SINGULAR QUADRATIC FUNCTIONALS*

BY

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Introduction. Singular quadratic functions of the type (1.1) have been briefly investigated by Kemble [1] in connection with Quantum Mechanics. The results of Kemble are of a relatively restricted character. On the other hand, Hardy, Littlewood, and Pólya,† in Chapter VII of [1] have studied special examples of these functionals employing special methods. The developments of this paper apparently give a systematic approach to the problem of minimizing singular quadratic functionals of the type (1.1). In particular, the results obtained include and generalize Theorems (254) and (253) of H.L.P. See Examples 9.2 and 12.1 of this paper.

The authors have admitted various classes of comparison curves. The results show in striking fashion how the existence of the minimum depends upon the classes of curves admitted.

The problem requires a remodeling of the conjugate point theory and an introduction of a new condition called the singularity condition. Lebesgue integrals or their extensions are used throughout. The results of this paper will be applied to extend the theory of characteristic roots and solutions of the related boundary problems.

I. One singular end point

1. The functional. We consider a function

$$f(x, y, y') = r(x)y'^2 + 2q(x)yy' + p(x)y^2, \tag{1.1}$$

where r, q, and p are single-valued, continuous functions of the real variable x on the open interval‡ $(0, d)$ and r is positive. We choose a constant b on the interval $(0, d)$ and consider the functional

$$J(y)\Big|_e^b = \int_e^b f(x, y, y')dx \qquad (0 < e < b < d). \tag{1.2}$$

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† We shall refer to these authors as H.L.P.

‡ In designating intervals it will be convenient to use the conventions:

$[a, b]$ means the interval $a \leqq x \leqq b$,
$(a, b]$ means the interval $a < x \leqq b$,
(a, b) means the interval $a < x < b$,
$[a, b)$ means the interval $a \leqq x < b$.

We shall term $y(x)$ and the curve $y=y(x)$ A-admissible on $[0, b]$ if

1. $y(x)$ is continuous on $[0, b]$ and $y(0)=y(b)=0$;
2. $y(x)$ is absolutely continuous and $y'^2(x)$ is in L on each closed subinterval of $(0, b]$.

On an A-admissible curve y'^2 will not in general be in L on $[0, b]$. For example,

$$y = x^{1/2} \cos x$$

is A-admissible on $[0, \pi]$ but y'^2 is not in L on this interval. Our A-admissible functions $y(x)$ are thus less restricted than the admissible functions of H.L.P.*

Observe that the segment $[0, b]$ of the x-axis is A-admissible and that on this segment $J=0$. We shall seek conditions under which

$$\liminf_{x=0} \int_x^b f(x, y, y')dx \geqq 0 \qquad (x > 0).\dagger \tag{1.3}$$

If (1.3) holds for a given class of curves $y=y(x)$ we shall say that $[0, b]$ affords a *minimum limit* to J among curves of the given class. This minimum limit will be termed *proper* if the equality sign in (1.3) holds at most when $y\equiv 0$.

As we shall see in later sections there are functionals J and intervals $[0, b]$ such that $[0, b]$ does not afford a minimum limit to J among A-admissible curves, but such that $[0, b]$ does afford a minimum limit to J if the class of A-admissible curves be further restricted. This fact is particularly important in the cases where the Euler equations have a regular singular point at $x=0$.

2. **Conjugate points.** We seek the analogue of the Jacobi condition. Classical definitions however of the point conjugate to a given point P break down in case P is the point (0, 0) and the theory of conjugate points requires additional development.

We begin with the Euler equation (written E.E.)

$$\frac{d}{dx}(ry' + qy) - (qy' + py) = 0. \tag{2.1}$$

In general the E.E. is singular at $x=0$. If $u(x)$ is a solution of the E.E. of class C^2 on $(0, d)$, we shall term the curve $y=u(x)$ an *extremal*.

We come to the definition of the first conjugate point of $x=0$. Let $x=a$

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* Hardy, Littlewood, and Pólya [1], Chapter VII.

† This restriction will be understood in the future.

be a point on $(0, b)$ and $x(a)$ the first conjugate point of $x=a$ following $x=a$ on the interval (a, d), if this conjugate point exists. If $x(a)$ exists for a value $a=a_1$, it also exists for $0<a<a_1$ and decreases with a; the *first conjugate point* $x=\alpha$ of $x=0$ on $[0, d)$ is then defined as the limit of $x(a)$ as a tends to zero. If $x(a)$ exists for no value of a on $(0, b)$, $x=0$ will be said to have no conjugate point on $[0, d)$. The first conjugate point of $x=0$ may coincide with $x=0$ as we shall see in §7.

We shall prove the following theorem.

THEOREM 2.1. *If the first conjugate point $x=\alpha$ of $x=0$ exists and $\alpha \neq 0$, there exists a solution $u(x)$ of the* E.E. *such that $u(\alpha)=0$ and $u(x)>0$ on $(0, \alpha)$. If there is no point on $[0, d)$ conjugate to $x=0$, there exists a solution $v(x)>0$ on $(0, d)$.*

It is sufficient to take $u(x)$ in the theorem as the solution for which

$$u(\alpha) = 0, \qquad u'(\alpha) = -1.$$

To determine a solution $v(x)$ which satisfies the theorem let a and b be two numbers such that $0<a<b<d$ and let $y(x, a)$ be a solution such that

$$y(a, a) = 0, \qquad y(b, a) = 1.$$

Observe that $y'(b, a)$ decreases as a decreases and tends to a limit λ as a tends to zero. The solution $v(x)$ such that

$$v(b) = 1, \qquad v'(b) = \lambda,$$

satisfies the theorem.

We seek to determine the first conjugate point of $x=0$ in terms of solutions of the E.E. characterized neighboring $x=0$. We begin with the following lemma.

LEMMA 2.1. *If there exist linearly independent solutions $u(x)$, $v(x)$ of the* E.E. *for which*

$$\lim_{x=0} \frac{u(x)}{v(x)} = k, \tag{2.2}$$

where k is a finite constant, the solution

$$w(x) = u(x) - kv(x) \tag{2.3}$$

has the following properties.

If $y(x)$ is a solution of the E.E. *which is linearly independent of $w(x)$,*

$$\lim_{x=0} \frac{w(x)}{y(x)} = 0 \tag{2.4}$$

and

$$\frac{w'(x)}{w(x)} - \frac{y'(x)}{y(x)} > 0 \tag{2.5}$$

for sufficiently small positive values of x.

Observe that

$$\lim_{x=0} \frac{w(x)}{v(x)} = 0, \tag{2.6}$$

so that $w(x)$ and $v(x)$ are linearly independent. Any solution $y(x)$ linearly independent of $w(x)$ can be represented in the form

$$y(x) = c_1 w(x) + c_2 v(x) \qquad (c_2 \neq 0),$$

and it follows from (2.6) that (2.4) holds as stated.

To establish (2.5) write the left member of (2.5) in the form

$$\frac{w'}{w} - \frac{c_1 w' + c_2 v'}{c_1 w + c_2 v} = \frac{c_2(w'v - wv')}{wy} = \frac{1}{2}\left(\frac{c_2 v}{y}\right)\left(\frac{v}{w}\right)^2 \frac{d}{dx}\left(\frac{w}{v}\right)^2. \tag{2.7}$$

The first parenthesis in (2.7) tends to one as x tends to zero. Moreover for x sufficiently small and positive

$$\frac{d}{dx}\left(\frac{w}{v}\right)^2 = 2\left(\frac{w}{v}\right)\frac{w'v - wv'}{v^2} \neq 0,$$

since the solutions $w(x)$ and $v(x)$ are independent. It follows from (2.6) that

$$\frac{d}{dx}\left(\frac{w}{v}\right)^2 > 0$$

for x sufficiently small and positive.

The proof of the lemma is complete.

We continue with the two following lemmas.

LEMMA 2.2. *If the solution $w(x)$ in the preceding lemma has a first positive zero $x=c$, any solution $y(x)$ which is independent of $w(x)$ vanishes on* $(0, c)$.

The solutions $w(x)$ and $y(x)$ satisfy (2.5) for x sufficiently small and positive. The lemma follows at once from a Sturm comparison theorem.

LEMMA 2.3. *If $x=0$ is not its own first conjugate point, the ratio u/v of any two linearly independent solutions tends to a finite limit or becomes infinite as x tends to zero.*

The lemma follows at once upon computing the derivative of u/v.

The preceding lemmas lead to the following theorem.

THEOREM 2.2. *If $x=0$ is not its own first conjugate point, there exists a solution $w(x)\not\equiv 0$ such that for each solution $y(x)$ independent of $w(x)$*

$$\lim_{x=0} \frac{w(x)}{y(x)} = 0. \tag{2.8}$$

Any solution with this property of $w(x)$ is linearly dependent on $w(x)$.

The first conjugate point of $x=0$, if it exists, is the first positive zero of $w(x)$, and the first positive zero of $w(x)$, if it exists, is the first conjugate point of $x=0$.

We see from Lemma 2.3 that there exist two linearly independent solutions u and v such that the ratio u/v tends to a finite limit, as x tends to zero. It follows from Lemma 2.1 that there exists a solution $w(x)$ for which (2.8) holds for each solution $y(x)$ independent of $w(x)$, and it follows from (2.8) that the only solutions with this property of $w(x)$ are dependent on $w(x)$.

It remains to prove the two concluding statements in the theorem.

