CONTINUOUS GROUPS AND SCHWARZ' LEMMA*

BY

MAX ZORN

Introduction

The famous lemma of H. A. Schwarz is doubtless one of the basic theorems in the theory of analytic functions. In this paper I propose to study the lemma from a topological point of view. The results have been announced, without proof, in a previous note.† Several changes, corrections, and additions have been made; I use the opportunity to state here my indebtedness to D. W. Hall for his inspiring interest and helpful criticism.

The theory to be presented is a by-product of a more comprehensive treatment of conformal mappings‡ which will be communicated elsewhere.

Like the theories of Kerékjártó§ and Stoïlow|| our investigations are made with a direct view to the characterization of conformal mappings. Yet both authors deal with the conformal mappings individually, whereas we aim more at the characterization of the system of all conformal mappings of a Riemann surface S in itself. As an equivalent to this simplification of the problem we attempt to keep the space S general as long as possible, whereas usually S is supposed to be locally plane from the outset.

The theory of Schwarz' lemma has been separated from the rest because of its independence and also because it seems to be of value for the study of the hardest characterization problem, the problem of Brouwer.

The present paper is divided into three parts. Part I is of a rather general nature and can be read without any topological preparation. For the other parts a certain familiarity with topological notions and theorems is necessary. Parts I and II together lead up to a theorem which is formally identical with the Schwarz lemma. In III, particularly in §8, we show that this formal identity is material identity; in §9 we derive, with the aid of the geometric theorems from II some interesting topological features of the underlying space.

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† *Sur le lemme de Schwarz*, Comptes Rendus de l'Académie des Sciences, vol. 206, p. 725.

‡ Cf. *Topological studies in the theory of analytic functions*, Bulletin of the American Mathematical Society, abstract 43-11-415.

§ Cf. B. de Kerékjártó, *Sur la structure des transformations topologiques* · · · , Enseignement Mathématique, vol. 35 (1936), p. 297.

|| Stoïlow, *Leçons sur les Principes Topologiques de la Théorie des Fonctions Analytiques*, Paris, 1938.

Notations. We use only italic letters; consequently, concepts of different logical types are often denoted by letters of the same alphabet: d_i, i, m_i, n, a are indices, d, i and m, n natural numbers, a is arbitrary; e, p, q, r, s, x, y, z, are points (most of them in S); S, A, C, E, K, L, O, U_x, V_y denote sets of points, usually contained in S; F, F^*, F', G, H_i, P_i, R, R_p, R_i, $R^{(x)}$ are transformations, usually continuous single-valued mappings of S in itself; N is a family of transformations; in general the transformations F, and so on, will belong to N.

If all x_i are in a set such as C, we call x_i a sequence from C. A "subsequence" of a sequence, say x_i, is formed by choosing an increasing sequence $(i_{n+1} > i_n)$ of indices; it is convenient to denote the new sequence by x_i'; a subsequence of the subsequence would be written x_i''.

Theorems and definitions are numbered together; a definition is indicated by brackets, a theorem by parentheses.

Part I

1. **Continuous transformations.** We make the following definition:

[1.1] *S is a topological space which we assume to be metrizable. The metric of the space does not occur explicitly, but we shall have to use limit relations like* $\lim x_n = x$, *functions like the closure* $\overline{A}$, *the boundary* Bd(A), *the frontier* Fr(A), *and properties like open, closed, connected, locally connected; the terms compact, limited are defined explicitly for obvious reasons.*

In Part I, however, we do not need all the consequences of the metrizability; it is sufficient to assume that S is an L^*-space as defined, for example, in Kuratowski's book on topology.†

[1.1.1] *S is an L^*-space if convergence of sequences is defined and satisfies the following conditions:*

I. *If* $\lim x_n = x$, *then* $\lim x_n' = x$.

II. *If* $x_n = x$, *then* $\lim x_n = x$.

III. *If for every subsequence x_n' of x_n a subsequence x_n'' with* $\lim x_n'' = x$ *can be found, then* $\lim x_n = x$.

[1.2] *A point x is a limit point of A if a sequence x_i from A exists such that* $\lim x_i = x$. *A point x is a limit point of a sequence x_i if a subsequence x_i' with* $\lim x_i' = x$ *exists.*

If every sequence x_i from A has at least one limit point, then A is called "limited."

A is "compact" if every sequence from A has a limit point in A.

† C. Kuratowski, *Topologie* I, Warsaw, 1933, pp. 76–77; cf. also the literature mentioned there.

In the sequel we shall be concerned mostly with a family N of single-valued, continuous transformations $F, G, H_i, I, P, R, \cdots$. The domain (of definition) is always S, the range (of values) $F(S)$ is a subset of S. The natural definition of continuity in L^*-spaces is the following:

[1.3] *F is continuous if* $\lim x_n = x$ *implies* $\lim F(x_n) = F(x)$.

Convergent sequences of (continuous) functions F will occur rather often; it seems that the type of convergence which has been introduced as "continuous convergence"† is the most appropriate one for the abstract theory of conformal mappings.

[1.4] *A sequence of transformations F_n is said to converge towards F, that is,* $\lim F_n = F$, *if* $\lim x_n = x$ *implies* $\lim F_n(x_n) = F(x)$.

Obviously this implies $\lim F_n(x) = F(x)$; but the converse is not true.

By virtue of the definition [1.4] any set of continuous mappings F forms an L^*-space. We shall have to use the corresponding property III in our proofs; hence we state explicitly the following theorem:

(1.4.1) *If every subsequence F_n' contains a subsequence F_n'' with* $\lim F_n'' = F$, *then* $\lim F_n = F$.

Indeed, let $\lim x_n = x$, and consider the sequence of points $F_n(x_n)$. From every subsequence $F_n'(x_n')$ we can select $F_n''(x_n'')$ such that $\lim F_n''(x_n'') = F(x)$. Consequently, $\lim F_n(x_n) = F(x)$, which implies $\lim F_n = F$.

[1.5] *A sequence F_n is called "properly divergent" if for no point x the sequence $F_n(x)$ has a limit point.*

We recall the usual notations and conventions about composition of functions:

[1.6] *The product $H = FG$ of F and G (in this order) is defined by $H(x) = FG(x) = F(G(x))$. The identity is the transformation I which leaves all points invariant, $I(x) = x$. A function G is the inverse of F if $GF = I$; it may not exist, but if it does, then it is unique and satisfies $FG = I$. Powers F^n are defined as usual; if the inverse exists, it is always written as F^{-1}.*

(1.7) *If* $\lim F_n = F$ *and* $\lim G_n = G$, *then* $\lim F_n G_n = FG$.

This is an immediate consequence of the "continuity" of the convergence.

Let $\lim x_n = x$. It follows from $\lim G_n(x_n) = G(x)$ that $\lim F_n(G_n(x_n)) = F(\lim G_n(x_n)) = F(G(x)) = FG(x)$.

