EXTENSIONS OF THE FOUR-VERTEX THEOREM*

BY

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1. Introduction. The "Vierscheitelsatz" states that an oval has at least four vertices, that is, that the curvature of an oval has at least four relative extrema.† It is the purpose of this paper to investigate the extent to which the theorem can be generalized.

In order that the theorem make geometric sense, it is essential that the curve under consideration be closed and have continuous curvature. Beyond these assumptions the only restriction placed on the curve is that it be regular. Initially, to be sure, it is assumed that the curve contains no rectilinear segments, but this restriction is eventually lifted.

By the angular measure of a regular curve shall be meant the algebraic angle through which the directed tangent turns when the curve is completely traced. The angular measure of a closed regular curve is $2n\pi$, where n is zero or an integer which can be made positive by a suitable choice of the sense in which the curve is traced. Closed regular curves may thus be divided into classes K_n $(n=0, 1, 2, \cdots)$, where the class K_n consists of all the curves of angular measure $2n\pi$.

A closed regular curve is an oval if (a) it has no points of inflection and either (b_1) it is simple or (b_2) it is of class K_1. Of the latter conditions we shall choose the condition (b_2). This condition is, in fact, weaker than the alternative condition (b_1). In other words, the simple closed regular curves constitute a subclass of the curves K_1.‡

It turns out that the four-vertex theorem is true for every simple closed regular curve except, of course, a circle. Fundamental in the establishment of this fact is the analysis of a certain kind of arc which, on account of the similarity of its shape to the curved portion of the Greek capital omega, is called an arc of type Ω. The most significant property, for the purpose in view, of an arc of this type is that, when it is traced so that its curvature is non-negative,

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† The theorem seems to have been first published by Mukhopadhyaya, *New methods in the geometry of a plane arc*, Bulletin of the Calcutta Mathematical Society, vol. 1 (1909), pp. 31–37. For other proofs, see, for example, Kneser, *Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven*, Heinrich Weber Festschrift (1912), pp. 170–180; Blaschke, *Die Minimalzahl der Scheitel einer geschlossenen konvexen Kurve*, Rendiconti di Palermo, vol. 36 (1913), pp. 220–222; Bieberbach, *Differentialgeometrie*, pp. 23–27.

‡ See, e.g., H. Hopf, *Über die Drehung der Tangenten und Sehnen ebener Kurven*, Compositio Mathematica, vol. 2 (1935), pp. 50–62.

the curvature has always at least one minimum interior to the arc (Theorem Ω, §4). By means of this property the four-vertex theorem is established for ovals which contain rectilinear segments (§6) as well as for ordinary ovals (§5). To establish the theorem for every simple closed regular curve it is found necessary to bring in, also, the theory of directed lines of support (§§7, 8).

Examples are given in §11 to show that the four-vertex theorem is not true for all the curves of any given class, K_n, of closed curves. However, there exists in each one of these classes a subclass of curves for which the theorem is true. When $n \geqq 2$, this subclass consists of all the curves which contain an arc of type Ω and possess points of inflection. If $n=0$ or $n=1$, the subclass consists of all the curves containing an arc of type Ω; the prescription of points of inflection is unnecessary in either of these cases. It is to be noted that the subclass of the class K_1 thus defined contains, in addition to the simple closed curves, many types of curves with double points.

In a recent paper* the four-vertex theorem for an oval was given a new form which is more discriminating and also more in keeping with the theorem as a result in the large. The theorem in its new form states that an oval has at least four *primary* vertices. By a primary vertex is meant an extremum of the curvature which is greater than or less than the average curvature according as it is a maximum or a minimum. The present paper, whose methods are largely qualitative rather than quantitative, deals with the theorem in its original form. Extension of the theorem in its new form is left an open question.

2. **Regular plane curves.** A plane curve admitting a parametric representation in terms of real single-valued functions of a real variable which are of class C'' in a certain interval and whose first derivatives never vanish simultaneously in this interval is known as a regular plane curve of class C'', and the parameter in question is called a regular parameter. For such a curve the arc s is a regular parameter and hence the curve has regular parametric equations of the form $x=x(s)$, $y=y(s)$.

If $\phi=\phi(s)$ is the single-valued function of s of class C' which measures the directed angle from the positive axis of x to the directed tangent to the curve at the point $P: s=s$, the curvature at P is given by $1/R=d\phi/ds$.

The curvature at P is positive or negative according as the curve in an immediate neighborhood of P lies to the left or to the right of the directed tangent at P, is zero if P is a point of inflection, and changes sign when the direction in which the arc is measured is reversed.