Suppose that $x=\alpha$ is a first conjugate point on $(0, b]$ of $x=0$. It follows from the definition of a conjugate point and from the Sturm Separation Theorem* that $w(x)$ vanishes on $(0, \alpha]$. But according to Theorem 2.1 there exists a solution $u(x)$ which vanishes at $x=\alpha$ but is positive on $(0, \alpha)$. It follows from Lemma 2.2 that $w(x)\neq 0$ on $(0, \alpha)$, for otherwise $u(x)$ would vanish on $(0, \alpha)$. Hence $w(\alpha)=0$, and we have shown that the first conjugate point of $x=0$, if it exists, is the first positive zero of $w(x)$.

Suppose finally that c is a first zero of $w(x)$ on $(0, d)$. Let $z(x)$ be a solution such that $z(x)\not\equiv 0$ and $z(a)=0$ where $c<a<d$. If a differs from c sufficiently little, $z(x)$ will be independent of $w(x)$ and by virtue of Lemma 2.2 must vanish on $(0, c)$. It follows from the definition of the first conjugate point of $x=0$ that $x=0$ possesses a first conjugate point $x=\alpha$. According to the result of the preceding paragraph $c=\alpha$, and we have shown that the first positive zero of $w(x)$, if it exists, is the first conjugate point of $x=0$.

The proof of the theorem is complete.

A solution $w(x)\not\equiv 0$ of the E.E. *such that for each solution $y(x)$ independent of $w(x)$,*

$$\lim_{x=0} \frac{w(x)}{y(x)} = 0$$

will be called a focal solution belonging to $x=0$.

As we have seen when $x=0$ is not its own first conjugate point, the first

* See Bôcher [1].

positive zero of a focal solution is identical with the first conjugate point of $x=0$.

We can now establish the analogue of the Jacobi necessary condition.

THEOREM 2.3. *If* $[0, b]$ *affords a minimum limit to* J *among* A*-admissible curves, there can be no point conjugate to* $x=0$ *on the interval* $[0, b)$.

Suppose $x=\alpha$ on $[0, b)$ is the first conjugate point of $x=0$. By virtue of the definition of such a conjugate point there will exist a solution $v(x)\not\equiv 0$ of the E.E. which vanishes at least twice on $(0, b)$, say at $x=x_0$ and $x=x_1$, where

$$0 < x_0 < x_1 < b.$$

But $[x_0, b]$ affords a minimum to J among neighboring curves of class D^1 which join its end points. Hence a point $x=x_1$ conjugate to $x=x_0$ cannot exist on (x_0, b).

The theorem follows at once.

3. **The Hilbert integral.**† Suppose there is no point conjugate to $x=0$ on $[0, b)$. The solution $w(x)$ such that

$$w(b) = 0, \qquad w'(b) = -1,$$

will not vanish on $(0, b)$. The family of extremals

$$y = \alpha w(x) \qquad (\alpha \text{ constant}) \tag{3.1}$$

will form a field in the region S of the xy-plane for which $0<x<b$. The slope $\lambda(x, y)$ of the extremal of the field through the point (x, y) is given by the equation

$$\lambda(x, y) = y\frac{w'(x)}{w(x)} \qquad (0 < x < b). \tag{3.2}$$

The *Hilbert integral* corresponding to the field (3.1) is a line integral of the form

$$H = \int [f(x, y, \lambda) - \lambda f_{y'}(x, y, \lambda)]dx + f_{y'}(x, y, \lambda)dy, \tag{3.3}$$

where $\lambda=\lambda(x, y)$. The equation (3.1) determines a transformation from the variables x, y to variables x, α. For x, y on S this transformation is one-to-one. In terms of x, α the Hilbert integral takes the form

$$H^* = \int A(x, \alpha)dx + B(x, \alpha)d\alpha, \tag{3.4}$$

† For a treatment of the Hilbert integral for non-singular functionals see Morse [1], p. 13 ff.

where

$$A(x, \alpha) = f[x, \alpha w(x), \alpha w'(x)],$$
$$B(x, \alpha) = w(x)f_{y'}[x, \alpha w(x), \alpha w'(x)].$$

When f is given by (1.1), the line integral (3.4) reduces to

$$H^* = \int \alpha^2(rw'^2 + 2qww' + pw^2)dx + 2\alpha(rww' + qw^2)d\alpha. \tag{3.5}$$

Let S^* denote the region of the $x\alpha$-plane in which $0<x<b$. Equation (3.1) defines a one-to-one transformation of S^* into S. Let S_b^* be the point set sum of S^* and the line $x=b$ in the $x\alpha$-plane. Under the transformation (3.1) every point (x, α) of the line $x=b$ is carried into the point $(b, 0)$ of the xy-plane.

The functions $A(x, \alpha)$ and $B(x, \alpha)$ are continuous in x and α for (x, α) on S_b^* and, as is well known (cf. Morse [1], p. 13 ff), there exists a function $K(x, \alpha)$ of class C^1 in x, α on S_b^* such that

$$dK = Adx + Bd\alpha.$$

Let γ be a continuous curve on S_b^* in the $x\alpha$-plane of the form

$$x = x(t), \qquad \alpha = \alpha(t) \qquad (t_1 \leqq t \leqq t_2).$$

If γ is rectifiable, the line integral (3.5) along γ exists as a Lebesgue line integral.† If further $x(t)$ and $\alpha(t)$ are absolutely continuous, this line integral equals the Lebesgue integral

$$\int_{t_1}^{t_2} \left\{A[x(t), \alpha(t)]\frac{dx}{dt} + B[x(t), \alpha(t)]\frac{d\alpha}{dt}\right\}dt.$$

Moreover H^* is independent of paths γ joining two fixed points in S_b^* provided $x(t)$ and $\alpha(t)$ are absolutely continuous. For

$$\frac{dK}{dt} = A\frac{dx}{dt} + B\frac{d\alpha}{dt}$$

almost everywhere on γ, and since $K[x(t), \alpha(t)]$ is absolutely continuous in t,

$$K[x(t), \alpha(t)]_{t_1}^{t_2} = \int_{t_1}^{t_2}\left[A\frac{dx}{dt} + B\frac{d\alpha}{dt}\right]dt = \int_\gamma Adx + Bd\alpha.$$

Thus along a curve γ for which $x(t)$ and $\alpha(t)$ are absolutely continuous and $0<x\leqq b$, the line integral H^* given by (3.5) *depends only on the end points of γ.*

† de la Vallée Poussin [1], p. 383.

Referring to (3.5) we see that for b fixed

$$B(b, \alpha) \equiv 0.$$

It follows that

$$K(b, \alpha) \equiv K(b, 0). \tag{3.6}$$

Thus for curves γ which start as a fixed point P on S^* and terminate at an arbitrary point Q on the line $x=b$, the value of H^* is independent of Q.

When interpreted in terms of the integral H in the xy-plane, the foregoing theory yields the following.

Let S_b be the point set sum of S and the point $(b, 0)$ in the xy-plane. Under the transformation (3.1) S_b^* goes into S_b. Let g be a curve on S_b of the form

$$x = x(t), \qquad y = y(t) \qquad (t_1 \leqq t \leqq t_2),$$

where $x(t)$ and $y(t)$ are absolutely continuous. If g lies on S, the Hilbert integral exists and depends only on the end points of g. If g terminates at the point $(b, 0)$, H still exists and depends only on its first end point provided

$$\frac{y(t)}{w[x(t)]} = \alpha(t)$$

is bounded as t tends to t_2. This follows from (3.6) and the continuity of $K(x, \alpha)$.

An extension of the Weierstrass formula. Let $y(x)$ be an A-admissible curve and $u(x)$ a solution of the E.E. not null neighboring $x=0$. Set

$$S[y, u] = -\frac{y^2}{u}(ru' + qu). \tag{3.7}$$

If, in particular, $u(x)$ is a solution which vanishes at $x=b$ but is not identically null, we set

$$S[y, u] = s[y, b],$$

observing that $S[y, u]$ does not depend upon the particular choice of a non-null solution $u(x)$ which vanishes at $x=b$. We term $s[y, b]$ the *singularity function* belonging to $x=b$.

Upon recalling the definition of the Weierstrass E-function† one finds that for our functional J,

$$E[x, y, \lambda, \mu] = r(x)(\lambda - \mu)^2.$$

We are ready for the following extension of the Weierstrass formula.

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† Morse [1], p. 5.

THEOREM 3.1. *If $y(x)$ is A-admissible on $[0, b]$ and of class C^1 neighboring $x=b$, if there is no point conjugate to $x=0$ on $[0, b)$, then*

$$J(y)\Big|_e^b = \int_e^b E[x, y(x), y'(x), p(x, y(x))]dx + s[y(e), b], \qquad (0 < e < b), \tag{3.8}$$

where $p(x, y)$ is the slope at the point (x, y) of the field F of extremals through the point $x=b$.

We set up the Hilbert integral H for the field F. Let g denote the curve $y=y(x)$ of the theorem. By virtue of the fact that g is of class C^1 neighboring $x=b$, H exists as a line integral when taken along g from $x=e$ to $x=b$. Let the value of H when taken along g from $x=e$ to $x=b$ be denoted by $H_y|_e^b$. Recalling the definition of the Weierstrass E-function, one then sees that

$$J(y)\Big|_e^b = \int_e^b E dx + H_y\Big|_e^b,$$

where the arguments are those of (3.8). We shall prove that

$$H_y\Big|_e^b = s[y(e), b]. \tag{3.9}$$

Let δ denote the line $x=e$. To establish (3.8) we consider the closed curve obtained by following δ from $y=0$ to $y=y(e)$, the curve $y=y(x)$ from $x=e$ to $x=b$, and finally the x-axis from $x=b$ to $x=e$. Taking the Hilbert integral H along this closed curve we find that

$$H_\delta\Big|_{y=0}^{y=y(e)} + H_y\Big|_{x=e}^{x=b} = 0. \tag{3.10}$$

But upon turning to the explicit definition of H we see that

$$H_\delta\Big|_0^{y(e)} = \int_0^{y(e)} 2\,\frac{rw' + qw}{w}\, y\, dy = -\, s[y(e), b]. \tag{3.11}$$

Equation (3.9) is a consequence of (3.10) and (3.11), and (3.8) is proved.