(1.8) *If* $\lim F_n = F$ *and* $\lim F_n^{-1} = G$, *then* $G = F^{-1}$.

In other words, if the inverses F_n^{-1} exist and converge towards a limit, then

† Cf. C. Carathéodory, *Conformal Representation*, Cambridge Tracts, no. 28.

$$\lim F_n^{-1} = (\lim F_n)^{-1}.$$

The proof is an algebraic consequence of (1.7), for $GF = (\lim F_n^{-1})(\lim F_n) = \lim F_n^{-1}F_n = I$.

(1.9) *F is called nilpotent if a point p exists such that* $\lim x_n = x$ *implies*

$$\lim F^n(x_n) = p.$$

[1.10] *The transformation which maps every point on the same point p is called "constant" and denoted by P.*

With this terminology we can say that F is nilpotent exactly if its powers converge towards a constant.

(1.11) *The point p is a fixed point of F,* $F(p) = p$. *It is also the only fixed point of F.*

Indeed, writing p in the form $\lim F^n(p)$, we obtain $F(p) = F(\lim F^n(p)) = \lim F^{n+1}(p) = \lim F^n(p) = p$.

If, on the other hand, $F(q) = q$, the relations $F^n(q) = q$, $\lim F^n(q) = q$, and $\lim F^n(q) = p$ give $q = p$.

2. **The family** N. In this section we introduce a group of definitions and assumptions which describe abstractly some features of the analytic mappings of the unit circle in itself.

[2.1] *The family N is a set of transformations with the following properties:*

I. Continuity. *The elements of N are single-valued continuous transformations of S into itself,* $F(S) \subset S$.

II. Composition, identity. *The identity I is in N, and if F and G are in N, then their product FG is in N.*

III. Cancellation. *If F, G, H are in N, and if F is not constant, then the equality* $GF = HF$ *implies* $G = H$.

IV. Normality. *Every sequence* F_n *from N contains a subsequence* F_n' *which is either properly divergent or else converges towards an element F of N.*

If we want S to be the unit circle of the complex number plane and N the set of all analytic mappings F, $F(S) \subset S$, we speak of "the classical case."

In the classical case, I–IV are fulfilled; I–III are elementary, whereas IV has perhaps a more advanced character and belongs to the theory of normal families.†

From these assumptions alone we shall derive a topological version of the Schwarz lemma. In Part II a geometrical formulation will be established on

† Cf. R. Montel, *Sur les Familles Normales de Fonctions Analytiques et leurs Applications*, Paris, Gauthier-Villars, 1927; cf. also Kerékjártó, loc. cit., p. 308; K. Szilárd, *Untersuchungen ueber die Grundlagen der Funktionentheorie*, Mathematische Zeitschrift, vol. 26, p. 653.

the basis of further restrictions on S and N. Finally we show how the abstract theorem yields the ordinary Schwarz lemma in the classical case.

The geometry in S will be provided by those elements of N which have an inverse in N. In particular, the analogue of the ordinary rotations is of use.

[2.2] *A transformation R is called a rotation if*

(a) *R is in N;*

(b) *the inverse R^{-1} exists;*

(c) *R^{-1} is in N;*

(d) *R has a fixed point p.*

We shall also say that R is a rotation "about p" or "with center p"; the fixed point will often be indicated by the subscript p: $R_p(p)=p$.

The point p, unless stated otherwise, may be considered fixed in advance. In particular, it will be fixed for the following definitions of "rotatory," "invariant," "circumference."

(2.3) *The rotations about p form a group.*

That means that R_1R_2 is a rotation, $(R_1R_2)R_3=R_1(R_2R_3)$, I is a rotation, and R_1^{-1} is a rotation satisfying $R_1^{-1}R_1=R_1R_1^{-1}=I$. (The proof is omitted.)

[2.4] *If the set A contains its image $R_p(A)$ under every rotation, it is called invariant. Since R_p^{-1} is also a rotation, we might have said $R_p(A)=A$.*

[2.5] *A set A is "rotatory" if for any two points q, r in A, there exists a rotation R_p such that $R_p(q)=r$.*

[2.6] *A set which contains a point q is called a circumference L_q if it is invariant and rotatory.*

If necessary, we say "circumference through q with center p."

This definition is justified by the fact that L_q consists of all points of the form $R_p(q)$.

The definitions and assumptions set forward in these two introductory paragraphs enable us now to formulate and prove the first (topological) version of Schwarz' lemma. The rotations will hereby play a quite important role; we shall establish first some of their properties.

3. **Rotations and circumferences.** We make the following assertion:

(3.1) *Let $\{F_a\}$ be a subset of N, the index a ranging over an arbitrary set of symbols. If for one single point p the set $\{F_a(p)\}$ is limited, then for every point x the set $\{F_a(x)\}$ is limited.*

We derive this from the normality property IV in the following more general form:

(3.1.1) *If* $\{F_a(p)\}$ *is limited, then every sequence* F_{a_i} *contains a subsequence* $F_{a_{i'}}$ *which converges towards an element of* N.

Indeed, we only have to select a subsequence $F_{a_{i'}}$ which is either convergent or properly divergent. The second possibility cannot arise, since $F_{a_{i'}}(p)$ has at least one limit point. Consequently, any sequence $F_{a_i}(x)$ has at least the limit point $\lim F_{a_{i'}}(x)$. From the theorem (3.1) we shall generally use the following special case:

(3.1.2) *If* $F_i(p) = p$, *then* F_i *has a convergent subsequence* F_i'.

Two other consequences are the following:

(3.1.3) *The circumferences* L_q *are limited.*

(3.1.4) *If the transformations* F_i *are in* N, *and if the sequence* F_i *converges "pointwise" towards* F, *that is, for every* x $\lim F_i(x) = F(x)$, *then* F *is in* N, *and the convergence*

$$\lim F_i = F$$

is continuous.

The theorem (3.1.3) is obvious since L_q consists of all points $R_p(q)$, and $R_p(p) = p$. We shall afterwards show that L_q is even compact. The second statement is based on (1.4.1), and we prove it in a more general form. We do not need the generalization; it is inserted merely as the abstract background of the theorems of Stieltjes-Porter-Vitali-Blaschke.†

(3.1.5) *Let* A *be such that* $F(x) = G(x)$ *for all* x *in* A *implies that* $F = G$ *in* S, *in case* F *and* G *are in* N. *Suppose that for all* x *in* A $\lim F_i(x)$ *exists. Then* $F_i(x)$ *converges for all* x *in* S, *towards say* $F(x)$; $F(x)$ *is in* N, *and we have* $\lim F_i = F$.

Indeed, for every subsequence F_n' there exists a convergent subsequence F_n'', with the limit F'' contained in N but formally dependent on the subsequence F_n''. Yet all these possible limit functions are identical on A; consequently, they are identical throughout. That is sufficient (cf. (1.4.1)) for the relation $\lim F_i = F$.

