ON INTERPOLATION BY FUNCTIONS ANALYTIC AND BOUNDED IN A GIVEN REGION*

BY

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The writer has recently formulated† the following problem, but without proving in detail any results on convergence of the sequences involved:

PROBLEM A. *Let the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$, not necessarily distinct, lie interior to the region R of the plane of the complex variable z. Let the function $f(z)$ be analytic in each point β_{nk}. Let $f_n(z)$ be the (or a) function which coincides with $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$, which is analytic in R, and the least upper bound M_n of whose modulus in R is a minimum. To study the functions $f_n(z)$, especially the approach to $f(z)$ of the sequence $f_n(z)$, and study the sequence M_n as n becomes infinite.*

A function $f_n(z)$ always exists (loc. cit.), and is unique if R is simply-connected.

It is the object of the present note to establish some results concerning Problem A, especially

THEOREM 1. *Let R be the interior of a Jordan curve C_1. Let each of the points β_{nk} lie on or interior to a Jordan curve C_2 interior to C_1, and let us suppose the relation*

$$(1)\qquad \lim_{n\to\infty} \left| (z-\beta_{n1})(z-\beta_{n2})\cdots(z-\beta_{nn}) \right|^{1/n} = e^{V_1(x,y)}, \qquad z = x+iy,$$

to hold at every point z exterior to C_2, uniformly on any closed bounded set exterior to C_2. Let $V_2(x, y)$ denote the function which coincides with $V_1(x, y)$ on C_1 and is harmonic interior to C_1, continuous in the corresponding closed region. Let us suppose the function $V(x, y) = V_1(x, y) - V_2(x, y)$ to be continuous in the closure $\overline{S}$ of the annular region S bounded by C_1 and C_2, and to take the constant value γ at every point of C_2. We denote generically by C_λ the locus $V(x, y) = \lambda$, $(\gamma < \lambda < 0)$, in R, so that C_λ is a Jordan curve separating C_1 and C_2; we denote by R_λ the interior of C_λ, and by $\overline{R}_\lambda$ the closed interior of C_λ.

Let the function $f(z)$ be analytic throughout the interior of R_ρ but not throughout the interior of any $R_{\rho'}$, $(\rho' > \rho)$. In the notation of Problem A, the sequence $f_n(z)$ converges uniformly to $f(z)$ on any closed set interior to R_ρ. Moreover we have $(\gamma < \sigma < \rho)$

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† Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 477–486.

$$\limsup_{n\to\infty} \left[\max \left| f(z) - f_n(z) \right|, \ z \text{ on } C_\sigma\right]^{1/n} = e^{\sigma-\rho}, \tag{2}$$

$$\limsup_{n\to\infty} \left[\text{l.u.b.} \left| f_n(z) \right|, \ z \text{ in } R\right]^{1/n} = e^{-\rho}. \tag{3}$$

1. **Proof of Theorem 1.** The technique of our study of Problem A is quite similar to the technique developed in a recent work* by the present writer, to which we shall make frequent reference.

The mere existence of the limit in (1) in R exterior to C_2 implies the uniformity of the limit on any closed bounded set exterior to C_2 (compare op. cit., p. 266). The function

$$U_n(x, y) = \frac{1}{n} \log \left| (z - \beta_{n1}) \cdots (z - \beta_{nn}) \right| \tag{4}$$

is harmonic exterior to C_2, so its limit $V_1(x, y)$ is also harmonic exterior to C_2. Consequently the function $V(x, y)$ is harmonic in S.

If Γ is an analytic Jordan curve separating C_1 and C_2, and if ν denotes the exterior normal for Γ, then the integral over Γ of $\partial U_n(x, y)/\partial\nu$ is 2π, whence (compare op. cit., p. 268)

$$2\pi = \int_\Gamma \frac{\partial V_1}{\partial \nu}\, ds = \int_\Gamma \frac{\partial V}{\partial \nu}\, ds. \tag{5}$$

A consequence of (5) is the inequality $\gamma < 0$.

Let C_1' be an analytic Jordan curve near C_1 containing C_1 in its interior. We shall eventually allow C_1' to approach C_1. Let $V_2'(x, y)$ denote the function which coincides with $V_1(x, y)$ on C_1' and is harmonic interior to C_1', continuous in the corresponding closed region. The function

$$V'(x, y) = V_1(x, y) - V_2'(x, y)$$

is continuous in the closure $\overline{S}'$ of the region S' bounded by C_1' and C_2 and vanishes on C_1'. As in the proof of (5) we have

$$2\pi = \int_{C_1'} \frac{\partial V_1}{\partial \nu}\, ds = \int_{C_1'} \frac{\partial V'}{\partial \nu}\, ds. \tag{6}$$

As in the book cited, §9.11 (p. 265), we may write the following equations for (x, y) interior to C_1'; the second of these equations is a consequence of the corresponding equation with V_1 replaced by U_n:

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* *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 20, New York, 1935. All references in the present note not otherwise indicated are to this book, to which the reader is also referred for terminology.

$$V_2'(x, y) = \frac{1}{2\pi}\int_{C_1'}\left(V_2' \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V_2'}{\partial \nu}\right) ds,$$

$$0 = \frac{1}{2\pi}\int_{C_1'}\left(V_1 \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V_1}{\partial \nu}\right) ds,$$

$$V_2'(x, y) = \frac{-1}{2\pi}\int_{C_1'}\left(V' \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V'}{\partial \nu}\right) ds,$$

$$(7) \qquad V_2'(x, y) = \frac{1}{2\pi}\int_{C_1'} \log r \frac{\partial V'}{\partial \nu} ds.$$

The integrals are to be taken in the counterclockwise sense, and ν indicates the exterior normal.

When C_1' approaches C_1, the function $V_2'(x, y)$ approaches $V_2(x, y)$ uniformly on and within C_1, by Lebesgue's results on harmonic functions in variable regions.* Then the function $V'(x, y)$ approaches $V(x, y)$ uniformly in $\bar{S}$, and on C_2 the function $V'(x, y)$ takes on values uniformly as near as desired to $\gamma<0$, provided merely that C_1' is sufficiently near to C_1. Thus when C_1' is sufficiently close to C_1, in S' we have $V'(x, y)<0$ because $V'(x, y)$ is zero on C_1' and negative on C_2, and on C_1' we have $\partial V'/\partial\nu \geqq 0$; the equality sign is excluded here by our choice of C_1' as an *analytic* Jordan curve.

Let now the points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{n,n-1}$ be chosen uniformly distributed on C_1' with respect to the parameter whose differential is the positive quantity $(\partial V'/\partial\nu)ds$ (compare op. cit., §§8.7 and 9.11). From (6) and (7) we have

$$\lim_{n\to\infty} \left| (z-\alpha_{n1})(z-\alpha_{n2}) \cdots (z-\alpha_{n,n-1}) \right|^{1/n} = e^{V_2'(x,y)},$$

uniformly on any closed set interior to C_1'; so by virtue of (1) we may write

$$(8) \qquad \lim_{n\to\infty} \left| \frac{(z-\beta_{n1}) \cdots (z-\beta_{nn})}{(z-\alpha_{n1}) \cdots (z-\alpha_{n,n-1})} \right|^{1/n} = e^{V'(x,y)},$$

uniformly on any closed set interior to S'.

