PARTIALLY ORDERED SETS*

BY

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1. **Introduction.** A relation, $\subset$, *orders* a set, K, of elements, $a, b, c, \cdots$, if Postulates 1.1 and 1.2 hold for any elements of K.

1.1 Postulate. *$a \subset b$ and $b \subset c$ imply $a \subset c$.*

1.2 Postulate. *$a \subset a$.*

Such a set need not possess simple or linear order and for this reason is commonly called a partially ordered set.‡ A large part of the theory of simply ordered sets applies, with little or no change, to partially ordered sets. Although the principal objectives of the paper are certain properties of partial order which have either a trivial or no counterpart in the theory of simple order, considerable use is made of the parallelism between the two theories without a systematic development of it. This parallelism makes it natural to call any set which satisfies Postulates 1.1 and 1.2 an *ordered set*, dropping the word "partially." This convention leads to no confusion in this paper as but scant reference is made to simple order. $a \subset b$ is read *a is contained in b*.

The relations of *equality* and *equivalence* order any set in which they occur. *Being a subset of* is a relation ordering the subsets of a given set. The relation *less than or equal to* orders the set of real numbers. A set of propositions is ordered by the relation of *implication*. An ordering relation can be assigned to any system, such as a Boolean algebra, which has an associative operation with respect to which every element is idempotent. The relations of *being homomorphic to* and *being a subsystem of* order any aggregate of classes with a common set of operations. These examples do not begin to exhaust even the

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† This paper contains the results of a doctoral dissertation, *Extensions of partially ordered sets*, accepted by Harvard University in May, 1935. It also contains further results obtained while the author was a Sterling Fellow at Yale University, 1935–36.

‡ Hausdorff (10), p. 139, Ore (18) p. 408, von Neumann (17), p. 94, and Alexandroff (1) p. 1650. Alexandroff (1) shows that discrete spaces in which $\overline{A}=\overline{B}$ implies $A=B$ are abstractly equivalent to partially ordered sets where $A \subset B$ if, and only if, A is an element of $\overline{B}$. Bennett (2) speaks of such sets as having semi-serial order. Tucker (26) calls such sets cell spaces and makes the relation $<$ (§2), instead of $\subset$, fundamental.

The numbers in parentheses refer to the bibliography at the end of the paper.

common instances of ordered sets in mathematics.*

This paper presupposes the calculus of classes without restriction as to cardinal number. The classes employed are defined by such properties that the admission or rejection of a candidate does not depend upon the disposition of candidates under consideration or not yet considered. Hence, the paradoxes connected with inconsistent aggregates do not arise and no recourse to the theory of types is necessary. In fact, no classes are employed where the admission or rejection of a candidate depends upon the disposition of elements already considered. The definition of such classes depends upon a particular well ordering of the candidates. For the classes here considered, the possibility of well ordering the candidates is not assumed.

The concept of a class or set, K, can be specialized by the imposition of correspondences between subsets of its elements. A correspondence can be considered as a subclass of the class of ordered pairs of subsets of the given set. Hence, the only new undefined concept needed for the introduction of correspondences is that of an ordered pair.† If, under a correspondence, a subset A corresponds to a subset B, $A \rightarrow B$, then B must be uniquely determined by A and the definition of the correspondence. A is called the *original* of B and B is called the *image* of A. The *domain* of a correspondence in K is the class of subsets which have an image in K. The *range*‡ is the class of subsets which have at least one original in K. The definition of a correspondence may restrict the subsets admissible to the domain both as to the particular elements and the number of elements they may contain. The correspondence may also impose such restrictions on the subsets admissible to the range. A set K is *closed* under a correspondence if the image of every admissible original exists in K. The ordering relation may be defined as a correspondence if the domain is restricted to single elements of K and the image of any element a is the set B of all elements b_j such that $a \subset b_j$. The closure of K under this correspondence is assured by Postulate 1.2. The relations, operations, and homomorphisms, the systematic exploitation of which is the object of this paper, are further instances of correspondences.

In terms of the ordering relation, new relations are defined in §2 and operations in §3. An ordered set is not necessarily closed under the opera-

* For example, the order preserving mappings, $a \rightarrow D(a)$, of an ordered set K upon the interval $[0, 1]$ of real numbers, such that $D(a)+D(b)=D(a+b)+D(ab)$, where addition and multiplication in K are defined in §3 and ordinary addition is used for the real numbers, have as special cases the theories of probability, measure, and dimensionality in projective geometries. See von Neumann (17) and Freudenthal (9).

† Quine (20), p. 10, for classes, p. 18 et seq.

‡ Stone (22), p. 33.

tions of §3, though closure under certain operations implies closure under others. The relations between closure under various subsets of the operations of §3 are worked out in §4 and §7. This forms a basis for the classification of ordered sets with respect to the operations under which they are closed, as equivalent types can now be recognized. Multiplicative systems, lattices or structures, and Boolean algebras are among the distinct types of ordered sets included in this classification.

The object of this classification is to establish content for the extensions which form the principal part of this paper. The purpose of an extension is to adjoin elements to a given set so as to increase the number of operations under which it is closed. This is usually accomplished by the construction of a new set with the desired closure properties and with a subset isomorphic with the given set. These constructions depend upon the theory of homomorphisms developed in §5. Methods of constructing and methods of combining extensions are discussed in §9. In the following sections, the adjunction of units, unrestricted sums and products, unrestricted distributive sums, and complements is attained. By successive application of these extensions, any ordered set can be imbedded in a complete Boolean algebra. §9 contains a more detailed statement of the program of extensions carried out in this paper.

2. **Definition of new relations.** In an ordered set, if $a \subset b$ does not hold, we write $a \not\subset b$ and read *a is not contained in b*. The proofs of theorems, in cases where they are practically automatic, have been omitted. The propositions which follow refer to a set, K, of elements, $a, b, c, \cdots$, ordered by the relation $\subset$.

2.1 THEOREM. *Every subset of an ordered set is an ordered set.*

2.2 DEFINITION. $a \supset b$ if $b \subset a$.

2.3 THEOREM. *The relation $\supset$ orders the set K.*

By virtue of Definition 2.2, an exact duality exists between the relations $\subset$ and $\supset$. Hence, any general proposition holding for the relation $\subset$ implies a dual proposition for the relation $\supset$. The relations $\subset$ and $\supset$ are not mutually exclusive. For example, $a \subset a$ and $a \supset a$. To obtain, mutually exclusive relations we give the following four definitions:

2.4 DEFINITION. $a = b$ if $a \subset b$ and $b \subset a$.
2.5 DEFINITION. $a < b$ if $a \subset b$ and $b \not\subset a$.
2.6 DEFINITION. $a > b$ if $a \not\subset b$ and $b \subset a$.
2.7 DEFINITION. $a \| b$ if $a \not\subset b$ and $b \not\subset a$.

2.8 THEOREM. *Exactly one of the relations $=$, $<$, $>$, and $\|$ holds for any ordered pair of K elements.*

These four mutually exclusive relations are usually used in defining ordered sets. An ordered set in which the relation $\|$ does not occur is a simply ordered set or a set with linear order. It frequently happens that the relations $<$ and $>$ do not occur in an ordered set. There is also the trivial case where all elements are equal. With these exceptions, all four relations occur in any ordered set. The equality defined by Definition 2.4 satisfies the usual postulates for equality and is an ordering relation.

2.9 THEOREM. *Postulates* 1.1 *and* 1.2 *are consistent and completely independent.*

Proof: The necessary examples can be constructed with three elements.

2.10 THEOREM. *If $a=b$ and $c=d$, then $a\subset c$ implies $b\subset d$ and $a\not\subset c$ implies $b\not\subset d$. More generally, equals may be substituted for equals in an ordered set without effecting the ordering relation.*

Let A be an ordered set of elements, a_i, to each distinct element of which an ordered set, $A_i\longleftrightarrow a_i$, of elements, a_{ij}, has been assigned. The set B of elements $b_{ij}\longleftrightarrow a_{ij}$ is defined as any set of elements in a one-to-one correspondence with the totality of elements in the sets A_i. A relation, $\subset$, for B is defined as follows: $b_{ij}\subset b_{k1}$ if either $a_i\subset a_k$ and $a_i\not\supset a_k$, or $a_i=a_k$ and $a_{ij}\subset a_{k1}$. This is the product (in the sense of Hausdorff (11), p. 46), ordered lexicographically, of two ordered sets. The sum (in the sense of Hausdorff (11), p. 44) is a special case of the product. This is an example of a property commonly restricted to simply ordered sets which generalizes at once to partially ordered sets.

2.11 THEOREM. *The set B, as defined above, is ordered by the relation $\subset$, as defined above, if, and only if, the sets A and A_i are ordered.*

3. **Definition of operations.** Given an ordered set, K, it is possible in various ways to define correspondences, $A\to B$, between subsets of K. For example, let $A\to B$ if b_j is in B if, and only if, $b_j\subset a_i$ for every a_i in A. In such situations, we can regard the correspondence as defining B as a function of A or as defining an operation upon A. We are particularly interested in certain correspondences where the elements of the set B are all necessarily equal. By Theorem 2.10, equals may be substituted for equals in ordered sets, so we can represent such sets by a single element. In general, correspondences between subsets and single elements will be referred to as operations. As a rule, these operations are relative to the set K for which they are defined and are not necessarily the same when defined for subsets of K. Thus, to avoid ambiguity, it is necessary to specify the set relative to which the operations are defined. We shall denote subsets of K by capital letters with or without subscripts and elements of these subsets by the same small letter with one

more subscript. For example, a_i is an element of A, b_{jk} of B_j, and so on. Because of the duality between the relations $\subset$ and $\supset$ each definition yields two operations, one for each relation. The operations have been defined for the relation $\subset$ and the changes necessary to obtain the dual have been indicated in parentheses. The following definitions of operations are made with no assumption as to the closure of the arbitrary set, K, under them. The question of closure is treated in later sections. In the theorems of this section it is tacitly assumed that the subsets involved lie in the domain of the operations involved.

3.1 Definition. a is a *product*, $\prod a_i$ (*sum*, $\sum a_i$) of A if $a \subset a_i$ ($a \supset a_i$) for every a_i in A and $x \subset a_i$ ($x \supset a_i$) for every a_i in A implies $x \subset a$ ($x \supset a$).†

3.2 Theorem. *If a and b are both products (sums) of A, then $a=b$.*

3.3 Theorem. *If $a=\prod a_i$ ($\sum a_i$) and $b=\prod b_j$ ($\sum b_j$) and, for each a_i in A, b_j exists in B such that $b_j \subset a_i$ ($b_j \supset a_i$), then $b \subset a$ ($b \supset a$).*

3.4 Corollary. *If $a=b$, then $ac=bc$ ($a+c=b+c$). More generally, equals may be substituted for equals in products (sums) without changing the value of the product (sum).*

3.5 Corollary. $a[bc]=[ab]c=abc$ $(a+[b+c]=[a+b]+c=a+b+c)$. *More generally, if each b_j is the product (sum) of a subset of A and, for each a_i in A, b_j exists in B such that $b_j \subset a_i$ ($b_j \supset a_i$), then $\prod a_i=\prod b_j$ ($\sum a_i=\sum b_j$).*

3.6 Corollary. $ab=ba$ $(a+b=b+a)$. *More generally, the value of a product (sum) is independent of any ordering or arrangement of the factors (summands).*

3.7 Corollary. $aa=a$ $(a+a=a)$. *More generally,* $\prod a=a$ ($\sum a=a$).

3.8 Definition. $\prod b_j$ ($\sum b_j$) *distributes* b with respect to $\prod a_i$ ($\sum a_i$) if $b=\prod b_j$ ($b=\sum b_j$) and, for each b_j, a_i exists such that $a_i \subset b_j$ ($a_i \supset b_j$).

3.9 Definition. ab ($a+b$) is *modular* with respect to a if every element, x, such that $ab \subset x \subset a$ ($a+b \supset x \supset a$) can be distributed with respect to ab ($a+b$).

3.10 Definition. $\prod a_i$ ($\sum a_i$) is *distributive* if every element, x, such that $\prod a_i \subset x$ ($\sum a_i \supset x$) can be distributed with respect to $\prod a_i$ ($\sum a_i$).

3.11 Theorem. *If $\prod a_i$ ($\sum a_i$) is distributive and $\prod a_i \subset b$ ($\sum a_i \supset b$), then $\prod b_j$ ($\sum b_j$) exists such that $\prod b_j$ ($\sum b_j$) is distributive and distributes b with respect to $\prod a_i$ ($\sum a_i$).*

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† For the case of two factors this goes back to Peirce (19), pp. 32–33. See also Schröder (21) p. 196, Huntington (12), and Bennett (2).

3.12 THEOREM. *If* $a=\prod a_i$ $(a=\sum a_i)$ *and each* $a_i=\prod a_{ij}$ $(a_i=\sum a_{ij})$ *and if all these products* (*sums*) *are distributive, then* $a=\prod a_{ij}$ $(a=\sum a_{ij})$ *is distributive.*

3.13 THEOREM. $a=\prod a_i$ $(a=\sum a_i)$ *and* $\prod a_i$ $(\sum a_i)$ *is distributive, if* A *is the set of all elements* $a_i \supset a$ $(a_i \subset a)$.

3.14 DEFINITION. 0 (I) is a *zero* (*one*) of K if $0 \subset a$ $(\mathrm{I} \supset a)$ for every a in K.

3.15 THEOREM. *If* 0 (I) *and* b *are both zeros* (*ones*) *of* K, *then* $0=b$ $(\mathrm{I}=b)$.

Collectively, the elements 0 and I are called the *units* of K.

3.16 THEOREM. $a0=0$ $(a+\mathrm{I}=\mathrm{I})$ *and* $a+0=a$ $(a\mathrm{I}=a)$. *Furthermore,* $a0$ $(a+\mathrm{I})$ *and* $a+0$ $(a\mathrm{I})$ *are distributive.*

3.17 DEFINITION. a' (a^*) is a *product* (*sum*) *complement*† of a if $aa'=0$ $(a+a^*=\mathrm{I})$ and $ax=0$ $(a+x=\mathrm{I})$ implies $x \subset a'$ $(x \supset a^*)$.

3.18 THEOREM. *If* a' (a^*) *and* b *are both product* (*sum*) *complements of* a, *then* $a'=b$ $(a^*=b)$.

3.19 THEOREM. $a \subset [a']'$ $(a \supset [a^*]^*)$.

3.20 THEOREM. $a \subset b$ *implies* $b' \subset a'$ *and* $b^* \subset a^*$.

3.21 DEFINITION. b is a *complement* of a with respect to c and d if $a+b=c$ and $ab=d$.

Complements, unlike sum and product complements, are not, in general, unique. Sum and product complements may be defined with respect to elements other than 0 and I. This further distinction, though not needed in this paper, would prove useful in developing Stone's generalized Boolean algebras along the lines of this paper.

3.22 THEOREM. *If* $a'=a^*$, *then* a' *is the unique complement of* a *with respect to* I *and* 0.

4. **Classification of ordered sets by closure under operations.** We proceed to the classification of ordered sets with respect to the operations under which they are closed. The discussion of the different types of complements is postponed until after the treatment of lattices. We further limit the number of types to be considered by distinguishing only between operations on finite and on infinite sets. The classification might be continued indefinitely by discriminating between different orders of infinity in restricting the subsets admissible to the domain of an operation. Under these restrictions, there remain to be considered six possibilities of closure with respect to multiplication, together with their duals for addition.

† Collectively called pseudo-complements by Birkhoff (3), p. 459.

4.1 POSTULATE. *If a and b are in K, then ab $(a+b)$ is in K.*

4.2 POSTULATE. *If ab $(a+b)$ is in K, it is modular with respect to a.*

4.3 POSTULATE. *If ab $(a+b)$ is in K, it is distributive.*

4.4 POSTULATE. *K contains a unit* 0 (I), *that is, the product (sum) of all its elements.*

4.5 POSTULATE. *If A is a subset of K, then $\prod a_i$ $(\sum a_i)$ is in K.*

4.6 POSTULATE. *If $\prod a_i$ $(\sum a_i)$ is in K, it is distributive.*

Clearly the dual postulates are not equivalent to the originals, so we shall denote them by asterisks. For example, Postulate 4.1 relates to products while Postulate 4.1* is the corresponding postulate relating to sums.

