T.

It is possible to use (x_2, y_2) as coordinates on SS^0 since

$$(35)\qquad \begin{aligned} x_2 &= x_2, \qquad y_2 = y_2, \\ y_1 &= (1 - |\lambda_2|^2 x_2^2 - y_2^2)^{1/2} \end{aligned}$$

is merely a parametric representation of (33) with (x_2, y_2) as the parameters. Then T can be expressed in terms of (x_2, y_2).

Assume that the stream line has its first crossing at $t=0$; then from (30) the coordinates of P are

$$(36)\qquad x_2 = \frac{a_2}{|\lambda_2|} \sin \theta_2, \qquad y_2 = a_2 \cos \theta_2,$$

and the coordinates of $P^{(k)}$ are

$$(37)\qquad \begin{aligned} x_2^{(k)} &= \frac{a_2}{|\lambda_2|} \sin\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi + \theta_2\right), \\ y_2^{(k)} &= a_2 \cos\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi + \theta_2\right). \end{aligned}$$

From (36), (37) we see that T has the invariant function $F = |\lambda_2|^2 x_2^2 + y_2^2$, and that each of the curves $F = a_2^2$ is a path curve of T. From (35) it follows that this path curve is the ellipse $y_1 = [1 - a_2^2]^{1/2}$ on SS^0. By letting a_2 take on all values on $0 \leqq a_2 \leqq 1$, we get a family of ellipses which fill up SS^0. The path curves and F exist because the dynamical system is integrable [Birkhoff 3, pp. 114–115]. There are two integrals of (28) besides (29):

$$(38)\qquad |\lambda_i|^2 x_i^2 + y_i^2 = a_i^2 \qquad (i = 1, 2).$$

Of the three integrals, only two are independent.

In order to see more clearly the nature of T, we transform to new parameters (ξ, η) by means of

$$(39)\qquad \xi = |\lambda_2|\, x_2, \qquad \eta = y_2.$$

Corresponding to (36), (37) the coordinates of P, $P^{(k)}$ are now

$$(40)\qquad \xi = a_2 \sin \theta_2, \qquad \eta = a_2 \cos \theta_2;$$

$$(41)\qquad \begin{aligned} \xi_k &= a_2 \sin\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi + \theta_2\right), \\ \eta_k &= a_2 \cos\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi + \theta_2\right). \end{aligned}$$

Expand the right hand member of (41) and substitute from (40). Then

$$
(42)\qquad
\begin{aligned}
\xi_k &= \quad \xi \cos\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi\right) + \eta \sin\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi\right),\\
\eta_k &= -\xi \sin\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi\right) + \eta \cos\left(\left|\frac{\lambda_2}{\lambda_1}\right| 2k\pi\right),
\end{aligned}
$$

i.e., T^k is a rotation about P^0: $\xi=\eta=0$ of SS^0 into itself. The point P^0 is therefore invariant under T; (39) shows that it corresponds to O_1^0. If $|\lambda_2|/|\lambda_1|$ is a rational fraction p/q, p and q without common factors, then T^q rotates SS^0 through p complete revolutions, and every point is invariant. In this case, every trajectory is closed and periodic. If $|\lambda_2|/|\lambda_1|$ is irrational, P^0 is the only invariant point of T and its iterates, and the only closed and periodic orbits of the system are O_1^0, O_2^0. These results prove the following theorem.

THEOREM 13. *The transformation T on SS^0 is a rotation. The center of rotation P^0 is an invariant point which corresponds to O_1^0. If $|\lambda_2|/|\lambda_1|$ is rational, every trajectory of the system is closed and periodic; if it is irrational, only O_1^0 and O_2^0 are closed and periodic.*

Consider T on E in S_2. The path curves $y_1^2 = a_1^2$ on SS^0 correspond to the curves

$$
(43)\qquad (1 - v^2)^{1/2} \arctan\left[\pm\left(\frac{1 - a_1^2 - v^2}{a_1^2}\right)^{1/2}\right] = \pi w.
$$

As a_1 varies from 0 to 1, we get a family of simple closed curves beginning with the ellipse (32) and shrinking down to its center. The center of rotation on SS^0 corresponds to the center of E, which is therefore an invariant point. We have previously shown that the stream line representing O_1^0 crosses E at its center. We may thus picture T on E as a distorted rotation which carries each of the curves (43) into itself.

8. **A formula in geometry.** We turn aside from our main subject to prove a formula that was used in the last section.

A 2-dimensional surface in 4-space is defined by

$$
(44)\qquad f(x_1, \cdots, x_4) = 0, \qquad \phi(x_1, \cdots, x_4) = 0,
$$

and a unit vector $C:(c_i)$ has its initial end at the point (x_i^0) of the surface. The problem is to obtain a formula for the angle which C makes with (44).