We denote by $r_n(z)$ the rational function of degree $n-1$ whose poles lie in the points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{n,n-1}$ and which interpolates to $f(z)$ in each of the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$; the sequence $r_n(z)$ has been studied in some detail (op. cit., §8.3), and in particular there can be established† the formula

$$(9) \qquad \limsup \left[\max \left| r_n(z) \right|, \ z \text{ on } C_\mu' \right]^{1/n} \leqq e^{\mu-\rho'},$$

* Rendiconti del Circolo Matematico di Palermo, vol. 24 (1907), pp. 371–402.

† Inequality (9) is an immediate consequence of equation (8) and the standard formula for $r_n(z)$ (op. cit., p. 186), which is valid even exterior to C_ρ. Indeed, the sign $\leqq$ in (9) can be replaced by the equality sign, as the writer expects to indicate in a forthcoming paper in these Transactions.

where C_λ' denotes generically the Jordan curve $V'(x, y)=\lambda$ in S', where $f(z)$ is analytic interior to $C_{\rho'}'$ but is not analytic throughout the interior of any $C_{\rho''}'$, $(\rho''>\rho')$, and where $\mu>\rho'$.

When C_1' approaches C_1, the locus C_λ' approaches uniformly the locus C_λ. Given any $\epsilon>0$, we can choose C_1' so near to C_1 that $|V'(x, y)-V(x, y)|<\epsilon$ uniformly in $\overline{S}$. For such a particular choice of C_1' we have $\rho'>\rho-\epsilon$; the curve C_1 lies interior to some C_μ', whence from (9)

$$\limsup_{n\to\infty}\,[\max\,|r_n(z)|,\ z \text{ on } C_1]^{1/n} \leqq e^{\mu-\rho+\epsilon} \leqq e^{-\rho+\epsilon}. \tag{10}$$

We have now exhibited functions $r_n(z)$ analytic in R, interpolating to $f(z)$ in the points β_{nk}, and satisfying (10). For the functions $f_n(z)$ whose least upper bound in R is a minimum we consequently have by (10)

$$\limsup_{n\to\infty}\,[\text{l.u.b.}\,|f_n(z)|,\ z \text{ in } R]^{1/n} \leqq e^{-\rho+\epsilon}. \tag{11}$$

A combination of (10) and (11) yields

$$\limsup_{n\to\infty}\,[\text{l.u.b.}\,|f_n(z)-r_n(z)|,\ z \text{ in } R]^{1/n} \leqq e^{-\rho+\epsilon},$$

whence for suitably chosen M,

$$|f_n(z)-r_n(z)| \leqq Me^{n(-\rho+2\epsilon)}, \qquad z \text{ in } R. \tag{12}$$

The function $f_n(z)-r_n(z)$ vanishes in each of the points β_{nk}; so the familiar reasoning used in the proof of Schwarz's lemma gives, for z interior to C_1,

$$\left|\,[f_n(z)-r_n(z)]\frac{(z-\alpha_{n1})\cdots(z-\alpha_{n,n-1})}{(z-\beta_{n1})\cdots(z-\beta_{nn})}\right|$$
$$\leqq Me^{n(-\rho+2\epsilon)}\Big/\left[\min\left|\frac{(z-\beta_{n1})\cdots(z-\beta_{nn})}{(z-\alpha_{n1})\cdots(z-\alpha_{n,n-1})}\right|,\ z \text{ on } C_1\right].$$

For z on C_1 we have $V=0$, $V'>-\epsilon$; for z on C_σ, $(\gamma<\sigma<\rho')$, we have $V'<\sigma+\epsilon$; then by (8) we may write

$$\limsup_{n\to\infty}\,[\max\,|f_n(z)-r_n(z)|,\ z \text{ on } C_\sigma]^{1/n} \leqq e^{\sigma-\rho+4\epsilon}. \tag{13}$$

But we know also (op. cit., p. 198) for $\sigma'<\rho'$

$$\limsup_{n\to\infty}\,[\max\,|f(z)-r_n(z)|,\ z \text{ on } C_{\sigma'}']^{1/n} \leqq e^{\sigma'-\rho'},$$

whence

$$\limsup_{n\to\infty}\,[\max\,|f(z)-r_n(z)|,\ z \text{ on } C_\sigma]^{1/n} \leqq e^{\sigma-\rho+2\epsilon}. \tag{14}$$

Inequalities (13) and (14) when combined now imply by letting ϵ approach zero $(\gamma<\sigma<\rho)$

(15) $$\limsup_{n\to\infty} [\max |f(z)-f_n(z)|, \ z \text{ on } C_\sigma]^{1/n} \leqq e^{\sigma-\rho}.$$

Likewise in (11) we may allow ϵ to approach zero:

(16) $$\limsup_{n\to\infty} [\text{l.u.b.} |f_n(z)|, \ z \text{ in } R]^{1/n} \leqq e^{-\rho}.$$

To complete the proof of Theorem 1, it remains merely to show that the inequality sign cannot hold in (15) or (16). The proof is indirect; let us assume for instance

(17) $$\limsup_{n\to\infty} [\text{l.u.b.} |f_n(z)|, \ z \text{ in } R]^{1/n} \leqq e^{-\rho_1}, \qquad \rho_1>\rho;$$

we shall reach a contradiction.

If $\eta>0$ is arbitrary, we have from (15) for n sufficiently large

$$|f_{n+1}(z)-f_n(z)| \leqq e^{(\sigma-\rho+\eta)n}, \qquad z \text{ on } C_\sigma,$$

and we have from (17) for n sufficiently large

$$|f_{n+1}(z)-f_n(z)| \leqq e^{(-\rho_1+\eta)n}, \qquad z \text{ in } R.$$

By an extension of Hadamard's Three-Circle Theorem* applied to the region bounded by C_1 and C_σ, we deduce for z on C_μ, $(\sigma<\mu<0)$,

(18) $$|f_{n+1}(z)-f_n(z)| \leqq \left[e^{(\sigma-\rho+\eta)n}\right]^{\mu/\sigma}\cdot\left[e^{(-\rho_1+\eta)n}\right]^{(\sigma-\mu)/\sigma} = e^{(\mu\sigma+\eta\sigma-\mu\rho-\rho_1\sigma+\mu\rho_1)n/\sigma}.$$

Since σ is negative, the sequence $f_n(z)$ converges uniformly on C_μ provided merely

$$\phi(\mu)=\mu\sigma+\eta\sigma-\mu\rho-\rho_1\sigma+\mu\rho_1>0.$$

For the value $\mu=\rho$ the continuous function $\phi(\mu)$ takes the value

$$\phi(\rho)=\rho\sigma+\eta\sigma-\rho^2-\rho_1\sigma+\rho\rho_1=(\sigma-\rho)(\rho-\rho_1)+\eta\sigma.$$

By virtue of $\rho_1>\rho$ and $\sigma<\rho$, it follows that when η is sufficiently small, $\phi(\rho)$ is positive. Consequently, $\phi(\mu)$ is positive also for suitably chosen values of μ *greater than* ρ. The limit of the sequence $f_n(z)$ is $f(z)$ interior to C_ρ, hence is the analytic function $f(z)$ throughout the interior of some curve C_μ, $(\mu>\rho)$, which contradicts our definition of ρ.