4. **The singularity condition.** The following example will show the inadequacy of the classical conditions.

EXAMPLE 4.1. Consider the functional

$$J = \int_e^1 \left(\frac{y'^2}{x} - \frac{y^2}{x^3}\right) dx.$$

The Euler equation is

$$\frac{d}{dx}\left(\frac{y'}{x}\right) + \frac{y}{x^3} = 0$$

and possesses the linearly independent solutions x and $x \log x$. The solution x is a "focal solution" belonging to $x=0$, so that $x=0$ has no first conjugate point in the sense of §2. Except for the singularity at $x=0$, the analogues of the ordinary conditions that the segment $[0, 1]$ afford a minimum limit to J are satisfied. Nevertheless an easy computation shows that along the curve $y=x(x-1)$,

$$\lim_{e=0} J(y) \Big|_e^1 = -\tfrac{1}{2}.$$

The minimum limit thus fails to exist along $[0, 1]$. We are accordingly led to seek further necessary and sufficient conditions for a minimum.

The following theorem introduces a new and important necessary condition.

THEOREM 4.1. *In order that $[0, b]$ furnish a minimum limit to J, it is necessary that*

$$\liminf_{x=0} s[y(x), b] \geqq 0 \tag{4.0}$$

for each A-admissible curve for which

$$\liminf_{e=0} J(y) \Big|_e^b \tag{4.1}$$

is finite.

We shall first prove the theorem for the case in which $y(x)$ is of class C^1 neighboring $x=b$, making this assumption in order that Theorem 3.1 may be applied. Theorem 4.1 will then follow for an arbitrary A-admissible curve g joining the end points of $[0, b]$, since g can be replaced by an A-admissible curve of class C^1 neighboring $x=b$ without altering g except in the neighborhood of $x=b$.

We suppose that the limit (4.1) is finite for the given curve $y=y(x)$. There accordingly exists a decreasing sequence of constants e_n such that

$$\lim_{n=\infty} e_n = 0 \qquad (0 < e_n < b)$$

and

$$\liminf_{e=0} J(y) \Big|_e^b = \lim_{n=\infty} J(y) \Big|_e^b.$$

We set $y(e_n)=c_n$ and let g_n denote the curve obtained by following an extremal from the point $(b, 0)$ to the point (e_n, c_n) and the curve $y=y(x)$ from (e_n, c_n) to $(0, 0)$. We apply (3.8) to g_n indicating the evaluation of J along g_n by adding the subscript n. We see that

$$J_n \Big|_{e_n}^{b} = s[c_n, b]. \tag{4.2}$$

But g_n is A-admissible so that

$$J_n \Big|_{e_n}^{b} + \lim_{r=\infty} J(y) \Big|_{e_n+r}^{e_n} \geqq 0 \qquad (r > 0).$$

Upon letting n become infinite we infer that

$$\lim_{n=\infty} J_n \Big|_{e_n}^{b} \geqq 0.$$

Combining this result with (4.2) we see that

$$\lim_{n=\infty} s[c_n, b] \geqq 0. \tag{4.2$'$}$$

We shall conclude the proof of the theorem by showing that

$$\lim_{n=\infty} s[c_n, b] = \liminf_{x=0} s[y(x), b]. \tag{4.3}$$

To that end we apply (3.8) to the curve $y=y(x)$ setting $e=e_n$. We find that

$$\lim_{n=\infty} \int_{e_n}^{b} E dx \tag{4.4}$$

exists and is finite. From the fact that $E \geqq 0$ we infer that

$$\lim_{e=0} \int_{e}^{b} E dx \tag{4.5}$$

exists, is finite, and equals the limit (4.4). Because of the finiteness of these limits, (3.8) yields the relations

$$\liminf_{e=0} J(y) \Big|_{e}^{b} = \lim_{e=0} \int_{e}^{b} E dx + \liminf_{e=0} s[y(e), b],$$

$$\lim_{n=\infty} J(y) \Big|_{e_n}^{b} = \lim_{e=0} \int_{e}^{b} E dx + \lim_{n=\infty} s[c_n, b],$$

from which (4.3) follows as stated.

Condition (4.0) is a consequence of (4.2) and (4.3), and the proof is complete.

EXAMPLE 4.2. The condition (4.1) is independent of the analogues of the classical Jacobi and Weierstrass conditions. For in the example at the beginning of this section the point $x=0$ has no conjugate point, and the Weierstrass condition holds on $(0, b]$ since $r(x)>0$ on $(0, b]$.

Let g denote the curve $y=x(x-1)$. The curve g is A-admissible on $[0, 1]$. Moreover the limit (4.1) is finite along g. Observe that $x \log x$ is a solution $u(x)$ of the E.E. which vanishes when $x=1$, and that when $y=x(x-1)$,

$$s[y, 1] = -r\frac{u'}{u}y^2 = -\frac{(1+\log x)}{\log x}(x-1)^2.$$

We see that

$$\lim_{x=0} s[y(x), 1] = -1,$$

so that the condition is not satisfied.

EXAMPLE 4.3. The provision in the theorem that the limit (4.1) be finite cannot be removed as an examination of the functional

$$\int_e^\pi (y'^2 - y^2)dx$$

will show.

It is clear that the limit (4.1) is not finite for the curve $y=x^{1/2}(x-\pi)$. The singularity function belonging to $x=\pi$ takes the form

$$s[y, \pi] = -y^2 \cot x.$$

Moreover $y=x^{1/2}(x-\pi)$ is an A-admissible curve g for which

$$\lim_{x=0} s[y(x), \pi] = -\pi^2,$$

so that (4.0) is not satisfied. Nevertheless we shall see in the next section that $[0, \pi]$ affords a minimum limit for this functional.

The condition of Theorem 4.1 that (4.0) hold for each A-admissible curve for which the limit (4.1) is not $+\infty$ will be termed the *singularity condition* belonging to the segment $[0, b]$.

5. **Sufficient conditions.** Let a and b be two constants such that $0<a<b<d$, and let g be an arbitrary extremal $y=g(x)$ defined on the interval $a \leqq x \leqq b$. Let $y=z(x)$ be an absolutely continuous curve which joins the end points of g and on which z'^2 is in L.

If there is no point conjugate to $x=a$ on $[a, b]$,

$$J(z)\Big|_a^b \geqq J(g)\Big|_a^b. \tag{5.1}$$

The proof of (5.1) follows classical lines. One covers g with a field F of extremals without singularity neighboring g, and makes use of the Hilbert integral theory as extended in §3. The proof of (5.1) is then completed in the usual way.

We turn to the question of the equality in (5.1). Let $p(x, y)$ be the slope of F at the point (x, y). The classical formula of Weierstrass

$$\Delta J = \int_a^b E dx$$

holds and shows that the equality in (5.1) prevails only if

$$\frac{dz}{dx} = p[x, z(x)] \tag{5.2}$$

almost everywhere on $[a, b]$. But $z(x)$ is absolutely continuous so that it follows from (5.2) that

$$z(x) - z(a) = \int_a^x p[x, z(x)] dx.$$

We infer that $z(x)$ is of class C^1. Returning to (5.2) we see that $z = z(x)$ must then be an extremal of F and hence identical with g.

We come to a principal theorem.

THEOREM 5.1. *Necessary and sufficient conditions that* $[0, b]$ *afford a minimum limit to* J *are that* $[0, b)$ *contain no point conjugate to* $x = 0$, *and that the singularity condition belonging to* $[0, b]$ *be satisfied.*

That the conditions of the theorem are necessary has already been established.

To prove the conditions of the theorem sufficient we shall begin with an A-admissible curve $y = y(x)$ of class C^1 neighboring $x = b$ and prove that

$$\liminf_{e=0} J(y) \Big|_e^b \geqq 0. \tag{5.3}$$

The relation (5.3) will then follow for an A-admissible curve h not of class C^1 neighboring $x = b$. For one can replace a segment of h terminating at $x = b$ on which x varies sufficiently little by an extremal joining its end points, obtaining thereby a new curve k of class C^1. Moreover it follows from (5.1) that

$$J(h) \Big|_e^b \geqq J(k) \Big|_e^b,$$

since there is no point conjugate to $x = b$ on the interval on which h is altered.

But in case $y(x)$ is of class C^1 neighboring $x=b$ we can use formula (3.8) and infer that (5.3) holds.

The proof of the theorem is complete.

The following theorem is of importance in the applications.

THEOREM 5.2. *If for every value of c for which $0<c<b$, and for every curve k which is A-admissible on* $[0, b]$,

$$\liminf_{e=0} J(k) \Big|_e^c \geqq 0,$$

then $[0, b]$ *will afford a minimum limit to J along curves which are A-admissible on* $[0, b]$.