DEFINITION. *The angle which C makes with the surface* (44) *is the complement of the angle between C and the normal to the surface which lies in the* 3-*plane containing C and the tangent* 2-*plane to the surface.*

We assume that the two 3-surfaces in (44) are not tangent at (x_i^0), i. e., we assume that the rank of the matrix

$$\left\| \begin{matrix} f_{x_i} \\ \phi_{x_i} \end{matrix} \right\| \tag{45}$$

is 2 [a subscript letter denotes a partial derivative with respect to that letter]. The tangent 2-plane to the surface at (x_i^0) is given by

$$f_{x_i}(x_i - x_i^0) = 0, \qquad \phi_{x_i}(x_i - x_i^0) = 0. \tag{46}$$

A repeated subscript in a product denotes a summation with respect to that subscript from 1 to 4. The two 3-planes in (46) are distinct since the rank of (45) is 2; taken together, therefore, they define a 2-plane.

Now determine the 3-plane which contains the tangent 2-plane (46) and the given vector C. All 3-planes which contain (46) are given by

$$Af_{x_i}(x_i - x_i^0) + B\phi_{x_i}(x_i - x_i^0) = 0. \tag{47}$$

This plane contains C if and only if it contains its end point $(x_i^0 + c_i)$. Substitute the coordinates of this point in (47) and solve for A, B; the result is

$$A = \phi_{x_i}c_i, \qquad B = -f_{x_i}c_i. \tag{48}$$

Now if A, B as given by (48) are both zero, we see that C lies in the tangent 2-plane (46) of the surface. Then C is tangent to the surface (44).

Assume henceforth that C does not lie in the tangent 2-plane; then A, B are not both zero and the required plane is

$$(\phi_{x_i}c_i)[f_{x_j}(x_j - x_j^0)] - (f_{x_i}c_i)[\phi_{x_j}(x_j - x_j^0)] = 0. \tag{49}$$

The ∞^1 normals to the surface at (x_i^0) have the direction components

$$\rho_1 f_{x_i} + \rho_2 \phi_{x_i}. \tag{50}$$

Now determine ρ_1, ρ_2 so that (50) lies in (49). A point on the vector (50) is $(x_i^0 + \rho_1 f_{x_i} + \rho_2 \phi_{x_i})$. Substitute in (49) and solve for ρ_1, ρ_2. The result is

$$\begin{aligned} \rho_1 &= (f_{x_i}\phi_{x_i})(\phi_{x_j}c_j) - (\phi_{x_i}\phi_{x_i})(f_{x_j}c_j), \\ \rho_2 &= -(f_{x_i}f_{x_i})(\phi_{x_j}c_j) + (\phi_{x_i}f_{x_i})(f_{x_j}c_j). \end{aligned} \tag{51}$$

Substitute these values for ρ_1, ρ_2 in (50), and we have the required normal vector $N:(n_i)$. Then if ψ is the angle at which C crosses the surface (44),

$$\cos\left(\frac{\pi}{2} - \psi\right) = \frac{(c_i n_i)}{(n_i n_i)^{1/2}}. \tag{52}$$

One detail remains. It must be shown that ρ_1, ρ_2 are not both zero, for if

they were, N would be a null vector. The desired result follows from (51) when we assume that (45) is of rank 2, and that A, B in (48) are not both zero.

9. **Some results on surfaces of section and surface transformations.** A common type of surface of section for reversible dynamical systems with two degrees of freedom is formed as follows: Take a closed orbit O without multiple points which either has no point in common with Z, or is an orbit traced with a backward and forward motion between two points of Z. Consider all points $(x_1, x_2; y_1, y_2)$ in M such that (x_1, x_2) is a point of O and (y_1, y_2) is a velocity vector which is tangent to O or lies on a certain specified side of it. These points form a surface Σ.

THEOREM 14. *The surface Σ is an analytic surface.*

Suppose first that O does not pass through a point of Z. Take any point P on O and rotate the axes so that the tangent at P is parallel to the x_2-axis. Then the equation of O near P can be written in the form $x_1 = \phi(x_2)$, where ϕ is analytic. Suppose Σ is formed with the velocity vectors for which $y_1 \geqq y_2\phi'(x_2)$, the prime here denoting a derivative with respect to x_2. Then Σ is defined by

$$\begin{aligned} M(x_1, x_2; y_1, y_2) &\equiv T - U - h/2 = 0, \\ x_1 = \phi(x_2), \qquad & y_1 \geqq y_2\phi'(x_2). \end{aligned} \tag{53}$$

We propose to show that one or the other of the sets (x_2, y_1), (x_2, y_2) can be taken as the parameters of an analytic representation of Σ in the neighborhood of P. Substitute from the second equation in (53) in the first. Then if the equation $M[\phi(x_2), x_2; y_1, y_2] = 0$ gives either y_1 or y_2 as an analytic function of the other two variables, the desired result follows. Now $M = 0$ can be solved for y_i if $\partial M/\partial y_i \equiv \partial T/\partial y_i \neq 0$. The desired result follows then unless both of these partial derivatives vanish. But $\partial T/\partial y_1$, $\partial T/\partial y_2$ vanish simultaneously only at points on Z, and O has no point in common with Z by hypothesis. Hence, Σ is analytic in the neighborhood of P, and since P was any point of O, Σ is analytic throughout.