* R. Nevanlinna, *Eindeutige Analytische Funktionen*, Berlin, 1936, p. 42. We are here using the Two-Constant Theorem (Zweikonstantensatz) in the form due to F. and R. Nevanlinna. A somewhat less precise form is due to Ostrowski. In the situation of Theorem 1 itself, but not in the more general situation described in §3, the Three-Circle Theorem can be applied after a conformal map by means of the function $w=\exp\{V(x,y)+iW(x,y)\}$, where $W(x,y)$ is conjugate to $V(x,y)$ interior to S.

We have now shown that the inequality sign in (16) is impossible. Precisely the same method shows that the inequality sign in (15) is impossible; so Theorem 1 is established.

A limiting case of (2) is also valid, namely

$$\limsup_{n\to\infty} [\max |f(z) - f_n(z)|, z \text{ on } C_2]^{1/n} = e^{\gamma-\rho};$$

indeed the obvious relation

$$\max [|f(z) - f_n(z)|, z \text{ on } C_2] \leqq \max [|f(z) - f_n(z)|, z \text{ on } C_\sigma]$$

by approach of σ to γ establishes the precise analogue of (15), and the previous method shows the impossibility of the inequality.

2. **Complements to Theorem 1.** A complement to Theorem 1 is the

COROLLARY. *Under the conditions of Theorem* 1 *we have* $(0>\mu\geqq\rho)$

$$\limsup_{n\to\infty} [\max |f_n(z)|, z \text{ on } C_\mu]^{1/n} = e^{\mu-\rho}.$$

From (15) and (16) respectively we have $(\sigma<\rho)$

$$\limsup_{n\to\infty} [\max |f_{n+1}(z) - f_n(z)|, z \text{ on } C_\sigma]^{1/n} \leqq e^{\sigma-\rho},$$

$$\limsup_{n\to\infty} [\text{l.u.b.} |f_{n+1}(z) - f_n(z)|, z \text{ in } R]^{1/n} \leqq e^{-\rho},$$

from which we deduce as in the proof of (18),

$$\limsup_{n\to\infty} [\max |f_{n+1}(z) - f_n(z)|, z \text{ on } C_\mu]^{1/n} \leqq e^{\mu-\rho}.$$

We are now at liberty to write

$$\limsup_{n\to\infty} [\max |f_n(z)|, z \text{ on } C_\mu]^{1/n} \leqq e^{\mu-\rho}.$$

The impossibility of the inequality sign here follows precisely as in (16) for $\mu>\rho$ and is trivial for $\mu=\rho$ (we should otherwise have $f_n(z)$ approaching zero uniformly interior to C_ρ); so the corollary is established.

It is of interest to note that when C_1 is a curve $V'(x, y)=\text{const.}$, it follows from (9) that the rational functions $r_n(z)$ have maximum modulus on C_μ, $(0>\mu>\rho)$, of the same order of magnitude as the maximum modulus of the extremal functions $f_n(z)$; a similar remark holds also of C_1. Under these conditions it is likewise true that max $|f(z)-r_n(z)|$ and max $|f(z)-f_n(z)|$ have the same order of magnitude on C_σ, $(\rho>\sigma>\gamma)$, and also on C_2.

A relation which essentially includes (2) (granted the convergence of $f_n(z)$ to $f(z)$ in R_ρ) as well as the corollary, and thereby unifies the preceding results is

$$\limsup_{n\to\infty} \left[\max \left| f_{n+1}(z) - f_n(z) \right|, \ z \text{ on } C_\sigma\right]^{1/n} = e^{\sigma-\rho},$$

$0>\sigma\geqq\rho$ or $\rho>\sigma>\gamma$. This relation with the equality sign replaced by $\leqq$ has been pointed out in the proof of the corollary; if the inequality sign were to hold we should have the inequality sign in (2) or in the corollary, according as $\sigma<\rho$ or $\sigma\geqq\rho$, which we know to be impossible. The corresponding limiting equations also hold and are similarly proved:

$$\limsup_{n\to\infty} \left[\text{l.u.b.} \left| f_{n+1}(z) - f_n(z) \right|, \ z \text{ in } R\right]^{1/n} = e^{-\rho},$$

$$\limsup_{n\to\infty} \left[\max \left| f_{n+1}(z) - f_n(z) \right|, \ z \text{ on } C_2\right]^{1/n} = e^{\gamma-\rho}.$$

It is an obvious consequence of Theorem 1 that under the hypothesis of that theorem there exists no sequence of functions $F_n(z)$ analytic in R and coinciding with $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$ such that we have

$$\limsup_{n\to\infty} \left[\text{l.u.b.} \left| F_n(z) \right|, \ z \text{ in } R\right]^{1/n} < e^{-\rho}.$$

We note too that Theorem 1 can be applied under the hypothesis of that theorem where C_μ, $(\mu>\rho)$, plays the role of the original C_1. The function $V(x, y)-\mu$ now takes the role of the original $V(x, y)$, and it follows from Theorem 1 that there exists a sequence of functions $F_n(z)$ analytic in R and coinciding with $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$, namely the extremal functions $f_n(z)$ pertaining to R, such that we have

$$\limsup_{n\to\infty} \left[\text{l.u.b.} \left| F_n(z) \right|, \ z \text{ in } R_\mu\right]^{1/n} = e^{\mu-\rho};$$

but there exists no sequence of functions $F_n(z)$ analytic in R_μ and coinciding with $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$ such that we have

$$\limsup_{n\to\infty} \left[\text{l.u.b.} \left| F_n(z) \right|, \ z \text{ in } R_\mu\right]^{1/n} < e^{\mu-\rho}.$$

Thus the extremal functions $f_n(z)$ of Theorem 1 have maximum moduli on C_μ, $(\mu>\rho)$, which are of the same order of magnitude as the least upper bounds of the corresponding extremal functions which pertain to R_μ itself.