Let A and B be the end points of $[0, b]$ and let g be an A-admissible curve which joins A to B. Let P be a point on g near B but not B. Let h be an extremal which joins P to B. If P is sufficiently near B, h will afford a minimum to J among A-admissible curves which join its end points. Let γ be a curve which follows g from A to P and h from P to B. Holding P fast let γ be altered by replacing B by a point $(c, 0)$ on the x-axis near B with $0<c<b$ and replacing h by an extremal joining P to $(c, 0)$. For the resulting curve μ,

$$\liminf_{e=0} J(\mu) \Big|_e^c \geqq 0 \tag{5.4}$$

by hypothesis. If the limit (5.4) is finite, we see that

$$\lim_{c=b} \liminf_{e=0} J(\mu) \Big|_e^c = \liminf_{e=0} J(\gamma) \Big|_e^b \geqq 0.$$

If the limit (5.4) is $+\infty$, we see that

$$\liminf_{e=0} J(\gamma) \Big|_e^b = +\infty.$$

The theorem follows readily.

In the following theorem and proof each inferior limit is an inferior limit as x tends to zero.

THEOREM 5.3. *If* $[0, e]$ *affords a minimum limit to J for any sufficiently small positive constant e, and if there exists a solution $u(x)$ of the* E.E. *for which the* $\liminf u(x) \neq 0$, *J assumes a minimum limit on any interval* $[0, b]$ *such that* $[0, b)$ *contains no point conjugate to* $x=0$.

Let $w(x)$ be a focal solution. We begin the proof by establishing statements (a) and (b).

(a) *If $v(x)$ is a solution of the* E.E. *independent of $w(x)$,* lim inf $v(x) \neq 0$.

If lim inf $w(x)=0$, (a) is true, for otherwise lim inf $u(x)$ would be null. If lim inf $w(x) \neq 0$, lim inf $v(x)$ must be infinite since lim $w/v=0$. Thus (a) is true in all cases.

(b) *If $y(x)$ is A-admissible,* lim inf $J(y)\big|_x^b$ *is finite, and $v(x)$ is independent of $w(x)$, then*

$$\liminf S[y(x), v(x)] \geqq 0. \tag{5.5}$$

If the constant e is sufficiently small, the solution $z(x)$ for which

$$z(e) = 0, \qquad z'(e) = -1$$

will have a non-null inferior limit. Since $[0, e]$ affords a minimum limit to J,

$$\liminf S[y(x), z(x)] \geqq 0 \tag{5.6}$$

by virtue of Theorem 4.1. But

$$S[y, v] - S[y, z] = \frac{y^2 r}{zv}[z'v - v'z] = \frac{ky^2}{zv}, \tag{5.7}$$

where k is a constant. Now lim inf $v(x) \neq 0$ by virtue of (a), and it follows from (5.7) that

$$\liminf S[y, v] = \liminf S[y, z]. \tag{5.8}$$

Statement (b) follows from (5.6) and (5.8).

Theorem 5.1 now shows that $[0, b]$ affords a minimum limit to J provided $x=0$ is not conjugate to $x=b$. That the theorem is valid in case $x=0$ is conjugate to $x=b$ follows from Theorem 5.2, and the proof is complete.

Example 4.2 shows that statement (b) is false if the phrase "$v(x)$ is independent of $w(x)$" is deleted.

We introduce the following theorem.

THEOREM 5.4. *If $p \geqq 0$ neighboring $x=0$, if $q \equiv 0$, and if there is no point on $[0, b)$ conjugate to $x=0$, J assumes a minimum limit on $[0, b]$ among A-admissible curves.*

Let e be a positive constant such that $p \geqq 0$ on $(0, e)$. Let $u(x)$ be a solution of the E.E. such that $u(e)=0$ and $u(x)>0$ on $(0, e)$. Observe that

$$J(u)\Big|_x^e = -r(x)u(x)u'(x) > 0.$$

Hence $u'(x)<0$ on $(0, e)$ and lim inf $u(x) \neq 0$. We infer that (5.5) holds as stated. The theorem then follows from Theorem 5.3.

We conclude with the following theorem.

THEOREM 5.5. *The minimum limits affirmed to exist in Theorems* 5.3 *and* 5.4 *are proper unless $x=b$ is a conjugate point of $x=0$. If $x=b$ is a conjugate point of $x=0$, the minimum limits may be proper or improper.*

Suppose $x=b$ is not conjugate to $x=0$. Under the hypotheses of the preceding theorems if $b<c<d$, and c differs sufficiently little from b, the functional J will assume a minimum limit on $[0, c]$ among curves which are A-admissible on $[0, c]$. A curve $y=u(x)$ which is A-admissible on $[0, b]$ and which gives a null inferior limit to J must be an extremal. A curve g which follows $y=u(x)$ from $x=0$ to $x=b$ and the x-axis from $x=b$ to $x=c$ will be A-admissible on $[0, c]$ and give a null inferior limit to J. The curve g must then be an extremal. This is possible only if $u(x)\equiv 0$. Hence $[0, b]$ affords a proper minimum limit to J unless $x=b$ is conjugate to $x=0$, and the proof is complete.

The segment $[0, \pi]$ of the x-axis furnishes an improper minimum limit to the integral

$$\int_0^\pi (y'^2 - y^2)dx.$$

On the other hand we shall see in §8 that there are examples in which $x=b$ is conjugate to $x=0$ and $[0, b]$ furnishes a proper minimum limit to J.

II. ONE SINGULAR END POINT; REGULAR CASE

6. **The integral.** Part II will be devoted to the study of integrals of the form

$$(6.1)\qquad \int_e^b [x^\alpha g(x)y'^2 - x^{\alpha-2}h(x)y^2]dx.$$

Here α is any real number, $g(x)$ and $h(x)$ are real analytic functions of x on the interval $[0, d)$, and $g>0$ on this interval. The corresponding E.E. may be given the form

$$(6.2)\qquad x^2gy'' + x(\alpha g + xg')y' + hy = 0.$$

It will be observed that $x=0$ is a regular* singular point of this differential equation.

Conversely, any differential equation of the form

$$(6.3)\qquad x^2\lambda(x)y'' + x\mu(x)y' + \nu(x)y = 0$$

* Most standard works on ordinary differential equations give a treatment of the regular singular point which is based on the methods of J. Frobenius. G. D. Birkhoff [1] gives an alternative simplified treatment of this question.

in which λ, μ, ν are real and analytic on $[0, d)$, and λ is positive on $[0, d)$, after multiplication by the factor

$$x^{\alpha-2}\frac{g(x)}{\lambda(x)}, \tag{6.4}$$

where

$$\begin{aligned} \alpha &= \frac{\mu(0)}{\lambda(0)}, \\ g(x) &= \exp\left[\int_0^x \frac{1}{x}\left(\frac{\mu(x)}{\lambda(x)} - \frac{\mu(0)}{\lambda(0)}\right)dx\right], \end{aligned} \tag{6.5}$$

becomes the E.E. of an integral of the form (6.1) in which α and g are defined by (6.5) and

$$h(x) = \frac{g(x)\nu(x)}{\lambda(x)}.$$

With the equation (6.2) one associates the so-called indicial equation

$$\rho^2 - \rho(1-\alpha) + \beta = 0, \tag{6.6}$$

where

$$\beta = \frac{h(0)}{g(0)}.$$

The roots σ and τ of (6.6) are called *indicial roots* associated with the point $x=0$. They satisfy the relations

$$\begin{aligned} \sigma + \tau &= 1 - \alpha, \\ \sigma\tau &= \beta. \end{aligned} \tag{6.7}$$

When σ and τ are real, we shall suppose that

$$\sigma \geqq \tau.$$

As is well known,* there exist two linearly independent solutions $w(x)$ and $v(x)$ of (6.2) of the respective forms

$$\begin{aligned} w(x) &= x^\sigma a(x), \\ v(x) &= x^\tau b(x) + cw(x)\log x, \qquad (0 < x < d), \end{aligned} \tag{6.8}$$

where $a(x)$ and $b(x)$ are analytic in x on $[0, d)$,

$$a(0)\cdot b(0) \neq 0,$$

* See, for example, Bieberbach [1], p. 214.

and c is a constant. If $\sigma-\tau$ is not an integer, $c=0$, and if $\sigma=\tau$, $c\neq 0$. If $\sigma-\tau$ is an integer $\neq 0$, c may or may not be zero depending on the differential equation.

Two cases arise according as the indicial roots are real or imaginary. Suppose the roots are of the form $k\pm hi$ where k and h are real and $h\neq 0$. The difference

$$\sigma - \tau = \pm 2hi$$

is not an integer so that $c=0$. The solutions (6.8) may then be replaced by the linearly independent solutions

$$\begin{aligned} W(x) &= x^k[A(x)\cos\log x^h + B(x)\sin\log x^h], \\ V(x) &= x^k[B(x)\cos\log x^h - A(x)\sin\log x^h], \quad (0 < x < d), \end{aligned} \tag{6.9}$$

where $A(x)$ and $B(x)$ are real and analytic on $[0, d)$ with

$$A^2(0) + B^2(0) \neq 0. \tag{6.10}$$

7. **Focal solutions and the singularity function.** We shall prove the following theorem.

THEOREM 7.1. *A necessary and sufficient condition that $x=0$ be its own first conjugate point is that the indicial roots be imaginary.*

If the roots are imaginary, a solution of the E.E. is given by $W(x)$ in (6.9). Since $W(x)$ vanishes infinitely often near $x=0$, $x=0$ must be its own first conjugate point. On the other hand, if the exponents of σ and τ are real, $w(x)$ in (6.8) gives a solution which vanishes near $x=0$ for no positive value of x so that in this case $x=0$ cannot be its own first conjugate point.

The proof of the theorem is complete.

We continue with the following theorem.