These postulates, together with the postulates for ordered sets give us a system of 14 postulates. We wish to investigate the independence and consistency of all subsets of this system. Not all these postulates are independent. In each case we must either establish a relation between some combination of these postulates or show their independence by examples. The consistency of the entire set is demonstrated by an example. Fortunately, the large number of examples required can all be constructed by combining a few simple examples in various ways. The following examples are the ones required for these combinations.

4.7 **Example.** K consists of a single element, a, such that $a \subset a$.

4.8 **Example.** K consists of two elements, a and b, such that $a \subset a$ and $b \subset b$ and no further relations hold.

4.9 **Example.** K consists of three elements, a, b, and c, such that $a \subset a$, $b \subset b$, and $c \subset c$ and no further relations hold.

4.10 **Example.** 4.9 with the additional relation $a \subset b$.

4.11 **Example.** K consists of the positive integers in their natural order.

4.12 **Example.** K consists of the positive integers where $m \subset 2n-1$, if $m \leqq 2n-1$ in the natural order; $2m \subset 2n$, if $m \leqq n$ in the natural order; and no further relations hold.

4.13 **Example.** K consists of three elements, a, b, and c, such that $a \subset a$, $a \subset b$, $b \subset b$, $b \subset c$, and $c \subset c$ and no additional relations hold.

4.14 **Example.** K consists of the positive integers where $m \subset n$ if $m < n$ in the natural order.

Examples 4.7, 4.8, 4.9, 4.10, and 4.13 are self dualistic. We denote the duals of the remaining examples with an asterisk. All the examples except 4.13 and 4.14 are ordered sets. These examples may be combined in sequences, such that the assigned relation holds between any two elements of the same example and such that all the elements of any example are contained in every element of every succeeding example. We shall enclose such sequences in brackets, for instance, [4.11, 4.7] consists of the integers in their natural order and a single element containing all of them. It follows from Theorem 2.11 that if the components of such a sequence are ordered sets, then the entire sequence is an ordered set.

The definitions of operations do not require that the relation $\subset$ be an ordering relation. Hence these definitions can be applied equally well to the examples which are not ordered sets and the postulates for closure under operations can be considered independently from those of ordering.

We shall study each postulate separately, determining all the combinations of the remaining postulates which imply it and constructing examples which show its independence from the remaining postulates by which it is not implied. By suitably combining these examples, an example can be constructed in which any given subset of the postulates, and those postulates shown to be implied by this subset, hold, while the remaining postulates do not hold.

4.15 THEOREM. *Postulate* 4.6 (4.6*) *is implied by no combination of the remaining postulates.*

Proof: Consider the example [4.7, 4.12*, 4.7]. Clearly, the product of the even integers is the zero of the set and every odd integer contains it, but no odd integer can be distributed with respect to it. The remaining postulates are satisfied.

4.16 LEMMA. *Postulates* 4.4 *and* 4.5* (4.4* *and* 4.5) *imply Postulate* 4.5 (4.5*).

Proof: If A is any subset of K and B is the set of all K elements such that $b_j \subset a_i$ for every element of A, then B is not void for it contains the element 0, which exists by 4.4, and $\sum b_j$ exists by 4.5*. But, by 3.1 and 3.1*, $\sum b_j = \prod a_i$. Therefore, 4.5 holds.

4.17 THEOREM. *Postulate* 4.5 (4.5*) *is implied by no combination of the remaining postulates except by Postulates* 4.4 *and* 4.5* (4.4* *and* 4.5) *jointly.*

Proof: Consider Example 4.11*. All the postulates except 4.4 and 4.5 are satisfied. Consider Example [4.11, 4.11*]. All the postulates except 4.5 and 4.5* are satisfied.

4.18 LEMMA. *Postulate* 4.5 (4.5*) *implies Postulate* 4.4 (4.4*).

Proof: The product of all the elements of K, which exists by 4.5, is the 0 of K.

4.19 THEOREM. *Postulate* 4.4 (4.4*) *is implied by no combination of the remaining postulates except by Postulate* 4.5 (4.5*).

Proof: Consider Example 4.11*. All the postulates except 4.4 and 4.5 are satisfied.

4.20 LEMMA. *Postulate* 4.5 (4.5*) *implies Postulate* 4.1 (4.1*).

4.21 THEOREM. *Postulate* 4.1 (4.1*) *is implied by no combination of the remaining postulates except by Postulate* 4.5 (4.5*) *and by Postulates* 4.4 *and* 4.5* (4.4* *and* 4.5) *which jointly imply Postulate* 4.5.

Proof: Consider Example [4.8, 4.7]. All the postulates except 4.1, 4.5, and 4.4 are satisfied. Consider Example [4.11, 4.8, 4.7]. All the postulates except 4.1, 4.5, and 4.5* are satisfied.

4.22 LEMMA. *Postulates* 4.1* *and* 4.2* (4.1 *and* 4.2) *imply Postulate* 4.2 (4.2*).

Proof: We assume that ab exists and that $ab\subset c\subset a$. We must distribute c with respect to ab. By 4.1*, $b+c$ exists. Let X be the set of all elements, x, such that $x\subset a$ and $x\subset b+c$. X is not void, for c is in X. We next show that x in X implies $x\subset c$. We have $c\subset c+x\subset b+c$, and the sums exist by 4.1*. By 4.2*, $c+x=c+y$, where $y\subset b$. Also, since $c\subset a$ and $x\subset a$, $c+x=c+y\subset a$ and $y\subset a$. But, $y\subset a$, $y\subset b$, and $ab\subset c$ imply $y\subset ab\subset c$. Hence, $c+x=c+y=c$ and $x\subset c$. By 3.1, $c=a(b+c)$. $a(b+c)$ exists, since c does, and distributes c with respect to ab.

4.23 LEMMA. *Postulates* 4.1* *and* 4.3* (4.1 *and* 4.3) *imply Postulate* 4.3 (4.3*).

Proof: We assume that ab exists and that $ab\subset c$. We must distribute c with respect to ab. By 4.1*, $a+c$ and $b+c$ exist. Let X be the set of all elements, x, such that $x\subset a+c$ and $x\subset b+c$. X is not void for c is in X. We next show that x in X implies $x\subset c$. We have, by 4.3*, since $x\subset a+c$, $x=u+v$ where $u\subset a$ and $v\subset c$ and, since $u\subset x\subset b+c$, $u=w+y$ where $w\subset b$ and $y\subset c$. But $w\subset b$, $w\subset u\subset a$, and $ab\subset c$ imply $w\subset ab\subset c$. Hence, $x=w+y+v\subset c$. By 3.1, $c=(a+c)(b+c)$. The product exists, since c does, and distributes c with respect to ab.

4.24 LEMMA. *Postulate* 4.3 (4.3*) *implies Postulate* 4.2 (4.2*).

4.25 LEMMA. *Postulate* 4.6 (4.6*) *implies Postulate* 4.3 (4.3*).

4.26 THEOREM. *Postulate* 4.3 (4.3*) *is implied by no combination of the remaining postulates except by Postulate* 4.6 (4.6*), *by Postulates* 4.1* *and* 4.3* (4.1 *and* 4.3) *jointly, and by combinations implying one of these.*

Proof: Consider Example [4.7, 4.9]. All the postulates except 4.3, 4.6, 4.1*, 4.4*, and 4.5* are satisfied. Consider Example [4.7, 4.9, 4.7]. All the postulates except 4.3, 4.6, 4.3*, and 4.6* are satisfied. Consider Example [4.7, 4.9, 4.12*]. All the postulates except 4.3, 4.5, 4.6, 4.1*, and 4.5* are satisfied.

4.27 THEOREM. *Postulate* 4.2 (4.2*) *is implied by no combination of the remaining postulates except by Postulate* 4.3 (4.3*), *by Postulates* 4.1* *and* 4.2* (4.1 *and* 4.2) *jointly, and by combinations implying one of these.*

Proof: Substitute 4.10 and 4.9 in the examples given for 4.26. In addition to the postulates listed in 4.26 as failing to hold in each case, 4.2 and, in the second case, 4.2* are no longer satisfied.

4.28 THEOREM. *Postulate* 1.1 *is not implied by any combination of the remaining postulates.*

Proof: Consider Example [4.7, 4.13, 4.7]. All the postulates except 1.1 are satisfied.

4.29 THEOREM. *Postulate* 1.2 *is not implied by any combination of the remaining postulates.*

Proof: Consider Example [4.7, 4.14*, 4.14, 4.7]. All the postulates except 1.2 are satisfied.

4.30 THEOREM. *Postulates* 1.1, 1.2, 4.1–4.6, *and* 4.1*–4.6* *are consistent.*

Proof: Consider Example [4.7, 4.7, 4.7]. All the postulates are satisfied.

Given any subset, Q (admitting the void set as a subset), of the postulates so far considered, the closure of Q is defined as the subset of the postulates which contains Q and all postulates implied by combinations of the postulates in Q. A subset, Q, of the postulates is said to be closed if it is identical with its closure. Clearly, the subset consisting of all the postulates which hold in any given ordered set, must be a closed subset. It remains to exhibit, for each closed subset, Q, of the postulates, an ordered set in which the postulates, Q, and no others, hold.

To do this we select examples from those used in proving the theorems of this section (specifically, 4.15, 4.17, 4.19, 4.21, 4.26, 4.27, and their duals) and arrange these examples in a sequence, ordering the result lexicographically as in 2.11. It is necessary that the examples selected have units, with the possible exceptions that the first example in the sequence may lack a zero and the last example may lack a one.

First, consider the case where Postulates 4.4 and 4.4* are in Q. Then for each postulate not in Q select an example in which it fails to hold and all the postulates in Q hold. Since Q is closed, this is always possible, as no postulate not in Q is implied by any combination of postulates in Q. Suppose there are n postulates not in Q, then we have n examples, A_i each with units 0_i, I_i, which we shall arrange in a sequence $A=[A_1, A_2, \cdots, A_i, \cdots, A_n]$. For this example, the postulates Q and no others hold. In the first place, by 2.11, A is ordered if, and only if, every A_i is ordered. Postulates 4.4 and 4.4* hold in A, since they are in Q and hence hold for A_1 and A_n. Next, consider any subset, B, of A. The elements of B come from the sets A_i. Suppose that j is the lowest index for which A_j contributes at least one element to B. Then $\prod b_k$, if it exists, is in A_j as 0_j is contained in this product and 0_{j+1} is not. Furthermore, every element of A appearing in some A_i, $i>j$, can be distributed with respect to $\prod b_k$. For this element contains I_j which contains some b_k by the selection of j. Hence the existence and distributive properties of $\prod b_j$ depends solely on the corresponding properties of the subset of B in A_j. Hence any of the postulates regarding the existence and distributive properties of products which holds for every A_i holds for A. A similar statement is true for sums. Furthermore, as the sum or product with respect to A of any subset of any A_i, being bounded between 0_i and I_i, cannot be in A_j for any $j\neq i$, hence the failure of a sum or product to exist or have certain distributive properties in some A_i guarantees the same failure in A. But every postulate of Q holds in every A_i, hence in A, and every postulate not in Q fails in some A_i and hence in A.

We now consider the case where Postulate 4.4 is not in Q but Postulate 4.4* is. If Postulate 4.5* is in Q, then the closure $\bar{Q}$ of the set formed by adjoining 4.4 to Q must contain 4.1 and 4.5. In no case need other postulates be adjoined to Q to obtain $\bar{Q}$. Since $\bar{Q}$ contains 4.4 and 4.4* an example, A, can be found for it under the case first considered. If 4.1 is not in Q, then $[4.8, A]$ is an example of Q. If 4.1 is in Q, then $[4.11^*, A]$ is an example of Q. Postulate 4.5 cannot be in Q as it would imply 4.4 in Q. By similar reasoning $[A, 4.8]$ and $[A, 4.11]$ are examples of the case where Q contains Postulate 4.4 and not Postulate 4.4* and A is an example of the closure of Q and Postulate 4.4*. If neither Postulate 4.4 nor Postulate 4.4* are in Q, and A is an example for the closure of Q, 4.4 and 4.4*, then $[4.8, A, 4.8]$, $[4.11^*, A, 4.8]$, $[4.8, A, 4.11]$ and $[4.11^*, A, 4.11]$ are the examples required.

5. **Homomorphisms, isomorphisms, automorphisms, and subsystems.** In the theory of ordered sets, so many operations are involved that it is necessary to formulate the concepts of homomorphism and subsystem with generality and precision to provide, without ambiguity, for the different cases which arise. A homomorphism is a correspondence between the elements, the ordering relations, and the operations of two ordered sets. Isomorphisms and automorphisms are particular types of homomorphism. We consider two

ordered sets, K and $\bar{K}$, and a set, Θ, of univocal operations defined by an ordering relation.

5.1 Definition. A *homomorphism* relative to Θ from K to $\bar{K}$ is a correspondence, $\rightarrow$, between the elements, ordering relation, and operations Θ of K and similar entities of $\bar{K}$ such that:

a. If a is in K, then a uniquely defined element, $\bar{a}$, exists in $\bar{K}$ such that $a\rightarrow\bar{a}$;

b. If $\bar{a}$ is in $\bar{K}$, then at least one element, a, exists in K such that $a\rightarrow\bar{a}$;

c. An ordering relation, $\bar{\subset}$, is defined for K such that $a\subset b$ in K implies $\bar{a}\,\bar{\subset}\,\bar{b}$ in $\bar{K}$;

d. If θ is in Θ and $\theta(a_i)$ exists in K, then $\bar{\theta}(\bar{a}_i)$ exists in $\bar{K}$ and $\theta(a_i)\rightarrow\bar{\theta}(\bar{a}_i)$.

$\bar{K}$ is homomorphic to K relative to Θ if there is a homomorphism relative to Θ from K to $\bar{K}$. Since the operations Θ are defined in terms of the relation $\subset$ and $\bar{\Theta}$ in terms of $\bar{\subset}$, the correspondence $\theta\rightarrow\bar{\theta}$ is determined by the correspondence $\subset\rightarrow\bar{\subset}$. Thus a homomorphism is determined by the correspondence between elements and between ordering relations. It can be classified by the operations Θ which it is said to preserve.

We have already remarked that the definition of operations is relative to the entire set to which they are applied and not just to the elements immediately involved. Thus, although any subset of an ordered set is an ordered set, there is no assurance that the operations defined for a subset will be identical with operations defined for the entire set when operating on elements of the subset. This fact motivates our definition of subsystems.

5.2 Definition. $\bar{K}$ is a subsystem of K relative to Θ if $\bar{K}$ is a subset of K and if, whenever $\theta(\bar{a}_i)$ exists in K, $\bar{\theta}(\bar{a}_i)$ exists in $\bar{K}$ and $\bar{\theta}(\bar{a}_i)=\theta(\bar{a}_i)$.

Nothing in the definitions of homomorphisms and subsystems requires that the sets involved be closed under any operations. In many cases the sets involved are not closed under the operations Θ. Every homomorphism preserves units. A homomorphism does not necessarily preserve sums, products, complements, or sum and product complements. If a subsystem preserves units, it preserves complements and sum and product complements. A subsystem does not necessarily preserve units, sums, and products.

Taking the complement is not always univocal† and, hence, is not always admissible to the set Θ. The other operations considered are always univocal. We next consider some of the properties of homomorphisms and subsystems.

5.3 Theorem. *A given aggregate of ordered sets, K_i, is ordered by the relation $\subset$, if $K_i\subset K_j$ whenever K_i is homomorphic to K_j relative to Θ.*

† See Theorems 7.28 and 7.29.

5.4 THEOREM. *A given aggregate of subsystems, K_i, relative to Θ of a given set is ordered by the relation $\subset$ if $K_i \subset K_j$ whenever K_i is a subsystem relative to Θ of K_j.*

5.5 THEOREM. *A given aggregate of ordered sets, K_i, is ordered by the relation $\subset$ if $K_i \subset K_j$ whenever K_i is homomorphic relative to Θ to a subsystem relative to Θ of K_j.*

5.6 DEFINITION. K and $\overline{K}$, two ordered sets, are isomorphic if a 1–1 correspondence exists between the elements of K and the elements of $\overline{K}$ such that $a \subset b$ implies $\bar{a} \mathrel{\bar{\subset}} \bar{b}$ and $\bar{a} \mathrel{\bar{\subset}} \bar{b}$ implies $a \subset b$.

We remark that an isomorphism is a symmetric relation. The expressions, an isomorphism exists between K and $\overline{K}$ and K is isomorphic to $\overline{K}$, are equivalent to K and $\overline{K}$ are isomorphic. All operations are preserved by an isomorphism, so it is not necessary to specify them.