Still another remark is appropriate in connection with Theorem 1, relative to functions $f(z)$ *analytic throughout* R. Under these conditions we can set $\rho=0$ in inequality (10), whence for the extremal functions $f_n(z)$ defined as in Theorem 1,

$$\limsup_{n\to\infty} \left[\text{l.u.b.} \left| f_n(z) \right|, \ z \text{ in } R\right]^{1/n} \leqq e^{\epsilon}.$$

Here we may allow ϵ to approach zero, whence

$$\limsup_{n\to\infty} [\text{l.u.b.} \,|\, f_n(z) \,|, \; z \text{ in } R]^{1/n} \leqq 1.$$

The inequality sign cannot hold here except in the trivial case $f(z)\equiv 0$, for the inequality sign implies that $f_n(z)$ approaches zero uniformly in R. As in the proof of (15) we have for every σ, $(\gamma<\sigma<0)$,

$$\limsup_{n\to\infty} [\max \,|\, f(z) - f_n(z) \,|, \; z \text{ on } C_\sigma]^{1/n} \leqq e^\sigma.$$

If $f(z)$ is analytic and bounded in R, the sequence $f_n(z)$ is uniformly bounded in R, for $f(z)$ itself satisfies the conditions of interpolation:

$$[\text{l.u.b.} \,|\, f_n(z) \,|, \; z \text{ in } R] \leqq [\text{l.u.b.} \,|\, f(z) \,|, \; z \text{ in } R].$$

There is evidence to indicate that the present methods alone do not enable us to determine the exact value of

$$\limsup_{n\to\infty} [\max \,|\, f(z) - f_n(z) \,|, \; z \text{ on } C_\sigma]^{1/n}, \qquad 0 > \sigma > \gamma,$$

when $f(z)$ is analytic throughout R. First, there are various comparison sequences $r_n(z)$ any one of which is adequate in the proof of Theorem 1 itself but which yield different results for

$$\limsup [\max \,|\, f(z) - r_n(z) \,|, \; z \text{ in } R]^{1/n}$$

when $f(z)$ is analytic throughout R. This is shown for instance by choosing $f(z)=1/(T-z)$, $(T>1)$, the β_{nk} as all zero, and the α_{nk} as the $(n-1)$st roots of A^{n-1}, where $1<A<T$, and by choosing $\beta_{nk}=0$ and C_1 as $|z|=1$. Equation (8) is fulfilled. The sequence $r_n(z)$ serves as a comparison sequence in the proof of Theorem 1 for an arbitrary function $f(z)$ satisfying the hypothesis of Theorem 1 without the necessity of allowing A to approach unity; that is to say, without the necessity of allowing C_1' to approach C_1: this is always the case when $V'(x, y)$ is constant on C_1. It follows (as in op. cit., p. 185) that we have with the special choice of $f(z)$

$$f(z) - r_n(z) = z^n(T^{n-1} - A^{n-1})/[T^n(z^{n-1} - A^{n-1})(T - z)],$$

where $r_n(z)$ is found by interpolation to $f(z)$ in the points β_{nk} and has the poles α_{nk}. Consequently we may write

$$\limsup_{n\to\infty} [\max \,|\, f(z) - r_n(z) \,|, \; \text{for } |\, z \,| = r \leqq 1]^{1/n} = r/A,$$

whereas A is completely arbitrary within the limits $1<A<T$, and its use is entirely accidental in the study of the functions $f_n(z)$.

Second, even when the singularities of the function $f(z)$ fall in the region in which (8) is valid, and when $V'(x, y)$ is constant on C_1 so that Theorem 1 itself can be established without varying the curve C_1' or the points α_{kn}, it is not true that the degree of convergence to $f(z)$ on C_σ, $(0>\sigma>\gamma)$, is necessarily the same for the sequences $r_n(z)$, $f_n(z)$. Let β be arbitrary, $(0<\beta<1)$, and set

$$f(z) = (z + \beta)/(1 + \beta z).$$

Well known methods (see for instance op. cit., §10.2) show that $f(z)$ is the unique function analytic and in modulus less than unity within R: $|z|<1$ which takes the value β for $z=0$ and has the derivative $1-\beta^2$ for the value $z=0$. In the notation of Theorem 1 we set $\beta_{nk}=0$; the extremal properties of $f(z)$ indicate that each of the functions $f_2(z), f_3(z), \cdots$ is identical with $f(z)$. Thus we have

$$\limsup_{n\to\infty} \left[\max |f(z) - f_n(z)|, \text{ for } |z| \leqq r < 1\right]^{1/n} = 0.$$

But the natural comparison sequence, according to the method of proof of Theorem 1 in somewhat simplified form, is found from the Taylor development of $f(z)$;* we take $r_n(z)$ as the sum of the first n terms of this development:

$$\limsup_{n\to\infty} \left[\max |f(z) - r_n(z)|, \text{ for } |z| \leqq r < 1\right]^{1/n} = r/\beta,$$

in contrast to the preceding relation.

3. **Extensions of Theorem 1; examples.** Merely for the sake of simplicity, we chose in Theorem 1 a region R bounded by a single Jordan curve. The theorem and corollary, together with their proofs, remain valid if R is an arbitrary limited region whose boundary consists of a finite number of mutually disjoint Jordan curves. Likewise the C_2 of Theorem 1 may be replaced by a finite number of mutually disjoint Jordan curves interior to R, no one of which separates any other from the boundary of R or separates any two components of the boundary of R. Under these conditions the locus C_λ also consists of a finite number of mutually disjoint Jordan curves in the region S bounded by C_1 and C_2, except that for certain values of λ the locus C_λ may have a finite number of multiple points, each shared by a finite number of Jordan curves.

The formal statement of this generalization of Theorem 1 lies immediately at hand, and is left to the reader. A number of special cases of this generalization are worth stating explicitly; in each case we use the notation of Problem A.

* We may equally well choose here the α_{nk} as the $(n-1)$st roots of A^{n-1}, with $A>1/\beta$. This choice does not alter the relation involving the functions $r_n(z)$.

(i) Let R be $|z|<1$, each $\beta_{nk}=0$, the function $f(z)$ analytic for $|z|<r<1$ but not for $|z|<r'$, with $r'>r$. Then the situation is analogous to that of Taylor's series; we have

$$\begin{aligned}
&\limsup_{n\to\infty}\,[\max|f(z)-f_n(z)|, \text{ for } |z|\leqq r_1<r]^{1/n}=r_1/r,\\
(19)\qquad &\limsup_{n\to\infty}\,[\text{l.u.b.}|f_n(z)|, \text{ for } |z|<1]^{1/n}=1/r,\\
&\limsup_{n\to\infty}\,[\max|f_n(z)|, \text{ for } |z|=r_2>r]^{1/n}=r_2/r,\ r_2<1.
\end{aligned}$$

(ii) Let R be $|z|<1$; let each $\beta_{nk}=\beta$, interior to R and independent of n and k; let the function $f(z)$ be analytic in the region

$$|(z-\beta)/(1-\bar\beta z)|<r<1$$

but not throughout any region

$$|(z-\beta)/(1-\bar\beta z)|<r'>r.$$

This represents a generalization of (i), and we have obvious equations analogous to (19).