THEOREM 7.2. *If the indicial roots are real and $\sigma\geqq\tau$, the solution $w=x^\sigma a(x)$ of* (6.8) *is a focal solution of the* E.E. *and the conjugate points of $x=0$ are the zeros of $a(x)$.*

In terms of the solutions $w(x)$ and $v(x)$ of (6.8) we see that

$$\lim_{x=0}\frac{w(x)}{v(x)} = 0.$$

If then $y(x)$ is any solution independent of $w(x)$,

$$\lim_{x=0}\frac{w(x)}{y(x)} = 0,$$

so that $w(x)$ is a focal solution of the E.E. That the conjugate points of $x=0$ are the zeros of $a(x)$ follows from Theorem 2.2.

We shall next verify three formulas useful in the evaluation of the singularity function $s[y, b]$ defined in §4. These formulas concern the solutions $w(x)$ and $v(x)$ of (6.8) together with any solution $u(x)$ of the form

$$u = c_1 w + c_2 v \qquad (c_2 \neq 0).$$

These formulas are as follows:

$$\frac{w'}{w} = \frac{1}{x}[\sigma + o(1)], \tag{7.1}$$

$$\frac{v'}{v} = \frac{1}{x}[\tau + o(1)], \tag{7.2}$$

$$\frac{u'}{u} = \frac{1}{x}[\tau + o(1)]. \tag{7.3}$$

Formula (7.1) is verified at once.

To establish (7.2) observe that

$$\frac{v'}{v} = \frac{1}{x}\left[\tau + \frac{xb'(x) + ck(x) + cxk'(x)\log x}{b(x) + ck(x)\log x}\right], \tag{7.4}$$

where

$$k(x) = x^{\sigma-\tau}a(x).$$

Upon separating the cases $ck(0)=0$ and $ck(0)\neq 0$, (7.2) appears as a consequence of (7.4).

Finally we see that

$$\frac{u'}{u} = \frac{c_1 w' + c_2 v'}{c_1 w + c_2 v} = \frac{c_1 \dfrac{w'}{w}\dfrac{w}{v} + c_2 \dfrac{v'}{v}}{c_1 \dfrac{w}{v} + c_2}. \tag{7.5}$$

Formula (7.3) follows from (7.5) upon making use of (7.1) and (7.2) together with the fact that w/v tends to zero as x tends to zero.

8. **Sufficient conditions in the regular singular case.** We consider the functional

$$J = \int [g(x)x^{\alpha}y'^2 - h(x)x^{\alpha-2}y^2]dx \tag{8.0}$$

of §6 excluding the trivial case $h(x)\equiv 0$. We are concerned with conditions that $[0, b]$ afford a minimum limit to J.

The exponent m. It will be convenient to rewrite J in the form

$$J = \int [g(x)x^{\alpha}y'^2 - k(x)x^{m-2}y^2]dx, \tag{8.1}$$

where $k(x)$ is analytic on the interval $[0, d)$ and $k(0)\neq 0$. We see that $m \geqq \alpha$.

Besides the conjugate point condition our sufficient conditions will involve the sign of $k(0)$ and the exponent m. To distinguish between proper and improper minima we shall need the roots σ and τ. Recall that σ and τ are the roots of the equation

$$\rho^2 - \rho(1 - \alpha) + \beta = 0 \tag{8.2}$$

with $\sigma \geqq \tau$ by convention when the roots are real.

We shall make use of the following lemma.

LEMMA 8.1. *If $y(x)$ is A-admissible and $x^{\alpha}y'^2(x)$ is in L on $[0, b]$, $x^{\alpha-1}y^2(x)$ tends to zero as a limit as x tends to zero.*

The lemma is trivial if $\alpha \geqq 1$. Suppose then that $\alpha < 1$. Let e and x be so chosen that $0 < e < x < b$. We note that

$$\left[y(x)\Big|_e^x\right]^2 = \left[\int_e^x y'dx\right]^2.$$

Upon using the Schwartz inequality we find that

$$\left[y(x)\Big|_e^x\right]^2 = \left[\int_e^x x^{-\alpha/2}(x^{\alpha/2}y')dx\right]^2 \leqq \int_e^x \frac{dx}{x^{\alpha}} \int_e^x x^{\alpha}y'^2dx.$$

If we let e tend to zero as a limit, this relation leads to the inequality

$$(1 - \alpha)x^{\alpha-1}y^2(x) \leqq \int_0^x x^{\alpha}y'^2(x)dx,$$

from which the lemma follows as stated.

We continue with the following lemma.

LEMMA 8.2. *If $y(x)$ is A-admissible and $x^{\alpha}y'^2(x)$ is in L on $[0, b]$, and if the indicial roots are real,*

$$\lim_{x=0} S[y(x), u(x)] = 0$$

for every A-admissible curve $y = y(x)$ and solution $u(x)\neq 0$.

Upon referring to the definition of the singularity function and making use of (7.1), (7.2), and (7.3) we see that

$$\left| S[y(x), u(x)]\right| \leqq x^{\alpha-1}y^2(x)O(1).$$

Lemma 8.2 follows from Lemma 8.1.

We come to a principal theorem.

THEOREM 8.1. *Necessary and sufficient conditions that* $[0, b]$ *afford a minimum limit to* J *among A-admissible curves are that* $[0, b)$ *contain no point conjugate to* $x=0$ *and that* $m>1$ *whenever* $k(0)>0$.

We have already proved that it is necessary that $[0, b)$ contain no point conjugate to $x=0$.

That the conditions of the theorem are sufficient when $k(0)<0$ follows from Theorems 5.4 and 2.3.

The case $m\leqq 1$, $k(0)>0$. We shall show that $[0, b)$ fails to afford a minimum limit to J in this case. Observe that the term in J of the form

$$-\int_x^b k(x)x^{m-2}y^2dx \tag{8.3}$$

will decrease without limit on a curve of the form $y=\text{constant} \neq 0$, as x tends to zero.

An A-admissible curve g for which

$$\lim_{e=0} J(g)\Big|_e^b \leqq -1 \tag{8.4}$$

can accordingly be constructed from an infinite succession of straight line segments

$$g_0, g_1, g_2, \cdots$$

as follows. We suppose g_0 terminates at the point $(b, 0)$, has the slope -1 and a length equal to b. Let (a_n, b_n) denote the initial point of g_n and suppose that $0<a_n<a_{n-1}$. For n odd, g_n shall terminate at (a_{n-1}, b_{n-1}) and have the slope zero. Its initial point (a_n, b_n) shall be so chosen that

$$J(g)\Big|_{a_n}^b < -1.$$

For n odd, g_{n+1} shall terminate at (a_n, b_n), have a slope b_n/a_n and a length equal to half the distance of (a_n, b_n) from the origin.

It is clear that g exists, is A-admissible and that on g (8.4) holds. If $[0, b]$ affords a minimum limit to J among A-admissible curves, it is accordingly necessary that $m>1$.

The case $m>1$, $k(0)>0$. In this case the integral (8.4) has a finite limit as e tends to zero. We distinguish between the case in which the limit

$$\lim_{e=0}\int_e^b x^\alpha y'^2dx \tag{8.5}$$

is finite and the case in which it is infinite. In the latter case it is clear that

$$\liminf_{e=0} J(y)\Big|_e^b > 0, \tag{8.6}$$

as desired. In case the limit in (8.5) is finite and $[0, b)$ contains no point conjugate to $x=0$, it follows from Lemma 8.2 that

$$\lim_{x=0} s[y(x), b] = 0.$$

We see from Theorem 3.1 that (8.6) holds in this case as well.

The proof of the theorem is complete.

We introduce the following lemma.

LEMMA 8.3. *If $[0, b]$ affords a minimum limit to J among A-admissible curves, the smaller exponent τ is never positive.*

We distinguish three cases.

CASE I. $m>\alpha$. In this case $\beta=0$ in (8.2). But $\beta=\sigma\tau$, and the lemma is clearly true.

CASE II. $m=\alpha$, $k(0)<0$. Here $\beta<0$ and the lemma is again true.

CASE III. $m=\alpha$, $k(0)>0$. By virtue of the theorem $m>1$ in this case. Hence

$$\alpha = 1 - \sigma - \tau > 1, \qquad \beta = \sigma\tau > 0,$$

and we infer that $\tau<0$.

The proof of the lemma is complete.

The following theorem completes the preceding theory.

THEOREM 8.2. *Necessary and sufficient conditions that $[0, b]$ afford an improper minimum limit to J among A-admissible curves are that $x=b$ be the first conjugate point of $x=0$, and that $m>1>\alpha$ in case $k(0)>0$, and that $\alpha=m$ or $\alpha<1$ in case $k(0)<0$.*

That it is necessary that $x=b$ be the first conjugate point of $x=0$ is proved as under Theorem 5.5. We assume the facts affirmed under Theorem 8.1 and continue with a proof of the following.

(a) *If $x=b$ is the first conjugate point of $x=0$, if $k(0)>0$ and $m>1$, the minimum limit assumed by J on $[0, b]$ is improper if and only if $\alpha<1$.*

We shall first prove that an improper minimum limit in (a) implies that $\alpha<1$. To that end observe that such a limit implies that the focal solution

$$w(x) = x^\sigma a(x)$$

is A-admissible, and hence $\sigma>0$. To continue we distinguish between two cases.

CASE I. $m=\alpha$. In this case $\sigma\tau>0$. But $\tau\leqq 0$ by virtue of Lemma 8.3 so that Case I is impossible.

CASE II. $m>\alpha$. In this case the indicial roots are $1-\alpha$ and 0. Thus $\alpha<1$ since $\sigma>0$.