* Graustein, *A new form of the four-vertex theorem*, Monatshefte für Mathematik und Physik, Wirtinger Festband (1936), pp. 381–384.

From the definition of ϕ, it follows that $dx/ds = \cos\phi$, $dy/ds = \sin\phi$. Hence, we obtain, as regular parametric equations of the curve,

$$(1) \qquad x = \int_0^s \cos\phi\, ds, \qquad y = \int_0^s \sin\phi\, ds,$$

where

$$(2) \qquad \phi = \int_0^s \frac{ds}{R},$$

provided merely that the point $P_0: s = 0$ is taken as the origin of coordinates and the directed tangent at P_0 as the positive axis of x.

The angle ϕ is, by definition, the directed arc of the tangent indicatrix, $\xi = \cos\phi$, $\eta = \sin\phi$, of the given curve. This arc is measured positively in the counterclockwise direction on the circle which bears the tangent indicatrix and negatively in the opposite direction.

By the *angular measure* of the arc P_1P_2 of the given curve, where P_1 and P_2 are the points $s = s_1$ and $s = s_2$ respectively, we shall mean the quantity $\phi_2 - \phi_1$, where ϕ_i is the value of ϕ for $s = s_i$, $i = 1, 2$. This angular measure may be thought of geometrically either as the algebraic angle through which the tangent to the curve at P turns when P traces the arc P_1P_2 from P_1 to P_2 or as the algebraic arc length of corresponding arc of the tangent indicatrix, when correspondingly traced. It may be positive, negative, or zero. The angular measure of a circle is, for example, $\pm 2\pi$, depending on the direction in which the circle is traced, and the angular measure of a lemniscate is zero.

3. **Arcs of non-negative curvature.** Consider a regular curvilinear arc of class C'' which contains no rectilinear segments and whose curvature is of one sign or zero. Trace the arc in the sense which makes the curvature non-negative, and denote the initial and terminal points of the arc so traced by A and B respectively. Measure the directed arc length from A and choose coordinate axes so that the complete arc, AB, is represented by equations (1), $0 \leqq s \leqq l$, where l is the length of the arc.

It will be convenient to use, instead of (1), the parametric representation of the arc AB in terms of the angle ϕ. Since $d\phi/ds \geqq 0$, $0 \leqq s \leqq l$, and the arc AB contains no rectilinear segments, $\phi = \phi(s)$ is a non-decreasing function of ϕ and $s = s(\phi)$ is a single-valued function of ϕ, defined over the interval $0 \leqq \phi \leqq \lambda$, where λ is the angular measure of the arc AB. Thus, the desired parametric representation is

$$(3) \qquad x = \int_0^\phi R(\phi) \cos\phi\, d\phi, \qquad y = \int_0^\phi R(\phi) \sin\phi\, d\phi, \qquad 0 \leqq \phi \leqq \lambda.$$

At the points of the interval $0 \leqq \phi \leqq \lambda$ at which $1/R=0$, the integrals in (3) are improper but convergent. Except at these points, $R=R(\phi)$ is a continuous function of ϕ and ϕ is a regular parameter for the curve.

LEMMA 1. *Let AB be an arc of non-negative curvature with no rectilinear segments, represented by equations of the form* (3), *and let $P_0=A, P_1, P_2, P_3, \cdots$, be the points on it for which $\phi=0$, $\phi=\pi/2$, $\phi=\pi$, $\phi=3\pi/2, \cdots$, respectively. Then the arc P_0P_1 is rising to the right, concave upward; the arc P_1P_2 is rising to the left, concave downward; the arc P_2P_3 is falling to the left, concave downward; the arc P_3P_4 is falling to the right, concave upward; and so forth.**

From the relations (3) we have $dx/d\phi=R\cos\phi$, $dy/d\phi=R\sin\phi$, and $d^2y/dx^2=\sec^3\phi/R$, and hence the lemma follows.

LEMMA 2. *The arc AB of Lemma* 1, *if it does not cut itself and never gets below the tangent L at A, has the character of an inwinding spiral. The only point on it, other than A, at which the tangent can coincide with L is the point P_4: $\phi=2\pi$, and P_4 is then to the left of A.*

Let the tangents at the points $P_0=A, P_1, P_2, \cdots$, defined in Lemma 1 be $T_0=L, T_1, T_2, \cdots$, respectively. By Lemma 1, T_2 lies above T_0 and T_4 lies below T_2. If T_4 were also below T_0, the arc P_2P_4 would cut T_0 and the hypothesis that AB never gets below T_0 would be contradicted. Thus, T_4 lies below T_2 and lies above or is coincident with T_0. Therefore, T_4 cuts the arc P_0P_2 in a unique point C, other than P_2.