Now suppose that O joins two points of Z. The proof given above applies to any interior point of O; hence, it will be sufficient to show that the part of Σ arising from points of O near Z is analytic. The orbit is given to us from the equations of motion in the parametric form

$$x_1 = x_1(t), \qquad x_2 = x_2(t). \tag{54}$$

If O touches Z at $P^0: (x_1^0, x_2^0)$, it is not regular there, i.e., both x_1' and x_2' vanish there. We shall show, however, that this state of affairs results from

the fact that the particle reverses its direction of motion there, and not from the nature of the curve itself.

Now if O passes through P^0 at time $t=0$, equations (54) are

$$(55)\qquad \begin{aligned} x_1 &= x_1^0 + a_2t^2 + a_4t^4 + \cdots, \\ x_2 &= x_2^0 + b_2t^2 + b_4t^4 + \cdots, \end{aligned}$$

only even powers of t occurring, because the equations of motion and the initial conditions

$$x_1 = x_1^0,\ y_1 = 0,\ x_2 = x_2^0,\ y_2 = 0, \qquad t = 0,$$

are unchanged when t is replaced by $-t$. Let T and U be

$$\begin{aligned} T &= \frac{1}{2}\sum_{i,j=1}^{2} T_{ij}(x_1, x_2)x_i' x_j', \qquad T_{ij} = T_{ji}, \\ U &= U(x_1, x_2). \end{aligned}$$

Since T is a positive definite quadratic form, we have

$$(56)\qquad |T_{ij}| > 0.$$

From the equations of motion, we find that a_2, b_2 satisfy the equations

$$(57)\qquad \begin{aligned} T_{11}(x_1^0, x_2^0)a_2 + T_{12}(x_1^0, x_2^0)b_2 &= U_{x_1}(x_1^0, x_2^0), \\ T_{21}(x_1^0, x_2^0)a_2 + T_{22}(x_1^0, x_2^0)b_2 &= U_{x_2}(x_1^0, x_2^0). \end{aligned}$$

But since there are no double points on Z by hypothesis [see (8)], we see that a_2, b_2 are not both zero. Suppose $b_2 \neq 0$. Then it is possible to solve the second equation in (55) for t^2, obtaining an analytic function of $(x_2 - x_2^0)$. Use this function to eliminate t^2 from the first equation in (55). We obtain

$$(58)\qquad x_1 = \phi(x_2 - x_2^0) \equiv x_1^0 + \frac{a_2}{b_2}(x_2 - x_2^0) + \cdots,$$

which defines a real analytic curve which crosses Z. The orbit is formed from the part of this curve which lies in R. The irregularity at P^0 is therefore due to the reversal of the direction of motion and not to the nature of the curve itself.

Furthermore, the curve (58) is not tangent to Z at P^0. Using the values of a_2, b_2 as given by (57), we find that the curves are tangent if and only if

$$(59)\qquad T_{22}U_{x_1}^2 - 2T_{12}U_{x_1}U_{x_2} + T_{11}U_{x_2}^2 = 0.$$

But this is impossible, because there are no double points on Z, and the quadratic form is positive definite. Also, (56) and (57) show that if we choose

the axes so that $U_{x_1}=0$ at P^0, then $b_2\neq 0$, and the equation of the curve along which O lies can be written in the form (58) near P^0.

We can now complete the proof that Σ is analytic. Rotate the axes so that Z is parallel to the x_1-axis at P^0. Then $U_{x_1}=0$ at P^0, and near this point the equation of the curve along which O lies can be written in the form (58). Furthermore Σ is defined by the equations

$$\begin{gathered} M(x_1, x_2; y_1, y_2) = 0, \\ x_1 = \phi(x_2 - x_2^0), \qquad y_1 \geqq y_2\phi'(x_2 - x_2^0). \end{gathered} \tag{60}$$

Then (y_1, y_2) can be taken as the parameters on Σ, for substitute from the second equation in (60) in the first. The resulting equation can be solved for x_2 if its partial derivative with respect to x_2 is not zero. At P^0 this partial derivative reduces to $U_{x_2}\neq 0$; hence, x_2 can be expressed analytically in terms of (y_1, y_2). Substitute now for x_2 in the second equation in (60), and we have x_1 also expressed analytically in terms of (y_1, y_2). Thus we have proved that Σ is analytic in all cases.