The foregoing theorems are now applied in the case of rotations (about p).

(3.2) *Every sequence* R_n *of rotations contains a convergent subsequence; the limit mapping is in* N.

This is again a special case of (3.1.2). But we can make the following stronger statement:

(3.3) *The limit of a sequence* R_i *of rotations is again a rotation* R.

† Cf. Bieberbach, *Funktionentheorie*, vol. 2, 1st edition, p. 158.

Anticipating the result, we write lim $R_i = R$. Since $R_i(p) = p$ implies $R(p) = p$, and R is (cf. (3.2)) in N, we have only to prove that R has an inverse in N.

Consider the sequence of rotations R_i^{-1}. There will be at least one convergent subsequence $R_i'^{-1}$, lim $R_i'^{-1} = G$, where G is N. Since the limit of the corresponding sequence R_i' is R, (1.8) yields that G is the inverse R^{-1} of R.

(3.4) *If* lim $R_i = R$, *then* lim $R_i^{-1} = R^{-1}$.

Take any subsequence $R_i'^{-1}$ of the sequence R_i^{-1}. The proof of the foregoing statement shows that we can select a convergent subsequence $R_i''^{-1}$ which converges towards R^{-1}. On account of (1.4.1) this implies lim $R_i^{-1} = R^{-1}$.

These theorems may be condensed into the statement that the rotations about p, under the continuous convergence, form a compact L^*-group.

(3.5) *The circumferences L_q are compact.*

Let q_i be a sequence from L_q; then by definition $q_i = R_i(q)$. Selecting a convergent subsequence R_i' with the limit R' we see that the corresponding subsequence $q_i' = R_i'(q)$ has the limit point $R'(q)$, which is in L_q.

(3.6) *If S has more than one point, then a rotation is not nilpotent.*

4. **Topological version of the lemma.** The theorem (4.1) is, in the classical case, one of the numerous consequences of Schwarz' lemma. It expresses as far as possible the tendency of a mapping F which has a fixed point p but is not a rotation, to move the points of S "nearer" p. Why we call it a topological version of the classical lemma will be evident afterwards, when the application to the classical case is made.

(4.1) *A transformation F in N with the fixed point p is either a rotation (about p) or is nilpotent.*

The proof is made in two steps, (4.2) and (4.3). We show first that F is already nilpotent if only one subsequence F^{n_i} of the sequence F^n converges towards the constant P. If then F is not nilpotent, there must be a convergent sequence F^{m_i} with a nonconstant limit F^* in N. It is shown in (4.3) that in this case F has an inverse F^{-1} in N, which is more or less explicitly constructed as a limit of a sequence of powers of F.

All transformations occurring in this paragraph will be in N, either by assumption (as in the case of F) or because they are limits of mappings in N.

(4.2) *If $F(p) = p$ and if a sequence F^{n_i}, where $n_{i+1} > n_i$, tends towards the constant P, then F is nilpotent and* lim $F^n = P$.

It is sufficient to show that every sequence F^{m_i}, $m_{i+1} > m_i$, contains a convergent subsequence $F^{m_i'}$ with the limit P.

We select two subsequences m_i', n_i' such that

(a) $d_i = n_i' - m_i'$ is increasing, and

(b) the sequence F^{d_i} is convergent with $\lim F^{d_i} = F'$.

Such a sequence exists; the first condition can be fulfilled because n_i and m_i are strictly increasing; the second condition, because $F(p) = p$, $F^n(p) = p$.

The relations $\lim F^{n_i'} = \lim F^{n_i} = P$, $\lim F^{d_i} = F'$ imply (cf. (1.7)) that $\lim F^{m'} = \lim F^{n_i'} F^{d_i} = PF' = P$.

We note that while the normality has been used freely, the cancellation property has not yet appeared in the proofs.

(4.3) *If $F(p) = p$ and if a convergent sequence F^{n_i} with the nonconstant limit F^* exists, then F has an inverse in N.*

The proof is somewhat similar to the preceding one, but F' is defined slightly differently and the cancellation property III is essential.

We select again a subsequence n_i' such that

(a) $d_i = n_{i+1}' - n_i' - 1$ is strictly increasing, and

(b) F^{d_i} is convergent with $\lim F^{d_i} = F'$.

From these assumptions we derive the equality $F^* = F^* F' F$, for

$$\lim F^{n_i} = \lim F^{n_{i+1}'} = \lim F^{n_i'} F^{d_i} F = (\lim F^{n_{i+1}'})(\lim F^{d_i}) F.$$

Writing this in the form $F^* I = F^*(F'F)$ and using the fact that F^* is not a constant, we obtain, by virtue of the cancellation law, $F'F = I$.

In F' we have, therefore, the inverse of F, the existence of which was asserted in our theorem.

The principal theorem follows now as indicated before. If $F(p) = p$, then $F^n(p) = p$ shows that a convergent sequence F^{n_i} exists. If the limit is constant, then (4.2) implies that F is nilpotent; if it is not, then (4.3) shows that F has an inverse and consequently is a rotation (about p).

As an interesting corollary we obtain the following:

(4.4) *A transformation with two different fixed points p and q is a rotation.*

In the classical case one knows more: the rotation F is the identity. This generalization suggests itself as an additional axiom, which (see the end of the paper) permits a more precise description of S and N. In view of the theory of Kerékjártó we call attention to the fact that instead of N we could have studied the subsystem formed by powers of F and their limits.

Part II

5. **New restrictions on S and N.** From now on we shall use more freely the topological terminology, indicated by the words open, neighborhood, closed; closure $\overline{A}$, boundary $\mathrm{Bd}(A)$ of a set A; connected, component, locally con-

nected; separate, cut point; semicompact, locally compact, (perfectly) separable, and metrizable.

The set AB is the common part and $A+B$ the union of the two sets, and $S-A$ is the complement of A in S.

[5.1] *S is now a metrizable space with the following additional properties:*

I. *S is connected and contains more than one point.*

II. *S is locally connected.*

III. *S has no cut points; that is, for all points x the set $S-\{x\}$ is connected.*

In Part II we shall use III generally for $x=p$, where p is arbitrary but fixed.

Before we set down the restrictions on N we define the geometrical concepts "circle" and "closed circle."

[5.2] *The component of $S-L_q$ which contains p is called the circle with center p determined by q and is denoted by C_q.*

In other words, the circle is the largest connected subset of the complement of the circumference L_q which contains p. If (and only if) q is identical with p, then C_q is empty.

We note that this describes the interior of the circular area determined by a circular curve in euclidean geometry, which has p as center and q on the curve.

[5.2.1] *The closure $\overline{C}_q$ of C_q, comprising C_q and all its limit points, is called a closed circle.*

The restrictions on N are now phrased as properties of circles and circumferences.