The point P_4 lies to the left of C on T_4 and hence to the left of P_0 if $T_4=T_0$ and $C=P_0$. For, if P_4 were to the right of C or coincided with C, P_4 would be outside the open region S_1 bounded by the arc P_0P_2, the lines T_0, T_2 and any line parallel to T_3 which is to the left of both T_3 and A, whereas the point P: $\phi=\pi+\epsilon$, $\epsilon>0$, is, for ϵ sufficiently small, in S_1; but, by Lemma 1, the arc P_2P_4 cannot leave S_1 except over the arc P_0P_2 and the arc AB would then have cut itself.

Consider, next, the open region S_2 bounded by the arc CP_2P_4 and the line segment P_4C. The arc P_4P_6 is entirely in S_2 since, by Lemma 1, it is above the line $T_4=P_4C$ and cannot intersect the arc CP_2P_4. In particular, then, T_5 is to the left of T_1, and T_6 is below T_2.

The argument now begins to repeat itself. Since P_6 is in S_2, T_6 cuts the arc P_2P_4 in a point D to the left of P_6, and the arc P_6P_8 lies in the open region S_3 bounded by the arc DP_4P_6 and the line segment P_6D, and so forth. Thus, the lemma is fully established.

* In using here, the terms "right," "left," "upward," "downward" and similar terms, we assume that the coordinate axes to which the arc AB is referred are so visualized that they appear horizontal and vertical respectively, with the x-axis directed to the right and the y-axis directed upward.

4. **Arcs of type Ω.** We shall say that an arc AB is of type Ω if (a) its curvature (when it is traced from A to B) is non-negative and it contains no rectilinear segments; (b) the tangents at A and B are identical; (c) it lies on one side of the common tangent, L, at A and B; and (d) it does not cut itself except that B may coincide with A.

As in §3, we shall assume that the arc AB is referred to A as origin of coordinates and to the directed tangent at A as axis of x, and we shall think of it as represented by equations of the form (3).

LEMMA 3. *The angular measure of an arc AB of type Ω is 2π and the terminal point B lies to the left of the initial point A, if it is not coincident with it.*

These facts follow directly from Lemma 2.

LEMMA 4. *The curvature of an arc AB of type Ω cannot be monotonic, unless it is constant.*

The points A and B have respectively the coordinates $y_A=0$ and

$$y_B = \int_0^\pi R \sin \phi \, d\phi + \int_\pi^{2\pi} R \sin \phi \, d\phi.$$

It is evident that, if R is not constant, y_B would be positive if R were non-increasing, and negative if R were non-decreasing. But, by property (b) of an arc of type Ω, $y_B=y_A=0$. Thus, the lemma is proved.

DEFINITION. An arc AB of angular measure 2π whose curvature (when the arc is traced from A to B) is non-negative is said to overlap itself with respect to a point K on it if, when K is taken as the origin of new coordinates $(\bar{x}, \bar{y})$ and the directed tangent at K is chosen as the $\bar{x}$-axis, the point B is horizontally to the left of the point A with respect to the new axes: $\bar{x}_A-\bar{x}_B>0$.

LEMMA 5. *An arc AB of type Ω fails to overlap itself with respect to every point on it for which $\pi/2\leqq\phi\leqq 3\pi/2$. It does overlap itself with respect to every point on it for which $0\leqq\phi<\pi/2$ or $3\pi/2<\phi\leqq 2\pi$, provided merely that B does not coincide with A.*

If $B=A$, there is evidently no overlap with respect to any point of the arc. If $B\neq A$, the results stated follow directly from the fact that the tangents at B and A are identical and B is to the left of A (Lemma 3).

LEMMA 6. *If the curvature of an arc AB of type Ω has just one extremum interior to the arc AB, this extremum must be a minimum.*

By an extremum of the curvature shall be meant a point of the arc AB, or of the corresponding interval $0\leqq\phi\leqq 2\pi$, at which the curvature has a relative extremum, or a segment of the arc, or the interval, for which the curvature has a constant relatively extreme value with respect to neighboring segments on either side of it. In the latter case, the extremum shall be said to be interior to the arc AB, or the interval $0 \leqq \phi \leqq 2\pi$, only if the segment in question contains neither end point of the arc, or the interval.