THEOREM 15. *The angle at which a trajectory crosses Σ is of the first order in the distance to the boundary.*

Now it can be shown that $(y_1-y_2\phi')$ is an infinitesimal of the first order in the distance from a point on Σ to the boundary. Also, Σ is defined by equations and inequalities of which (53) are typical. The direction components of a stream line are dx_1/dt, dx_2/dt, dy_1/dt, dy_2/dt. Let ψ be the angle which the stream line makes with Σ at the point of crossing. Now the two 3-dimensional surfaces in (53) and (60) are never tangent since O is never tangent to Z. Then the formula developed in §8 can be used for determining ψ. Remembering that $dM/dt\equiv 0$ because $M=0$ is the integral of energy, we find by a straightforward calculation that

$$\sin\psi = \frac{-(M_{x_1}^2 + M_{x_2}^2 + M_{y_1}^2 + M_{y_2}^2)(y_1 - y_2\phi')}{\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2\right]^{1/2}(m_i m_i)^{1/2}}. \tag{61}$$

Here $(m_1, \cdots, m_4)$ denotes a vector which differs from the vector $(n_1, \cdots, n_4)$ of §8 only by a factor. An examination shows that it is not a null vector, not even on the boundary of Σ. Since there are no double points on Z, the first factor in the numerator is not zero. For the same reason, the stream lines are regular curves at every point, and the first radical in the denominator does not vanish. Then the theorem follows immediately from the fact that $(y_1-y_2\phi')$ measures the distance to the boundary of Σ.

If O does not touch Z, Σ is bounded by two closed stream lines corresponding to O traced in the two directions, and it is homeomorphic to a ring bounded by two concentric circles. If O joins two points of Z, Σ is bounded by a single closed stream line which corresponds to O, and it is homeomorphic to a circular disc. In all cases Σ is an analytic surface; the angle at which a stream line crosses Σ is of the first order in the distance to the boundary; and all stream lines which cross Σ cross it in the same sense. Then if it can be shown that every trajectory of the system crosses Σ at least once in a given interval of time, it follows from the definition that Σ is a surface of section.

We proceed to the proof of a theorem which gives an important qualitative result on the nature of the orbits of a reversible dynamical system.

THEOREM 16. *If there exists a periodic orbit O joining two points of Z from which a surface of section SS of type Σ can be formed, and if R is homeomorphic to a circular disc, there exists at least one further periodic orbit joining two points of Z.*

In the first place, since O joins two points of Z, SS is bounded by a single closed stream line and is homeomorphic to a circular disc. Since SS is a surface of section, there is an analytic transformation T on it defined in the usual way. Furthermore, T has a certain number of invariant points [Birkhoff 1, p. 287]. To each invariant point P_i an integer δ_i is assigned as follows: Draw the vector from a given point Q in the neighborhood of P_i to its image Q' under T. Then when Q describes a small circle about P_i in the positive direction, the vector rotates through the angle $2\delta_i\pi$. Now the sum of the δ_i for all the invariant points on SS is 1 [Birkhoff 1, p. 290; note that the formula should be $(2q+d-2)$].

Assume now that the theorem is false, i.e., assume that O is the only periodic orbit which joins two points of Z. Then each invariant point of T arises from a closed orbit which does not touch Z. Corresponding to such orbits there are two stream lines in M and they are distinct [Theorem 2]. Since O divides R into two regions, each of these stream lines crosses SS and gives rise to the same number of invariant points of T. Hence, to each invariant point P_i there is a unique second invariant point Q_i; the number of invariant points is even.

Now if it can be shown that P_i and Q_i have the same number δ_i, it will follow that the sum of the δ_i is an even number in contradiction to the fact that it is 1. But δ_i is determined from the equations of variation and is the same for P_i and Q_i. The theorem follows.

At the same time we have proved that under the hypotheses of Theorem 16, the following theorem is true also.

THEOREM 17. *The invariant points of T and its powers are paired, the two points of a pair being distinct unless the corresponding orbit joins two points of Z.*

10. **Analytic continuation of the surface of section.** For the present assume only that $\lambda_1 \neq \lambda_2$. Then by Corollary 1, §5, at least one of the two orbits O_1^0, O_2^0 can be continued analytically for $\mu > 0$. Suppose the notation is so chosen that it is O_2^0 which can be continued. Then for $\mu = 0$ the system has the surface of section SS^0 as described in §7.

For $\mu = 0$ the periodic orbit O_2 reduces to O_2^0, which lies along the line $x_1 = 0$; since O_2 varies analytically with μ, it follows that for μ sufficiently small its equation can be written in the form

$$x_1 = \phi(x_2, \mu), \qquad \phi(x_2, 0) \equiv 0. \tag{62}$$

We shall now show that for each value of μ the surface in M defined by (62) and the inequality $y_1 \geqq y_2 \partial\phi/\partial x_2$ forms a surface of section SS which is the analytic continuation of SS^0.