[5.3] *N is from now on a family of transformations which has not only the properties* I–IV *of* [2.1] *but the following:*

V. *If $q\neq p$, then L_q separates S; that is, $S-L_q$ is not connected.*

VI. *The space S is not representable as a finite sum of circles (with possibly different centers).*†

These two axioms constitute very heavy restrictions on N, but in exchange we obtain a quite rich geometry (topologically speaking) for S.

6. **Circles and circumferences.** We can make the following assertion:

(6.1) *The circles C_q are open and connected.*

† This property was not contained in the before mentioned note; my proof for the central theorem, loc. cit. (II, 5), contained a mistake which was pointed out to me by Mr. Hall and which I was not able to correct without a new assumption. The particular form of VI has been chosen since it is also useful for the justification theorem (§8).

If $q=p$, then $L_q=\{p\}$, and $S-L_q$ does not contain p. In this case we have to interpret C_q as the empty set, which may be considered open and connected.

If $q\neq p$, then p is in $S-L_q$, since $R(q)=p$ implies $q=R^{-1}(p)=p$.

L_q is closed (even compact); its complement $S-L_q$ is consequently open. Now S is locally connected; that is, in every neighborhood U_x (open set containing x) there exists a neighborhood V_x which is connected.

It follows that a component (largest connected subset) of any open set in a locally connected space is open; C_q is such a component, hence it is open; it is connected by definition. If it is not empty, it contains p.

(6.2) *The circles C_q and their boundaries* $\mathrm{Bd}(C_q)$ *are invariant.*

This is a consequence of the following group of statements:

(6.2.1) $R(A+B)=R(A)+R(B)$; $R(AB)=R(A)R(B)$; $R(S-A)=S-R(A)$.

This holds for subsets A, B of S and for any (1-1) mapping of S on itself, in particular, for a rotation.

(6.2.2) $R(\overline{A})=\overline{R(A)}$.

This holds at least for topological mappings (where R and R^{-1} are continuous).

(6.2.3) $R(\mathrm{Bd}(A))=\mathrm{Bd}(R(A))$.

The boundary, as the set of all points which are limit points of sequences from A but not in A, can be written as

$$\mathrm{Bd}(A) = \overline{A} - A.$$

(6.2.3) follows algebraically from this definition and the preceding identities.

(6.2.4) *Any function of invariant sets A, B, C which is composed from sums, products, complements, and closures is invariant.*

For example the "frontier $\mathrm{Fr}(A)$ of A" is equal to $R(\mathrm{Fr}(A))$ because by definition

$$\mathrm{Fr}(A) = \overline{A}(\overline{S - A}), \qquad R(\overline{A}\cdot\overline{S - A}) = R(\overline{A})R(\overline{S - A}),$$

$$R(\overline{A}) = \overline{R(A)} = \overline{A}, \qquad R(\overline{S - A}) = \overline{R(S - A)} = \overline{R(S) - R(A)} = \overline{S - A}.$$

Also

$$R(\mathrm{Bd}(A)) = R(\overline{A} - A) = R(\overline{A}) - R(A) = \overline{R(A)} - R(A) = \mathrm{Bd}(R(A)).$$

In order to derive (6.2) we have only to go back to the definition of C_q. The set L_q is invariant; hence $S-L_q$ is invariant; a rotation R maps $S-L_q$

topologically on itself, a connected subset on a connected subset, a largest connected subset on a (possibly different) subset of the same character, and, since $R(p)=p$, the component C_q of p on itself.

The boundary $\mathrm{Bd}(C_q)$ is invariant as a function of an invariant set; this invariance we use now for the determination of $\mathrm{Bd}(C_q)$.

(6.3) *The boundary* $\mathrm{Bd}(C_q)$ *is exactly* L_q, *if* $q \neq p$; *if* $q=p$ *it is, of course, empty.*

If $q=p$, then $C_q=0$; hence $\mathrm{Fr}(C_q)=\mathrm{Fr}(0)=0$. Hence we assume $C_q \neq 0$. Since q is in L_q, q is not in C_q and C_q is not equal to S.

The set C_q could not be closed, for an open and closed set in a connected space S is either 0 or S. Hence there is a point which is limit point for C_q but not in C_q; let r be such a boundary point. The point r cannot be in $S-L_q$, for C_q is a component of $S-L_q$; hence it contains all its limit points in $S-L_q$, and it is "relatively closed" with respect to $S-L_q$. The point r, that is, any boundary point of C_q, is therefore in L_q.

The boundary is not only a non-empty subset of L_q, it is also invariant. Since 0 and L_q are the only invariant subsets of L_q, the boundary C_q is exactly L_q.

The connectedness of S and C will be used so often in the proofs to come that we deem it advisable to insert the following theorem:

(6.3.1) *The connectedness of a space is equivalent to the following implications*:

(a) *If a set* A *is open and closed, it is either* 0 *or the whole space.*

(b) *If an open set* A *has no boundary, then it is* 0 *or the whole space.*

(c) *If one knows, for an open set* A, *that* $\mathrm{Bd}(A) \subset A$, *then* A *is* 0 *or the whole space.*

(d) *The space is not the sum of two disjoint open proper subsets.*

These statements are trivial consequences of the following definition:

[6.3.2] *A space* S *is connected if* $A+B=S$ *and* $\overline{A}\overline{B}=0$ *imply that either* $\overline{A}$ *or* $\overline{B}$ *is empty.*

Since $\mathrm{Bd}(A)=\overline{A}-A$, we get from (6.3) the corollary:

(6.3.3) $\overline{C}_q=C_q+L_q$, *if* $p\neq q$.

For $p=q$ this is not true since $\overline{C}_q=0$; but $\overline{C}_q \subset C_q+L_q$ is always true.

(6.3.4) L_q *is also, for* $q\neq p$, *equal to the frontier* $\mathrm{Fr}(C_q)$.

We show that every point of L_q is a limit point of $S-\overline{C}_q$. Since $S-\overline{C}_q$ is invariant, we need this for one single point r of L_q.

We know that C_q is an open and closed set with respect to $S-L_q$; its complement in $S-L_q$ is exactly $(S-L_q)-C_q=S-(L_q+C_q)=S-\overline{C}_q$; such a complement is also open and closed in $S-L_q$. Therefore $S-\overline{C}_q$ has no limit points in C_q. It is not empty since, because of property V, $S-L_q$ is not connected whereas C_q is connected.

In S itself $S-\overline{C}_q$ is open; it could not be closed because it is neither empty nor equal to S. There must be a boundary point r, and this point is necessarily on L_q.

We shall now have to derive a series of relations between different circles and circumferences; it will be convenient to write L_1, L_2, L_i, C_1, C_2, C_i instead of L_{q_1}, C_{q_i}, and so on; it is always understood that C_i is the circle determined by L_i.

(6.4) *The product* L_1C_2 *is either empty or* L_1.