Hence an improper minimum limit in (a) implies that $\alpha<1$.

Conversely, we shall assume that $\alpha<1$ in (a) and prove that $y=w(x)$ is A-admissible and

$$\lim_{x=0} J(w)\Big|_x^b = 0. \tag{8.7}$$

First, observe that

$$1-\sigma-\tau<1$$

when $\alpha<1$, so that $\sigma>0$. It follows from (8.7) that

$$\begin{aligned} J(w)\Big|_x^b &= -\,g(x)x^\alpha ww' = O[x^\alpha x^\sigma x^{\sigma-1}] \\ &= O[x^{\sigma-\tau}]. \end{aligned} \tag{8.8}$$

But $\tau\leqq 0$ by virtue of Lemma 8.3 so that $\sigma-\tau>0$. Relation (8.7) follows from (8.8), and the proof of (a) is complete.

We continue with a proof of statement (b).

(b) *If $x=b$ is the first conjugate point of $x=0$ and if $k(0)<0$, the minimum limit assumed by J on $[0, b]$ is improper if and only if $m=\alpha$ or $\alpha<1$.*

As in the proof of (a) we see that the minimum limit in (b) is improper only if $\sigma>0$. If $m>\alpha$, the indicial roots are $1-\alpha$ and 0, and if further $\sigma>0$, it is necessary that $\alpha<1$. Hence the minimum limit in (b) is improper only if $m=\alpha$ or $\alpha<1$.

If, conversely, $\alpha=m$ and $k(0)<0$, we infer that $\sigma\tau<0$ and hence $\sigma-\tau>0$. Thus (8.7) follows from (8.8) when $\alpha=m$ in (b). But on the other hand if $m>\alpha$, the indicial roots are $1-\alpha$ and 0, and if in addition $\alpha<1$, $\sigma=1-\alpha>0$ so that (8.8) again follows from (8.7).

The proof of (b) and of the theorem is complete.

For completeness we add the following.

The minimum limit zero in Theorem 8.2 *is assumed by J only on solutions dependent on focal solutions.*

In the regular singular case $m\geqq\alpha$. We shall prove the following theorem.

THEOREM 8.3. *If $m<\alpha$ and $k(0)>0$ in* (8.1), $[0, b]$ *fails to afford a minimum limit to J among A-admissible curves.*

Let c be a positive constant so large that the indicial roots belonging to the E.E. of the functional

$$J^* = \int_0^b [x^\alpha g y'^2 - c x^{\alpha-2} k y^2] dx$$

are imaginary. Under the hypotheses of the theorem there exists a positive constant $\eta<b$ such that

$$k(x) > 0, \qquad c x^{\alpha-m} < 1$$

on $[0, \eta]$. For such a choice of η,

$$J^*(y) \Big|_e^\eta \geqq J(y) \Big|_e^\eta \qquad (0 < e < \eta) \tag{8.9}$$

for every $y(x)$ which is A-admissible on $[0, \eta]$. But since the indicial roots belonging to J^* are imaginary, an A-admissible function $y(x)$ exists for which the left member of (8.9) has a negative inferior limit as e tends to zero.

The theorem follows from (8.9).

9. **Other types of admissibility.** We continue with the regular singular case with the functional (8.0).

Curves α-admissible. A curve $y=y(x)$ which is A-admissible on $[0, b]$ will be said to be *α-admissible* on $[0, b]$ if the limit

$$\lim_{e=0} \int_e^b x^\alpha y'^2 dx \tag{9.1}$$

is finite. It follows from Lemma 8.2 that

$$\lim_{x=0} S[y(x), u(x)] = 0 \tag{9.2}$$

whenever the indicial roots are real, $y(x)$ is A-admissible, and $u(x)$ is a solution of the E.E. not identically zero.

We continue with the following lemma.

LEMMA 9.1. *If $y(x)$ is α-admissible, $x^{\alpha-2}y^2$ is in L on $[0, b]$ except at most when $\alpha=1$.*

To prove this lemma we consider the functional

$$I = \int_e^x [x^\alpha y'^2 - K x^{\alpha-2} y^2] dx \qquad (\alpha \neq 1), \tag{9.3}$$

where K is a positive constant. Since $\alpha \neq 1$, the constant K can be chosen so small that the indicial roots corresponding to I are real. We see from (3.8) and (9.2) that

$$\int_0^b x^\alpha y'^2 dx \geqq \lim_{e=0} K \int_e^x x^{\alpha-2} y^2 dx.$$

The lemma follows directly.

EXAMPLE 9.1. The lemma would be false were not the case $\alpha = 1$ excepted as is shown by the following example. Neighboring $x = 0$, we suppose that

$$y(x) = [-\log x]^{-1/2}.$$

We find that

$$\int_e^x xy'^2 dx = -\frac{1}{4}\int_e^x (\log x)^{-3} d \log x = \frac{1}{8}[\log x]^{-2}\Big|_e^x,$$

$$\int_e^x x^{-1} y^2 dx = -\int_e^x [\log x]^{-1} d \log x = -[\log(-\log x)]_e^x.$$

From these relations we see that xy'^2 is in L on $[0, 1]$ while $x^{-1}y^2$ is not in L.

We are considering the functional J given by (8.0). When $y(x)$ is α-admissible, it follows from the preceding lemma that the limit

$$\lim_{e=0} J(y)\Big|_e^b$$

is finite except at most when $\alpha = 1$. When $\alpha = 1$, this limit may be finite or infinite.

The principal theorem follows.

THEOREM 9.1. *A necessary and sufficient condition that $[0, b]$ afford a minimum limit to J among α-admissible curves is that $[0, b)$ contain no point conjugate to $x = 0$. This minimum limit is improper if and only if $x = b$ is the first conjugate point of $x = 0$, if $\sigma > 0$ and $\sigma > \tau$.*

The theorem follows readily upon abstracting the appropriate arguments from the proofs in §8. In particular the reader should refer to (8.8) in proving the concluding affirmation.

Curves F-admissible. If

(1) $y(x)$ is continuous on $(0, b]$ and $y(b) = 0$,

(2) $y(x)$ is absolutely continuous and $y'^2(x)$ is in L on each closed subinterval of $(0, b]$, the curve $y = y(x)$ will be said to be *F-admissible* on $[0, b]$. We are concerned with the functional J of (8.0) and (8.1) and state the following theorem.

THEOREM 9.2. *In order that* $(0, b]$ *afford a minimum limit to* J *among F-admissible curves, it is sufficient that* $[0, b)$ *contain no point conjugate to* $x=0$ *and that* $\tau<0$ *or* $\sigma=\tau=0$. *Under these conditions the minimum limit is improper if and only if* $x=b$ *is the first conjugate point of* $x=0$ *and* $\sigma>0$.

In case $\tau<0$ or $\sigma=\tau=0$, the solution $v(x)$ of (6.8) is of one sign neighboring $x=0$. If $v(x)$ is chosen so as to be positive, its derivative $v'(x)$ will be negative neighboring $x=0$. It follows that

$$S[y(x), v(x)] \geqq 0 \qquad (x > 0)$$

for all values of x sufficiently near zero. The first statement of the theorem is thus a consequence of (3.8) when $x=b$ is not conjugate to $x=0$, and follows when $x=b$ is conjugate to $x=0$ by the limiting process used in the proof of Theorem 5.2. The second statement of the theorem is a consequence of earlier remarks and of formula (8.8).

The proof of the theorem is complete.

The "general" case in which $(0, b]$ affords a minimum limit to J is described in the preceding theorem. There are known special cases. To give final necessary and sufficient conditions would require the introduction of a new element, the first focal point of the y-axis. For the cases included in the theorem, this focal point is identical with the first conjugate point of $x=0$. We omit further details.

Curves admissible relative to an extremal g. Let $y=u(x)$ be an extremal g which joins the points $(0, 0)$ and (b, c). We assume that $J(u)\big|_e^b$ tends to a finite limit as e tends to zero. Relative to the extremal g we admit comparison curves of the form

$$y = u(x) + \eta(x),$$

where $\eta(x)$ is X-admissible on $[0, b]$, X symbolizing A, α, or F.

If for each such curve

$$\liminf_{e=0} J(y)\Big|_e^b \geqq J(u)\Big|_0^b,$$

we shall say that g affords a minimum limit to J among the particular class of curves admitted.

A simple integration by parts shows that

$$J(y)\Big|_e^b = J(u)\Big|_e^b + J(\eta)\Big|_e^b + T[\eta(e), u(e)], \tag{9.4}$$

where

$$T[\eta, u] = 2\eta[ru' + qu].$$

If our knowledge of the functional T is sufficient, the preceding theorems about $J(\eta)$ will lead to theorems about $J(y)$. As an example we shall translate Theorem 9.1.

The solution $u(x)$ will vanish at $x=0$ and $J(u)\big|_e^b$ will tend to a finite limit if and only if $u(x)$ is a focal extremal and $\sigma>0$. Under these conditions $x^\alpha u'^2$ will be in L on $[0, b]$, and $x^\alpha(u'+\eta')^2$ will be in L on $[0, b]$ if and only if $x^\alpha \eta'^2$ is in L on $[0, b]$.

THEOREM 9.3. *A necessary and sufficient condition that a focal extremal*

$$y = x^\sigma a(x) \qquad (\sigma > 0;\, 0 \leqq x \leqq b)$$

afford a minimum limit to J relative to curves $y=u(x)+\eta(x)$ for which $\eta(x)$ is α-admissible on $[0, b]$ is that $[0, b)$ contain no point conjugate to $x=0$. This minimum limit is improper if and only if $x=b$ is the first conjugate point of $x=0$ and $\sigma>\tau$. The value of the minimum limit is $g(b)u(b)u'(b)$.