5.7 THEOREM. *If K and $\overline{K}$ are isomorphic, then each is homomorphic to the other relative to any set of operations.*

For homomorphisms and isomorphisms it is not necessary that the elements of K and $\overline{K}$ be distinct. Furthermore, if K and $\overline{K}$ have elements in common, it is not necessary that their ordering relations be the same in their common part. We call an isomorphism between K and $\overline{K}$ an automorphism if the elements of K and $\overline{K}$ are the same, though the ordering relations need not be the same. Among automorphisms, we are particularly interested in dual automorphisms.

5.8 DEFINITION. An isomorphism between K and $\overline{K}$ is a dual automorphism if the elements of K and $\overline{K}$ are the same and the ordering relation of $\overline{K}$ is the dual of the ordering relation of K.

5.9 THEOREM. *If K is an ordered set and $\overline{K}$ is any subset of K such that a in $\overline{K}$ and $b \subset a$ ($b \supset a$) imply b in $\overline{K}$, then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.10 COROLLARY. *If A is a subset of K and x is in $\overline{K}$ if, and only if, $x \subset a_i$ ($x \supset a_i$) for every a_i in A, then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.11 COROLLARY. *If a is in K and x is in $\overline{K}$ if, and only if, $x \subset a$ ($x \supset a$), then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.12 THEOREM. *If A is any subset of an ordered set K and $\overline{K}$ is the subset of K such that x is in $\overline{K}$ if, and only if, $a_i \subset x$ ($a_i \supset x$) for every a_i in A, then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.13 COROLLARY. *If a is in K and x is in $\overline{K}$ if, and only if, $a \subset x$ $(a \supset x)$, then $\overline{K}$ is a subsystem of K with respect to multiplication* (*addition*).

5.14 THEOREM. *Given a set of subsystems, $\overline{K}_i$, of K with respect to multiplication* (*addition*) *and $\overline{K}$ the section, or common part, of the subsystems $\overline{K}_i$, then $\overline{K}$ is a subsystem of K with respect to multiplication* (*addition*).

5.15 COROLLARY. *If A and B are subsets of K such that $a_i \subset b_j$ for every a_i in A and every b_j in B and x is in $\overline{K}$ if, and only if, $a_i \subset x \subset b_j$ for every a_i in A and every b_j in B, then $\overline{K}$ is a subsystem of K with respect to addition and multiplication.*

5.16 COROLLARY. *If a and b are in K, $a \subset b$, and x in $\overline{K}$ if, and only if, $a \subset x \subset b$, then $\overline{K}$ is a subsystem of K with respect to addition and multiplication.*

5.17 THEOREM. *If $\overline{K}$ is homomorphic to K with respect to certain types, Θ, of addition and multiplication and A is the set of elements, a_i, in K which have the same image, $\bar{a}$, in $\overline{K}$, then A is a subsystem of K with respect to Θ.*

5.18 COROLLARY. *If K is closed under a subset Φ of Θ then A is closed under Φ.*

6. **Multiplicative systems and lattices.** An ordered set which satisfies Postulate 4.1 for the existence of finite products is called a *multiplicative system*. If Postulate 4.5 for the existence of unrestricted products is satisfied, the ordered set is said to be a *complete multiplicative system*. In this section, the dual theorems apply to systems in which the dual postulates are satisfied.

6.1 THEOREM. *If K is a multiplicative system and a, b, and c are elements in K, then*

(a) $a=a$;
(b) $a=b$ *implies* $b=a$;
(c) $a=b$ *and* $b=c$ *implies* $a=c$;
(d) ab $(a+b)$ *exists in* K;
(e) $a=b$ *implies* $ac=bc$ $(a+c=b+c)$;
(f) $ab=ba$ $(a+b=b+a)$;
(g) $a[bc]=[ab]c$ $(a+[b+c]=[a+b]+c)$;
(h) $aa=a$ $(a+a=a)$;
(i) $ab=a$ $(a+b=a)$ *implies* $a\subset b$ $(a\supset b)$, *and conversely*;
(j) $ab=a$ $(a+b=a)$ *implies that* $a+b$ (ab) *exists and that* $a+b=b$ $(ab=b)$.

6.2 THEOREM. *If* 6.1 (b–h) *are taken as postulates and* $\subset$ $(\supset)$ *is defined by* 6.1 (i), *then a set, K, satisfying* 6.1 (b–h) *is ordered by the relation* $\subset$ $(\supset)$, *is a multiplicative system, and the equality and multiplication* (*addition*) *defined by* $\subset$ *can be identified with the equality and multiplication* (*addition*) *postulated by* 6.1 (b–h).

Theorems 6.1 and 6.2 establish the equivalence between the approach to multiplicative systems in which equality and multiplication are made fundamental and that in which the ordering relation is made fundamental. Theorem 6.1 serves to show the greater simplicity of the latter method of approach and gives the more important elementary identities of such systems.

6.3 THEOREM. *If, in a multiplicative system,* $\sum a_i$ $(\prod a_i)$ *exists and is distributive, then* $\sum a_i b = [\sum a_i]b$ $(\prod [a_i+b] = \prod a_i + b)$.

Proof: $a_j \subset \sum a_i$ for every a_j in $\sum a_i$, hence, $a_j b \subset [\sum a_i]b$ and $\sum a_i b \subset [\sum a_i]b$. On the other hand, since $\sum a_i$ is distributive and $[\sum a_i]b \subset \sum a_i$, $[\sum a_i]b = \sum c_k$ where, for each c_j in $\sum c_k$, a_j exists such that $c_j \subset a_j$. Also, $c_j \subset [\sum a_i]b$, hence, $c_j \subset a_j[\sum a_i]b = a_j b$. It follows that $[\sum a_i]b = \sum c_k \subset \sum a_i b$. Therefore, $\sum a_i b = [\sum a_i]b$.

6.4 COROLLARY. *If, in a multiplicative system,* $a+b$ *exists and is distributive, then* $ac+bc = [a+b]c$ $([a+c][b+c] = ab+c)$.

6.5 THEOREM. *If, in a multiplicative system,* $\sum a_i$ $(\prod a_i)$ *exists and for every element,* b, *in the system* $\sum a_i b = [\sum a_i]b$ $(\prod [a_i+b] = \prod a_i + b)$, *then* $\sum a_i$ $(\prod a_i)$ *is distributive.*

6.6 THEOREM. *If, in a multiplicative system,* $\sum a_i$ $(\prod a_i)$ *exists and is distributive, then* $\sum a_i b$ $(\prod [a_i+b])$ *exists and is distributive.*

Proof: $\sum a_i b = [\sum a_i]b$ which exists. If $x \subset \sum a_i b$, then, since $\sum a_i b = [\sum a_i]b$, $x \subset \sum a_i$ and $x \subset b$. Since $\sum a_i$ is distributive, $x = [\sum a_i]x = \sum a_i x$. Since $x \subset b$, $x = bx$, hence, $x = \sum a_i bx$ and $\sum a_i b$ is distributive.

6.7 THEOREM. *If, in a multiplicative system,* $a+b$ (bc) *exists, is modular with respect to* a (c), *and* $a \subset c$, *then* $a+bc = [a+b]c$.

Theorem 6.7 is stated so as to exhibit the intrinsic self duality of the modular property.

6.8 THEOREM. *If, in a multiplicative system,* $a+b$ (bc) *exists and for every element* c (a) *such that* $a \subset c$, $a+bc = [a+b]c$, *then* $a+b$ (bc) *is modular with respect to* a (c).

An ordered set which is a multiplicative system with respect to both the relations $\subset$ and $\supset$ is called a *lattice*.† If the multiplicative systems defined by the relations $\subset$ and $\supset$ are complete, then the lattice is *complete*. If but one of these systems is complete, we speak of a lattice with unrestricted products or sums. However, these latter categories are not very general as Lemma

† Birkhoff (3), p. 442. Ore (18) calls such sets structures; Klein (14 and 15), Verbände; Dedekind (6), Dualgruppe. Grell (9), Menger (16), and von Neumann (17) deal with special cases of such sets.

4.16* implies that any complete multiplicative system with a unit I is a complete lattice.

If Postulate 4.2 holds in a lattice, the lattice is *modular*. Lemma 4.22 implies that in a lattice Postulates 4.2 and 4.2* are equivalent. Theorem 6.9 is self dualistic.

6.9 THEOREM. *A lattice is modular if, and only if,* $a \subset c$ *implies* $a+bc = [a+b]c$.

If Postulate 4.3 holds in a lattice, the lattice is *distributive*. In a lattice, Postulates 4.3 and 4.3* are equivalent, by Lemma 4.23.

6.10 THEOREM. *A lattice is distributive if, and only if,* $ac+bc=[a+b]c$ $([a+c][b+c]=ab+c)$.

Actually, $ac+bc \subset [a+b]c$ is sufficient to imply that a lattice be distributive, for $ac+bc \subset [a+b]c$ in any lattice.

A complete lattice in which Postulate 4.6 (4.6*) holds is called a lattice with *completely distributive products* (*sums*). The situation in the complete case is not analogous to that in the finite case, for Postulate 4.6 does not imply Postulate 4.6* in a complete lattice. If both Postulates 4.6 and 4.6* hold, the lattice is said to be *completely distributive*.

6.11 THEOREM. *A lattice has completely distributive products* (*sums*) *if, and only if,* $\prod [a_i+b]=\prod a_i+b$ $([\sum a_i]b=\sum a_i b)$.

Some properties of subsystems and homomorphisms in multiplicative systems and lattices are now considered.†

6.12 THEOREM. *If K is a multiplicative system and $\overline{K}$ is a subset of K belonging to one of the types considered in* 5.10–5.16, *then $\overline{K}$ is a subsystem of K with respect to multiplication, addition, modular addition, and distributive addition.*

Proof: By 5.10–5.16, 6.5, and 6.8.

6.13 COROLLARY. *$\overline{K}$ is a multiplicative system and if K is complete, so is $\overline{K}$.*

6.14 COROLLARY. *If K is a lattice, modular lattice, distributive lattice, complete lattice, or lattice with completely distributive sums* (*products*), *then so is $\overline{K}$.*

A subsystem with respect to multiplication of a multiplicative system will be called a *multiplicative subsystem*. Similarly, a subsystem with respect to addition and multiplication of a lattice will be called a *sublattice*.

6.15 THEOREM. *If K is a multiplicative system and b is an element of K, then the correspondence $x \rightarrow bx$, carrying K into the multiplicative subsystem $\overline{K}$*

† Compare Ore (18), p. 416 et seq.

of elements $y \subset b$, *is a homomorphism with respect to multiplication and distributive addition. The sum* $\sum a_i$ *is preserved if, and only if,* $[\sum a_i]b = \sum a_i b$.

Proof: $a \subset c$ implies $ab \subset cb$. $ac \to acb = abcb$. $\sum a_i \to [\sum a_i]b = \sum a_i b$ if, and only if, $[\sum a_i]b = \sum a_i b$. If $\sum a_i$ is distributive, then $[\sum a_i]b = \sum a_i b$ and $\sum a_i b$ is distributive, by 6.3 and 6.6.

6.16 COROLLARY. *If* K *is a modular lattice, then all sums* $a+c$, *where* $c \subset b$, *are preserved as modular sums.*

6.17 THEOREM. *If* K *is a modular lattice,* X *the sublattice of all elements* x_i *such that* $a \subset x_i \subset a+b$, *and* Y *the sublattice of all elements* y_j *such that* $ab \subset y_j \subset b$, *then the correspondence* $x_i \to bx_i$ *between* X *and* Y *is the inverse of the correspondence* $y_j \to a+y_j$ *between* Y *and* X. *Together, these correspondences define an isomorphism between* X *and* Y.

Proof: $x_i \to bx_i \to a+bx_i = x_i$ and $x_i \subset x_j$ implies $bx_i \subset bx_j$. Similarly, $y_j \to a+y_j \to b[a+y_j] = y_j$ and $y_i \subset y_j$ implies $a+y_i \subset a+y_j$.

6.18 COROLLARY.† *The correspondence* $x_i \longleftrightarrow bx_i$, *or* $y_j \longleftrightarrow a+y_j$, *between* X *and* Y *preserves all types of sums and products defined relative to* K.

Proof: An isomorphism preserves all operations. Hence, relative to X and Y, all types of sums and products are preserved. However, X and Y are subsystems of K with respect to these operations by 5.16 and 6.14.

7. **Complements.** We are now ready to continue the classification of ordered sets with respect to the existence of complements. We consider the following postulates.

7.1 POSTULATE. *If* a *is in* K, *then* a' (a^*) *is in* K.

7.2 POSTULATE. *A dual automorphism exists in* K.

7.3 POSTULATE. *The correspondence* $a \longleftrightarrow a'$ $(a \longleftrightarrow a^*)$ *is a dual automorphism.*

7.4 LEMMA. *Postulate* 7.1 (7.1*) *implies Postulates* 4.4 *and* 4.4*.

Proof: The existence of a product complement demands a 0. The product complement of 0 is I.

7.5 THEOREM. *If* K *belongs to any of the classifications satisfying Postulates* 4.4 *and* 4.4* *considered in* §4, *then an example of the same classification in* §4 *can be found in which Postulate* 7.1 (7.1*) *holds.*

Proof: Consider Example [4.7, K].

† This is a generalization of Ore (18), p. 418, Theorem 2, as it shows that infinite, as well as finite, sums and products are preserved.

7.6 LEMMA. *Postulates* 4.4, 4.5*, *and* 4.6* (4.4*, 4.5, *and* 4.6) *imply Postulate* 7.1 (7.1*).

Proof: 0 exists by 4.4, hence, the set B of all elements b_j such that $ab_j=0$, for any fixed a in K, is not void. By 4.5*, $\sum b_j$ exists and, by 4.6*, $a(\sum b_j)=\sum ab_j=0$. Therefore, $\sum b_j$ is the product complement of a.

7.7 **Example.** K consists of the finite, or void, subsets of the positive integers with the relation $a \subset b$ if every integer in a is an integer in b.

7.8 THEOREM. *Postulate* 7.1 (7.1*) *is implied by no combination of the postulates of* §1 *and* §4 *except by Postulates* 4.4, 4.5*, *and* 4.6* (4.4*, 4.5, *and* 4.6) *jointly and combinations implying these postulates.*

Proof: Consider Example 4.11*. Postulates 4.4, 4.5, and 7.1 are not satisfied. Consider Example 7.7. Postulates 4.4*, 4.5*, 7.1, and 7.1* are not satisfied. Consider Example [7.7, 4.11*]. Postulates 4.5, 4.5*, and 7.1 are not satisfied. Consider Example [7.7, 4.7]. Postulates 4.6* and 7.1 are not satisfied. In each case the remaining postulates of §1 and §4 are satisfied.

7.9 LEMMA. *If the corresponding* $a\rightarrow\bar{a}$ *is a dual automorphism and* $\sum a_i$ ($\prod a_i$) *exists, then* $\prod \bar{a}_i$ ($\sum \bar{a}_i$) *exists and* $\overline{[\sum a_i]}=\prod \bar{a}_i$ ($\overline{[\prod a_i]}=\sum \bar{a}_i$).

Proof: $a_i\subset\sum a_i$, hence $\overline{[\sum a_i]}\subset\bar{a}_i$. If $y\subset\bar{a}_i$ for every a_i in $\sum a_i$, then x is uniquely determined so that $x\leftrightarrow y$, or $y=\bar{x}$, and $a_i\subset x$. Therefore, $\sum a_i\subset x$ and $y=\bar{x}\subset \overline{[\sum a_i]}$.

7.10 THEOREM. *If Postulate* 7.2 *holds, then the dual of any other postulate which holds is also satisfied.*

Proof: Apply 7.9.

7.11 THEOREM. *Except for duals, no new postulates are implied by combining Postulate* 7.2 *with a given set of the remaining postulates.*

Proof: Consider any subset of the postulates, excluding 7.2 and 7.3, and adjoin the duals. Next adjoin all postulates implied by these. The dual of each such postulate will be such a one and thus adjoined at the same time. By the methods of §4 and this section an example, K, can be constructed in which just these postulates hold. The Example [K, K^*] also has these properties and in addition satisfies 7.2.

7.12 THEOREM. *Postulate* 7.2 *is implied by no combination of the remaining postulates except Postulate* 7.3.

Proof: Consider Example [4.7, 4.7, 4.8, 4.7]. All the postulates except 7.2 and 7.3 are satisfied.