The surface SS is a surface of the type Σ studied in §9. It is an analytic surface in M which varies analytically with μ and reduces to SS^0 for $\mu = 0$. As shown in §9, it forms a surface of section if every trajectory cuts it at least once in a fixed interval θ of time. It was shown in §7 that this requirement is satisfied for $\mu = 0$. Now the intersections of a given stream line with SS are determined by the intersections of the corresponding orbit with (62). These orbits intersect at an angle different from zero for $\mu = 0$, and since they vary analytically with μ, they continue to intersect at least for $\mu > 0$ but small, with the length of time between successive crossings of SS uniformly bounded. Conceivably this argument might fail in the neighborhood of the boundary of SS, but here we have recourse to the equations of variation for the trajectory corresponding to (62). A detailed consideration shows that every trajectory continues to cross SS for μ sufficiently small, and that the length of time between successive crossings is uniformly bounded; hence, SS is a surface of section as stated.

The definition of T on SS is the same as in previous cases. Now R is homeomorphic to a circular disc [§6], and all the other hypotheses of Theorem 16 are satisfied. This theorem proves that the equations corresponding to (24) have a solution for each value of μ so long as the surface of section exists. Continuation of O_1^0 is therefore possible, but our results do not show that this continuation is unique.

THEOREM 18. *If $\lambda_1 \neq \lambda_2$, there exist at least two periodic orbits joining two points of Z for $\mu > 0$ but small.*

Suppose now that neither λ_1/λ_2 nor λ_2/λ_1 is an integer. Then by Theorem 8 both O_1^0 and O_2^0 can be continued analytically. There is an invariant point P on SS which corresponds to O_1, the continuation of O_1^0. For $\mu=0$, T is a rotation about P^0 through an angle not an integral multiple of 2π. Then δ for P^0 is 1, and P^0 is said to be a simple and stable invariant point [Birkhoff 1, pp. 287–288]. Now the nature of T about P is determined by the characteristic exponents for the corresponding trajectory [Poincaré 1, vol. I, chap. IV]. The characteristic exponents are continuous functions of μ; hence, for $\mu>0$ but small, P is a stable invariant point. Then it is possible to consider SS a ring surface with P the inner boundary. We shall show that under certain conditions Poincaré's Last Geometric Theorem can be applied to the transformation on this ring [Poincaré 2; Birkhoff 2, and 1, p. 294].

In the first place, T is an analytic transformation of the ring into itself which carries the boundaries into themselves. In the second place, T has an invariant area integral as a result of Theorem 11 [Birkhoff 1, p. 285]. Finally, there are rotation numbers α_B and α_P associated with the transformation on the two boundaries of the ring [Birkhoff 3, pp. 87–88]. We have shown in §7 that for $\mu=0$ the two rotation numbers are equal. We shall show by means of an example, however, that α_B and α_P are functions of μ and in general are not equal. Then Poincaré's considerations prove the following theorem [Poincaré 2, §3; Birkhoff 1, pp. 297–298].

THEOREM 19. *If $\alpha_B\neq\alpha_P$, there exist infinitely many periodic orbits.*

The limiting integrable system affords a good example to show that the conclusion of this theorem may not hold if $\alpha_B=\alpha_P$ [see §7].

As an example to show that α_B and α_P are not identically equal, consider the system for which the equations of motion and the integral of energy are

$$\begin{aligned} x_1'' &= -|\lambda_1|^2 x_1 + 2\mu^2 x_1^3, \\ x_2'' &= -|\lambda_2|^2 x_2, \\ y_1^2 + y_2^2 &+ |\lambda_1|^2 x_1^2 + |\lambda_2|^2 x_2^2 - \mu^2 x_1^4 = 1. \end{aligned} \tag{63}$$

For all values of μ there are two periodic orbits O_1, O_2 joining two points of Z, and they lie along the two axes. The surface of section SS is defined by $x_1=0$, $y_1\geqq 0$. As in the limiting system $\mu=0$, we may take (x_2, y_2) as the coordinates on the surface.

For all values of μ the point $x_2=y_2=0$ is the invariant point P on SS and corresponds to O_1. The equation of O_1 is $x_1=x_1(t, \mu)$, $x_2=0$; its period is $[2\pi/|\lambda_1|+\tau_1(\mu)]$, where $\tau_1(0)=0$. The period is independent of the amplitude only for simple harmonic motion, however; hence, $\tau_1(\mu)\not\equiv 0$.