That the conjugate point condition insures a minimum limit will follow from (9.4) and Theorem 9.1 once we have shown that

$$\lim_{e=0} T[\eta(e), u(e)] = 0. \tag{9.5}$$

Observe that

$$T(\eta, u) = \eta(x) x^\alpha x^{\sigma-1} O(1).$$

But η is α-admissible, and it follows from Lemma 8.1 that

$$\eta = x^{(1-\alpha)/2} o(1).$$

Recalling that $\alpha = 1-\sigma-\tau$ we see that

$$T(\eta, u) = x^{(\sigma-\tau)/2} o(1),$$

so that (9.5) holds as stated.

The remainder of the theorem follows from (9.4) and Theorem 9.1.

EXAMPLE 9.2. We shall now derive a theorem of H.L.P. With a slight variation which accords with the present notation this theorem is as follows.

If $\mu>4$, $y(0)=0$, $y(1)=1$, $y(x)$ is absolutely continuous and y' is in L^2, then

$$J(y) = \int_0^1 \left[\mu y'^2 - \frac{y^2}{x^2}\right] dx \geqq \mu\sigma.$$

The only case of equality occurs when $y=x^\sigma$.

The indicial equation is

$$\rho^2 - \rho + \frac{1}{\mu} = 0.$$

If $\mu > 4$, the roots are real and distinct with $\sigma > 0$. The focal solution x^σ vanishes only when $x = 0$ so that there is no point conjugate to $x = 0$. The minimum limit afforded by x^σ is accordingly proper. Its value is

$$g(1)u(1)u'(1) = \mu\sigma.$$

III. Two singular end points

10. **Necessary conditions.** We return to a function f of the form (1.1) and a functional of the form

$$J(y)\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} f(x, y, y')dx, \tag{10.1}$$

where x_1 and x_2 are constants such that

$$0 < x_1 < x_2 < d. \tag{10.2}$$

Functions $y(x)$ which are A-, α-, or F-admissible on $[0, b]$ have already been defined. A function $y(x)$ will be termed A-, α-, or F-admissible respectively on $[0, d]$ if for each constant b on $(0, d)$, $y(x)$ satisfies the conditions for A-, α-, or F-admissibility on $[0, b]$ excepting at most the condition that $y(b) = 0$.

The transformation

$$x^* = d - x \tag{10.3}$$

interchanges the points $x = 0$ and $x = d$. Functions which result from curves A-, α-, or F-admissible on the interval $[0, b]$ or $[0, d]$ upon applying the transformation (10.3) will be termed A^*-, α^*-, or F^*-admissible respectively on $[0, b]$ or $[0, d]$.

Let X denote one of the symbols A, α, and F, and Y one of the symbols A^*, α^*, and F^*. A function $y(x)$ will be termed (X, Y)-*admissible* on $[0, d]$ if by an alteration on an arbitrarily small neighborhood of $x = d$, $y(x)$ can be made X-admissible on $[0, d]$, and if by an alteration on an arbitrarily small neighborhood of $x = 0$, $y(x)$ can be made Y-admissible on $[0, d]$. A curve $y = y(x)$ will be termed (X, Y)-admissible if $y(x)$ is (X, Y)-admissible. We thus have nine different types of (X, Y)-admissibility.

We shall begin by seeking conditions under which

$$\liminf_{x_1=0,\, x_2=d} \int_{x_1}^{x_2} f(x, y, y')dx \geqq 0 \tag{10.4}$$

for curves which are (A, A^*)-admissible on $[0, d]$. If (10.4) holds, we shall say that $[0, d]$ affords a *minimum limit* to J among curves (A, A^*)-admissible on $[0, d]$.

In (10.2) we defined the first conjugate point of $x=0$ on $[0, d)$. Under the transformation (10.3), J will be carried into a functional J^*. Relative to J the first conjugate point of $x=d$ will be defined as the image under (10.3) of the first conjugate point of $x^*=0$ relative to J^*.

A solution $w(x)\not\equiv 0$ of the E.E. such that for each solution $y(x)$ independent of $w(x)$

$$\lim_{x=d} \frac{w(x)}{y(x)} = 0$$

will be called a *focal solution* belonging to $x=d$. A focal solution belonging to $x=d$ will always exist, if $x=d$ is not its own first conjugate point on $(0, d]$. If $x=d$ is not its own first conjugate point on $(0, d]$, the first conjugate point of $x=d$ preceding $x=d$, if it exists, will be given by the first zero preceding $x=d$ of a focal solution belonging to $x=d$. These results and others follow from the results of §2 upon applying the transformation (10.3) to J.

If $[0, d]$ affords a minimum limit to J among curves (A, A^*)-admissible on $[0, d]$ and if b is any constant on $(0, d)$, $[0, b]$ will afford a minimum limit to J relative to curves A-admissible on $[0, b]$. We are thereby led to the following theorem.

THEOREM 10.1. *If $[0, d]$ affords a minimum limit to J among curves (A, A^*)-admissible on $[0, d]$, there can be no first conjugate point of $x=0$ on $[0, d)$ or first conjugate point of $x=d$ on $(0, d]$.*

The conclusions of the theorem are not independent, for it follows from Lemma 2.2 that if $x=0$ has no first conjugate point on $[0, d]$, $x=d$ will have no first conjugate point on $(0, d]$. Moreover the roles of $x=0$ and $x=d$ can be interchanged.

In §4 we have defined the singularity condition at $x=0$ belonging to $[0, b]$. The condition that

$$\liminf_{x=d} s[y(x), b] \leqq 0 \qquad (0 < b < d)$$

for each A^*-admissible curve for which

$$\liminf_{x=d} J(y)\Big|_b^x \neq +\infty$$

will be called the *singularity condition* at $x=d$ belonging to $[b, d]$. With this understood we have the following theorem.

THEOREM 10.2. *If $[0, d]$ affords a minimum limit to J, the singularity condition at $x=0$ belonging to $[0, b]$ and the singularity condition at $x=d$ belonging to $[b, d]$ must be satisfied by each curve $y=y(x)$ (A, A^*)-admissible on $[0, d]$ and each constant b on $(0, d)$.*

11. **Sufficient conditions.** If $u(x)$ is a solution of the E.E. which does not vanish on an interval $I=(0, d)$ and $y(x)$ is a function (A, A^*)-admissible on $[0, d]$, it will be convenient to set

$$E[x, y(x), y'(x), p[x, y(x)]] = \mathcal{E}[y(x), u(x)] \qquad \text{(on } I\text{)}, \tag{11.0}$$

where $p(x, y)$ is the slope of the field $y=cu(x)$ at the point (x, y). With this understood we state an extension of Theorem 3.1.

THEOREM 11.1. *If $y(x)$ is (A, A^*)-admissible on $[0, d]$ and $u(x)$ is a solution of the* E.E. *which does not vanish on $(0, d]$, then for $0<x_1<x_2<d$,*

$$J(y)\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \mathcal{E}[y(x), u(x)]dx + S[y(x), u(x)]_{x_2}^{x_1}. \tag{11.1}$$

The proof of (11.1) is similar to the proof of (3.1). It is based on the formula

$$J(y)\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \mathcal{E}dx + H_y\Big|_{x_1}^{x_2},$$

where H is the Hilbert integral set up for the field $y=cu(x)$. As in §3 one shows that

$$H_y\Big|_{x_1}^{x_2} = S[y(x), u(x)]_{x_2}^{x_1}$$

and (11.1) follows at once.

We are led to the following theorem.

THEOREM 11.2. *In order that $[0, d]$ afford a minimum limit to J, it is sufficient that there exist a solution $u(x)$ of the* E.E. *such that $u(x)\neq 0$ on $(0, d)$ and such that*

$$\liminf_{x_1=0,\, x_2=d} S[y(x), u(x)]_{x_2}^{x_1} \geqq 0$$

for every (A, A^*)-admissible curve $y=y(x)$.

Theorem 11.2 follows at once from (11.1).

An inequality. Let $z(x)$ and $\mu(x)$ be two solutions of the E.E. such that $z(b)=\mu(b)$ for some value of b on $(0, d)$ and such that $z(x)>0$ on $(0, b]$ and $\mu(x)>0$ on $[b, d)$. Let $y=y(x)$ be a curve which is (A, A^*)-admissible on $[0, d]$. By a proof similar to that of (11.1) we find that

$$J(y)\Big|_{x_1}^{b} = \int_{x_1}^{b} \mathcal{E}[y(x), z(x)]dx + S[y(x), z(x)]_{b}^{x_1}, \tag{11.2}$$

$$J(y)\Big|_{b}^{x_2} = \int_{b}^{x_2} \mathcal{E}[y(x), \mu(x)]dx + S[y(x), \mu(x)]_{x_2}^{b}. \tag{11.3}$$

We thus arrive at the following inequality

$$J(y)\Big|_{x_1}^{x_2} \geqq S[y(x),\, z(x)]_b^{x_1} + S[y(x),\, \mu(x)]_{x_2}^{b}. \tag{11.4}$$

We shall use (11.4) in proving the following theorem.

THEOREM 11.3. *If in the case $q(x)\equiv 0$ there exists a solution $u(x)$ of the* E.E. *for which*

$$u'(x) \leqq 0, \qquad u(x) > 0, \qquad (x > 0),$$

neighboring $x=0$ and a solution $v(x)$ such that

$$v'(x) \geqq 0, \qquad v(x) > 0, \qquad (x < d),$$

neighboring $x=d$, and if $x=0$ possesses no conjugate point on $[0, d)$, J assumes a minimum limit on $[0, d]$ among curves which are (A, A^*)-admissible.