7.13 THEOREM. *Postulate* 7.3 (7.3*) *implies* $[a']'=a$ ($[a^*]^*=a$).

Proof: Since $a'a=0$, $a \subset [a']'$. Similarly, $a' \subset [(a')']'$, and by the dual automorphism, $[a']' \subset a$. Therefore, $[a']'=a$.

7.14 THEOREM. *Postulates* 7.3 (7.3*) *and* 1.1 *imply Postulate* 1.2.

Proof: $a \subset [a']'$ and $[a']' \subset a$ imply $a \subset a$.

7.15 LEMMA. *Postulate* 7.3 (7.3*) *implies Postulates* 7.1, 7.1*, 4.4, 4.4*, 7.2, *and* 7.3*.

7.16 THEOREM. *If Postulates* 7.1 (7.1*), 1.1 *and* 1.2 *are satisfied and* $\sum a_i(\prod a_i)$ *exists, then* $\prod a_i'\,(\sum a_i^*)$ *exists and* $\prod a_i' = [\sum a_i]'(\sum a_i^* = [\prod a_i]^*)$.

Proof: Since $[\sum a_i]'$ exists by hypothesis, the existence of $\prod a_i'$ follows from the equality. For every a_i, $a_i[\sum a_i]' \subset \sum a_i[\sum a_i]'=0$ and $[\sum a_i]' \subset a_i'$. If, for every a_i, $x \subset a_i'$, then $xa_i=0$, $a_i \subset x'$, $\sum a_i \subset x'$, and $x \subset [\sum a_i]'$.

When $a \longleftrightarrow a'$ $(a \longleftrightarrow a^*)$ is a dual automorphism 7.16 is a special case of 7.9.

7.17 THEOREM. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then* $a'=a^*$.

Proof: If $a \subset x$ and $a' \subset x$, then $x' \subset a'$, $x' \subset [a']'$, $x' \subset a'[a']'=0$, and $x=\mathrm{I}$. Therefore, $a+a'=\mathrm{I}$. On the other hand, if $a+y=\mathrm{I}$, then, by 7.16, $a'y'=0$ and $y' \subset [a']'$. Therefore, $a' \subset y$.

7.18 COROLLARY. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then a' is the unique complement of a with respect to* I *and* 0.

7.19 THEOREM. *Postulates* 7.3 (7.3*) *and* 1.1 *imply Postulate* 4.6* (4.6).

Proof: If $\sum a_i$ exists and b is any element such that $b \subset \sum a_i$, then let B consist of all elements b_k such that $b_k \subset b$ and $b_k \subset a_i$ for some a_i, and let X consist of all elements x_i such that $b_k \subset x_i$ for every b_k in B. If we show that $b \subset x_i$ for every x_i in X, then $b=\sum b_k$ and the theorem is proved. It remains to prove that $b \subset x_i$ for every x_i in X. Let x be any element of X and y any element such that $y \subset b$ and $y \subset x'$. Then $ya_i=0$ for every a_i, since $z \subset y$ and $z \subset a_i$ imply z in B, $z \subset x$, $z \subset x'$, and $z=0$. Hence $y \subset a_i'$ and $y \subset \prod a_i' = [\sum a_i]'$. But $y \subset b \subset \sum a_i$. Therefore, $y=0$, $bx'=0$, and $b \subset x$. Since this holds for any x in X, there can be no x for which it fails, and it must be true for every x in X.

7.20 COROLLARY. *Postulates* 7.3 (7.3*) *and* 1.1 *imply Postulates* 4.2, 4.2*, 4.3, 4.3*, *and* 4.6 (4.6*).

7.21 LEMMA. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then either Postulate* 4.5 *or* 4.5* *implies the other and* 4.1 *and* 4.1*, *and either* 4.1 or 4.1* *implies the other*.

7.22 THEOREM. *None of the Postulates* 4.1, 4.1*, 4.5, *and* 4.5* *are implied by Postulates* 7.3 (7.3*) *and* 1.1.

Proof: Consider the following set, K, consisting of units, 0 and I and four elements a_i, $i=1, 2, 3, 4$, and their product complements a_i' where $a_i' \subset a_j$ for $i \neq j$. $a_i a_j$ and $a_i' + a_j'$, for $i \neq j$, do not exist, therefore none of the Postulates 4.1, 4.1*, 4.5, and 4.5* hold. However, Postulates 7.3 and 1.1 are satisfied.

7.23 THEOREM. *Postulates* 4.5 *and* 4.5* *are not implied by Postulates* 7.3 (7.3*), 1.1, 4.1, *and* 4.1*.

Proof: Consider the set, K, of finite collections of intervals with rational end points in the closed interval $[0, 1]$, where $a \subset b$ if every point of a is a point of b. $[0, r_i]$ and $[r_i, 1]$, where R consists of all rational numbers less than $(1/2)^{1/2}$, do not exist. Hence Postulates 4.5 and 4.5* are not satisfied. Postulates 7.3, 1.1, 4.1, and 4.1* are fulfilled.

7.24 THEOREM. *No subset of the postulates of* §1, §4, *and the remaining postulates of* §7 *imply Postulate* 7.3 (7.3*).

Proof: Consider Example [4.7, 4.7, 4.7].

7.25 THEOREM. *All the postulates of* §1, §4, *and* §7 *are consistent.*

Proof: Consider Example [4.7, 4.8, 4.7].

7.26 THEOREM. *If Postulates* 7.1 (7.1*) *and* 1.1 *are satisfied and* $[a']' \subset a$ $(a \subset [a^*]^*)$, *then Postulate* 7.3 (7.3*) *is satisfied.*

Proof: Apply 3.18 and 3.19.

7.27 THEOREM. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then* $a \subset b$ *if, and only if,* $ab'=0$ $(a^*+b=\mathrm{I})$.

Proof: If $a \subset b$, then $a=ab$ and $ab'=abb'=a0=0$. If $ab'=0$, then $a \subset (b')'=b$.

Sum complements and product complements are unique. We now consider the necessary and sufficient conditions that complements be unique.

7.28 THEOREM. *If K is a distributive lattice and b and $\bar{b}$ are complements of a with respect to c and d, then* $b=\bar{b}$.

Proof: By hypothesis, $a+b=c=a+\bar{b}$ and $ab=d=a\bar{b}$. Hence, $b=b(a+b)$ $=b(a+\bar{b})=ba+b\bar{b}=a\bar{b}+b\bar{b}=(a+b)\bar{b}=(a+\bar{b})\bar{b}=\bar{b}$.

7.29 THEOREM. *If K is a lattice, but not distributive, then K contains a set of elements a, b, $\bar{b}$, c, and d, such that b and $\bar{b}$ are complements of a with respect to c and d, but* $b \neq \bar{b}$.

Proof: If K is modular, it contains a subset of the form [4.7, 4.9, 4.7].† If K is not modular, it contains a subset of the form [4.7, 4.10, 4.7].†

† Birkhoff (4), p. 617, Theorem 4 and Corollary.

7.30 THEOREM. *In any distributive lattice, with units, if a has a complement b with respect to* I *and* 0, *then b is the product* (*sum*) *complement of a.*

Proof: $ab=0$. If b is not the product complement of a, then c exists in K such that $ac=0$ but $c \not\subset b$. This implies that $c+b \neq b$ and that $c+b$ is also a complement of a, contrary to Theorem 7.28.

7.31 THEOREM. *If, for every element a_i of a subset, A, of an ordered set, K, the product* (*sum*) *complement a_i'* (a_i^*) *exists, $\sum a_i$ and $\prod a_i'$* ($\prod a_i$ *and* $\sum a_i^*$) *exist, and $\sum a_i$* ($\prod a_i$) *is distributive, then $\prod a_i'$* ($\sum a_i^*$) *is the product* (*sum*) *complement of $\sum a_i$* ($\prod a_i$).

Proof: $[\sum a_i]\prod a_i' = \sum [a_i \prod a_i'] = 0$. If $[\sum a_i]x=0$, then $a_i x=0$, $x \subset a_i'$ for every a_i', and $x \subset \prod a_i'$.

7.32 THEOREM. *If K is an ordered set and A is a subset of K such that each element a_i in A has a complement b_i with respect to c and d and $\prod a_i$ and $\sum b_i$* ($\sum a_i$ *and* $\prod b_i$) *exist and are distributive, then $\prod a_i$ and $\sum b_i$* ($\sum a_i$ *and* $\prod b_i$) *are complementary with respect to c and d.*

Proof: By hypothesis, $a_i+b_i=c$ and $a_ib_i=d$. Hence, $a_i \subset c$, $b_i \subset c$, and $\prod a_i + \sum b_i \subset c$. On the other hand, $\prod a_i + \sum b_i = \prod (a_i + \sum b_i) \supset \prod (a_i+b_i)=c$. Therefore, $\prod a_i + \sum b_i = c$. Similarly, $\prod a_i\ (\sum b_i) = d$.

8. **Boolean algebras.** Systems satisfying Postulates 1.1, 4.1, and 7.3 are considered next. These Postulates imply all the postulates considered in §1, §4, and §7, except Postulates 4.5 and 4.5*. Hence, such a system is a distributive structure with units and, by Theorem 7.18, complements. The postulates for a Boolean algebra of Huntington,† Stone, and Tarski can be readily verified in such a system. Moreover, Postulates 1.1, 4.1, and 7.3 are well known properties of a Boolean algebra. Therefore Theorem 8.1 follows.

8.1 THEOREM. *For a set, K, to be a Boolean algebra, it is necessary and sufficient that Postulates* 1.1, 4.1, *and* 7.3 *be satisfied.*

It follows from Theorem 7.23 that Postulate 4.5 does not necessarily hold in a Boolean algebra. If Postulate 4.5 does hold, the Boolean algebra is said to be *complete*. Since Postulate 4.5 implies Postulate 4.1, it is no longer necessary to affirm Postulate 4.1 and Theorem 8.2 follows.

8.2 THEOREM. *For a set, K, to be a complete Boolean algebra, it is necessary and sufficient that Postulates* 1.1, 4.5, *and* 7.3 *be satisfied.*

It is interesting to note that Postulates 4.6 and 4.6* are always satisfied in a Boolean algebra, hence, it is not possible to classify Boolean algebras according to the extent to which the distributive property holds, as was done

† Huntington (12 and 13), Stone (23), and Tarski (25).

in the case of lattices. Stone† considers generalized Boolean algebras which are ordered sets with a unit 0 and such that for every b in the set, the subsystem of elements, x, such that $x \subset b$ is a Boolean algebra.

9. **Extensions of ordered sets.** The object of an extension is to complete or close a given ordered set with respect to specified operations. Whenever this can be done in some uniquely defined manner it becomes unnecessary to postulate closure under these operations as closure may be obtained by an extension. To secure the uniqueness of an extension, we require that it be minimal in a sense made precise by Definition 9.4. Furthermore, it is important that a minimal extension be determined by the given ordered set and the operations under which it is to be closed and not depend upon the particular method of extension employed. By achieving this latter goal, we are able to make an extension by any sequence of steps we desire without affecting the uniqueness of the result. Naturally, an extension must preserve in the given set the operations under which it is to be closed. With these points in mind, we are lead to the following definitions and theorems.

9.1 Definition. If K and L_i are ordered sets and Φ is a set of univocal operations defined by an ordering relation, then L_i is an extension to Φ of K provided that:

a. L_i is closed under Φ;

b. L_i contains a subsystem, K_i, relative to Φ such that K and K_i are isomorphic.

We now seek an ordering relation for the set of extensions, L_i, to Φ of a given set, K. It is with respect to this ordering relation that we require an extension to be minimal. Let L_i and L_j be extensions to Φ of K.

9.2 Definition. $L_i \subset L_j$ if a correspondence exists such that:

a. L_i is isomorphic to a subset of L_j;

b. K_i is isomorphic to K_j.

We remark that the image of L_i in L_j need not be a subsystem relative to Φ of L_j for the relation of Definition 9.2 to hold. In fact, Definition 9.2 and also Theorem 9.3 will hold equally well if L_i and L_j are not required to be extensions to Φ of K, but merely closed under Φ and containing a subset isomorphic to K.

9.3 Theorem. *The relation* $\subset$ *of Definition* 9.2 *orders the set of extensions to* Φ *of* K.

9.4 Definition. L_0 is a *minimal extension* to Φ of K if L_0 is a zero of the ordered set of extensions to Φ of K.

† Stone (23 and 24).

If L_0 and $\bar{L}_0$ are both zeros of the set of extensions to Φ of K, then they are equal. But the equality in this case means only that each is isomorphic to a subset of the other under certain restrictions. We desire a stricter equality, namely, that L_0 and $\bar{L}_0$ be isomorphic. This result is established by considering required elements.

9.5 Definition. An element of L_i, an extension to Φ of K, is *required* by Φ and K if it can be represented by a finite succession of operations of Φ on subsets of K_i.

From the definition it is clear that the set of elements required by Φ and K depends only upon Φ and K and not upon the particular extension, L_i, under consideration. Furthermore, each L_i contains the set of elements required by Φ and K. The representation of required elements in terms of Φ and K and the ordering relation of K impose certain restrictions on their ordering. Such restrictions we call required relations and define precisely in Definition 9.6.

9.6 Definition. The *relation* of a to b, elements of an extension to Φ of K and required by Φ and K, is *required* if either $a \subset b$ or $a \not\subset b$ is implied by their representation in terms of Φ and K.

If the relation of a to b is not required we say that it is optional. It is a characteristic property of minimal extensions that whenever the relation of a to b is optional then $a \subset b$. At the other extreme, we have free extensions which are characterized by the property that if the relation of a to b is optional then $a \not\subset b$. For examples of required elements and relations let addition and multiplication be in Φ and let a, b, and c be unrelated elements of K, then $a+b$ and ab are required elements and $ab \subset a+b$ and $a+b \not\subset ab$ are required relations. The relation of ab to ac is optional.

In each case which we consider we are able to find an extension in which all the elements and relations are required. In fact, we are able to find a unique representation for each element of each extension such that all the relations are required by the unique representations and the given set. We call such an extension a canonical extension.

9.7 Definition. An extension, L, to Φ of K is *canonical* if it consists entirely of required elements and a unique representation can be assigned to each of its formally different elements such that all the relations of L are required by this representation and K.

It should be noted that the representation need be unique only for elements which differ in form. It may well be that some of the required relations will introduce additional equalities.

9.8 THEOREM. *If L is a canonical extension to Φ of K, then L is a minimal extension to Φ of K.*

Proof: Let L_i be any extension to Φ of K. Then L_i contains all elements required by Φ and K. In particular, L_i contains elements with the representations assigned to elements of L. To each element of L let correspond the element of L_i with the same representation. Since all the relations of L are required, this 1–1 correspondence between L and a subset of L_i is an isomorphism. Furthermore, under this correspondence, the subsets of L and L_i which correspond to K are isomorphic. Hence, L is a minimal extension to Φ of K.

The proof of Theorem 9.8 goes through equally well if we do not demand that L_i be an extension to Φ of K, but merely that L_i be closed under Φ and contain a subset isomorphic to K. Since the ordering relation for extensions applies to this type of set too, we can assert that L is minimal with respect to all sets closed under Φ and containing a subset isomorphic with K.

9.9 LEMMA. *If L is a canonical extension to Φ of K, then any isomorphism which carries L into L', a subset of itself, so that the image of K corresponds to itself, element for element, carries each element of L into itself and L' is the entire set L.*

Proof: Since the image of K is preserved, element for element, and since all the relations of L are required by the unique representations and K, each element of L must correspond to itself.

9.10 THEOREM. *If L is a canonical extension to Φ of K and L_0 is a minimal extension to Φ of K, then L and L_0 are isomorphic.*

Proof: Since L_0 is a minimal extension to Φ of K, it is isomorphic to a subset of L, and the images of K in L and L_0 are also isomorphic under this correspondence. Using this isomorphism between the images of K, an isomorphism can be constructed between L and a subset of L_0 by letting each element of L correspond to the element of L_0 with the same representation. Applying this isomorphism from L to a subset of L_0 and carrying this subset back into L by the isomorphism from L_0 to a subset of L, we have an isomorphism between L and a subset of L which preserves the image of K, element for element. By Lemma 9.9, this subset of L must be the set L, itself. Thus every element of L is the image of some element of L_0 under the isomorphism from L_0 to a subset of L and the subset of L in question is the entire set L. Hence, L and L_0 are isomorphic.