Consider α_P first. A nearby orbit to O_1 is $x_1 = x_1(t, \mu) + \xi_1$, $x_2 = \xi_2$, and from (63) we find that the equations of variation are

$$\begin{aligned} \xi_1'' &= -|\lambda_1|^2\xi_1 + 6\mu^2[x_1(t, \mu)]^2\xi_1, \\ \xi_2'' &= -|\lambda_2|^2\xi_2. \end{aligned} \tag{64}$$

From the second of these equations we obtain

$$\begin{aligned} \xi_2 &= \xi_2^0 \cos|\lambda_2|t + (\eta_2^0/|\lambda_2|)\sin|\lambda_2|t, \\ \eta_2 &= -|\lambda_2|\xi_2^0 \sin|\lambda_2|t + \eta_2^0 \cos|\lambda_2|t. \end{aligned} \tag{65}$$

Here (ξ_2, η_2) are coordinates in the neighborhood of P and correspond to (x_2, y_2), and (ξ_2^0, η_2^0) is the point on SS through which the stream line passes at time $t=0$. To find the point into which it is carried by T, we have only to set $t = [2\pi/|\lambda_1| + \tau_1(\mu)]$ in (65). By introducing new coordinates as was done in §7, we show that the limiting transformation at P is a rotation through the angle

$$-|\lambda_2|\,[2\pi/|\lambda_1| + \tau_1(\mu)]. \tag{66}$$

Then α_P is given by (66).

Now consider α_B. The orbit corresponding to the stream line which forms the boundary B is $x_1 = 0$, $x_2 = \sin|\lambda_2|t$, which is periodic with period $2\pi/|\lambda_2|$. The equations of variation are found in the usual way to be

$$\begin{aligned} \xi_1'' &= -|\lambda_1|^2\xi_1, \\ \xi_2'' &= -|\lambda_2|^2\xi_2. \end{aligned} \tag{67}$$

The first equation determines the intersections of the varied stream line with SS; the stream line crosses SS when $\xi_1 = 0$, $\eta_1 \geqq 0$. As in previous cases, we can use (ξ_2, η_2) as coordinates on SS near B. Then equations (65) hold. A stream line which crosses SS at $t=0$ has its next crossing at $t = 2\pi/|\lambda_1|$. Then (65) show that T on B is essentially a rotation through the angle

$$-2\pi|\lambda_2|/|\lambda_1|. \tag{68}$$

Then α_B is given by (68).

Comparing (66) and (68), we see that α_P and α_B are in general not equal since $\tau_1(\mu) \not\equiv 0$. Our conclusion is the following: *If the system really depends on μ, i.e., if it is not identical with the limiting integrable system for all values of μ, then α_P and α_B are not equal in general for $\mu > 0$.*

11. Symmetric systems. We shall now suppose that the system is symmetric in the origin on the characteristic surface, i.e., we assume that (9) holds. As we have already seen, a system of this kind has special properties, which we shall now study in greater detail.

Assume now that neither $|\lambda_1|/|\lambda_2|$ nor $|\lambda_2|/|\lambda_1|$ is an odd integer. Then by Theorem 9 both O_1^0 and O_2^0 can be continued analytically for $\mu>0$; for each value of μ these orbits pass through the origin and are symmetric in this point. A surface of section SS can be formed from either one of these orbits; let it be formed from O_2. All the results of §10 apply in the present case, but the symmetry leads to special properties of T. In order to state the results more easily, we employ the representation of M in S_2 [see §6].

It is clear that R can be deformed into $C: u^2+v^2 \leqq 1$ in such a way that symmetric points are carried into points symmetric in the center of C, and so that O_1 and O_2 lie along the diameters $v=0$ and $u=0$ of C respectively. Then the stream lines in S_2 have the essential properties of symmetry possessed by the stream lines in M; also SS is represented in S_2 by the ellipse E in the plane $u=0$.

Now it was shown in §2 that the stream lines are related by fours. The significance of this fact is that there are two transformations V_1 and V_2 which when applied to a stream line and its transforms by V_1 and V_2 yield four and only four stream lines. In S_2 the transformation V_1 is

$$\begin{aligned} u' &= u, \qquad v' = v, \\ w' &= w + [1 - (u^2 + v^2)]^{1/2}, \end{aligned} \tag{69}$$

coupled with a reversal of the direction of flow [see Theorem 1 and (27)]. By reversal of the direction of flow, we mean the following: if the flow proceeds from P to Q on the given stream line T^*, it proceeds from Q' to P' on V_1T^*. If necessary w' in (69) is to be reduced modulo $2[1-(u^2+v^2)]^{1/2}$. As shown in §2, T^* and V_1T^* correspond to a single orbit traced in the two directions. Obviously $V_1^2=I$, the identity.

In S_2 we find that V_2 is defined by

$$\begin{aligned} u' &= -u, \qquad v' = -v, \\ w' &= w + [1 - (u^2 + v^2)]^{1/2}, \end{aligned} \tag{70}$$

without reversal of the direction of flow [see Theorem 4 and (27)]. Again w' is to be reduced modulo $2[1-(u^2+v^2)]^{1/2}$ when necessary. Then $V_2^2=I$.