For L_1C_2, as a product of invariant sets, is invariant; 0 and L_1 are the only invariant subsets of L_1.

(6.4.1) $L_x \subset C_y$ *and* $x \in C_y$ *are equivalent.*

A non-trivial statement is the following:

(6.5) $L_1C_2=0$ *implies* $C_2 \subset C_1$.

We shall derive this by showing that the product C_1C_2 is equal to C_2. If C_2 is empty, then $C_2 \subset C_1$ is trivially true. If not, we shall see that C_1C_2 is a non-vanishing open and relatively closed subset of C_2; $C_1C_2=C_2$ follows because C_2, as a circle, is connected.

To this purpose we determine the relative boundary of C_1C_2 in C_2, that is, the set of all limit points of C_1C_2 which are in C_2 but not in C_1C_2; in other terms, the product $C_2\mathrm{Bd}(C_1C_2)$. Here and later we shall often use the following formulas:

(6.5.1) $\mathrm{Bd}(A+B) \subset \mathrm{Bd}(A)+\mathrm{Bd}(B)$; $\mathrm{Bd}(AB) \subset \mathrm{Bd}(A)+\mathrm{Bd}(B)$.

Now we have $\mathrm{Bd}(C_1C_2) \subset \mathrm{Bd}(C_1)+\mathrm{Bd}(C_2) \subset L_1+L_2$; consequently,

$$C_2\mathrm{Bd}(C_1C_2) \subset C_2L_1 + C_2L_2.$$

The set C_2L_2 is always empty, $C_2 \subset S-L_2$; C_2L_1 is empty by assumption. Hence $C_2\mathrm{Bd}(C_1C_2)=0$. Since C_1C_2, absolutely open, as a product of open sets in S, is a fortiori relatively open in C_2, it is either empty or equal to C_2. How could C_1C_2 be empty? Only if one of the factors is empty, for otherwise both will contain the point p. The case that C_2 is empty has been disposed of; if C_1 were empty, $L_1=\{p\}$ would imply $L_1C_2=L_1\neq 0$, contrary to our assumption.

Property VI has not been used yet.

(6.6) $L_1C_2=L_1$ *implies* $C_1\subset C_2$.

Considering (6.5) we see that it suffices to prove $L_2C_1=0$. The proof is indirect and based on property VI.

Suppose that $L_2C_1\neq 0$; then it is equal to L_2 and $L_2\subset C_1$.

Now consider the (open) set C_1+C_2 and in particular, its boundary $\text{Bd}(C_1+C_2)$. The relation

$$\text{Bd}(C_1+C_2)\subset \text{Bd}(C_1)+\text{Bd}(C_2)\subset L_1+L_2$$

implies together with

$$L_1\subset C_2,\qquad L_2\subset C_1,\qquad L_1+L_2\subset C_1+C_2$$

the fact that the open set C_1+C_2 contains its boundary. Hence it is equal to S or to 0. Since L_1 is in C_2, C_2, and a fortiori C_1+C_2, are not empty, and in this way we have derived from the assumption $L_2C_1\neq 0$ that the space S is a sum of a finite number of circles $S=C_1+C_2$. That is excluded by property VI; hence $L_2C_1\neq 0$ is wrong, $L_2C_1=0$ is true, and that implies $C_1\subset C_2$, as we know from the preceding theorem.

As an immediate formal consequence of (6.4), (6.5), and (6.6) we obtain the next theorem:

(6.7) *If C_1 and C_2 are two circles (as always with center p), then at least one of the inclusions $C_1\subset C_2$, $C_2\subset C_1$ is true.*

The next theorem states the equivalence of several other inclusion relations, which we have to use later on:

(6.8) *The following properties are equivalent*:
(a) $L_x\subset C_y$ (*we know that this is equivalent to* $x\in C_y$).
(b) $\overline{C}_x\subset C_y$, *and* C_y *is not empty*.
(c) $L_y\subset S-\overline{C}_x$, *and if* $x=p$ *then* $y\neq x$.
(d) $C_y\not\subset C_x$.

We show that every one of these relations implies the succeeding one and that the last implies the first.

(a) *implies* (b). $L_x\subset C_y$ shows that C_y is not empty. From (6.6) we get $C_x\subset C_y$; consequently, $\overline{C}_x=C_x+\text{Bd}(C_x)\subset C_x+L_x\subset C_y+C_y=C_y$.

(b) *implies* (c). C_y is not empty; hence if $x=p$, y is not equal to x, for C_q is empty. In both cases the set $L_y\overline{C}_x$ is invariant, and hence either 0 or L_y. If it is L_y, then $L_y\subset\overline{C}_x$, $\overline{C}_x\subset C_y$ would yield the contradiction $L_y\subset C_y$. Hence $L_y\overline{C}_x=0$ or $L_y\subset S-\overline{C}_x$.

(c) *implies* (d). In view of (6.7) let us show that $C_y\subset C_x$ is impossible. Indeed, if $x=p$, C_x is empty and $C_y\subset C_x$ would make C_y empty, whereas y is not x. If $x\neq p$ and $y\neq p$, then $L_y\subset S-\overline{C}_x$, $L_y\subset \overline{C}_y\subset \overline{C}_x$ constitutes a contradiction. If $x\neq p$ and if $y=p$, $L_y\subset C_x$ would contradict the assumption $L_y\subset S-\overline{C}_x$.

(d) *implies* (a). From $C_y\not\subset C_x$ we infer that $C_x\subset C_y$, but not $C_x=C_y$, also that C_y is not empty, $\overline{C}_y\supset L_y$. Consequently,

$$\overline{C}_x\subset\overline{C}_y,\qquad L_x\subset\overline{C}_y=C_y+L_y.$$

Hence we get for L_x

$$L_x=L_xC_y+L_xL_y.$$

The set L_xL_y must be empty; for in the opposite case $L_x=L_y$, $C_x=C_y$, $C_y\subset C_x$ would ensue. It follows that $L_x=L_xC_y$, which is (a).

Abstract absolute values, symbols of the form $|x|$, where x is a point in S, and the number 0 are now introduced by the following definition:

[6.9] $|x|<|y|$ or $|y|>|x|$ *shall mean* $L_x\subset C_y$; $|x|=|y|$ *shall mean* $L_x=L_y$; $|x|=0$ *shall mean* $x=p$; $|x|>0$ *shall mean* $x\neq p$; $|x|\geqq|y|$ *shall mean* $|x|>|y|$ *or* $|x|=|y|$.

(6.10.1) *For any two points x, y exactly one of the relations $|x|<|y|$, $|x|=|y|$, $|x|>|y|$ is true.*

Suppose that neither $|x|<|y|$ nor $|x|>|y|$ is true; in other terms, neither $L_x\subset C_y$ nor $L_y\subset C_x$ is true. On account of (6.4) we have then $L_xC_y=L_yC_x=0$; from (6.5) we conclude $C_x\subset C_y$ and $C_y\subset C_x$; hence $C_x=C_y$, $L_x=L_y$, or $|x|=|y|$, which was to be shown.