The independence of a canonical extension from the particular steps by which it is constructed is established by repeated applications of the following theorem.

9.11 THEOREM. *If L_i is a canonical extension to Φ_i of K; L_{ij}, a canonical extension to Φ_j of L_i where Φ_i is a subset of Φ_j; and K is a subsystem relative to Φ_j of L_i: then L_{ij} is a canonical extension to Φ_j of K.*

9.12 THEOREM. *If K is closed under Φ, then K is a canonical extension to Φ of itself.*

In particular, it follows from either Theorem 9.11 or 9.12 that the iterated application of a canonical extension yields nothing new. Thus a canonical extension has all the properties which we require.

It is not in general true that a canonical extension exists for arbitrary Φ and K. For instance, a lattice, K, which is not distributive cannot be extended to a distributive lattice, L, preserving both addition and multiplication, for some $a(b+c)\neq ab+ac$ exists in K, while the preservation, in L, of this inequality denies the distributive law. On the other hand, if only the distributive sums and distributive products of K are preserved in L, then the extension is not uniquely defined. We find, however, that the problem has a unique solution if all products, but only distributive sums, are preserved. Furthermore, the form of an extension may require that the given set be closed under certain operations and it is natural to require that the extension be closed under these operations, too. Hence, for an extension, we are given two suitably chosen sets of operations, Θ and Φ, of which Θ is a subset of Φ, and an arbitrary ordered set, K, closed under Θ. We are to construct a set, L, which is a canonical extension to Φ of K.

Theorem 9.11 gives us the conditions under which canonical extensions may be combined. We now seek the conditions under which canonical extensions may be split up.

9.13 THEOREM. *If L is a canonical extension to Φ of K, where the only restrictions on K are that it be an ordered set closed under Θ, a specified subset of Φ, and Φ', a subset of Φ, contains Θ, then the set L' of elements of L required by Φ' and K is a canonical extension to Φ' of K, and K is a subsystem relative to Φ of L'.*

Theorems 9.11 and 9.13 enable us to derive a large number of extensions from the few which we actually give. It is our aim to give extensions which may be used as a basis for all extensions possible under the operations which we consider, in that all such extensions may be derived from the given ones. We fall short of this goal in a few cases.

An ordered set is said to be *given* if its elements and ordering relation are defined. The operations, which we consider, are determined by this information. We assume that this determination can actually be carried out. In performing a sequence of extensions it is necessary to work out the internal properties of each before proceeding to the next. An ordered set is said to be *constructed* when its elements and ordering relation have been defined, that is, it can be used as a given set for successive constructions.

The elements of an extension, L, to Φ of a given set, K, closed under Θ are usually suitably restricted subsets of K, such that the ordering relation of L can be defined in terms of the known properties of the elements of L immediately involved. We shall characterize three types of extension by the methods allowed in defining these subsets. The first type, an *absolute* extension, is algebraic in nature and is characterized by the fact that the elements required by Φ and a subsystem of K relative to Φ and closed under Θ can be defined as subsets within the subsystem. Such an extension has the property that the extension of a subsystem of K relative to Φ and closed under Θ is isomorphic with the corresponding subsystem of the extension of the entire set. An absolute extension does not require the calculus of classes. The second type, a *relative* extension, depends upon the calculus of classes, including the theory of infinite sections, but does not involve the well ordering hypothesis. That is, elements of L are classes of K where the classes of K are so defined that the admission, or rejection, of an element to a class does not depend upon the disposition of elements already considered. On the other hand, the subsets of K by which the elements of L are represented involve the entire set K in their definition and the property of absolute extensions with respect to the invariance of subsystems need not hold. The third type of extension employs the well ordering hypothesis† in the definition of subclasses of K to represent elements of L.

Each successive type of extension permits greater liberty in the definition of classes or, to put it in another way, requires stronger assumptions as to the existence of classes. Each type of extension includes its predecessors. In order to get by with as few assumptions as possible, we give preference to an absolute extension. If an absolute extension is not possible, we have recourse to a relative extension. In no case do we require the well ordering hypothesis.

In a canonical extension it frequently happens that equal elements of L can be represented by different subsets of K. If a uniquely defined member is selected to represent each class of equal elements in L, then the extension has been reduced to normal form. In other words, if the elements of an extension, L, have been reduced to normal form, then elements are equal only if the subsets of K by which they are represented are identical. A normal form is found for each extension which follows. In each case the normal form is neces-

† Fraenkel (7) and Zermelo (27).

sarily relative to the entire set K, even though the extension may be absolute. Here again we have no recourse to the axiom of choice.

In the light of this discussion we now list the steps which are required in the construction of a canonical extension. In each case we consider the construction as de facto evidence that such an extension is possible. We then take up the cases where an extension cannot be made and point out the cases which have not, as yet, been solved. In presenting the extensions which follow, we shall pass from step to step without comment, assuming that the motives behind the various steps have been made sufficiently clear in this section.

9.14 CONSTRUCTION. A canonical extension to Φ of K, where K is closed under a subset, Θ, of Φ has been constructed when the following steps have been completed:

a. The elements of L have been defined;

b. The ordering relation of L has been defined;

c. The closure of L under Φ has been established;

d. An isomorphism has been established between K and a subset of L such that the image of K is a subsystem of L relative to Φ;

e. All operations not in Φ relative to which the image of K is a subsystem of L have been found; (Strictly speaking, this information is not needed for the construction of a canonical extension, but it is needed in applying 9.11.)

f. A unique representation, required by Φ and K has been assigned to each element of L;

g. All the relations of L have been shown to be required by K and the unique representations of the elements of L;

h. The possibility of adjoining new operations to both Θ and Φ has been investigated; (Strictly speaking, this is not necessary except to show the full power of the method of extension under consideration. Theorem 9.13 shows merely the possibility of adjoining operations of Φ to Θ.)

i. If the extension is relative, the impossibility of an absolute extension has been shown;

j. A normal form has been found for the elements of L.

10. **The adjunction of units.** It has been shown in §4 that Postulates 1.1 and 1.2 for an ordered set do not imply Postulates 4.4 and 4.4* for units. The adjunction of units may be done directly and presents no difficulty. However, we make a precise statement of the construction in order to avoid ambiguity in the sections which follow. We are given an ordered set, K, closed under specified operations, but not satisfying Postulate 4.4. An extension, L, satisfying Postulate 4.4 is constructed as follows.

10.1 Definition. The elements of L are the elements of K and an element 0.

10.2 Definition. $a \subset b$ in L if a and b are in K and $a \subset b$ in K and $0 \subset a$ in L for every element, a, in L.

10.3 Theorem. *L is an ordered set, closed under the operations for which the closure of K is specified, and satisfies Postulate* 4.4.

10.4 Theorem. *If every element of K corresponds to itself in L, then all the operations of* §3 *which can be performed in K are preserved in L.*

10.5 Theorem. *Every set, closed under the operations for which the closure of K is specified, which satisfies Postulate* 4.4 *and has a subset isomorphic with K has a subset of isomorphic with L.*

It is clear that if K is a multiplicative system, a lattice, or a modular or distributive lattice, then so is L. If K does not satisfy Postulate 4.4*, then a unit I can be adjoined in an entirely analogous manner.

11. **Imbedding an ordered set in a complete lattice.** Given an arbitrary ordered set, K, with units, we construct a canonical extension, L, of K to a complete lattice. The assumption that K have units involves no loss of generality, for units can be adjoined to any ordered set by the method of §10. The extension to a complete lattice is accomplished by means of a generalization of the cuts used by Dedekind† to define irrational numbers. This method has the advantage of preserving in form the essential dualism between sums and products. A cut can be interpreted either as a sum or as a product. This extension is necessarily relative.

11.1 Definition. A *cut* $[A, B]$, of K consists of two sets, A and B, of elements, a_i and b_j, of K which satisfy the following conditions:

a. $a_i \subset b_j$ for every a_i in A and every b_j in B,

b. $a_i \subset x$ for every a_i in A implies x in B,

c. $y \subset b_j$ for every b_j in B implies y in A.

The unit 0 of K is always in the initial set, A, of a cut and the unit I is always in the final set, B, of a cut. By assuming that K has units we have eliminated the possibility that either of the sets of a cut be void. Furthermore, if the two sets of a cut have elements in common, the common elements are equal.

11.2 Definition. The *section*, $|A_i|$, of an aggregate of sets, A_i, is the set of elements common to every A_i.

11.3 Lemma. *If $[A_i, B_i]$ is any set of cuts in K, A is $|A_i|$ (B is $|B_i|$), and*

† Dedekind (5).

B (A) is the set of all elements b_k (a_j) such that $a_j \subset b_k$ for every a_j in A $(b_k$ in $B)$, then $[A, B]$ is a cut.

Proof: $[A, B]$ satisfies 11.1a and 11.1b by definition of B. Since $[A_i, B_i]$ is a cut and A is a subset of A_i, $a_j \subset b_{ik}$ for every a_j in A and every b_{ik} in B_i. Therefore, B_i is a subset of B. Hence, $y \subset b_k$ for every b_k in B implies $y \subset b_{ik}$ for every b_{ik} in B_i and, since $[A_i, B_i]$ is a cut, y is in A_i. This holds for every A_i. Therefore, y is in A and $[A, B]$ satisfies 11.1c.

We are now ready to construct the complete lattice extension, L, of the given ordered set, K, with units.

11.4 Definition. The elements of L are the cuts in K.

11.5 Definition. $[A, B] \subset [C, D]$ if every element of A is an element of C.

11.6 Theorem. *$[A, B] \subset [C, D]$ if, and only if, every element of D is an element of B.*

Proof: Since D is a subset of B and $[A, B]$ is a cut, $a_i \subset d_j$ for every a_i in A and every d_j in D. Hence, since $[C, D]$ is a cut, A is a subset of C. On the other hand, if D is not a subset of B, then $a_i \not\subset d_j$ for some a_i and d_j, and A is not a subset of C.

11.7 Theorem. *L is a complete lattice.*

Proof: $[A, B] \subset [C, D]$ and $[C, D] \subset [E, F]$ imply $[A, B] \subset [E, F]$, for if A is a subset of C and C, of E, then A is a subset of E. $[A, B] \subset [A, B]$, for every element of A is an element of A. Given any set of cuts $[A_i, B_i]$ and, if $[A, B]$ is the cut defined in 11.3, then $[A, B] = \prod [A_i, B_i]$. For, since every element of A is an element of A_i for every A_i, $[A, B] \subset [A_i, B_i]$ for every $[A_i, B_i]$ and, since any subset of every A_i is a subset of A, $[X, Y] \subset [A_i, B_i]$ for every $[A_i, B_i]$ implies $[X, Y] \subset [A, B]$. Similarly, if $[A, B]$ is the cut defined in 11.3*, then $[A, B] = \sum [A_i, B_i]$.

11.8 Lemma. *If a is any element of K, A consists of all elements $a_i \subset a$, and $\overline{A}$ consists of all elements $a \subset \bar{a}_j$, then $[A, \overline{A}]$ is a cut.*

11.9 Theorem. *The correspondence $a \longleftrightarrow [A, \overline{A}]$, where $[A, \overline{A}]$ is defined as in 11.8, establishes an isomorphism between K and a subset of L which preserves the relation $\subset$, units, sums, and products.*

Proof: $[A, \overline{A}] \subset [B, \overline{B}]$ if, and only if, $a \subset b$. For $a_i \subset a \subset b$ implies $a_i \subset b$ and a_i in B, for every a_i in A. Conversely, if every a_i in A is in B; then, in particular, a is in A; hence, in B; and $a \subset b$. $0 \longleftrightarrow [0, K]$ and $\mathrm{I} \longleftrightarrow [K, \mathrm{I}]$, preserving units. $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$ if, and only if, $b = \prod a_i$. For $b = \prod a_i$ implies $b \subset a_i$ and $[B, \overline{B}] \subset [A_i, \overline{A}_i]$. If $[X, Y] \subset [A_i, \overline{A}_i]$, then $x_j \subset a_i$, $x_j \subset b$, and x_j

is in B. Hence, $[X, Y] \subset [B, \overline{B}]$. Conversely, if $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$, then $b \subset a_i$ and $x \subset a_i$ implies $[X, \overline{X}] \subset [A_i, \overline{A}_i]$, $[X, \overline{X}] \subset [B, \overline{B}]$, and $x \subset b$. An analogous proof holds for sums.

11.10 LEMMA. $[A, B] = \prod [B_i, \overline{B}_i]$ ($[A, B] = \sum [A, \overline{A}_i]$) *and* $\prod [B_i, \overline{B}_i]$ ($\sum [A_i, \overline{A}_i]$) *is distributive.*

11.11 THEOREM. *The correspondence* $a \longleftrightarrow [A, \overline{A}]$ *preserves modular and distributive sums and products, complements, and sum and product complements.*

Proof: $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$ is distributive if, and only if, $b = \prod a_i$ is distributive. For, if $[B, \overline{B}] \subset [X, Y]$, then $[B, \overline{B}] \subset [Y_j, \overline{Y}_j]$ and $b \subset y_j$. Therefore, since $\prod a_i$ is distributive, $y_j = \prod z_{jk}$ where, for each z_{jk}, a_i exists such that $a_i \subset z_{jk}$. But, since $[X, Y] = \prod [Y_j, \overline{Y}_j]$, $[X, Y] = \prod\prod [Z_{jk}, \overline{Z}_{jk}]$ and, for each $[Z_{jk}, \overline{Z}_{jk}]$, $[A_i, \overline{A}_i]$ exists such that $[A_i, \overline{A}_i] \subset [Z_{jk}, \overline{Z}_{jk}]$. Therefore, $\prod [A_i, \overline{A}_i]$ is distributive. Conversely, if $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$ is distributive and $b \subset x$, then $[B, \overline{B}] \subset [X, \overline{X}]$ and x can be distributed since $[X, \overline{X}]$ can. An analogous proof holds for distributive sums. By a similar proof it can be shown that $[A, \overline{A}][B, \overline{B}]$ is modular with respect to $[A, \overline{A}]$ if, and only if, ab is modular with respect to a. In this case we have $[A, \overline{A}][B, \overline{B}] \subset [X, Y] \subset [A, \overline{A}]$. Hence, by deleting from $\prod [Y_j, \overline{Y}_j]$ all factors such that $[A, \overline{A}] \subset [Y_j, \overline{Y}_j]$, the proof goes through as before. An analogous proof holds for modular sums. Complements are preserved, since sums, products, and units are. Let $a \longleftrightarrow [A, \overline{A}]$ and $a' \longleftrightarrow [A', \overline{A}']$. Then $[A', \overline{A}']$ is the product complement of $[A, \overline{A}]$ if, and only if, a' is the product complement of a. If a' is the product complement of a, then $aa' = 0$ and $[A, \overline{A}][A', \overline{A}'] = [0, K]$. Furthermore, if $[A, \overline{A}][X, Y] = [0, K]$ then $[A, \overline{A}][X_i, \overline{X}_i] = [0, K]$, $ax_i = 0$, $x_i \subset a'$, and $[X_i, \overline{X}_i] \subset [A', \overline{A}']$. Since $[X, Y] = \sum [X_i, \overline{X}_i]$ by 11.8*, $[X, Y] \subset [A', \overline{A}']$ follows. Hence, $[A', \overline{A}']$ is the product complement of $[A, \overline{A}]$. Conversely, if $[A', \overline{A}']$ is the product complement of $[A, \overline{A}]$ and $ax = 0$, then $[A, \overline{A}][X, \overline{X}] = [0, K]$, $[X, \overline{X}] \subset [A', \overline{A}']$, $x \subset a'$, and a' is the product complement of a.

From Lemma 11.10 we see that any cut $[A, B]$ can be represented in terms of Φ and K either as $\prod b_i$ or $\sum a_i$. We choose $\sum a_i$ as the unique representation of $[A, B]$.

11.12 THEOREM. *All the relations of L are required by K and the representation* $\sum a_i$ *of* $[A, B]$.

Proof: If $[A, B] \subset [C, D]$, then A is a subset of C and $\sum a_i \subset \sum c_j$. If $\sum a_i \subset \sum c_j$, then d_k in D implies $c_j \subset d_k$, $\sum c_j \subset d_k$, $\sum a_i \subset d_k$, and $a_i \subset d_k$. Therefore, d_k is in B, D is a subset of B, and $[A, B] \subset [C, D]$.