Now V_1 and V_2 generate a group with the four distinct transformations I, V_1, V_2, V_1V_2. We see that V_1V_2 is merely a reflection in the w-axis in S_2 with reversal of the direction of flow, and that $(V_1V_2)^2=I$. By applying the transformations of this group to T^*, we obtain three others. The four stream lines are permuted among themselves by any transformation of the group. Now the four are not always distinct [see §2]. If the orbit corresponding to any one of them is its own symmetric image or is a curve touching Z, there are

at most two distinct stream lines; if it is both, the four are identical. In all other cases the four are distinct.

Now if T^* is closed, the other three stream lines of a group are closed; hence, with the obvious convention in case they are not all distinct, we have the following theorem.

THEOREM 20. *The invariant points of T and its powers and the closed periodic trajectories of the system occur in groups of four.*

Now SS is represented by E in the plane $u=0$; hence, T may be expressed in terms of the coordinates (v, w). Since (V_1V_2) carries a stream line into a stream line with reversal of the direction of flow, we see that *if T^k carries (v_0, w_0) into (v_1, w_1), then T^k also carries $(-v_1, w_1)$ into $(-v_0, w_0)$.* It follows that $(V_1V_2)T(V_1V_2)T=I$. Then $(V_1V_2)T$ is a transformation U with period 2: $(V_1V_2)T=U$, $U^2=I$. Hence, $T=(V_1V_2)U$. We have thus proved the following theorem.

THEOREM 21. *The transformation T is the product of two transformations, one of which is a reflection in the w-axis, and both of which have the period* 2.

Now suppose that T^k carries a point $(0, w_0)$ into $(0, w_1)$; then by the italicized statement above, T^k also carries $(0, w_1)$ into $(0, w_0)$, and both points are invariant under T^{2k}. They correspond to a single closed stream line. Let the segment $v=0$ on E be denoted by AB. A point on AB corresponds to an orbit passing through the center of symmetry, and the above statement is equivalent to Theorem 5. Thus, if we can prove that there are points on AB which are transformed into points on AB, we can conclude that there exist closed periodic orbits passing through the center of symmetry.

In the first place, there exists an invariant point P of T on AB. It is the center of E, the point at which the stream line corresponding to O_1 crosses E. Now consider the images of AB on E under T and its powers. If the rotation numbers α_P and α_B [see §10] are unequal, the image of AB under T and its powers is a spiral, and we can assert that there is an integer N such that the image of AB under T^k for $k \geqq N$ intersects AB in points distinct from P. This is proved as follows. The rotation numbers for T^k on the boundary B of E and at P are $k\alpha_B$ and $k\alpha_P$. Then for k sufficiently large, say $k=N$, $k\alpha_B$ and $k\alpha_P$ correspond to transformations differing by at least one complete cycle; for $k=\rho N$, they differ by at least ρ cycles. Hence, the image of AB under T^k intersects AB at least for $k \geqq N$, and the number of such intersections becomes infinite with k.

Suppose that Q_1 on AB is carried into Q_2, distinct from Q_1, on AB by T^m, and that m is the smallest power of T for which this happens. Then Q_1, Q_2

are invariant under T^{2m} and correspond to a periodic orbit O which passes twice through the center of symmetry; the two branches there have distinct tangents since Q_1, Q_2 are distinct. The corresponding stream line crosses SS $2m$ times. Since R is divided into two parts by O_2, we see that O crosses O_2 twice for each crossing of SS by the stream line, i.e., O crosses O_2 $4m$ times. We therefore say this orbit is of type O_{4m}. An orbit O_{4m} cannot touch Z, because it has two distinct branches at the center of symmetry. We have thus proved the following theorem.

THEOREM 22. *If $\alpha_B \neq \alpha_P$, there exists an infinite number of closed periodic orbits of type O_{4m}, there being one or more for each $m \geqq N$.*

We proceed to establish the existence of an infinite number of periodic orbits of a second type.

The transformation

$$x_i' = x_i, \qquad y_i' = -y_i \qquad (i = 1, 2), \tag{71}$$

transforms M into itself and in particular carries SS into a surface SS' which is also a surface of section. The surfaces SS and SS' are bounded by the same closed stream line, and taken together they form a surface homeomorphic to a 2-sphere. In S_2 the transformation (71) is V_1; hence, SS' is represented in S_2 by E', the part of the circle $u=0$ which lies outside of E. By means of (71) we extend the definition of T on SS to the combined surface $SS+SS'$. Each half of this surface is a surface of section; hence, the stream lines define a transformation of it into itself. Since O_2 divides R into two parts, a stream line which crosses SS (SS') at Q has its first succeeding crossing of the surface at Q' on SS' (SS). Then the new transformation is that which carries Q into Q'; we designate it by $T^{1/2}$ since it has the obvious property that its square is T. Also $T^{1/2}$ is not a sense-preserving transformation; it has the nature of a reflection.