(iii) Let R be $|z|<1$; let the numbers $\beta_{n1},\cdots,\beta_{nn}$ be the first n numbers of the sequence $\beta_1,\beta_2,\cdots,\beta_l,\beta_1,\beta_2,\cdots,\beta_l,\beta_1,\beta_2,\cdots$, with each β_k interior to R; let the function $f(z)$ be analytic on the set $|p(z)|<r<1$ but not throughout any set $|p(z)|<r'>r$, where

$$p(z)=\prod_{j=1}^{l}\frac{z-\beta_j}{1-\bar\beta_j z}\cdot$$

The point set $|p(z)|<r$ is not necessarily connected. The situation is analogous to that of a certain series of interpolation (op. cit., §9.5). The equations corresponding to (19) are

$$\begin{aligned}
&\limsup_{n\to\infty}\,[\max|f(z)|-f_n(z)|, \text{ for } |p(z)|\leqq r_1<r]^{l/n}=r_1/r,\\
(20)\qquad &\limsup_{n\to\infty}\,[\text{l.u.b.}|f_n(z)|, \text{ for } |p(z)|<1]^{l/n}=1/r,\\
&\limsup_{n\to\infty}\,[\max|f_n(z)|, \text{ for } |p(z)|=r_2>r]^{l/n}=r_2/r, r_2<1.
\end{aligned}$$

(iv) Let R be $|p(z)|<1$, where $p(z)=q(z-\beta_1)(z-\beta_2)\cdots(z-\beta_l)$; let the numbers $\beta_{n1},\beta_{n2},\cdots,\beta_{nn}$ be the first n numbers of the sequence $\beta_1,\beta_2,\cdots,\beta_l,\beta_1,\beta_2,\cdots,\beta_l,\beta_1,\beta_2,\cdots$, with each β_k interior to R; let the function $f(z)$ be analytic on the set $|p(z)|<r<1$ but not throughout any set $|p(z)|<r'>r$; the set $|p(z)|<r$ is not necessarily connected. The situation

is analogous to that of the series of interpolation related to the Jacobi series (op. cit., §3.4). Equations (20) are valid also in the present case.

(v) Let R be $|z|<1$; let the set $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$ be the roots of $z^n-b_n^n=0$, $(|b_n|\leqq b<1)$; let the function $f(z)$ be analytic for $|z|\leqq r>b$, $r<1$, but not analytic throughout $|z|=r'$ with $r'>r$. Then equations (19) are valid provided merely $r_1\geqq b$.

(vi) In the statement of Theorem 1, let C_1 and C_2 be arbitrary (satisfying the conditions imposed), and let $V(x, y)$ denote a function harmonic in S, continuous in the corresponding closed region, taking on the values zero and $\gamma<0$ on C_1 and C_2, respectively, where γ is so chosen that the integral of the normal derivative of $V(x, y)$ over an analytic Jordan curve separating C_1 and C_2 is 2π. Let the points β_{nk} be uniformly distributed on C_2 with respect to the function conjugate to $V(x, y)$ in S. Then (op. cit., §8.7) all the conditions of Theorem 1 are fulfilled. This situation is a generalization of (v) if $|b_n|=b$.

Theorem 1 can be extended, as we have indicated, by lessening the restrictions on C_1 and C_2. Still another extension of Theorem 1 (and of the more general results outlined) is obtained by requiring the limit (1) to hold not at every point exterior to C_2, but to hold at every point exterior to C_2 except at the points of a set T having no limit point exterior to C_2, and to hold uniformly on any closed set exterior to C_2 having no point in common with T. The points β_{nk} are no longer required to lie on or interior to C_2, but must lie in R. No modification need be made in the proofs already given to meet this new hypothesis, except that in the proof of such a relation as (13) where C_σ passes through a point of T, we give the proof first with σ replaced by $\sigma_1>\sigma$, where C_{σ_1} does not pass through a point of T, and then allow σ_1 to approach σ. With this new requirement on (1), it is not always essential to suppose *all* the points β_{nk} interior to the region R_ρ in which $f(z)$ is assumed defined and analytic; methods for the study of the corresponding sequence $r_n(z)$ are developed in the book already referred to (chap. 11); those methods, together with the present ones, apply directly to the study of Problem A.

We state but a single illustration of the remark just made. Let R be the region $|z|<1$; let the sequence $\beta_1, \beta_2, \cdots$ lie interior to $|z|=1$ and approach zero as its limit, and let us identify $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$ with $\beta_1, \beta_2, \cdots, \beta_n$; let the function $f(z)$ be analytic for $|z|<r<1$ but not throughout $|z|<r'$ with $r'>r$. If some of the points β_k lie on or exterior to $|z|=r$, the prescription that $f_n(z)$ shall interpolate to $f(z)$ in those points may be interpreted as requiring that $f_n(z)$ shall interpolate to any function, analytic or not, but not depending on n, in those particular points β_k. The equations (19) are valid.

4. **Invariant properties of Theorem 1.** Problem A as formulated is invariant under an arbitrary one-to-one conformal transformation. Thus each

of the special situations (i)-(vi) yields, by such a transformation, a new result which the reader can easily express in invariant terms. Theorem 1 itself, especially with regard to condition (1),* has no invariant properties that are obvious, but does have certain relations to invariance, as we shall now proceed to show. The following theorem, previously suggested (op. cit., p. 276) for formulation and proof, is analogous to a theorem already established (op. cit., p. 272, Theorem 20):

THEOREM 2. *Let C' be a Jordan curve of the $w(=u+iv)$-plane, let the points $w=\beta'_{nk}$ lie on or within C', and let us suppose*

$$\lim_{n\to\infty} \left| (w-\beta'_{n1})(w-\beta'_{n2})\cdots(w-\beta'_{nn}) \right|^{1/n} = e^{U(u,v)} \tag{21}$$

exterior to C', uniformly on any closed bounded set exterior to C'. Let a bounded region D' containing C' in its interior be transformed conformally and one-to-one into a bounded region D of the $z(=x+iy)$-plane by the transformation $w=\phi(z)$, $z=\psi(w)$, with C' transformed into the Jordan curve C and the points β'_{nk} transformed into the points $\beta_{nk}=\psi(\beta'_{nk})$ interior to C. Then the limit

$$\lim_{n\to\infty} \left| (z-\beta_{n1})\cdots(z-\beta_{nn}) \right|^{1/n} = e^{W(x,y)} \tag{22}$$

exists in every finite point exterior to C, uniformly on any closed bounded set exterior to C.