Since K is an arbitrary ordered set with units, modular and distributive lattices are eligible for extension by cuts. The question as to whether the extension by cuts of a modular or distributive lattice is itself a modular or distributive lattice has not yet been answered. The corresponding question for Boolean algebras is answered by the next theorem.

11.13 THEOREM. *If K is a Boolean algebra, then the extension, L, by cuts of K is a complete Boolean algebra.*

Proof: Postulates 1.1 and 4.5 hold by Theorem 11.7. If $[A, B]$ is a cut and A' and B' consist of the complements of the elements of A and B, then $[B', A']$ is a cut, the product complement of $[A, B]$ and $[A, B] \longleftrightarrow [B', A']$ is a dual automorphism.

From Theorem 9.13 we find that the canonical extension of K to a multiplicative system is the subset of L consisting of the image of K and products of finite numbers of these elements. The canonical extension of K to a complete multiplicative system is the subset of L consisting of the image of K and products of any number of these elements. The subset of L consisting of the image of K and all sums and products of finite numbers of these elements is the canonical extension of K to a lattice. However, since a complete multiplicative system with a unit I is a complete lattice, the canonical extension of K to a complete multiplicative system is the entire set L.

11.14 THEOREM. *The extension to a complete lattice of an ordered set with units is necessarily relative.*

Proof: Consider the set of elements a, b, c, and d satisfying 1.2 and no additional relations except $d \subset a$ and $d \subset b$. In the entire set, $ab = d \neq abc$. In the extension of the subset a, b, and c, $ab = abc$.

The elements of this extension are already in normal form.

12. **Extension of a multiplicative system to a distributive lattice.** The next problem is the construction of a canonical extension, L, to a lattice with completely distributive sums of an arbitrary multiplicative system with units, K. To be precise, the operations Θ under which K is closed are units and finite products. The operations Φ under which L is to be closed are units, finite products, and unrestricted distributive sums. As usual, subsets of K are denoted by capital letters.

12.1 DEFINITION. The elements of L are the subsets of K.

12.2 DEFINITION. $A \subset B$ if, for every a_i in A, $a_i = \sum_j a_i b_j$, where the sum is taken over every b_j in B, and $\sum_j a_i b_j$ is distributive.

12.3 THEOREM. *The relation $\subset$ of Definition 12.2 orders the elements of L.*

Proof: If $A \subset B$ and $B \subset C$, then $a_i = \sum_j a_i b_j$ for every a_i in A and

$b_j=\sum_k b_j c_k$ for every b_j in B. Substitute for each b_j in the first set of equations its value as a sum from the second set of equations, then use the first set of equations multiplied through by an element of C and distributed to eliminate the elements of B. That is, $a_i=\sum_j a_i b_j=\sum_j\sum_k a_i b_j c_k=\sum_k a_i c_k$. Since $\sum_j a_i b_j$ and $\sum_k b_j c_k$ are distributive, $\sum_k a_i b_j c_k$, $\sum_j\sum_k a_i b_j c_k$, and $\sum_k a_i c_k$ are distributive. Therefore, $A \subset C$. Clearly, $A \subset A$. Hence, L is ordered by the relation $\subset$.

12.4 THEOREM. *L contains units,* 0 *and* I, *namely, the elements of L consisting solely of the* 0 *or* I *of K.*

12.5 THEOREM. *$B=\prod A_i$ for any finite set of elements A_i of L, if B is the set of all products with one component from each A_i as factors.*

Proof: For each b_k in B there exists some a_{ik} in each A_i such that $b_k a_{ij}=b_k$, for each b_k has a factor from each A_i. Therefore, $b_k=\sum_j b_k a_{ij}$ and $\sum_j b_k a_{ij}$ is distributive, since b_k appears as a summand. This implies $B \subset A_i$ for every A_i. If $X \subset A_i$ for every A_i, then $x_m=\sum_j x_m a_{ij}$ and $\sum_j x_m a_{ij}$ is distributive. Hence, $x_m=\prod_i\sum_j x_m a_{ij}=\sum x_m\prod_i a_{ij}=\sum_k x_m b_k$ and $\sum_k x_m b_k$ is distributive. This implies $X \subset B$. Therefore, $B=\prod A_i$.

12.6 THEOREM. *$B=\sum A_i$ for any set of elements A_i of L and $\sum A_i$ is distributive, if B is the set of all elements which are components of some A_i.*

Proof: Since every component a_{ij} of every A_i is an element b_k of B, $a_{ij}=\sum_k a_{ij} b_k$ and $\sum_k a_{ij} b_k$ is distributive, for a_{ij} appears as a b_k. This implies $A_i \subset B$ for every A_i. If $A_i \subset X$ for every A_i, then $a_{ij}=\sum_m a_{ij} x_m$ and $\sum_m a_{ij} x_m$ is distributive for every a_{ij} in every A_i. But every b_k is an a_{ij}, hence, $b_k=\sum_m b_k x_m$ and $\sum_m b_k x_m$ is distributive for every b_k in B. This implies $B \subset X$. Therefore, $B=\sum A_i$. $X \subset \sum A_i$ implies $x_m=\sum_i\sum_j x_m a_{ij}$ and that $\sum_i\sum_j x_m a_{ij}$ is distributive. It follows that $x_m=\sum_i\sum_j\sum_k x_m x_k a_{ij}$ and that $\sum_i\sum_j\sum_k x_m x_k a_{ij}$ is distributive. Hence $X \subset \sum_i XA_i$, but $\sum_i XA_i \subset X$ in any ordered set. Therefore, $X=\sum_i XA_i$ and $\sum A_i$ is distributive.

Since L is closed under unrestricted sums and has a unit 0, it is a complete lattice. That is, L is closed under unrestricted products. If K is a complete multiplicative system and the multiplicative axiom is assumed, then Theorem 12.5 holds for unrestricted products and we have a representation of all products in terms of the components of their factors. However, we make no use of this representation. Furthermore, by Lemma 7.6, every element of L has a product complement.

12.7 THEOREM. *The correspondence $a \longleftrightarrow A$, where A is the subset of K consisting solely of the element a, is an isomorphism between K and a subset of L such that units, finite products, and unrestricted distributive sums are preserved.*

Proof: $a \subset b$ implies $a = ab$ which is trivially distributive. Hence, $A \subset B$, where $a \longleftrightarrow A$ and $b \longleftrightarrow B$. Conversely, $A \subset B$ implies $a = ab$ and $a \subset b$. Units are preserved, as shown in 12.4, and finite products are preserved, as shown in 12.5. Distributive sums are preserved, for, if $a = \sum a_i$ is distributive and $a \longleftrightarrow A$ and $a_i \longleftrightarrow A_i$, then $a_i \subset a$ and $A_i \subset A$. If $A_i \subset X$ for every A_i, then $a_i = \sum_m a_i x_m$ and $\sum_m a_i x_m$ is distributive. Adding and applying the distributive property of $\sum a_i$, $a = \sum_m a x_m$ and $\sum_m a x_m$ is distributive. Therefore $A \subset X$ and $A = \sum A_i$.

12.8 THEOREM. *The isomorphism of Theorem* 12.7 *preserves unrestricted products and product complements.*

Proof: If $a = \prod a_i$, then $a \subset a_i$ and $A \subset A_i$. If $X \subset A_i$ for every A_i, then $x_m = x_m a_i$ for every x_m and a_i. Multiplying, $x_m = x_m a$ for every x_m and $X \subset A$. Therefore, $A = \prod A_i$. If a' is the product complement of a and $a \longleftrightarrow A$ and $a' \longleftrightarrow A'$, then $AA' = 0$. By Theorem 12.5, $AX = 0$ implies $ax_m = 0$ for every x_m in X. Hence, $x_m \subset a'$, $x_m = x_m a'$, and $X \subset A'$. Therefore, A' is the product complement of A.

12.9 THEOREM. *The isomorphism of Theorem* 12.7 *does not preserve, in general, sums which are not distributive or sum complements.*

Proof: Consider Example [4.7, 4.10; 4.7]. This set is a multiplicative system. When extended to a distributive lattice, neither $a+c$ nor c^*, of 4.10, are preserved.

Complements are not necessarily preserved, since sums are not preserved, in general. From Theorem 12.6, it follows that each element of L can be represented as the distributive sum of its components. We choose this representation. Thus an arbitrary element, X, of L is represented by $\sum x_m$, where $\sum x_m$ is distributive.

12.10 THEOREM. *All the relations of L are required by K and the representation $\sum x_m$, where $\sum x_m$ is distributive, of X.*

Proof: If $\sum x_m \subset \sum y_n$, then, since $x_i \subset \sum x_m$ for every x_i in X, $x_i \subset \sum y_n$ and $x_i = x_i \sum y_n = \sum_n x_i y_n$, by Theorem 6.3, and $\sum_n x_i y_n$ is distributive, by Theorem 6.6. Hence, $X \subset Y$. If $X \subset Y$, then $x_i = \sum_n x_i y_n = x_i \sum y_n$ and $x_i \subset \sum y_n$. Therefore, $\sum x_m \subset \sum y_n$.

Since L is closed for unrestricted products and product complements and the isomorphism of Theorem 12.7 preserves these operations, they may perfectly well be added to the set Φ of operations to which K has been extended, without any further changes. Since the isomorphism of Theorem 12.7 does not, in general, preserve non-distributive sums and sum complements, these operations may not be added to the set Φ.

If the finite subsets of K are taken as the elements of L, the canonical extension of K to a distributive lattice is obtained.

This is an absolute extension.

We now seek a normal form for this extension. Although it is desirable that a normal form preserve the finiteness of an element which has a finite representation, in this case it is easy to construct examples to show that this is not possible without the introduction of a finite descending chain condition or some other type of well ordering. For example, let K be the set of proper subsets of the positive integers where $a \subset b$ if a is a subset of b. K does not have a unit I. The extension, L, has a unit I which can be represented in infinitely many ways as the sum of two or any finite number of elements of K, but, lacking any form of the axiom of choice, we are unable to choose one from among them. The normal form is obtained by selecting a subsystem of L in which the converse of the following lemma is true.

12.11 LEMMA. *If A and B are in L and A is a subset of B, then $A \subset B$.*

12.12 DEFINITION. An element A of L is in normal form if $x = \sum a_i x$ and $\sum a_i x$ distributive imply x in A.

$\bar{L}$ denotes the subset of L containing all the elements of L which are in normal form. If an element is in $\bar{L}$, it will be denoted with a bar as $\bar{A}$. $\bar{L}$, being a subset of L, is ordered by the ordering relation of L.

12.13 LEMMA. *If $\bar{A} \subset \bar{B}$, then $\bar{A}$ is a subset of $\bar{B}$.*

12.14 THEOREM. *$\bar{A} \subset \bar{B}$ if, and only if, $\bar{A}$ is a subset of $\bar{B}$.*

12.15 COROLLARY. *$\bar{A} = \bar{B}$ if, and only if, $\bar{A}$ and $\bar{B}$ are identical.*

12.16 THEOREM. *If A is in L and C consists of all elements c_k such that $c_k = \sum a_i c_k$ where $\sum a_i c_k$ is distributive, then C is in normal form and $A = C$.*

Proof: If $x = \sum c_k x$ and $\sum c_k x$ is distributive, then $x = \sum_k \sum_i a_i c_k x = \sum_i \sum_k a_i c_k x = \sum_i a_i x$ and $\sum a_i x$ is distributive. Hence, C is in normal form. $C \subset A$ by definition of C and 12.2 and, since $a_k = \sum_i a_i a_k$ where $\sum_i a_i a_k$ is distributive, A is a subset of C and $A \subset C$ by 12.11. Therefore, $A = C$.

The elements of $\bar{L}$ can be characterized as follows.

12.17 DEFINITION. A subset, A, in K is an *ideal* in K, where K is any partially ordered set, if a_i in A and $x \subset a_i$ imply x in A and $x = \sum_j a_{ij}$, where $\sum_j a_{ij}$ is the distributive sum of a subset A_i of A, implies x in A.

12.18 THEOREM. *If K is a multiplicative system with unit* 0, *then $\bar{L}$ is the set of ideals in K.*

Proof: If A is an ideal, then $x = \sum a_i x$ and $\sum a_i x$ distributive imply x in A, since, for each a_i, $a_ix \subset a_i$. Hence, A is in $\bar{L}$. If A is in $\bar{L}$, then $x \subset a_i$ implies $x = \sum a_ix$, where $\sum a_ix$ is distributive, and $x = \sum_j a_{ij}$, where $\sum_j a_{ij}$ is distributive and A_i is a subset of A, implies $x = \sum a_ix$ and $\sum a_ix$ distributive. Hence, A is an ideal.

The term ideal has been used because, if A is an ideal in K, then the product of any element in A with any element in K, if it exists in K, is in A and the sum, interpreted as distributive sum, of any subset of A, if it exists in K, is in A. The extension of any partially ordered set K, with unit 0, to a lattice with completely distributive sums can be effected at once by letting L be the set of ideals in K and defining $A \subset B$ if A is a subset of B. The initial set of a cut is an ideal. In fact, by Theorem 5.17, the sum, not necessarily distributive, of any subset of the initial set of a cut, if it exists in K, is in the initial set of the cut. However, cuts cannot be so characterized.

Extension by ideals is necessarily relative.

13. **Extension of a distributive lattice to a Boolean algebra.** We now construct the canonical extension to a Boolean algebra of a distributive lattice with units. The first step is to imbed the given distributive lattice, K, in a multiplicative system, K', where the product complements of the image of K exist and are isomorphic to K with respect to the dual relation. The second step is the application to K' of the previously established extension to a distributive lattice of a multiplicative system. The resulting distributive lattice is shown to be the sought for Boolean algebra.

13.1 Definition. The elements of K' are the ordered pairs, $(a, \bar{a})$, where a and $\bar{a}$ are arbitrary elements of K.

13.2 Definition. $(a, \bar{a}) \subset (b, \bar{b})$ if $a \subset \bar{a}+b$ and $a\bar{b} \subset \bar{a}$.

13.3 Theorem. *The relation* $\subset$ *of Definition* 13.2 *orders the elements of* K'.

Proof: $(a, \bar{a}) \subset (b, \bar{b})$ and $(b, \bar{b}) \subset (c, \bar{c})$ imply $a \subset \bar{a}+b$, $a\bar{b} \subset \bar{a}$, $b \subset \bar{b}+c$, and $b\bar{c} \subset \bar{b}$. Hence, $a \subset \bar{a}+b \subset \bar{a}+\bar{b}+c$, $a = a(\bar{a}+\bar{b}+c) = a\bar{a}+a\bar{b}+ac \subset \bar{a}+a\bar{b}+c = \bar{a}+c$, and $a\bar{c} = a\bar{c}(\bar{a}+b) = a\bar{a}\bar{c}+ab\bar{c} \subset a\bar{a}\bar{c}+a\bar{b} \subset a\bar{a}\bar{c}+\bar{a} = \bar{a}$. Therefore, $(a, \bar{a}) \subset (c, \bar{c})$. $(a, \bar{a}) \subset (a, \bar{a})$, since $a \subset \bar{a}+a$ and $a\bar{a} \subset \bar{a}$.

13.4 Theorem. $\prod(a_i, \bar{a}_i) = (\prod a_i, \sum \bar{a}_i)$, *if* $\prod a_i$ *and* $\sum \bar{a}_i$ *exist in* K *and are distributive.*

Proof: $\prod a_i \subset \sum \bar{a}_i + a_n$ and $\bar{a}_n \prod a_i \subset \bar{a}_n \subset \sum \bar{a}_i$ for every a_n and $\bar{a}_n$. If $(x, \bar{x}) \subset (a_n, \bar{a}_n)$ for every element in $\prod(a_i, \bar{a}_i)$, then $x \subset \bar{x}+a_n$ and $x\bar{a}_n \subset \bar{x}$. Multiplying or adding and distributing we get $x \subset \prod(\bar{x}+a_i) = \bar{x}+\prod a_i$ and $x\sum \bar{a}_i = \sum x\bar{a}_i \subset \bar{x}$.

13.5 Corollary. *K' is a multiplicative system.*

In general, sums do not exist in K', so a theorem for sums corresponding to Theorem 13.4 for products must necessarily be much more restricted. Theorem 13.6 is not the most general theorem possible, but it covers the cases which we require.