We shall establish the desired conclusion by showing that the opposite one leads to a contradiction. If the single extremum of $1/R$ is a maximum, $1/R$ can vanish only at A or at B. Hence, R is continuous interior to the arc AB, has a single extremum, a minimum, interior to the arc AB, and may become infinite at A or at B.

The curve which consists of the graph of the function $R = R(\phi)$, $0 \leqq \phi \leqq 2\pi$, and the portions of the lines $\phi = 0$ and $\phi = 2\pi$ which lie respectively above the points $\phi = 0$ and $\phi = 2\pi$ of this graph when these points exist, is a continuous curve and hence possesses a horizontal chord of length π. It follows from the conditions on $R(\phi)$ that, if c is the height of this chord above the y-axis, $R(\phi) \leqq c$ for the portion of the graph having the same projection on the ϕ-axis as the chord and $R(\phi) \geqq c$ for the remainder of the graph. Hence, if γ is the abscissa of the midpoint of the chord,

$$I = \int_0^{2\pi} R(\phi) \cos (\phi - \gamma) d\phi < 0.$$

When we set $\phi - \gamma = \bar{\phi}$, the integral I becomes

$$I = \int_0^{2\pi-\gamma} \bar{R}(\bar{\phi}) \cos \bar{\phi}\, d\bar{\phi} - \int_0^{-\gamma} \bar{R}(\bar{\phi}) \cos \bar{\phi}\, d\bar{\phi} = \bar{x}_B - \bar{x}_A,$$

where $\bar{x}_A$ and $\bar{x}_B$ are respectively new abscissas of the points A and B, referred to the point $K: \phi = \gamma$ on the arc AB as origin and the directed tangent at K as axis of $\bar{x}$. Hence, $\bar{x}_A - \bar{x}_B > 0$. By construction, $\pi/2 \leqq \gamma \leqq 3\pi/2$. Thus, the arc AB overlaps itself with respect to a point on it for which $\pi/2 \leqq \phi \leqq 3\pi/2$. But, herewith we have the desired contradiction to Lemma 5.

THEOREM Ω. *The (non-negative) curvature of an arc of type Ω has at least one minimum interior to the arc or is constant throughout the arc.*

Assuming that the curvature is not constant, we conclude that it has extrema in the closed interval $0 \leqq \phi \leqq 2\pi$, since it is continuous in this interval. Furthermore, it is not monotonic, by Lemma 4, and hence has at least one extremum in the open interval. If it has just one, this must be a minimum, by Lemma 6, and if it has more than one, at least one must be a minimum in any case.

The arc (3) for which

$$R = a + \sin (\phi/2), \qquad a > 1, \qquad 0 \leqq \phi \leqq 2\pi,$$

is readily shown to be an arc of type Ω with a single extremum, a minimum, interior to it.

5. **Ovals.** By an oval we shall mean here a closed regular curve of class C'' with no rectilinear segments whose curvature is always of one sign or zero and whose tangent indicatrix is a circle traced just once. We shall assume that the oval is so traced that the curvature is non-negative. The angular measure is then 2π.

LEMMA 7. *An oval is an arc of type Ω with respect to every point on it.*

If A is a point of an oval, the oval may be considered as an arc AB of non-negative curvature, where $B=A$. It will follow that this arc is of type Ω if it can be shown that it lies to the left of the tangent at A and does not cut itself. But these properties follow without difficulty by application of Lemma 1.

By a vertex of a closed curve we shall mean an extremum of the curvature, as defined in §4.

THEOREM 1. *An oval, not a circle, has at least four vertices.*

The curvature of the oval has in any case two extrema, a maximum and a minimum. Let A be the point of the oval, or a point of the segment thereof, for which the curvature has a minimum. By Lemma 7, the arc AB, where $B=A$, is of type Ω, and hence, by Theorem Ω, its curvature has at least one minimum interior to it. Thus, the curvature of the oval has at least two minima and therefore at least two maxima.

6. **Flattened ovals.** A closed regular curve of class C'' which contains one or more rectilinear segments and, when properly traced, has non-negative curvature and angular measure 2π, we shall call a flattened oval.

It may be readily proved by use of an extension of Lemma 1 covering the case of an arc of non-negative curvature with rectilinear segments that a flattened oval lies to the left of every directed tangent and does not cut itself. Hence, we conclude the following proposition.

