EXPONENT TRAJECTORIES IN SYMBOLIC DYNAMICS*

BY

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1. **Introduction.** Morse, Hedlund,† and others have developed the theory of dynamics from the symbolic point of view. This theory is concerned in the main with the periodicity, recurrency, and transitivity properties of symbolic trajectories and rays. Morse has made use of exponents on symbols. Unless a trajectory T is of a very special type, it can be shown that the exponents on the symbols in a symbolic trajectory T form a symbolic trajectory T_e termed the "exponent trajectory" of T. The trajectory T_e is uniquely determined by T. Similar considerations hold for rays. In the present paper we are concerned with relations between a trajectory or ray and the associated exponent trajectory or ray. In particular we prove that a periodic or recurrent trajectory T has a periodic or recurrent exponent trajectory T_e respectively, while a transitive ray R has an exponent ray R_e which is in a sense also transitive. Further, if a trajectory T is periodic, T is distinct from its exponent trajectory. There exist, however, trajectories identical with their exponent trajectories, and in the case of trajectories generated by the symbols 1, 2 only, there is *one and only one such trajectory*. The term "identical" is used here in the usual sense, and will be defined explicitly in the next section. In the paper referred to above, Morse and Hedlund have given some methods of constructing recurrent trajectories from a given recurrent trajectory. The introduction of exponent trajectories yields another method of constructing such trajectories. Whether or not there exist recurrent trajectories identical with their exponent trajectories is still an open question.

2. **Definitions and conventions.** We shall use the term "symbolic trajectory" in a slightly more general sense than that employed by Morse and Hedlund in that we shall allow an infinite set of generating symbols. Let S_1 denote a sequence $abc \cdots$ of symbols $a, b, c, \cdots$ which may or may not be taken from a finite set of distinct symbols, and let S_2 denote a second such sequence $\alpha\beta\gamma \cdots$. Let S_2^{-1} denote the sequence $\cdots \gamma\beta\alpha$ of symbols obtained from S_2 by reversing the order of the symbols in S_2. The sequence $S_2^{-1}S_1$, given by $\cdots \gamma\beta\alpha abc \cdots$, is termed a *symbolic trajectory*, or simply a *trajectory*. The sequence S_1 (also S_2^{-1}) is termed a *ray*. The symbol a in S_1 is termed the *initial symbol* of the ray S_1. We shall have occasion to use the notation

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† Marston Morse and Gustav A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics, vol. 60 (1938), pp. 815–866.

$S_1=abc\cdots$ meaning that S_1 is the sequence $abc\cdots$. A finite sequence $ab\cdots k$ of symbols is termed a *block*. If there are n symbols in the set $a, b, \cdots, k$, the block $ab\cdots k$ is said to be of *length* n, and will be called an *n-block*. If B is a block, the length of B will be denoted by $l(B)$. We shall write $B=ab\cdots k$ to indicate that B is the block $ab\cdots k$. If $B_1=a_1\cdots a_m$, $B_2=b_1\cdots b_n$, then B_1B_2 is the block $a_1\cdots a_mb_1\cdots b_n$. The blocks B_1 and B_2 are *the same* if $m=n$ and the symbol a_i is identical with the symbol b_i for each i in the range $1, 2, \cdots, n$. In a block $C=a_{-n}\cdots a_{-1}a_0a_1\cdots a_n$ of odd length, we term a_0 the *central symbol* of C. A trajectory T can be written as

$$\cdots a_{-2}a_{-1}a_0a_1a_2\cdots.$$

The symbols a_i and a_j are said to be in *different positions* in T if $i\neq j$. If $i=j$ these elements are in the *same position* in T. Let a_0 denote a symbol in a fixed position in a trajectory T_1. The trajectory T_1 is said to be *identical* with a trajectory T_2 if T_2 contains the symbol a_0 in a fixed position so that for each n the block A_n in T_1 of length $2n+1$ containing a_0 as central symbol is identical with the $(2n+1)$-block B_n of T_2 containing a_0 as central symbol.

Sequences of consecutive symbols of a trajectory T (or ray R or block B) which form a block or ray we term a *subblock* or *subray* of T (or R or B), and they are said to be *contained* in T (or R or B). As remarked above the symbols in a trajectory T (or ray R or block B) are taken from a finite or infinite set S of distinct symbols, which will be termed the *generating symbols* of T (or R or B). A block $a\cdots a$ formed by repeating the symbol a n times is written as a^n. The symbol n in a^n is termed the *exponent* of a in a^n, and a is termed the *base* in a^n. We term a^n a *power*. We write a block B as a sequence of powers such that the bases in consecutive powers are distinct. The exponents then form the *exponent block* B_e of B. Unless a trajectory T contains a subray formed by only one generating symbol, T can be written as a sequence

$$\cdots a^pb^qc^r\cdots, \tag{1}$$

where no two consecutive bases are identical. The exponents in (1) form a trajectory $\cdots pqr\cdots$, which we term the *exponent trajectory* T_e of T. Similarly, if a ray R does not contain a subray formed by one generating symbol, the ray R can be written as $a^pb^qc^r\cdots$, where consecutive bases are distinct. The exponents then form the *exponent ray* R_e of R. A trajectory T (or ray R) will be termed *admissible* if it has an exponent trajectory (or ray); that is, T (or R) does not contain a subray of the form $aaa\cdots$ or $\cdots aaa$.