(6.10.2) $|x|<|y|$, $|y|<|z|$ *imply* $|x|<|z|$.

We know $L_x\subset C_y$ and (cf. (6.8)) $\overline{C}_y\subset C_z$; we have a fortiori $C_y\subset C_z$; hence $L_x\subset C_z$ or $|x|<|z|$ by definition.

(6.11) $\lim x_i=p$ *is true if and only if for every $|e|>0$ an index i^* can be found such that for $i>i^*$, $|x_i|<|e|$.*

For the set of all points x with $|x|<|e|$ is the circle C_e, which is, because of the relation $|e|>0$, a neighborhood of p, and must contain almost all points of any sequence which converges towards p.

7. **Geometrical version of the lemma.** The following statement is of use:

(7.1) *If $|x|<|y|$, then a z exists which satisfies $|x|<|z|<|y|$.*

The relation $|x|<|y|$ implies, as we know, $\overline{C}_x\subset C_y$, and C_y is not empty. We maintain that $C_y-\overline{C}_x$ is not empty; for otherwise the open set C_y, neither empty nor S, would be equal to the closed set $\overline{C}_x$, which is impossible.

It is also impossible that $C_y-\overline{C}_x$ is equal to the one-point set $\{p\}$, for $\{p\}$ is closed and a difference "open minus closed" is open. Since $x\neq y$, the set $\{p\}$ is not equal to S. Hence we see that $C_y-\overline{C}_x$ is not only not empty but contains a point z which is not p. Any such z will do in (7.1) because $z \in C_y$ gives $L_z\subset C_y$ and $|z|<|y|$. On the other hand, z is in $S-\overline{C}_x$, hence $L_z\subset S-\overline{C}_x$; and if $x=p$, then $z\neq x$, for we took $z\neq p$; (6.8c) reveals this as as an equivalent of $|x|<|z|$.

(7.1.1) *For every x there exists a y with $|y|>|x|$, if S has more than one point.*

If $|x|=0$, take $y\neq p$; if $|x|>|p|$, take any point from $S-\overline{C}_x$, which is not empty since $S-L_x$ is not connected.

(7.2) *If* $\lim x_i=x$, $\lim y_i=y$, $|x|<|y|$, *then there exists an index i^* such that for $i>i^*$, $|x_i|<|y_i|$.*

Choose a z exactly as before; then $|x|<|z|$ yields $x \in C_z$, and $|z|<|y|$ implies $y \in S-\overline{C}_z$. The sets C_z and $S-\overline{C}_z$ are open; consequently, there exists an index i^* such that for $i>i^*$

$$x_i \in C_z, \qquad y_i \in S-\overline{C}_z.$$

The first formula is equivalent to $|x_i|<|z|$, the second to $|z|<|y|$ since $z\neq p$. The transitive law (6.10.2) furnishes $|x_i|<|y_i|$, which was to be proved.

We may state the following corollary:

(7.2.1) *If the sequences x_i, y_i are convergent, $|x_i|=|y_i|$ for all i implies* $|\lim x_i|=|\lim y_i|$.

[7.3] *If F is a (single-valued) mapping of S in itself, then $S=S_1+S_2+S_3$, where S_1, S_2, S_3 in this order are defined by the relations $|F(x)|<|x|$, $|F(x)|=|x|$, $|F(x)|>|x|$.*

The geometric version of Schwarz' lemma is a statement about the S_i of a transformation F in N with $F(p)=p$. We derive first, with the aid of (7.2), a simple statement for continuous transformations.

(7.4) *If F in* [7.3] *is continuous, then S_1 and S_3 are open sets.*

We prove that S_1 is open; the proof for S_3 is virtually the same.

For a point x in S_1 we have, by definition, $|F(x)|<|x|$. Let $\lim x_i=x$; then we have to show that for almost all indices i, $|F(x_i)|<|x_i|$. This follows from (7.2) if we define $y=F(x)$, $y_i=F(x_i)$, and use the relation (continuity) $\lim F(x_i)=F(x)$.

Again we note without proof that S_2 is closed.

(7.5) *If F is a continuous mapping of S in itself and if neither S_1 nor S_3 is empty, then there exist at least two points p, q with $p \neq q$ in S_2.*

This is a well known theorem about continuous functions coupled with the fact that S has no cut points. If S_2 were empty, $S = S_1 + S_3$ would be a non-trivial decomposition of S into two disjoint open sets, which does not exist in a connected space. If $S_2 = \{p\}$, then $S_1 + S_3$ would be a non-trivial decomposition of $S - \{p\}$ into open sets, and p would be a cut point of S.

(7.6) *Let F be in N, $F(p) = p$, such that S_2 contains p. If now S_2 contains another point q, $q \neq p$, $|F(q)| = |q|$, then F is a rotation.*

For a rotation $|F(x)| = |x|$ is identically true and S_1 and S_3 are both empty.

Proof. Since $F(q)$ and q are in the same circumference, there exists a rotation R such that $R(F(q)) = q$. What do we know about the transformation RF? The relations $RF(p) = R(p) = p$, $RF(q) = q$ show that RF, which is in N, has two different fixed points. The corollary (4.4) tells us that RF is a rotation R_1. From $RF = R_1$ we get $F = R^{-1}R_1$, which is a rotation since it is the product of two rotations.

(7.7) *If F is in N and $F(p) = p$, then one of the sets S_1 and S_3 is empty.*

Indeed, if none were empty there would exist two different points in S_2 (cf. statement (7.5)), and F would have to be a rotation; S_1 and S_3 would be empty.

(7.8) *If F is in N, $F(p) = p$, then S_3 is empty. In other words, $|F(x)| \leqq |x|$ for all x.*

This is the geometrical version of the Schwarz lemma.

Proof. If S_3 is not empty, then S_1 is empty on account of (7.7). The set S_2 is not empty, for it contains the point p. It does not contain any others, for in that case F would be a rotation and S_3 would be empty (as well as S_1). Therefore the inequality $|F(x)| > |x|$ would hold whenever $|x| \neq 0$. But this contradicts the topological alternative (cf. (4.1)) that F is either a rotation or nilpotent. Indeed, F is not a rotation, and $|F(x)| > |x|$ for all $|x| > 0$ is incompatible with nilpotency. For $|x| > 0$ implies (by mathematical induction) $|F^n(x)| \neq 0$ and $|F^n(x)| \geqq |x|$; and this would show that the Cauchy condition (6.11) for $\lim F^n(x) = p$ cannot be fulfilled with $|e| = |x|$. If our theorem were wrong, we should have a contradiction; therefore, assertion (7.8) is true.

Combining (7.6) and (7.8), we formulate the final geometrical theorem, (7.9), which corresponds to the classical Schwarz lemma together with its standard corollary.