LEMMA 8. *A flattened oval with only one rectilinear segment may be thought of as consisting of an arc AB of type Ω and the line segment BA.*

The curvature of a flattened oval of this type has a minimum on the line segment BA and, by Theorem Ω, at least one minimum interior to the arc AB; and that of a flattened oval with more than one rectilinear segment has at least as many minima as there are rectilinear segments. Thus, we pass to the following result.

THEOREM 2. *A flattened oval has at least four vertices.*

7. **Lines of support.** Let C be a closed regular curve of class C'' without component line segments.

LEMMA 9. *If C, or an open arc of C, lies wholly on one side of a line L except for one or more points on L, then L is a non-inflectional tangent to C at each of these points.*

By hypothesis, C has neither "corners" nor cusps. Hence, in passing through a point P, C crosses every line through P, including the tangent at P, unless this tangent is non-inflectional.

DEFINITION. A directed line L shall be called a line of support of the curve C if C is tangent to L and lies wholly to the left of L except for its one or more points of contact with L.

LEMMA 10. *The curve C has a unique line of support in every oriented direction.*

A pencil of parallel and similarly directed lines admits a unique dichotomy such that all points of C are to the left of every line of the first class and at least one point of C lies on, or to the right of, an arbitrarily chosen but fixed line of the second class. Clearly, there is no "last" line in the first class. There is then a "first" line, L, in the second class. Evidently, C lies wholly to the left of L except for one or more points on it and, therefore, by Lemma 9, C is tangent to L at each of these points. Thus, L is the unique line of support in the given oriented direction.

DEFINITION. A line of support of the curve C shall be called a simple, or a multiple, line of support, according as it is tangent to C in one point or more than one point.

LEMMA 11. *If the directed tangent T_0 to the directed curve C at a point P_0 is a simple line of support, the directed tangent to C at every point in a certain neighborhood of P_0 is a simple line of support. In this neighborhood of P_0 there exists a parametric representation of C in terms of the directed angle from T_0 to an arbitrary line of support and this parametric representation is identical with a representation in terms of the arc of the tangent indicatrix.*

We refer C to P_0 as origin and T_0 as x-axis, and measure the arc s from P_0. By hypothesis, the curvature of C at P_0 is positive or zero, and, if zero, does not change sign at P_0. Hence $\epsilon>0$ exists so that at all points of the arc $P_{-\epsilon}P_0P_\epsilon$, that is, at every point P_s of C for which $-\epsilon \leqq s \leqq \epsilon$, the curvature is non-negative. It follows, by Lemma 1, that, if T_s is the directed tangent at the arbitrary point P_s of this arc, the arc lies to the left of T_s except for the point P_s, provided that ϵ is so chosen that the angular measure of the arc is less than π.

The directed distance $D(s, \bar{s})$ from T_s to a variable point on the complementary arc $P_\epsilon P_{-\epsilon}$, that is, to an arbitrary point $P_{\bar{s}}$ for which $\epsilon \leqq \bar{s} \leqq l-\epsilon$, where l is the length of C, is a continuous function of s and $\bar{s}$. Moreover, it follows from the fact that T_0 is a simple line of support that $D(0, \bar{s})$, the directed distance from T_0 to $P_{\bar{s}}$, is bounded away from zero for $\epsilon \leqq \bar{s} \leqq l-\epsilon$. Hence, a positive constant $\delta \leqq \epsilon$ exists so that for every T_s for which $-\delta < s < \delta$, $D(s, \bar{s})$ fails to vanish for $\epsilon \leqq \bar{s} \leqq l-\epsilon$. Consequently, the tangent T_s at every point P_s for which $-\delta < s < \delta$ is a simple line of support, and the first part of the lemma is proved.

The arc $P_{-\delta}P_0P_\delta$, since its curvature is non-negative, admits a representation in terms of the directed angle ϕ from the positive x-axis to an arbitrary directed tangent; see §2. But the directed tangents to the arc are all simple lines of support of C and constitute precisely the lines of support of C in the oriented directions of these tangents. Hence, the angle ϕ can be thought of as the angle from the positive axis of x to an arbitrary one of these lines of support, and the second part of the lemma is established.

8. **Simple closed curves.** We shall mean by a simple closed curve a closed regular curve of class C'' which contains no rectilinear segments and no double points.

LEMMA 12. *A simple closed curve all of whose lines of support are simple is an oval.*

Lemma 11 implies in this case that the lines of support of the given curve C envelope a closed continuous arc of C which when properly traced has non-negative curvature and angular measure 2π; in other words, an oval. But C is a connected curve without double points and consists, therefore, only of this oval.

LEMMA 13. *A simple closed curve, not an oval, has at least one multiple line of support.*

This follows directly from Lemma 12.