13.6 THEOREM. *$\sum(a_i, \bar{a}_i) = (\sum a_i, \prod \bar{a}_i)$ and $\sum(a_i, \bar{a}_i)$ is distributive if $\sum a_i$ and $\prod \bar{a}_i$ exist in K and are distributive and $a_j\bar{a}_j \subset a_k + \bar{a}_k$ whenever $(a_j, \bar{a}_j)$ and $(a_k, \bar{a}_k)$ are summands of $\sum(a_i, \bar{a}_i)$.*

Proof: Since $a_j \subset \sum a_i + \bar{a}_j$ and $a_j \prod \bar{a}_i \subset \bar{a}_j$ whenever $(a_j, \bar{a}_j)$ is a summand of $\sum(a_i, a_i)$, $(a_j, \bar{a}_j) \subset (\sum a_i, \prod \bar{a}_i)$. If $(a_j, \bar{a}_j) \subset (x, \bar{x})$ for every summand of $\sum(a_i, a_i)$, then $a_j \subset \bar{a}_j + x$ and $a_j\bar{x} \subset \bar{a}_j$. Hence, $a_j = a_j\bar{a}_j + a_jx \subset a_j\bar{a}_j + x \subset a_k + \bar{a}_k + x \subset \bar{a}_k + x$, $\sum a_i \subset \bar{a}_k + x$, and $\sum a_i \subset \prod[\bar{a}_i + x] = \prod \bar{a}_i + x$. Similarly, $a_j\bar{x} \subset a_j\bar{a}_j\bar{x} \subset [a_k + \bar{a}_k]\bar{x} \subset a_k\bar{x} + \bar{a}_k\bar{x} \subset \bar{a}_k$, $[\sum a_i]\bar{x} = \sum a_i\bar{x} \subset \bar{a}_k$, and $[\sum a_i]\bar{x} \subset \prod \bar{a}_i$. Therefore, $(\sum a_i, \prod \bar{a}_i) \subset (x, \bar{x})$ and $(\sum a_i, \prod \bar{a}_i) = \sum(a_i, \bar{a}_i)$. Furthermore, by Theorems 6.6 and 6.6*, $\sum a_i b$ and $\prod[\bar{a}_i + \bar{b}]$ exist and are distributive and $a_j\bar{a}_j \subset a_k + \bar{a}_k$ implies $ba_j[\bar{b} + \bar{a}_j] \subset ba_k + \bar{b} + \bar{a}_k$. Therefore, $(b\sum a_i, \bar{b} + \prod \bar{a}_i) = (\sum ba_i, \prod[\bar{b} + \bar{a}_i]) = \sum(ba_i, \bar{b} + \bar{a}_i)$ and $\sum(a_i, \bar{a}_i)$ is distributive.

13.7 COROLLARY. *$\sum(a_i, \bar{a}_i) = (\sum a_i, \prod \bar{a}_i)$ if $\sum a_i$ and $\prod \bar{a}_i$ exist and are distributive and if, for every two different summands $(a_j, \bar{a}_j)$ and $(a_k, \bar{a}_k)$ of $\sum(a_i, \bar{a}_i)$, at least one of the relations $a_j \subset a_k$, $a_j \subset \bar{a}_k$, $\bar{a}_j \subset a_k$, or $\bar{a}_j \subset \bar{a}_k$ holds.*

13.8 COROLLARY. *$\sum(a_i, \bar{a}) = (a, \bar{a})$ and $\sum(a_i, \bar{a})$ is distributive if $a = \sum a_i$ and $\sum a_i$ is distributive.*

13.9 COROLLARY. *$\sum(a, \bar{a}_i) = (a, \bar{a})$ and $\sum(a, \bar{a}_i)$ is distributive if $\bar{a} = \prod \bar{a}_i$ and $\prod \bar{a}_i$ is distributive.*

13.10 COROLLARY. *$(a + \bar{a}, \bar{a}) = (a, \bar{a}) = (a, a\bar{a})$.*

13.11 COROLLARY. *$(a, \bar{a}) + (\bar{a}, \bar{\bar{a}}) = (a + \bar{a}, \bar{a}\bar{\bar{a}})$ and $(a, \bar{a}) + (\bar{a}, \bar{\bar{a}})$ is distributive. In particular, if $\bar{\bar{a}} \subset \bar{a} \subset a$, then $(a + \bar{a}, \bar{a}\bar{\bar{a}}) = (a, \bar{\bar{a}})$.*

13.12 THEOREM. *$(0, 0)$ is a zero and $(\mathrm{I}, 0)$ is a one of K'.*

Proof: $0 \subset 0 + a$, $0\bar{a} \subset 0$, $a \subset \bar{a} + \mathrm{I}$, and $a0 \subset \bar{a}$, where $(a, \bar{a})$ is any element of K'.

It follows from Corollary 13.10 that we could have restricted the elements, $(a, \bar{a})$, of K' to those ordered pairs of K elements which satisfy the condition $\bar{a} \subset a$. We make use of this fact later in finding a normal representation of this extension.

13.13 THEOREM. *The correspondence $a \longleftrightarrow (a, 0)$ is an isomorphism between K and a subset of K' which preserves units, distributive sums, and complements.*

Proof: $a \subset b$ implies $(a, 0) \subset (b, 0)$ and conversely. $0 \longleftrightarrow (0, 0)$ and $\mathrm{I} \longleftrightarrow (\mathrm{I}, 0)$.

Distributive sums are preserved, by 13.8. If a and b are complementary in K with respect to I and 0, then $ab=0$ and $a+b=\mathrm{I}$. Since K is a distributive lattice, ab, $a+b$, and $0+0$ are distributive. Hence, by 13.4, $(a, 0)(b, 0)=(ab, 0)=(0, 0)$, the zero of K' and, by 13.8, $(a, 0)+(b, 0)=(a+b, 0)=(\mathrm{I}, 0)$ the one of K'.

13.14 THEOREM. *The correspondence $a \longleftrightarrow (a, 0)$ preserves distributive products in K as products in K'.*

Proof: If $a=\prod a_i$ and $\prod a_i$ is distributive, then, by 13.4, since $\sum 0=0$ is distributive, $\prod(a_i, 0)=(a, 0)$.

The images in K' of distributive products in K are not, in general, distributive. Sum and product complements in K are not, in general, preserved in K'.

13.15 THEOREM. *(I, a) and $(a, 0)$ are mutually complementary with respect to* I *and* 0, *and each is the product complement of the other.*

Proof: $(\mathrm{I}, a)(a, 0)=(a, a)=(0, 0)$. By 13.11, $(\mathrm{I}, a)+(a, 0)=(\mathrm{I}, 0)$. If $(a, 0)(y, \bar{y})=(0, 0)$, then $ay \subset \bar{y}$ implying $(y, \bar{y}) \subset (\mathrm{I}, a)$, and if $(\mathrm{I}, a)(z, \bar{z})=(0, 0)$, then $z \subset a+\bar{z}$ implying $(z, \bar{z}) \subset (a, 0)$.

13.16 THEOREM. *The correspondence $(a, 0) \longleftrightarrow (\mathrm{I}, a)$ is an isomorphism between the image of K and its product complements ordered by the dual relation.*

Proof: $a \subset b$ is necessary and sufficient for both $(a, 0) \subset (b, 0)$ and $(\mathrm{I}, b) \subset (\mathrm{I}, a)$.

Since $(a, \bar{a})=(a, 0)(\mathrm{I}, \bar{a})$, any element, $(a, \bar{a})$, in K' can be represented in terms of K, products, and product complements as $a\bar{a}'$, where $\bar{a}'$ is the product complement of $\bar{a}$. We choose this as the unique representation of the elements of K' in terms of K and the operations of a Boolean algebra. We recollect that in a Boolean algebra complements are identified with product complements and any element is equal to the product complement of its product complement.

13.17 THEOREM. *All the relations of K' are required by K and the representation $a\bar{a}'$ of $(a, \bar{a})$.*

Proof: If $a\bar{a}' \subset b\bar{b}'$, then, multiplying through by b' and by $\bar{b}$, $a\bar{a}'b'=0$ and $a\bar{a}'\bar{b}=0$. These conditions imply $a \subset \bar{a}+b$ and $a\bar{b} \subset \bar{a}$, by 7.13, 7.16, and 7.30. Hence, $(a, \bar{a}) \subset (b, \bar{b})$. If $(a, \bar{a}) \subset (b, \bar{b})$, then $a \subset \bar{a}+b$, $a\bar{b} \subset \bar{a}$, $a\bar{a}'b'=0$, $a\bar{a}'\bar{b}=0$, $a\bar{a}' \subset b$, $a\bar{a}' \subset \bar{b}'$, and $a\bar{a}' \subset b\bar{b}'$.

We have shown that every Boolean algebra with a subset isomorphic with K has a subset isomorphic with K'. Since K' is not necessarily closed with respect to product complements, it is not, in general, a Boolean algebra. Let

L be the canonical extension of K' to a distributive lattice. Then the elements of L can be represented as distributive sums of finite subsets of K', and L satisfies Postulates 1.1 and 4.1 of the postulates for a Boolean algebra. Furthermore, since a Boolean algebra is a distributive lattice, every Boolean algebra with a subset isomorphic with K has a subset isomorphic with L. In particular, if L is a Boolean algebra, then it is the sought for extension of K.

13.18 THEOREM. *L, the canonical extension to a distributive lattice of K', is a Boolean algebra.*

Proof: Postulates 1.1 and 4.1 hold in L. Since the extension from K' to L preserves distributive sums and products, 13.11 and 13.15 imply that $(a, 0)$ and (I, a) are mutually complementary with respect to I and 0 in L even though the extension from K' to L need not, in general, preserve complements. Since $(a, \bar{a}) = (a, 0)(\mathrm{I}, \bar{a})$, $(a, \bar{a})$ has a complement in L, by 7.32 remembering that all finite products in L are distributive. Again applying 7.32, all finite sums, $\sum(a_i, \bar{a}_i)$, have complements in L. But these sums include all of L. By 7.30, each of two complementary elements in L is the product complement of the other. Hence, each element of L has a product complement and, applying 3.20, 7.3 holds. Therefore, L is a Boolean algebra.

The extension from K to L is absolute.

Applying Theorem 11.13, L can be extended by cuts to a complete Boolean algebra. This can also be done directly by extending K' to a lattice with completely distributive sums. Both these extensions are canonical and are, therefore, isomorphic. The first of these methods is a relative extension, the second, absolute. We can now state the following general theorem.

13.19 THEOREM. *Any ordered set, K, can be extended by a canonical extension to a complete Boolean algebra, L, so as to preserve units, finite products and unrestricted distributive sums.*

Theorem 13.19 is not a complete statement of the results of the extensions which we have considered, for further restrictions on K or fewer restrictions on L may make it possible to preserve in L more of the properties of K. To to ascertain this, it is necessary to refer to the operations preserved by each extension, separately.

We now seek a normal form for the extension from a distributive lattice, K, to a Boolean algebra, L. That is, the elements $\sum(a_i, \bar{a}_i)$ of L can be divided into mutually exclusive and jointly exhaustive classes of equal elements, and we propose to select a unique representative from each of these classes. Since the elements of L are finite sums of elements of K', we may use the integers from 0 to n as subscripts for an element of $n+1$ terms. We first show that the subset, $\bar{L}$, of elements $\sum_{i=0}^{n}(a_i, \bar{a}_i)$ of L which satisfy the conditions $a_j \subset \bar{a}_k \subset a_k$ for $k<j$ is an extension of K to a Boolean algebra and must, therefore, contain elements from each class of equal elements in L. Elements of $\bar{L}$ will be denoted with a minus sign, $\sum_{i=0}^{n}(a_i - \bar{a}_i)$. This is in keeping with the fact that elements of $\bar{L}$ may be considered as the symmetric difference of the elements of K involved. Abstractly, the elements of $\bar{L}$† are the set of all finite descending sequences of K elements with an even number of terms.

13.20 Theorem. *If $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ is in $\bar{L}$, then $\sum_{i=0}^{n+1}(\bar{a}_{i-1} - a_i)$, where $\bar{a}_{-1} = \mathrm{I}$ and $a_{n+1} = 0$, is in $\bar{L}$ and $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ and $\sum_{i=0}^{n+1}(\bar{a}_{i-1} - a_i)$ are mutually complementary in L.*

Proof: $\sum_{i=0}^{n}(a_i - \bar{a}_i) + \sum_{i=0}^{n+1}(\bar{a}_{i-1} - a_i) = (\mathrm{I} - 0)$, by repeated application of 13.11.

$$\left[\sum_{i=0}^{n}(a_i - \bar{a}_i)\right]\left[\sum_{i=0}^{n+1}(\bar{a}_{i-1} - a_i)\right] = \sum_{i=0}^{n}\sum_{j=0}^{n+1}(a_i\bar{a}_{j-1}, \bar{a}_i + a_j) = (0 - 0),$$

since $a_i\bar{a}_{j-1} \subset \bar{a}_i + a_j$ for all values of i and j.

13.21 Theorem. *If $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ and $\sum_{j=0}^{m}(b_j - \bar{b}_j)$ are in $\bar{L}$, then $\sum_{k=0}^{m+n}(c_k - \bar{c}_k)$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ and $\bar{c}_k = \sum_{i=0}^{k}(\bar{a}_i b_{k-i} + a_i \bar{b}_{k-i})$, is in $\bar{L}$ and $[\sum_{i=0}^{n}(a_i - \bar{a}_i)][\sum_{j=0}^{m}(b_j - \bar{b}_j)] = \sum_{k=0}^{m+n}(c_k - \bar{c}_k)$.*

Proof: $\bar{c}_k \subset c_k$, for $\bar{a}_i b_{k-i} \subset a_i b_{k-i}$ and $a_i \bar{b}_{k-i} \subset a_i b_{k-i}$. $c_j \subset \bar{c}_k$ if $k<j$, for $a_i b_{j-i} \subset a_i \bar{b}_{k-i} \subset \bar{a}_i b_{k-i} + a_i \bar{b}_{k-i}$. Hence $\sum(c_k - \bar{c}_k)$ is in $\bar{L}$. Furthermore,

$$\sum_{i=0}^{k}(a_i b_{k-i}, \bar{a}_i + \bar{b}_{k-i}) = \left(\sum_{i=0}^{k} a_i b_{k-i}, \prod_{i=0}^{k}[\bar{a}_i + \bar{b}_{k-i}]\right) = (c_k - \bar{c}_k), \quad \text{by } 13.7,$$

since $a_i b_{k-i} \subset \bar{a}_j + \bar{b}_{k-j}$ for $i = j$, and by 13.10, since $[\sum_{i=0}^{k} a_i b_{k-i}]\prod_{i=0}^{k}[\bar{a}_i + \bar{b}_{k-i}] = \bar{c}_k$. Hence,

$$\left[\sum_{i=0}^{n}(a_i - \bar{a}_i)\right]\left[\sum_{j=0}^{m}(b_j - \bar{b}_j)\right] = \sum_{i=0}^{n}\sum_{j=0}^{m}(a_i b_j, \bar{a}_i + \bar{b}_j) = \sum_{k=0}^{m+n}(c_k - \bar{c}_k).$$

We remark that if $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ is represented symbolically as the ordered pair of polynomials $(A - \bar{A})$, where $A = a_0x^n + a_1x^{n-1} + \cdots + a^n$ and $\bar{A} = \bar{a}_0x^n + \bar{a}_1x^{n-1} + \cdots + \bar{a}_n$, then, using the usual laws for the addition and multiplication of polynomials, $(A - \bar{A})(B - \bar{B}) = (AB - [\bar{A}B + A\bar{B}])$. This

† Hausdorff (11), p. 78 defines $\bar{L}$ in this way for the case where the elements of K are subsets of a set of points.

symbolic formula is given merely as an aid to carrying out multiplications in $\bar{L}$.

By Theorem 7.30, we know that complements and product complements are identified in L. Hence, since $\bar{L}$ is a subsystem of the Boolean algebra L and is closed under finite multiplication and product complementation, it is a Boolean algebra. Since $a \leftrightarrow (a-0)$, the image of K is in $\bar{L}$, $\bar{L}$ is a canonical extension of K to a Boolean algebra, and L is isomorphic to $\bar{L}$. It is easy to show directly that each element of L is equal to an element of $\bar{L}$. For each element of K' this is true by 13.10. Hence,

$$\sum_{i=0}^{n} (a_i, \bar{a}_i) = \sum_{i=0}^{n} (a_i, a_i\bar{a}_i) = \left[\prod_{i=0}^{n} [(1 - a_i) + (a_i\bar{a}_i - 0)] \right]',$$

where the latter expression can be reduced to the required form by 13.20 and 13.21. We next make a direct attack on the ordering relation in L and $\bar{L}$.