We introduce the notation

$$U_n(x, y) = \frac{1}{n}\sum_{k=1}^{n} \log \left| \phi(z) - \phi(\beta_{nk}) \right|, \qquad U'_n(x, y) = \frac{1}{n}\sum_{k=1}^{n} \log \left| z - \beta_{nk} \right|,$$

$$U''_n(x, y) = \frac{1}{n}\sum_{k=1}^{n} \log \left| \frac{\phi(z) - \phi(\beta_{nk})}{z - \beta_{nk}} \right|,$$

whence $U_n(x, y) = U'_n(x, y) + U''_n(x, y)$. Let Γ denote an arbitrary analytic Jordan curve in D containing C in its interior. Then we have (op. cit., p. 266, Lemma IV) for (x, y) exterior to Γ

$$U'_n(x, y) = \frac{1}{2\pi}\int_\Gamma \left(U'_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U'_n}{\partial \nu} \right) ds.$$

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* Thus if the points β_{nk} are the n roots of unity, equation (1) holds exterior to C_2: $|z|=1$ with $V_1(x, y)=\log |z|$. Under the transformation $z=(w-\beta)/(1-\bar\beta w)$, with $|\beta|<1$, the points β_{nk} correspond to the roots of the equation $[(w-\beta)/(1-\bar\beta w)]^n-1=0$, and the analogue of (1) is for $|w|>1$

$$\lim_{n\to\infty} \left| \left[\left(\frac{w-\beta}{1-\bar\beta w} \right)^n - 1 \right] \frac{(1-\bar\beta w)^n}{1-(-\bar\beta)^n} \right|^{1/n} = |w-\beta|.$$

The function $U_n''(x, y)$ is harmonic without exception on and interior to Γ (when suitably defined in the points $z=\beta_{nk}$); so we have for (x, y) in D exterior to Γ

$$0 = \frac{1}{2\pi}\int_\Gamma\left(U_n'' \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U_n''}{\partial \nu}\right) ds;$$

by addition we write for (x, y) in D exterior to Γ

$$U_n'(x, y) = \frac{1}{2\pi}\int_\Gamma\left(U_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U_n}{\partial \nu}\right) ds. \tag{23}$$

By hypothesis (21) holds; so the function $U_n(x, y)$ approaches uniformly on Γ the function $U(x, y)$, the transform in the (x, y)-plane of the function $U(u, v)$ in the w-plane; moreover the derivatives of $U_n(x, y)$ on Γ approach uniformly the corresponding derivatives of $U(x, y)$; so by (23) the limit (22) exists, with the relation

$$W(x, y) = \frac{1}{2\pi}\int_\Gamma\left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu}\right) ds, \tag{24}$$

where it is understood that Γ shall be chosen to contain C but not (x, y) in its interior. Equation (22) is first proved for (x, y) exterior to Γ but interior to D; however (see op. cit., p. 266) the sequence $U_n'(x, y)$ is a normal family of harmonic functions in the region exterior to C; when (22) holds in a subregion, that relation holds uniformly on any closed bounded set exterior to C. Theorem 2 is established.

Theorem 2 extends at once to the more general situation outlined at the beginning of §3.

The significance of Theorem 2 in connection with Theorem 1 lies in two remarks. (i) Although condition (1) is not itself invariant under conformal transformation, and to that extent is unsuited to a discussion of Problem A, condition (1) is shown by Theorem 2 to have certain properties related to invariance, and thereby to be a not unreasonable hypothesis to use. Thus the geometric configuration of Theorem 1 may be subjected to a transformation which carries the closed interior of C_1 into the closed interior of another Jordan curve C_1' conformally and one-to-one. Theorem 1 applies also to the new configuration. (ii) If there is given a region R of simple or multiple connectivity, such that a single Jordan curve or a set of Jordan curves C_2 contains the points β_{nk} not on C_2 in its interior with (1) satisfied, but if R is infinite or if the boundary of R consists not of Jordan curves but of a finite number of other continua, none of which is a single point, then R can be mapped conformally onto a finite region bounded by a finite number of mutually disjoint analytic Jordan curves, so that condition (1) persists in character, and hence the extension of Theorem 1 applies.

5. **Invariant formulation of Theorem 1.** Even though Theorem 1 itself is not expressed in form invariant under arbitrary one-to-one conformal transformation, an equivalent result can be so expressed with relative ease, as we shall now proceed to indicate. But our immediate methods apply rather to Theorem 1 itself than to the extension of Theorem 1 to multiply-connected regions R.

THEOREM 3. *Let R be a simply connected region of the extended plane whose boundary C_1 consists of more than two points, and let the function $w=\phi(z)$ map R conformally and one-to-one onto $|w|<1$. Let C_2 be a Jordan curve interior to R, let C_2 separate the points β_{nk} not lying on C_2 itself from C_1, and let*

$$\lim_{n\to\infty}\left|\frac{[\phi(z)-\phi(\beta_{n1})]\cdots[\phi(z)-\phi(\beta_{nn})]}{[\bar{\phi}(\beta_{n1})\phi(z)-1]\cdots[\bar{\phi}(\beta_{nn})\phi(z)-1]}\right|^{1/n}=e^{U(x,y)} \tag{25}$$

hold at every point of the annular region S bounded by C_1 and C_2, uniformly on any closed set interior to S. Let the function $U(x, y)$ be continuous in $\bar{S}$ and take the constant value γ on the curve C_2. We denote generically by C_λ the locus $U(x, y)=\lambda$, $(\gamma<\lambda<0)$, in R, so that C_λ is a Jordan curve separating C_1 and C_2; we denote by R_λ the region bounded by C_λ containing C_2 in its interior, and by $\bar{R}_\lambda$, the closure of R_λ.

Let the function $f(z)$ be analytic throughout the interior of R_ρ but not throughout the interior of any $R_{\rho'}$, $(\rho'>\rho)$. In the notation of Problem A, *the sequence $f_n(z)$ converges uniformly to $f(z)$ on any closed set interior to R_ρ. Moreover we have (for $\gamma<\sigma<\rho$) equations* (2) *and* (3).

The functions harmonic in R except in the points β_{nk},

$$U_n(x, y)=\frac{1}{n}\log\left|\frac{[\phi(z)-\phi(\beta_{n1})]\cdots[\phi(z)-\phi(\beta_{nn})]}{[\bar{\phi}(\beta_{n1})\phi(z)-1]\cdots[\bar{\phi}(\beta_{nn})\phi(z)-1]}\right|,$$

when suitably defined on C_1 are all continuous in the two-dimensional sense on C_1, except of course that the functions need not be defined exterior to R, and they take the value zero on C_1. Their uniform convergence on a curve C_λ therefore implies their uniform convergence in the closed region bounded by C_1 and C_λ; so $U(x, y)$ also is continuous in the two-dimensional sense on C_1 and vanishes there. Of course $U_n(x, y)$ is negative in R, and indeed by the hypothesis on the β_{nk} is uniformly bounded from zero on any closed set interior to S; so the relation $\gamma<0$ can be made a matter of proof rather than hypothesis.

Our discussion of Theorem 3 is quite similar to the proof of Theorem 2. Let us transform R conformally without change of notation so that it becomes the interior of an arbitrary Jordan curve C_1. We introduce the notation

$$U_n'(x, y) = \frac{1}{n}\sum_{k=1}^{n} \log |z - \beta_{nk}|,$$

$$U_n''(x, y) = \frac{1}{n}\sum_{k=1}^{n} \log \frac{|\phi(z) - \phi(\beta_{nk})|}{|[z - \beta_{nk}][\bar{\phi}(\beta_{nk})\phi(z) - 1]|},$$

whence $U_n(x, y) = U_n'(x, y) + U_n''(x, y)$.