LEMMA 14. *The directions of a directed simple closed curve at two points of contact with a multiple line of support are the same and the number of inflections in an arc of the curve joining the two points, unless infinite, is even or zero.*

Consider the open region S bounded by the multiple line of support, L, and one of the two arcs of the given curve C joining the two given points of contact of C with L. If the directions of C at the two given points were opposite, there would exist two points neighboring respectively to the given points and belonging to the second arc of C joining the given points, one of which is inside S while the other is outside S, and hence the second arc of C

would have to cross either the first arc or L. In either case an hypothesis would be contradicted. Thus, C has the same direction at the two given points.

Let C now be directed so that the common direction at the two points is that of L. Then C lies to the left of its directed tangents at the two points. Moreover, since L is a line of support, neither of the points is an inflection. Therefore, in sufficiently restricted neighborhoods of the two points, the curvature of C is non-negative and hence the number of inflections in each arc of C joining the two points, if not infinite, is even or zero.

LEMMA 15. *If an arc of a simple closed curve which joins two points of contact of the curve with a multiple line of support contains no points of inflection, the arc, when suitably traced, is of type* Ω.

Let the given curve C be directed as in the paragraph preceding the lemma, and denote by A and B respectively the initial and terminal points of the arc of C in question. Then the arc AB has non-negative curvature, and obviously has the remaining properties required of an arc of type Ω.

LEMMA 16. *A simple closed curve with no points of inflection is an oval.*

It suffices to show that all the lines of support of the given curve C are simple, for Lemma 12 will then apply. Suppose there were a multiple line of support, L, tangent to C at the distinct points A and B. Then, if C were properly traced, both of the arcs AB and BA would be of type Ω, by Lemma 15. Consequently, if L were thought of as horizontal, each of the points A and B would be to the left of the other on L, by Lemma 3. This is absurd, and accordingly C has only simple lines of support.

9. **The four-vertex theorem for simple closed curves.** We prove the following theorem:

THEOREM 3. *A simple closed curve, not a circle, has at least four vertices.*

A simple closed curve C obviously has no inflections, an even number, or infinitely many. Inasmuch as there is at least one vertex between each two points of inflection, the theorem is true, though trivial, if there are more than two points of inflection.

If there are no points of inflection, the curve C is an oval, by Lemma 16,* and hence has, according to Theorem 1, at least four vertices, unless it is a circle.

There remains the case in which there are just two points of inflection.

* It seems preferable to employ here the simple Lemma 16 rather than the more powerful proposition that every simple closed curve, properly traced, has angular measure 2π (see, e.g., Hopf, loc. cit.), for thereby the proof of Theorem 3 is effected without appeal to this proposition.

The curve C cannot, then, be an oval, and hence has, by Lemma 13, at least one multiple line of support, L. Let A and B be two points of contact of C with L, and consider the arcs AB and BA in which they divide C. According to Lemma 14, one of these arcs must contain both points of inflection of C. The other arc contains, then, no point of inflection and is therefore, by Lemma 15, of type Ω. Let this be the arc AB, traced from A to B, and denote the complementary arc now by BI_1I_2A, where I_1 and I_2 are the two points of inflection.

Since we are now tracing C so that the direction at A is that of the directed line L, the curvature of the arc I_2ABI_1 is non-negative, whereas that of the arc I_1I_2 is non-positive.

Inasmuch as the arc AB is of type Ω, it follows, from Theorem Ω, that $1/R$ has a minimum interior to the arc AB and hence interior to the arc I_2ABI_1. But $1/R$ surely has a minimum interior to the arc I_1I_2. Thus, $1/R$ has at least two minima and so must have at least two maxima. Consequently, C has at least four vertices.

10. **The four-vertex theorem for flattened simple closed curves.** A closed regular curve of class C'' without double points which contains one or more rectilinear segments we shall call a flattened simple closed curve.

THEOREM 4. *A flattened simple closed curve has at least four vertices.*

The proof of Theorem 3 rests ultimately on Lemmas 1–6, 9–16 and Theorem Ω. Theorem 4 will be established if it can be shown that these propositions remain valid in all essentials when rectilinear segments are admitted as component parts of the curves and arcs, including arcs of type Ω, which are discussed in them. This is indeed the case, as inspection, in view of the following remarks on the more pertinent points at issue, will show.