Let $w(x)$ be a solution of the E.E. which is positive on $(0, d)$. Let b be a constant on $(0, d)$. Let $z(x)$ be a solution such that

$$z(b) = w(b), \qquad z'(b) = w'(b) - e \qquad (e > 0). \tag{11.5}$$

Note that

$$z(x) \geqq w(x) > 0 \qquad (0 < x \leqq b).$$

Similarly let $\mu(x)$ be a solution such that

$$\mu(b) = w(b), \qquad \mu'(b) = w'(b) + e. \tag{11.6}$$

Observe that

$$\mu(x) \geqq w(x) > 0 \qquad (b \leqq x \leqq d).$$

Let $y=y(x)$ be an (A, A^*)-admissible curve. We turn to the relation (11.4).

If the above constant e is sufficiently small, neither $z(x)$ nor $\mu(x)$ will be a focal solution belonging to $x=0$ or $x=d$. It follows as in the proof of Theorem 5.3 that

$$\liminf_{x_1=0} S[y(x),\, z(x)] \geqq 0, \tag{11.7}$$

$$\liminf_{x_2=d} S[y(x),\, \mu(x)] \leqq 0. \tag{11.8}$$

From (11.4) we see that

$$\liminf_{x_1=0,\, x_2=d} J(y)\Big|_{x_1}^{x_2} \geqq S[y(b),\, \mu(b)] - S[y(b),\, z(b)]. \tag{11.9}$$

Upon letting the constant e approach zero the right member of (11.9) tends to zero, and the theorem follows directly.

The following theorem is an easy consequence of the preceding (cf. proof of Theorem 5.4).

THEOREM 11.4. *If $p \geqq 0$ neighboring $x=0$ and $x=d$, if $q\equiv 0$, and if there is no point on $[0, d)$ conjugate to $x=0$, $[0, d]$ affords a minimum limit to J among (A, A^*)-admissible curves.*

12. **The regular case: two singular points.** We now suppose that the integrand f in (10.1) may be given the form of the integrand in (8.0) or (8.1) neighboring $x=0$, and that neighboring $x=d$ it may be given the form

$$
\begin{aligned}
(12.1)\qquad &(x-d)^{\alpha^*}g^*(x)y'^2-(x-d)^{\alpha^*-2}h^*(x)y^2\\
&\qquad\equiv (x-d)^{\alpha^*}g^*(x)y'^2-(x-d)^{m^*-2}k^*(x)y^2,
\end{aligned}
$$

where $g^*(x)$, $h^*(x)$, and $k^*(x)$ are analytic in x neighboring $x=d$, with $g^*(x)>0$ and $k^*(d)\neq 0$. The E.E. corresponding to such a functional will have a regular singular point at $x=0$ and at $x=d$. The indicial roots corresponding to the singular point $x=d$ will be denoted by σ^*, and τ^*, with $\sigma^*\geqq\tau^*$ when σ^* and τ^* are real.

Our first theorem is as follows.

THEOREM 12.1. *Necessary and sufficient conditions that $[0, d]$ afford a minimum limit to J among curves which are (A, A^*)-admissible are that $[0, d]$ contain no point conjugate to $x=0$ and that $m>1$ if $k(0)>0$ and $m^*>1$ if $k^*(0)>0$.*

That the conditions of the theorem are necessary follows from the theorems for one singular point. One considers the segments $[0, b]$ and $[b, d]$ with b on $(0, d)$.

To prove the conditions sufficient we consider the limits

$$
(12.2)\qquad \lim_{x_1=0}\int_{x_1}^{b} x^{\alpha}y'^2dx,\qquad \lim_{x_2=d}\int_{b}^{x_2}(x-d)^{\alpha^*}y'^2dx,
$$

where $y(x)$ is (A, A^*)-admissible on $[0, d]$ and $(0<b<d)$. If one or both of the limits (12.2) are infinite, it is clear that

$$
(12.3)\qquad \liminf_{x_1=0,\, x_2=d} J(y)\Big|_{x_1}^{x_2} > 0,
$$

as desired. Suppose both of the limits (12.2) are finite. It follows from Lemma 8.1 that $S[y(x), u(x)]$ tends to zero as x tends to zero or d where $u(x)$ is any solution of the E.E. which is not zero on $(0, d)$.

The theorem follows from Theorem 11.2.

Theorem 12.1 has the following corollary.

COROLLARY. *If* $[0, d)$ *contains no point conjugate to* $x=0$, *necessary and sufficient conditions that* J *assume a minimum limit among* (A, A^*)*-admissible curves on* $[0, d]$ *are that for each constant* b *on* $(0, d)$ J *assumes a minimum limit on* $[0, b]$ *and* $[b, d]$ *respectively among curves* A*-admissible and* A^**-admissible on* $[0, b]$ *and* $[b, d]$ *respectively.*

We next consider the possibility of improper minimum limits. Theorem 8.2 includes the conditions

$$(12.4)' \qquad m > 1, \qquad \alpha < 1 \text{ if } k(0) > 0,$$

$$(12.4)'' \qquad \alpha = m, \text{ or } \alpha > 1 \text{ if } k(0) < 0.$$

We term these conditions the *finite conditions* in Theorem 8.2. Analogous finite conditions at the point $x=d$ are obtained by placing an asterisk above m, α, and k in (12.4). With this understood we state the following theorem.

THEOREM 12.2. *Necessary and sufficient conditions that* $[0, d]$ *afford an improper minimum limit to* J *among curves which are* (A, A^*)*-admissible on* $[0, d]$ *are* (1) *that a focal solution belonging to* $x=0$ *be a focal solution belonging to* $x=d$ *and* (2) *that the finite conditions* (12.4) *hold at* $x=0$ *and the analogous conditions hold at* $x=d$.

We begin by proving condition (1) necessary.

If $[0, d]$ affords an improper minimum limit to J, there exists an admissible curve $y=y(x)\neq 0$ such that

$$(12.5) \qquad \liminf_{x_1=0,\, x_2=d} J(y)\Big|_{x_1}^{x_2} = 0.$$

Suppose in particular that $y(b)>0$ where $0<b<d$. Let $z(x)$ and $\mu(x)$ be focal solutions belonging to $x=0$ and $x=d$ respectively such that

$$(12.6) \qquad z(b) = \mu(b) = y(b).$$

If condition (1) is not necessary, as we now assume, $z(x)\not\equiv\mu(x)$. It follows that

$$(12.7) \qquad z'(b) > \mu'(b)$$

as one sees from the properties of focal solutions.

We shall make use of the relation (11.4). If (12.5) holds, it is impossible for either of the limits (12.2) to be infinite. It follows from Lemma 8.1, as we have noted previously, that (11.7) and (11.8) hold, and accordingly (11.9). Upon making use of the definitions of $z(x)$ and $\mu(x)$ and the relation (12.7), (11.9) takes the form

$$\liminf_{x_1=0,\, x_2=d} J(y) \Big|_{x_1}^{x_2} \geqq r(b)y(b)[z'(b) - \mu'(b)] > 0 \tag{12.8}$$

contrary to (12.5).

We infer that $z(x) \equiv \mu(x)$, and condition (1) of the theorem is proved necessary.

To prove conditions (2) necessary we trace (12.8) back to equations (11.2) and (11.3) and observe that the equality holds in (12.8) only if

$$y(x) \equiv z(x) \equiv \mu(x);$$

that is, if $y(x)$ is a focal solution belonging to $x=0$ and $x=d$. That conditions (2) are necessary and that conditions (1) and (2) are sufficient follow now as in the proof of Theorem 8.2.

It is clear that we can use theorems on minimum limits when the functional has one singular point to suggest theorems on minimum limits for the various types of (X, Y)-admissibility. The methods of proof will combine the methods of §9 and the present section. We give one additional theorem suggested by Theorems 9.1 and 9.2.

THEOREM 12.3. *Sufficient conditions that* $[0, d]$ *afford a minimum limit to* J *among curves which are* (α, F)*-admissible are that* $[0, d)$ *contain no point conjugate to* $x=0$ *and that* $\tau^*<0$ *or* $\sigma^*=\tau^*=0$. *This minimum limit is improper if and only if a focal solution belonging to* $x=0$ *is a focal solution belonging to* $x=d$, *if* $\sigma>0$, $\sigma-\tau>0$, *and* $\sigma^*>0$.

EXAMPLE 12.1. The following theorem is stated in H.L.P. It follows from Theorem 12.3.

If y' *belongs to* $L^2(0, \infty)$, $y_0=0$, *and* y *is not always zero, then*

$$J(y) = \int_0^\infty \left(4y'^2 - \frac{y^2}{x^2}\right) dx > 0. \tag{12.9}$$

The curves admitted by H.L.P. are a sub-class of those (α, F)-admissible on $[0, \infty]$. To apply the preceding theorem we make the transformation

$$t = \frac{x}{x+1}, \qquad x = \frac{t}{1-t}.$$

The transformed functional will have singular points at $t=0$ and $t=1$ with $\sigma=\tau=\frac{1}{2}$ and $\alpha^*=m^*=2$. Since

$$-1 = 1 - \alpha^* = \sigma^* + \tau^*,$$

we infer that $\tau^*<0$. The E.E. corresponding to (12.9) has the focal solution

$x^{1/2}$ so that the conjugate point condition is satisfied. The minimum limit afforded by $[0, \infty]$ is proper since $\sigma = \tau$.

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