The function $U_n''(x, y)$, when a suitable definition is provided in the points β_{nk}, is harmonic throughout the interior of R; so if Γ_2 is an analytic Jordan curve containing C_2 in its interior, but to which (x, y) is exterior, we have (op. cit., p. 265, Lemma III) for (x, y) either in S or on or exterior to C_1

$$0 = \frac{1}{2\pi}\int_{\Gamma_2}\left(U_n'' \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U_n''}{\partial \nu}\right) ds,$$

where ν indicates the interior normal for Γ_2 and the integral is taken in the clockwise sense. Under these circumstances we also have (op. cit., p. 266, Lemma IV) for (x, y) either in S or even on or exterior to C_1

$$U_n'(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(U_n' \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U_n'}{\partial \nu}\right) ds,$$

whence for (x, y) anywhere exterior to C_2,

$$U_n'(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(U_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U_n}{\partial \nu}\right) ds. \tag{26}$$

The sequence $U_n(x, y)$ converges uniformly to $U(x, y)$ on Γ_2, and the derivatives of $U_n(x, y)$ converge uniformly on Γ_2 to the corresponding derivatives of $U(x, y)$; so it follows from (26) that $U_n'(x, y)$ converges at every finite point exterior to C_2, uniformly on any closed limited set exterior to C_2, to the function

$$U'(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu}\right) ds, \tag{27}$$

where it is understood that Γ_2 is so chosen that (x, y) lies exterior to Γ_2, and C_2 interior to Γ_2. With this understanding, the functions $U_n'(x, y)$ and $U'(x, y)$ defined by (26) and (27) are harmonic at every finite point of the plane even exterior to C_1, and are independent of the particular curve Γ_2 (depending on (x, y)) which is chosen.

Let Γ_1 denote an arbitrary analytic Jordan curve containing in its interior both C_2 and the point (x, y). Then we have

$$U_n''(x, y) = \frac{1}{2\pi}\int_{\Gamma_1}\left(U_n''\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n''}{\partial \nu}\right)ds,$$

where ν indicates exterior normal for Γ_1 and the integral is taken in the counterclockwise sense. We also have (op. cit., p. 265, Lemma II)

$$0 = \frac{1}{2\pi}\int_{\Gamma_1}\left(U_n'\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n'}{\partial \nu}\right)ds,$$

whence for (x, y) interior to Γ_1,

$$U_n''(x, y) = \frac{1}{2\pi}\int_{\Gamma_1}\left(U_n\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n}{\partial \nu}\right)ds. \tag{28}$$

The sequence $U_n(x, y)$ converges uniformly to $U(x, y)$ on Γ_1, and the various derivatives of $U_n(x, y)$ converge uniformly on Γ_1 to the corresponding derivatives of $U(x, y)$; so it follows from (28) that $U_n''(x, y)$ converges at every point interior to C_1, uniformly on any closed set interior to C_1, even interior to C_2, to the function

$$U''(x, y) = \frac{1}{2\pi}\int_{\Gamma_1}\left(U\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U}{\partial \nu}\right)ds. \tag{29}$$

It is of course understood that Γ_1 is chosen interior to C_1, with both (x, y) and C_2 in its interior. With this understanding, the functions $U_n''(x, y)$ and $U''(x, y)$ expressed by (28) and (29) are analytic throughout the interior of C_1, and are independent of the particular curve Γ_1 (depending on (x, y)) chosen.

The function $U'(x, y)$, harmonic at every point of the plane exterior to C_2, can now be identified with the function $V_1(x, y)$ of Theorem 1. From

$$U(x, y) = U'(x, y) + U''(x, y),$$

valid interior to S, and from the continuity of $U(x, y)$ and $U'(x, y)$ on C_1, it follows that $U''(x, y)$ when suitably defined on C_1 also is continuous on C_1, and takes on the values $-U'(x, y)$ there. Then $U''(x, y)$ is precisely the negative of the function $V_2(x, y)$ of Theorem 1. That is to say, we have shown that under the conditions of Theorem 3 with R the interior of a Jordan curve, the hypothesis of Theorem 1 is satisfied, with $V(x, y)$ of Theorem 1 equal to $U(x, y)$ of Theorem 3; this first yields a proof* of Theorem 3, and second

* A much shorter proof of Theorem 3, which however does not tend to show the equivalence of Theorems 1 and 3, can be given from Theorem 1 by use of the substitution $w=\phi(z)$ in (25).

shows part of the equivalence of Theorems 1 and 2. The complete equivalence of Theorems 1 and 2 will be established by our showing now that the hypothesis of Theorem 1 implies condition (25).

We interpret $U_n''(x, y)$ as the unique function harmonic in R and continuous in the corresponding closed region which equals $-U_n'(x, y)$ on C_1. By hypothesis* the functions $U_n'(x, y)$ converge uniformly on C_1 to the function $V_1(x, y)$; then the functions $U_n''(x, y)$ converge uniformly on C_1 to the function $-V_1(x, y)$, and hence converge uniformly in the closed region $R+C_1$, to some function $-V_2(x, y)$ harmonic interior to R, continuous in $R+C_1$, and equal to $-V_1(x, y)$ on C_1. Then the functions $U_n(x, y)$ converge uniformly on any closed set interior to S, to the function $V_1(x, y)-V_2(x, y)$. Consequently, equation (25) is satisfied with $U(x, y)$ equal to the function $V(x, y)$ of Theorem 1, as we desired to show.

Theorem 3, like Theorem 1, applies without further change in proof even if C_2 consists no longer of a single Jordan curve but of several mutually disjoint Jordan curves interior to R, no one of which separates any other from C_1 or separates any two of the components of C_1; of course C_2 must separate the β_{nk} not lying on C_2 from C_1; the region S is bounded by C_1 and C_2. The expression of the examples (i)—(vi) in invariant form already suggested is the formulation of several special cases of this extension of Theorem 3.

To Theorem 1 corresponds an expression in form invariant under conformal transformation, namely Theorem 3. Similarly the extension of Theorem 1 to a multiply-connected region R can be expressed in a form invariant under conformal mapping, provided that the connectivity of R is finite and that no component of the boundary C_1 of R consists of a single point; we continue the lighter conditions on C_2. But here we replace condition (25) by the condition that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} G(x, y; \beta_{nk}) = U(x, y), \tag{30}$$

uniformly on any closed set in the region S bounded by C_1 and C_2, where $G(x, y; \beta)$ denotes generically Green's function for R with pole in the point β interior to R, and with running coordinates x and y. Condition (30) is a generalization of condition (25), for if R is simply-connected we have the relation

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* In the hypothesis of Theorem 1 it is sufficient to assume that (1) holds uniformly merely in S, by virtue of the equation

$$U_n'(x, y) = \frac{1}{2\pi} \int_{\Gamma_2} \left(U_n' \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U_n'}{\partial \nu} \right) ds$$

used in the proof of Theorem 3.

$$G(x, y; \beta_{nk}) = \log \left| \frac{\phi(z) - \phi(\beta_{nk})}{\overline{\phi(\beta_{nk})}\phi(z) - 1} \right|;$$

the right-hand member is harmonic interior to R except at β_{nk}, is continuous and equal to zero on C_1, and when diminished by $\log |z-\beta_{nk}|$ is bounded in the neighborhood of the point $z=\beta_{nk}$.