(7.9) *If F is in N, $F(p)=p$, then for all x in S we have $|F(x)| \leqq |x|$. If equality holds for one point distinct from p, then it holds throughout. In the latter case F has an inverse which is an element of N.*

The classical lemma would be a consequence provided we know that the abstract relation $|x| < |y|$ is equivalent to the analytically defined inequality $|x| < |y|$. In §8 we shall prove a theorem to the effect that the analyticity of linear homogeneous functions together with simple topological properties of the euclidean circles make the abstract and analytical order relations equivalent.

PART III

8. **Characterization of circumferences.** Our definitions of absolute value relations are such that if S is the unit circle in the plane of the complex numbers, p the origin, L_q the circular curve through q with center p, and C_q the interior of the corresponding circular area, then $|x| < |y|$ is equivalent to saying that the classical absolute value of x is less than the classical absolute value of y.

But we wish to know if the euclidean circumferences are circumferences in the sense of our definition. Of course it is well known that in the classical case an abstract rotation is an ordinary rotation; but this is usually shown as an application of the Schwarz lemma, or at least derived in an analogous fashion.

Let us therefore denote a euclidean circumference with K and the corresponding circle with E; and let us discuss the case where K contains a point z but not the point p.

If we use the analyticity of linear homogeneous transformations, we see immediately that, N being the set of all analytical mappings of S in itself, K is rotatory; that is, that there exists a topological mapping in N which carries p into itself and a preassigned x on K into an arbitrary y on K. Applying some elementary topology of the euclidean plane, we can make the following assertion:

(8.1) (a) *$K \subset S$ is not empty; it contains a point z but not the point p.*
(b) *$S-K$ is not connected; the component of $S-K$ which contains p is E.*
(c) *K is the boundary* $\mathrm{Bd}(E)$ *of E.*
(d) *K is rotatory.*
(e) *$\overline{E}=E+K$ is compact.*

We maintain that from these statements and the properties I–VI of N and I–III of S it follows that K is a circumference. (The case $K=\{p\}$ is trivial, since $L_p=p$.)

Let us forget the euclidean origin of (8.1) and make the following definition:

[8.2] "*(K, E, z) is circular*" *shall mean that the sets $K \subset S$, $E \subset S$, and the point z satisfy the relations* (8.1).

The "justification theorem" in question is now simply the following:

(8.3) *Let N and S be as in Part* II. *If (K, E, z) is circular, then K is a circumference and E a circle; in short $K = L_z$, $E = C_z$.*

Due to the definition (in (8.1a)) of E and [5.2] of C_z it is sufficient to show $K = L_z$.

The proof is arranged backwards:

(8.4) *If (K, E, z) is circular and if no point of the circumference L_z is in E, then $K = L_z$.*

Consider the set C_zE; this set, the product of two open sets, is open. (The set E is open since K, being a boundary, is closed.) The set C_zE is not empty because p is in C_z and in E.

We study, as we always did in questions of this type, the relative boundary of C_zE, this time with respect to both C_z and E.

Note that K is a subset of L_z, for it contains z and is rotatory. We get $\text{Bd}(C_zE) \subset \text{Bd}(C_z) + \text{Bd}(E) \subset L_z + K \subset L_z$. Of course we cannot conclude directly that equality holds, for we do not know yet that K is invariant. But at least we can say that the relative boundaries $C_z\text{Bd}(C_zE)$ and $E\text{Bd}(C_zE)$ are empty. That $C_zL_z = 0$ follows from the definition of C_z; whereas $L_zE = 0$ is an assumption of our theorem. Hence the relative boundaries of C_zE with respect to the (connected) sets C_z and E are empty as subsets of C_zL_z and EL_z, respectively. Since C_zE is not empty, we obtain $C_zE = C_z$, $C_zE = E$; hence $C_z = E$. Taking boundaries on both sides, we have $L_z = K$, which was to be proved.

(8.5) *If (K, E, z) is circular and if S is not compact, then L_z has no point in common with E.*

The proof is indirect: If q is in L_zE, then $L_q = L_z$, and to every point x in $L_q = L_z$ there will exist a rotation $R^{(x)}$ such that $R^{(x)}(q) = x$. The open set E is transformed into open sets $R^{(x)}(E)$, and $q \in E$ implies $R^{(x)}(q) = x \in R^{(x)}(E)$. In other terms,

$$L_z \subset \sum_{x \in L_z} R^{(x)}(E).$$

Now we have to use, for the first time, the metrizability of the space S. Since

L_z is a compact subset (cf. (3.5)) of a metrizable space, the Heine-Pincherle-Borel-Lebesgue† theorem is valid, and already a finite number of sets $R^{(x)}(E)$, say $R_1(E), \cdots, R_n(E)$ covers L_z; that is,

$$L_z \subset \sum_1^n R_i(E).$$

We set $S' = \sum_1^n R_i(E)$ and propose to show that $S' = S$. This is again done with the standard device based on the connectedness of S.

The set S' is open as a sum of open sets; it is not empty because it contains L_z. What is its boundary? We obtain

$$\mathrm{Bd}(S') = \mathrm{Bd}\left(\sum_1^n R_i(E)\right) \subset \sum_1^n \mathrm{Bd}(R_i(E)) = \sum_1^n R_i(\mathrm{Bd}(E)) \subset \sum_1^n R_i(L_z) \subset L_z.$$

(We have applied (6.6.1), (6.2.4), $K = \mathrm{Bd}(E)$, $K \subset L_z$, and $R_i(L_z) \subset L_z$.) Isolating the first and the last terms, we have $\mathrm{Bd}(S') \subset L_z$; and since $L_z \subset S'$ we see that the open, non-empty set S' contains its boundary; S is connected, hence (cf. (6.3.1)) $S' = S$.

From $S = \sum_1^n R_i(E)$ we obtain a fortiori $S = \sum_1^n R_i(\overline{E})$. Since (K, E, z) is circular, $\overline{E}$ and its topological images $R_i(\overline{E})$ are compact; the sum of a finite number of compact sets is compact; hence S is compact, which contradicts the assumption of the theorem. Hence we have seen, indirectly, that if S is compact, $L_zE = 0$, which was to be shown.

Finally, we remove, in (8.6), the last condition.

(8.6) *S is not compact.*

For if it were, it would have to be bicompact, being metrizable. Consider the covering which is defined by assigning to p the open set C_q and to every other point x the circle with center x determined by p. If S were bicompact, a finite number of these circles would have the sum S, which is excluded by property VI.

With (8.6) the proof of the justification theorem (8.3) is completed.

9. **Separability and local compactness of S.** If we use the foregoing theory for variable centers p, we see that every point is contained in arbitrarily small neighborhoods with compact, metrizable and hence separable boundaries. From a theorem of F. B. Jones‡ we could infer the next theorem:

† A space S is called bicompact or the Heine-Pincherle-Borel-Lebesgue theorem holds in S if from every covering of S by open sets a finite set of elements (open sets) can be extracted which has S as its sum.