13.22 Lemma. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) = (0, 0)$ if, and only if, $a_i \subset \bar{a}_i$ for $i = 0, \cdots, n$.*

Proof: If $a_i \subset \bar{a}_i$, then $(a_i, \bar{a}_i) = (0, 0)$ and $\sum_{i=0}^{n}(a_i, \bar{a}_i) = (0, 0)$. If $\sum_{i=0}^{n}(a_i, \bar{a}_i) = (0, 0)$, then $(a_i, \bar{a}_i) \subset (0, 0)$ and $a_i \subset \bar{a}_i$.

13.23 Theorem. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(b_j, \bar{b}_j)$ if, and only if, $a_i\prod\bar{b}_j \subset \bar{a}_i + \sum b_j$ for $i = 0, \cdots, n$ and for all combinations of the indices $j = 0, \cdots, m$ such that each index occurs either in $\bar{b}_j$ or in b_j but not in both, and $\prod\bar{b}_j = \mathrm{I}$ if no index occurs on the left and $\sum b_j = 0$ if no index occurs on the right.*

Proof: By 7.27, $\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(b_j, \bar{b}_j)$ if, and only if, $[\sum_{i=0}^{n}(a_i, \bar{a}_i)] \prod_{j=0}^{m}(\mathrm{I}, b_j) + (\bar{b}_j, 0)] = (0, 0)$. Distributing this product,

$$\left[\sum_{i=0}^{n} (a_i, \bar{a}_i) \right] \prod_{j=0}^{m} [(\mathrm{I}, b_j) + (\bar{b}_j, 0)] = \sum (a_i \prod \bar{b}_j, \bar{a}_i + \sum b_j),$$

where the sum is taken over i from 0 to n and over all combinations of j as specified in the statement of the theorem. Applying 13.22, $\sum(a_i\prod\bar{b}_j, \bar{a}_i + \sum b_j) = (0, 0)$ if, and only if, $a_i\prod\bar{b}_j \subset \bar{a}_i + \sum b_j$ for the combinations of indices specified.

13.24 Corollary. *$\sum_{i=0}^{n}(a_i - \bar{a}_i) \subset \sum_{j=0}^{m}(b_j - \bar{b}_j)$ if, and only if, $\sum_{i=0}^{k} a_i\bar{b}_{k-i-1} \subset \sum_{i=0}^{k}(\bar{a}_i\bar{b}_{k-i-1} + a_i b_{k-i})$ for $k = 0, 1, \cdots, m+n+1$.*

13.25 Corollary. *$\sum_{i=0}^{n}(a_i - \bar{a}_i) \subset \sum_{j=0}^{m}(b_j - \bar{b}_j)$ if, and only if, $a_i\bar{b}_{j-1} \subset \bar{a}_i + b_j$ for $i = 0, \cdots, n$ and $j = 0, \cdots, m+1$.*

13.26 Corollary. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(b_j, \bar{b}_j)$ if, and only if, for any two elements x and y of K, $x \subset y$ whenever $x \subset a_i$, $\bar{a}_i \subset y$ for some subscript i and either $x \subset \bar{b}_j$ or $b_j \subset y$ for every subscript j.*

Corollary 13.26 is in a form which extends immediately to infinite sums as it depends only on the ordering of K, whereas Theorem 13.23 depends on sums and products in K which might not exist in the infinite case.

13.27 THEOREM. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) = (a, \bar{a})$, where $a = \sum_{i=0}^{n} a_i$ and $\bar{a} = \prod_{i=0}^{n} \bar{a}_i$, if, and only if, for any two elements x and y of K, $x \subset y$ whenever $x \subset a$, $\bar{a} \subset y$ and either $x \subset \bar{a}_i$ or $a_i \subset y$ for each subscript i.*

If $\sum a_i$ and $\prod \bar{a}_i$ exist and are distributive, Theorem 13.27 can be generalized at once to the infinite case yielding both necessary and sufficient conditions for the validity of the addition formula of Theorem 13.6.

If we define the *degree* of $\sum_{i=0}^{n}(a_i, \bar{a}_i)$ as n, then the degree of $[\sum_{i=0}^{n}(a_i, \bar{a}_i)]'$ is $2^{n+1}-1$, that of $\sum_{i=0}^{n}(a_i, \bar{a}_i) + \sum_{j=0}^{m}(b_j, \bar{b}_j)$ is $m+n+1$, that of $[\sum_{i=0}^{n}(a_i, \bar{a}_i)][\sum_{j=0}^{m}(b_j, \bar{b}_j)]$ is $(m+1)(n+1)-1$, and the number of conditions required to establish $\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(b_j, \bar{b}_j)$ is $(n+1)2^{m+1}$, assuming no reductions are made. However, in $\bar{L}$, the degree of $[\sum_{i=0}^{n}(a_i - \bar{a}_i)]'$ is only $n+1$, that of $[\sum_{i=0}^{n}(a_i - \bar{a}_i)][\sum_{j=0}^{m}(b_j - \bar{b}_j)]$ is $m+n$, and the number of conditions required to establish $\sum_{i=0}^{n}(a_i - \bar{a}_i) \subset \sum_{j=0}^{m}(b_j - \bar{b}_j)$ is $m+n+2$, though it is frequently easier, in practice, to apply the $(n+1)(m+2)$ conditions of Corollary 13.25. We now agree to make the following *elementary reductions* in the degree of $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ whenever possible. Whenever $\sum_{i=0}^{n}(a_i - \bar{a}_i) = (0, 0)$, by Lemma 13.22, it will be written $(0, 0)$. Whenever $\sum_{i=0}^{n}(a_i - \bar{a}_i) \neq (0, 0)$, $(a_i - \bar{a}_i)$ is omitted if $a_i = \bar{a}_i$ as, in this case, $(a_i - \bar{a}_i) = (0, 0)$ and $(a_i - \bar{a}_i) + (a_{i+1} - \bar{a}_{i+1})$ is written $(a_i - \bar{a}_{i+1})$ if $\bar{a}_i = a_{i+1}$, applying Corollary 13.11. The order in which these latter two reductions are applied does not effect the final result which is to strengthen the relations on the a's to $a_k < \bar{a}_j < a_j$ for $j < k$. When these reductions have been performed, it follows that the degree of $\sum_{i=0}^{n}(a_i - \bar{a}_i) + \sum_{j=0}^{m}(b_j - \bar{b}_j)$ is not greater than $m+n+1$ and that for every element of L an equal element can be found in $\bar{L}$ of equal or lower degree. Hence the apparent advantages of operating in $\bar{L}$, derived from considerations of degree, are real. We remark that the elementary reductions in $\bar{L}$ are so called because they are the only reductions which are obtainable from the application of the operations and ordering relation of K upon the elements a_i and $\bar{a}_i$. In the reduction so far we have preserved the absolute character of the extension, the final step is necessarily relative to the entire set, K.

To continue with the reduction to normal form, we must assume that K has completely distributive products. This involves no real loss in generality, since any distributive lattice may be so extended by the dual of the extension from a multiplicative system to a lattice with completely distributive sums and this extension can be put in normal form. As a preliminary to finding a normal form we discuss the theory of minimal covers.

13.28 DEFINITION. An element, c, of K is a *minimal cover* of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ in $\bar{L}$ if $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset(c-0)$ and if $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset(x-0)$ implies $c\subset x$.

13.29 THEOREM. *If c and d are minimal covers of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, then $c=d$.*

13.30 THEOREM. *If $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ is in $\bar{L}$, then c, the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, exists in K.*

Proof: Let $c=\prod x_j$ where the product is taken over all elements x_j in K such that $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset(x_j-0)$. Since $\prod x_j$ is distributive, it is preserved, by 13.14, in K' and hence in L and $\bar{L}$. Therefore, $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset(c-0)$. Also, $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset(x-0)$ implies $c\subset x$, for every such x is an x_j. The set x_j is not void, for I is an x_j.

13.31 THEOREM. *If c is the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, then $c\subset a_0$.*

Proof: $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset(a_0-0)$.

13.32 THEOREM. *If c is the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, d the minimal cover of $\sum_{j=0}^{n}(b_j-\bar{b}_j)$, and $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset\sum_{j=0}^{m}(b_j-\bar{b}_j)$, then $c\subset d$.*

Proof: $\sum_{i=0}^{n}(a_i-\bar{a}_i)\subset\sum_{j=0}^{m}(b_j-\bar{b}_j)\subset(d-0)$.

13.33 COROLLARY. *If $\sum_{i=0}^{n}(a_i-\bar{a}_i)=\sum_{j=0}^{m}(b_j-\bar{b}_j)$, then $c=d$.*

13.34 COROLLARY. *$c+d$ is the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)+\sum_{j=0}^{m}(b_j-\bar{b}_j)$.*

13.35 COROLLARY. *$f\subset cd$, where f is the minimal cover of $[\sum_{i=0}^{n}(a_i-\bar{a}_i)]$ $[\sum_{j=0}^{m}(b_j-\bar{b}_j)]$.*

Thus the correspondence $\sum_{i=0}^{n}(a_i-\bar{a}_i)\rightarrow c$ from $\bar{L}$ to K is a homomorphism which preserves sums but not, in general, products.

13.36 DEFINITION. $\sum_{k=0}^{p}(c_k-\bar{c}_k)$ is the normal form of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ $\neq(0-0)$ if c_k is the minimal cover of $\sum_{i=k}^{n}(\bar{c}_{k-1}a_i-\bar{c}_{k-1}\bar{a}_i)$, $\bar{c}_k$ is the minimal cover of $\sum_{i=k}^{n}(c_k\bar{a}_i-c_ka_{i+1})$, and p is the greatest subscript for which $c_p\neq 0$. $(0-0)$ is the normal form of zero.

13.37 THEOREM. *$\sum_{k=0}^{p}(c_k-\bar{c}_k)$ is in $\bar{L}$ and $p\leqq n$.*

Proof: By 13.31, $c_k\subset\bar{c}_{k-1}a_k\subset\bar{c}_{k-1}$ and $\bar{c}_k\subset c_ka_k\subset c_k$. Also $p\leqq n$ for $c_{n+1}=0$.

13.38 LEMMA. *If $\sum_{i=k}^{n}(\bar{c}_{k-1}a_i-\bar{c}_{k-1}\bar{a}_i)=\sum_{j=k}^{m}(\bar{c}_{k-1}b_j-\bar{c}_{k-1}\bar{b}_j)$, then $\sum_{i=k}^{n}(c_k\bar{a}_i-c_ka_{i+1})=\sum_{j=k}^{m}(c_k\bar{b}_j-c_kb_{j+1})$.*

Proof: By 13.33, c_k is the minimal cover of both sides of the hypothesis. Also, $c_k\subset\bar{c}_{k-1}$ and $c_ka_k=c_kb_k=c_k$. Taking complements multiplying through by (c_k-0) on both sides, the first term on each side drops out, and the conclusion remains.

13.39 LEMMA. *If* $\sum_{i=k}^{n}(c_k\bar{a}_i - c_k a_{i+1}) = \sum_{j=k}^{m}(c_k\bar{b}_j - c_k b_{j+1})$, *then* $\sum_{i=k+1}^{n}(\bar{c}_k a_i - \bar{c}_k\bar{a}_i) = \sum_{j=k+1}^{m}(\bar{c}_k b_j - \bar{c}_k\bar{b}_j)$.

Proof: The proof is analogous to that of 13.38.

13.40 THEOREM. *If* $\sum_{k=0}^{p}(c_k - \bar{c}_k)$ *is the normal form of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$, $\sum_{k=0}^{q}(d_k - \bar{d}_k)$ *is the normal form of* $\sum_{j=0}^{m}(b_j - \bar{b}_j)$, *and* $\sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{j=0}^{m}(b_j - \bar{b}_j)$, *then* $c_k = d_k$, $\bar{c}_k = \bar{d}_k$, *and* $p = q$.

Proof: The proof is by induction. Since $\sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{j=0}^{m}(b_j - \bar{b}_j)$, $c_0 = d_0$ and the hypothesis of 13.38 is satisfied for $k=0$. Hence, the conclusion of 13.38 for $k=0$, which is in turn the hypothesis, for $k=0$, of 13.39, is fulfilled. In general, if for any value of k, $\bar{c}_{k-1} = \bar{d}_{k-1}$ and the hypothesis of 13.38 holds, then $c_k = d_k$ and the hypothesis of 13.39 holds for k. In an analogous manner, if for any value of k, $c_k = d_k$ and the hypothesis of 13.39 holds, then $\bar{c}_k = \bar{d}_k$ and the hypothesis of 13.38 holds for $k+1$. Since $c_p \neq 0$ and $c_{p+1} = 0$, $d_p \neq 0$ and $d_{p+1} = 0$. Hence, $p = q$.

13.41. COROLLARY. *If* $\sum_{k=0}^{p}(c_k - \bar{c}_k)$ *and* $\sum_{k=0}^{q}(d_k - \bar{d}_k)$ *are normal forms of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$, *then* $c_k = d_k$, $\bar{c}_k = \bar{d}_k$, *and* $p = q$.

13.42 LEMMA. *Let* $\sum_{j=0}^{p}(c_j - \bar{c}_j)$ *be the normal form of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ *and define expressions* $A(k)$ *and* $B(k)$ *as follows*:

$$A(k) \qquad \left[\sum_{i=0}^{n}(a_i - \bar{a}_i)\right]' = \sum_{j=0}^{k}(\bar{c}_{j-1} - c_j) + \sum_{i=k}^{n}(c_k\bar{a}_i - c_k a_{i+1}),$$

$$B(k) \qquad \sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{j=0}^{k}(c_j - \bar{c}_j) + \sum_{i=k+1}^{n}(\bar{c}_k a_i - \bar{c}_k\bar{a}_i).$$

Then $A(k)$ *implies* $B(k)$ *and* $B(k)$ *implies* $A(k+1)$.

Proof: By 13.36, $\sum_{i=k}^{n}(c_k\bar{a}_i - c_k a_{i+1}) = \sum_{i=k}^{n}(\bar{c}_k\bar{a}_i - \bar{c}_k a_{i+1})$. Substituting this in $A(k)$ and taking the complement, since $\bar{c}_k\bar{a}_k = \bar{c}_k$, $B(k)$ results. Similarly, since $\sum_{i=k+1}^{n}(\bar{c}_k a_i - \bar{c}_k\bar{a}_i) = \sum_{i=k+1}^{n}(c_{k+1}a_i - c_{k+1}\bar{a}_i)$ and $c_{k+1}a_{k+1} = c_{k+1}$, $A(k+1)$ results from substituting and taking the complement.

13.43 THEOREM. *If* $\sum_{j=0}^{p}(c_j - \bar{c}_j)$ *is the normal form of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$, *then* $\sum_{j=0}^{p}(c_j - \bar{c}_j) = \sum_{i=0}^{n}(a_i - \bar{a}_i)$.

Proof: Since $\sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{i=0}^{n}(c_0 a_i - c_0\bar{a}_i)$ and $c_0 a_0 = c_0$, by substituting and taking the complement $A(0)$ is established. $B(p)$ follows from this by induction employing 13.42. But $\sum_{i=p+1}^{n}(\bar{c}_p a_i - \bar{c}_p\bar{a}_i) = \sum_{i=p+1}^{n}(c_{p+1}a_i - c_{p+1}\bar{a}_i) = (0-0)$, since $c_{p+1} = 0$. Hence, $B(p)$ reduces to the required result.

In each class of equal elements in $\bar{L}$ there must be at least one element of degree not greater than that of any other element in the class. The normal form of this element falls within the class by Theorem 13.43, hence cannot be of lower degree than the element of least degree. However, by Theorem 13.37, its degree cannot be greater. Furthermore, each class of equal elements has but one normal form, by Theorem 13.40. Hence, we conclude that the degree of an element in normal form is not greater than the degree of any equal element. This implies that no elementary reductions can be performed on a normal form.

In general, there is no upper bound for the degrees of elements in normal form in $\bar{L}$. Consider Example [4.11, 4.7]. After elementary reductions have been made, each element is in normal form and such elements of any degree can be constructed.

Since the reduction of $\bar{L}$ to normal form also reduces K' to normal form, either of the methods of extending K to a complete Boolean algebra can be put into normal form by using the normal forms of the component extensions employed.

If the given distributive lattice, K, does not contain a unit I, then the extension to $\bar{L}$, with trivial changes, yields a Boolean ring. However, the dependence of the extension upon the unit 0 is essential.

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