Though the given curve or arc does not admit, along the component rectilinear segments, a parametric representation in terms of the arc ϕ of the tangent indicatrix, the angle ϕ may still be thought of as a parameter for the curve or arc, provided it is agreed that to certain values of ϕ shall correspond, not points, but segments of straight lines. In keeping with this agreement, the meaning of the term "point" must be broadened, on occasion, to include rectilinear segments. In particular, by a "point of inflection" may be meant either a point at which the curvature changes sign or a rectilinear segment along which the curvature changes sign with respect to neighboring arcs, one on each side of the segment. Again, a line of support will be simple if it is tangent to the given curve either in just one point or along just one rectilinear segment. In this connection, it should be noted that the parametric representation mentioned in Lemma 11 may fail in the sense above described.

However, the only proposition dependent on Lemma 11, namely, Lemma 12, is still valid. Of course, this proposition, and also Lemma 16, has to do with a *flattened* oval as well as with a flattened simple closed curve.

Inasmuch as the parametric representation (3) is employed to establish Lemmas 4 and 6, the possibility of the extension of these lemmas must be particularly scrutinized. Both lemmas deal with an arc AB of type Ω. There can be no rectilinear segment *interior* to AB by the hypothesis of Lemma 4, and, if there were one in the case of Lemma 6, this lemma would be obvious. It may, therefore, be assumed in both cases that a rectilinear segment occurs only at an end of the arc AB. When such segments are suppressed, the proofs of the original lemmas are valid and guarantee the desired extensions.

11. **Further extensions of the theorem.** In this paragraph we shall understand by a closed curve any closed regular curve of class C'', with or without rectilinear segments.

Curves of class K_1. We have shown that every simple closed curve, other than a circle, has at least four vertices. As is well known,* the simple closed curves form a subclass of the class K_1, consisting of all closed curves with angular measure 2π.†

It is not true that every curve of the class K_1 has at least four vertices, as will be evident shortly from an example. However, there exists a subclass of K_1 which includes, besides all simple closed curves, many types of curves with double points, for which the four-vertex theorem holds, namely, the subclass of curves which contain arcs of type Ω with or without rectilinear segments.

THEOREM 5a. *A closed curve of angular measure 2π which contains an arc of type Ω has at least four vertices or is a circle.*

Since the angular measure of an arc of type Ω, when properly traced, is 2π, the given curve is either an oval or has inflections. In the former case Theorems 1 and 2 apply, and in the latter, the general argument in the proof of Theorem 3 in the case of inflections is valid.

For the purpose of giving examples of curves of class K_1 which have just two vertices, we need conditions of closure for a curve when it is given by its intrinsic equation.

It is known that, if $1/R$ is a real, single-valued, continuous function of s in the interval $-\infty < s < \infty$, and

$$\phi = \phi(s) = \int_0^s \frac{ds}{R}, \tag{4}$$

* See, e.g., H. Hopf, loc. cit.

† Here, and later, we assume that every closed curve is so traced that its angular measure is non-negative.

then the equations

$$x = \int_0^s \cos\phi\, ds, \qquad y = \int_0^s \sin\phi\, ds \tag{5}$$

represent a regular plane curve of class C'' for which s is the measure of the arc and $1/R$ is the curvature. This curve is closed and of length l if the functions $x(s)$ and $y(s)$ in (5) are periodic with l as their smallest positive common period.

LEMMA 17. *A necessary and sufficient condition that the curve* (5) *be closed and of length l is that* $1/R$ *be a periodic function of s with the period l such that*

$$\phi(l) = 2n\pi, \qquad n \text{ an integer or zero}, \tag{6}$$

$$\int_0^l \cos\phi\, ds = 0, \qquad \int_0^l \sin\phi\, ds = 0, \tag{7}$$

and that l be the smallest positive period with these properties.

Suppose that the curve is closed and l is its length. Then, $x(l) = y(l) = 0$ and equations (7) hold. Furthermore, differentiation of the identities

$$x(s + l) \equiv x(s), \qquad y(s + l) \equiv y(s) \tag{8}$$

leads to the relation,

$$\phi(s + l) \equiv \phi(s) + 2n\pi, \tag{9}$$

whence it follows that $\phi(l) = 2n\pi$ and, on differentiation, that $1/R$ is periodic of period l.

Conversely, the assumption that $1/R$ is periodic of period l such that (6) holds yields (9) and, by means of (9) and (7), the identities (8) are readily established, and the curve is closed. Moreover, since l is the smallest positive period of $1/R$ for which (6) and (7) are valid, l is the length of the curve.

It is evident that $\phi(l) = 2n\pi$ is the angular measure of the curve. In other words, the curve is of class K_n.