The methods already set forth above show that condition (30) implies the hypothesis of Theorem 1 extended, provided R is a limited region bounded by a finite number of mutually disjoint Jordan curves, and that conversely condition (30) is a consequence of the hypothesis of Theorem 1 extended. We do not emphasize (30) further, however, for it is apparently much more difficult to apply than (25), in the absence of a simple formula for $G(x, y; \beta_{nk})$ when R is multiply-connected.

A consequence of the remark just made is that Theorem 1 extends not merely to a region R bounded by a finite number of mutually disjoint Jordan curves, but also to an arbitrary region R of finite connectivity each component of whose boundary C_1 consists of more than a single point; we still suppose C_2 to consist of a finite number of mutually exterior Jordan curves which separate each of the points β_{nk} not lying on C_2 from the point at infinity. If R is finite, our hypothesis (1) implies, by the reasoning already given in connection with (25) and (30), that equation (30) is valid uniformly on any closed set in S; consequently Theorem 3 in its extended form applies, and so also does the conclusion of Theorem 1. If R is infinite, we may replace (1) by the condition

$$\lim_{n\to\infty} \left| \frac{(z-\beta_{n1}) \cdots (z-\beta_{nn})}{(z-\beta)^n} \right|^{1/n} = \frac{e^{V_1(x,y)}}{|z-\beta|}, \tag{31}$$

where β is an arbitrary fixed point separated by C_2 from the point at infinity. The function

$$W_n(x, y) = \frac{1}{n} \log \left| \frac{(z-\beta_{n1}) \cdots (z-\beta_{nn})}{(z-\beta)^n} \right|$$

is harmonic even at infinity, when suitably defined there, and the sequence $W_n(x, y)$ converges to the harmonic function $V_1(x, y) - \log |z-\beta|$ (also suitably defined at infinity), uniformly on any closed set *bounded or unbounded* exterior to C_2. Denote by $g_n(x, y)$ the function harmonic interior to R, continuous in the corresponding closed region, which coincides with $W_n(x, y)$ on C_1; the sequence $g_n(x, y)$ converges uniformly on C_1, and hence converges uniformly in the closed region $R+C_1$, to a function $g(x, y)$ harmonic in R, continuous in $R+C_1$, equal to $V_1(x, y)-\log|z-\beta|$ on C_1. We obviously have in the notation of (30)

$$\frac{1}{n}\sum_{k=1}^{n} G(x, y; \beta_{nk}) = W_n(x, y) - G(x, y; \beta) - g_n(x, y);$$

so equation (30) is satisfied uniformly on any closed set in S with

$$U(x, y) = V_1(x, y) - \log|z-\beta| - G(x, y; \beta) - g(x, y). \tag{32}$$

Consequently Theorem 3 in its extended form applies, and so also does the conclusion of Theorem 1, if we identify $U(x, y)$ as defined by (32) with the function $V(x, y)$ of Theorem 1.

Of course the Corollary to Theorem 1 has an exact analogue in the situation of Theorem 3 extended.

6. **Supplementing a given incomplete sequence β_{nk}.** It is to be noted that such relations as (2) and (3) involve the superior limit as n takes on *all* the values $1, 2, 3, \cdots$. Our proofs remain essentially valid if the β_{nk} are defined merely for an infinite sequence of indices n_j, $(j=1, 2, \cdots)$, with $n_{j+1}>n_j$, provided the difference $n_{j+1}-n_j$ is bounded. But the proofs are no longer valid if the difference $n_{j+1}-n_j$ is not bounded, and (in the absence of specific examples) the analogy with Taylor's series suggests that the conclusions do not remain true. It seems therefore of interest to be able to start with a set β_{nk} satisfying (1) for a suitable sequence of indices n, and to enlarge the set so that (1) is fulfilled for the entire sequence $n=1, 2, \cdots$. Methods of solving this problem lie now at hand, as we proceed to indicate.

By our present hypothesis, namely (1) for a suitably chosen sequence of indices n, the function $V_1(x, y)$ is harmonic at every point exterior to C_2. We define $U_n(x, y)$ by means of (4). Let Γ_2 be an analytic Jordan curve containing C_2 in its interior, but to which (x, y) is exterior. Then we have (op. cit., p. 266, Lemma IV)

$$U_n(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(U_n\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n}{\partial \nu}\right)ds,$$

where the integral is taken in the clockwise sense and ν indicates interior normal for Γ_2. The function $U_n(x, y)$ approaches $V_1(x, y)$ uniformly on Γ_2, and the derivatives of $U_n(x, y)$ approach uniformly the corresponding derivatives of $V_1(x, y)$; so we have for (x, y) exterior to Γ_2

$$V_1(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(V_1\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial V_1}{\partial \nu}\right)ds.$$

By the harmonic character of $V_2(x, y)$ on and within Γ_2, we may write (op. cit., p. 265, Lemma III) for (x, y) exterior to Γ_2

$$0 = \frac{1}{2\pi}\int_{\Gamma_2}\left(V_2 \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V_2}{\partial \nu}\right) ds;$$

so for (x, y) exterior to Γ_2 we have

$$V_1(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(V \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V}{\partial \nu}\right) ds.$$

If C_2 is an analytic Jordan curve, this integral can be taken over C_2 itself; by the constancy of $V(x, y)$, now assumed on C_2, we have for (x, y) exterior to C_2

$$V_1(x, y) = \frac{-1}{2\pi}\int_{C_2} \log r \frac{\partial V}{\partial \nu} ds. \tag{33}$$

Even if the Jordan curve C_2 is not analytic, equation (33) is valid if the integral is taken in an extended sense (op. cit., §7.6). If the points β'_{nk} are uniformly distributed on C_2 with respect to the parameter σ, where

$$d\sigma = -\frac{\partial V}{\partial \nu} ds,$$

it follows from (33) and the equation

$$\int_{C_2} d\sigma = 2\pi,$$

a consequence of (5), that

$$\lim_{n\to\infty} \left| (z - \beta'_{n1}) \cdots (z - \beta'_{nn}) \right|^{1/n} = e^{V_1(x,y)} \tag{34}$$

for (x, y) exterior to C_2, uniformly on any closed limited set exterior to C_2.

If now the given β_{nk} do not appear in (1) for every n, we need merely set $\beta_{nk} = \beta'_{nk}$ for the omitted values of n. Then the new set β_{nk} is defined for every n, and it follows from (1) and (34) that (1) holds uniformly on any closed bounded set exterior to C_2, when n takes all the values $1, 2, 3, \cdots$. Such equations as (2) and (3) apply to the new set β_{nk}.

These remarks on supplementing a given incomplete sequence β_{nk} apply without essential change to the more general situation outlined at the beginning of §3.

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