We return to the representation in S_2 in order to simplify the exposition. The stream line which crosses E at P also crosses E' at P'. Then $P' = T^{1/2}(P)$. Associated with the transformation of P into P' by T there is a rotation number which is obviously $\alpha_P/2$; the common boundary B of E and E' is transformed into itself by $T^{1/2}$, and the corresponding rotation number is $\alpha_B/2$.

Suppose $T^{k+1/2}$, k an integer, carries (v_0, w_0) into (v_1, w_1). Apply the transformation (V_1V_2), and we see that $T^{k+1/2}$ also carries $(-v_1, w_1)$ into $(-v_0, w_0)$. Then as before $(V_1V_2)T^{1/2}(V_1V_2)T^{1/2} = I$. Set $(V_1V_2)T^{1/2} = V$. Then $T^{1/2} = (V_1V_2)V$, where $V^2 = I$. Thus by defining T on the entire surface $SS+SS'$

we are able to factor T into the product of four factors, each with the period $2: T=(V_1V_2)V(V_1V_2)V$.

Now if $T^{k+1/2}$ carries $(0, w_0)$ into $(0, w_1)$, it also carries $(0, w_1)$ into $(0, w_0)$, and both points are invariant under T^{2k+1}. One of these points is on E. Let $A'B'$ designate the line segment $v=0$ on E'. Then if we can show that the image of AB under $T^{k+1/2}$ intersects $A'B'$, we can prove the existence of further periodic orbits. Now the image of AB under $T^{k+1/2}$ is a spiral which always intersects $A'B'$ at P'. For k sufficiently large, and at least for $k>N$, this spiral intersects $A'B'$ at points other than P'. The number of such intersections becomes infinite with k. Each such intersection gives an invariant point on E under T^{2k+1}.

Suppose that Q is such an invariant point on E under T^{2m+1}, and that this is the lowest power of T under which it is invariant. The corresponding orbit O passes twice through the center of symmetry and crosses O_2 $(4m+2)$ times. This orbit is therefore said to be of type O_{4m+2}. The orbits of this type may or may not touch Z. The orbit O_1, which corresponds to P on E, is included in this class with $m=0$. We have thus proved the following theorem.

THEOREM 23. *If $\alpha_B \neq \alpha_P$, there exist an infinite number of periodic orbits of type O_{4m+2}, there being one or more for each $m \geqq N$.*

Similar results can be obtained if it is assumed that the system is symmetric in one or both of the axes on the characteristic surface.

BIBLIOGRAPHY

BIRKHOFF, G. D.

1. *Dynamical systems with two degrees of freedom*, these Transactions, vol. 18 (1917), pp. 199–300.
2. *Proof of Poincaré's geometric theorem*, these Transactions, vol. 14 (1913), pp. 14–22.
3. *Surface transformations and their dynamical applications*, Acta Mathematica, vol. 43 (1922), pp. 1–119.
4. *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. 9 (1927).
5. *The restricted problem of three bodies*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 265–334.
6. *On the periodic motions of dynamical systems*, Acta Mathematica, vol. 50 (1927), pp. 359–379.

HORN, J.

1. *Beiträge zur Theorie der kleinen Schwingungen*, Zeitschrift für Mathematik und Physik, vol. 48 (1902–03), pp. 400–434.

MORSE, M.

1. *The foundations of a theory in the calculus of variations in the large*, these Transactions, vol. 30 (1928), pp. 213–274.
2. *On a one-to-one representation of geodesics on a surface of negative curvature*, American Journal of Mathematics, vol. 43 (1921), pp. 33–51.
3. *Recurrent geodesics on a surface of negative curvature*, these Transactions, vol. 22 (1921), pp. 84–100.

MOULTON, F. R.

1. *Differential Equations*, Macmillan, 1930.

PAINLEVÉ, P.

1. *Sur les petits mouvements périodiques des systèmes*, Comptes Rendus, vol. 124 (1897), pp. 1222–1225.

POINCARÉ, H.

1. *Méthodes Nouvelles de la Mécanique Céleste*, vol. I (1892), vol. III (1899), Gauthier-Villars.

2. *Sur un théorème de géométrie*, Rendiconti del Circolo Matematico di Palermo, vol. 33 (1912), pp. 375–407.

PRICE, G. B.

1. *A class of dynamical systems on surfaces of revolution*, American Journal of Mathematics, vol. 54 (1932), pp. 753–768.

2. *On the Strömgren-Wintner natural termination principle*, American Journal of Mathematics, vol. 55 (1933), pp. 303–308.

WHITTAKER, E. T.

1. *Analytical Dynamics*, 3d edition (1927), Cambridge University Press.

UNIVERSITY OF ROCHESTER,
ROCHESTER, N. Y.