‡ F. B. Jones, *A theorem concerning locally peripherally separable spaces*, Bulletin of the American Mathematical Society, vol. 41 (1936), p. 437.

(9.1) *The space S is (perfectly) separable.*

This result will also appear as a corollary of the theorem (9.8). Independently from (9.1) we are going to show that the closed circles $\overline{C}_q$ are compact, and that the space S is representable as the sum of a countable number of circles.

(9.2) *Let x_i be a sequence of points such that a point x, a subsequence x_i' and a sequence of rotations R_i can be found with* $\lim R_i(x_i') = x$. *Then there exists also a limit point for the sequence x_i.*

We select corresponding subsequences R_i', x_i'' such that $\lim R_i' = R$ exists; we know then that $\lim R_i'^{-1} = R^{-1}$, and from $\lim R_i'(x_i'') = x$ it follows that $x_i'' = R_i'(R_i'^{-1}(x_i''))$ is convergent (with the limit $R^{-1}(x)$).

(9.3) *Suppose that the sequence x_i is such that sequences x_i', R_i, as described in* (9.2), *do not exist. Then for every point y in S there exists a neighborhood U_y and an index i^* such that for $i > i^*$, U_y is completely in C_{x_i} or completely in the exterior of C_{x_i}.*

In other terms, $i > i^*$ implies that either $U_y \subset C_i$ or $U_y \subset S - \overline{C}_{x_i}$.

As before, we shall write C_i for C_{x_i}, L_i for L_{x_i}.

We first choose a neighborhood V_y and an index i^* such that for all indices $i > i^*$ $L_i V_y = 0$, and in addition for $i > i^*$, $C_i \neq 0$. Such a V_y exists; for L_i consists exactly of the points $R(x_i)$, where R is arbitrary. With the first countability axiom of Hausdorff (a trivial consequence of the metrizability of S) a sequence $R_i(x_i')$ with limit y could be constructed. If $C_i' = 0$ for a subsequence C_i', then $x_i' = p$, $R_i = I$ yields $\lim R_i(x_i') = p$.

Since S is locally connected, V_y contains a connected neighborhood U_y of y. Now consider the formula

$$U_y = U_y \overline{C}_i + U_y(S - \overline{C}_i);$$

if $i > i^*$, then $C_i \neq 0$ and $\overline{C}_i = C_i + L_i$; hence

$$U_y = U_y C_i + U_i L_i + U_y(S - \overline{C}_i).$$

For $i > i^*$, $U_i L_i$ is 0; the resulting equation

$$U_y = U_y C_i + U_y(S - \overline{C}_i)$$

is a decomposition of U_y into two disjoint open sets. One of these must be empty, since U_y is connected and not empty; but that means that either $U_y \subset C_i$ or $U_y \subset S - \overline{C}_i$, which was to be shown.

(9.4) *Let the sequence x_i be such that to every y in S there belongs a neighborhood U_y and an index i^* such that for $i > i^*$ either $U_y \subset C_i$ or $U_y \subset S - \overline{C}_i$ is true. Then the set $S' = \sum C_i$ (which is trivially open) is closed.*

In order to prove this we show that if y_i is a convergent sequence from S', its limit y is also an element of S'.

Without loss of generality we may assume $y_i \varepsilon C_i$; this corresponds to the deletion of some C's and introduction of a new index, which does not affect the validity of our theorem.

Now let i^* be such that for $i>i^*$ (a) $y_i \varepsilon U_y$ and (b) either $U_y \subset C_i$ or $U_y \subset S-\overline{C}_i$.

(a) may be satisfied since $\lim y_i=y$; (b) has been explicitly assumed. Since for $i>i^*$, $y_i \varepsilon U_y$, $y_i \varepsilon C_i$, we see that $y_i \varepsilon U_yC_i$; that decides the alternative (b) in favor of $U_y \subset C_i$; but $U_y \subset C_i$ (any special case such as $i=i^*+1$ will do) implies $y \varepsilon C_i$ and a fortiori $y \varepsilon \sum C_i = S'$.

(9.5) *If a sequence x_i has no limit point, then $S=\sum C_{x_i}$ $(=\sum C_i)$.*

Since $x_i=p$ can be true only a finite number of times, almost all C_i are not empty; a fortiori the open set $S'=\sum C_i$ is not empty.

(9.2), (9.3), (9.4) together guarantee that S' is closed; the connectedness argument yields $S'=S$.

(9.6) *The circles C_q are limited, and their closures $\overline{C}_q$ compact.*

The proof is indirect. Let x_i be a sequence from C_q. If it had no limit point, we would have $S=\sum C_{xi}$. On the other hand, $x_i \varepsilon C_q$ implies $C_{xi} \subset C_q$, (cf. (6.6)) since $C_q \neq 0$. That would lead to the contradiction $S \subset C_q$, since q is not in C_q. Hence every sequence x_i from C_q must have a limit point, which was to be proved.

We could express and slightly generalize this in the following familiar form:

(9.6.1) *If all points x of a set satisfy $|x| \leqq |q|$, then every infinite sequence x_i has a limit point.*

As a consequence we have the statement:

(9.7) *S is locally compact.*

For if x is an arbitrary point, there exists a point y such that $|x| < |y|$, $x \varepsilon C_y$. Hence every point is contained in a limited open set.

(9.8) *S is semicompact; that is, it is the sum of a sequence of compact sets $\overline{C}_i$. (They will be closed circles.)*

If S were compact (which is excluded by (8.6)), then it would be trivially semicompact. If it is not, then there exists a sequence x_i without limit points. In that case we have (cf. (9.5)) $\sum C_{x_i}=S$ and a fortiori $\sum \overline{C}_i = S$; and the $\overline{C}_i$ are now known to be compact.

From the theorem (9.8) (all theorems in this section are proved without recourse to (9.1)) we get (9.1) as a trivial consequence, using metrizability.

Conclusion. It would be possible to obtain valuable new properties of S and N by adjunction of new postulates. We could demand that the circumferences be connected; this would permit us to conclude that for every pair x, y a center p and a rotation R exist such that $R_p(x) = y$. If we postulate that a transformation with two fixed points is the identity, L_q would be homeomorphic to a connected compact continuous group. These groups are rather well known, and together with the fact that the abstract absolute values can be interpreted as real numbers, this additional axiom would heavily restrict the structure of the space S. If, finally, S is supposed to be homeomorphic to the euclidean plane, the application of a theorem of Hilbert would show that the invertible transformations in N induce an absolute, that is, either euclidean or hyperbolic, geometry in S. The decision as to whether these axioms together with a maximality axiom are categoric will largely depend, we believe, on the better understanding and proper generalization of the Schwarz lemma.

University of California at Los Angeles,
Los Angeles, Calif.