Consider, now, the curve with the intrinsic equation

$$\frac{1}{R} = a \cos s + 1, \qquad a > 0. \tag{10}$$

Inasmuch as $1/R$ is periodic with period 2π and has just two extrema in a period interval, and since

$$\phi(s) = a \sin s + s,$$

so that $\phi(2\pi) = 2\pi$, this curve is a closed curve of class K_1 with just two vertices provided merely that a be so chosen that

$$I_1 = \int_0^{2\pi} \cos (a \sin s + s) ds = 0, \qquad I_2 = \int_0^{2\pi} \sin (a \sin s + s) ds = 0.$$

It is found that $I_2 \equiv 0$ and that $I_1 \equiv -4F(a)$, where

$$F(a) = \int_0^{\pi/2} \sin (a \sin s) \sin s \, ds.$$

When $\sin s$ is replaced by t and the integral is integrated by parts, it turns out that $F(a) = af(a)$, where

$$f(a) = \int_0^1 (1 - t^2)^{1/2} \cos at \, dt.$$

Hence equation (10) represents a closed curve of the desired type for every value of a for which $f(a) = 0$.

It is readily shown that the function $f(a)$ satisfies the differential equation

$$f''(a) + \frac{3}{a} f'(a) + f(a) = 0$$

and that $|f'(a)| \leqq 1/3$. Hence $|f''(a) + f(a)| \leqq 1/a$, and it follows that $f(a)$ is an oscillating function, with infinitely many zeros.

The smallest zero is approximately $(11/9)\pi$; the corresponding closed curve (10) has the shape of a figure eight with a loop in one lobe, interior to the lobe. The next zero is about $(41/18)\pi$ and the corresponding curve looks like a figure eight with two loops, one within the other, in the one lobe and a single loop in the other lobe. The curve corresponding to the nth zero has n loops in the one lobe, and $n-1$ in the other. All the curves are *analytic* and are symmetric in the y-axis.

Since a curve of class K_1 with no inflections is an oval and one with more than two inflections surely has at least four vertices, a necessary condition that a curve of class K_1 have just two vertices is that it have just two points of inflection, with the two vertices separated by them. It would appear, then, that every curve of class K_1 with just two vertices has the general character of one of the infinite set of the curves just described.

Curves of class K_0. The counterpart of Theorem 5a is true also in this case.

THEOREM 5b. *A closed curve of angular measure zero which contains an arc of type Ω has at least four vertices.*

For, since an arc of type Ω, properly traced, has angular measure 2π, the given curve has inflections and therefore the general argument in the proof of Theorem 3 applies.

A lemniscate is a simple example of a curve of class K_0 which has just two vertices.

Curves of class K_n, $n \geqq 2$. The analog of Theorem 5b (or 5a) is not true for a curve of angular measure greater than 2π, inasmuch as in this case it does not follow from the hypothesis of the theorem that the curve has inflections, and there actually exist curves without inflections which have only two vertices and yet contain an arc of type Ω. A limaçon with a double point is, for example, a curve of class K_2 with these properties.

It is, however, true, as is readily verified, that the following variation of Theorem 5b (or 5a) holds.

THEOREM 5c. *A closed curve of angular measure greater than 2π which has inflections and contains an arc of type Ω has at least four vertices.*

It is not difficult to give an example of a closed curve of positive curvature belonging to any prescribed class $K_n (n \geqq 2)$ and having just two vertices. For this purpose we note that, since we are assuming $1/R$ positive, equations (5) may be replaced by the equations

$$x = \int_0^\phi R \cos \phi \, d\phi, \qquad y = \int_0^\phi R \sin \phi \, d\phi, \qquad -\infty < \phi < \infty, \tag{11}$$

and that Lemma 17 then becomes:

LEMMA 18. *A necessary and sufficient condition that the curve of positive curvature* (11) *be a closed curve of class K_n is that n be the smallest positive integer such that $R = R(\phi)$ is periodic of period $2n\pi$ and*

$$\int_0^{2n\pi} R \cos \phi \, d\phi = 0, \qquad \int_0^{2n\pi} R \sin \phi \, d\phi = 0. \tag{12}$$

By means of this lemma it is readily verified that the curve with the intrinsic equation

$$R = a + \cos \frac{\phi}{n}, \qquad a > 1,$$

where n is an integer greater than unity, is a closed curve of class K_n. That the curve has just two vertices is obvious. In particular, if $n = 2$, the curve has the same general character as the limaçon with a loop.

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