A trajectory T is *periodic* if it can be written as a sequence

$$\cdots BBB\cdots \tag{2}$$

of blocks identical with a block B. If B is a block of shortest length such that T can be written as (2), the block B is said to be a *period block* of T, and its length is termed the *period* of T. A trajectory T is termed *recurrent* if for each n there exists an m such that each block of length n in T is contained in each m-block of T. If T is recurrent, for each n there exists a least m such that each m-block of T contains each n-block of T. We write $R(n) = m$, and term $R(n)$ the *recurrency function* of T. A ray R is said to be *transitive* if every possible block that can be formed from the generating symbols of R is a subblock of R.

3. **Periodicity, recurrence, and transitivity of exponent trajectories.** We shall now prove the following theorem.

THEOREM 1. *If a trajectory T in two or more generating symbols is periodic, T is admissible and the exponent trajectory T_e is periodic.*

Let B represent a period block of T so that T is given by (2). Suppose that B begins with the symbol a and is preceded by a in T. Then B is of the form $a^r b^s \cdots c^t a^u$, where no two consecutive symbols in the set $a, b, \cdots, c, a$ are identical. The block $C = a^w b^s \cdots c^t$, where $w = u + r$, is then also a period block of T. The block $C_e = ws \cdots t$ thus occurs in T_e, and T_e is of the form $\cdots C_e C_e C_e \cdots$, whence the theorem is proved.

THEOREM 2. *The exponent trajectory T_e of an admissible periodic trajectory T is distinct from T.*

As noted above T contains a period block $C = a^w b^s \cdots c^t$, where $a \neq c$, and the exponent block $C_e = ws \cdots t$ of C is a subblock of T_e. Evidently C_e or a subblock of C_e is a period block of T_e. The period of T is $\omega = w + s \cdots + t$. The period of T_e is no greater than the length L of C_e. If at least one of the symbols in C_e is greater than 1, we have $\omega > L$. If all of the symbols in C_e equal 1, the period of T_e is 1 and certainly less than ω.

Morse and Hedlund* have exhibited a nonperiodic recurrent trajectory T in four symbols with the property that consecutive symbols in T are distinct. If follows that in this case T_e is of the form

$$\cdots 111 \cdots . \tag{3}$$

Since (3) is periodic, there exist nonperiodic trajectories whose exponent trajectories are periodic. That this is not true of trajectories with two generating symbols is stated in the theorem which follows.

THEOREM 3. *An admissible trajectory T with two generating symbols is periodic if and only if its exponent trajectory T_e is periodic.*

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* See the reference to Morse and Hedlund above, p. 844.

Let the generating symbols be denoted by 1, 2. Let the period of T_e be denoted by ξ, and a period block of T_e by $B_e = a_1 \cdots a_\xi$. Let B be a block of T with exponent block B_e. If ξ is even, the first and last symbols of B are distinct, for $B = 1^{a_1}2^{a_2}1^{a_3} \cdots 2^{a_\xi}$ or $2^{a_1}1^{a_2}2^{a_3} \cdots 1^{a_\xi}$. Hence T is given by (2), and T is periodic. If ξ is odd, the first and last symbols of B are identical. It follows that T is given by

$$\cdots B_1B_2B_1B_2B_1 \cdots , \tag{4}$$

where $B_1 = 1^{a_1}2^{a_2}1^{a_3} \cdots 1^{a_\xi}$, $B_2 = 2^{a_1}1^{a_2}2^{a_3} \cdots 2^{a_\xi}$. Hence T is periodic.

THEOREM 4. *If the exponent trajectory T_e of a periodic trajectory T in two generating symbols has the period block $a_1a_2 \cdots a_\xi$, the trajectory T has the period ω, where*

$$\omega = \sum_{j=1}^{\xi} a_j, \tag{5}$$

$$\omega = 2\left(\sum_{j=1}^{\xi} a_j\right), \tag{6}$$

according as ξ is even or odd.

From the proof of Theorem 3 it follows in the case where ξ is even that the trajectory T is given by (2), where $B = 1^{a_1}2^{a_2}1^{a_3} \cdots 2^{a_\xi}$ or $2^{a_1}1^{a_2}2^{a_3} \cdots 1^{a_\xi}$. Hence $\omega \leqq (a_1 + \cdots + a_\xi)$. It is no restriction to suppose that $B = 1^{a_1}2^{a_2}1^{a_3} \cdots 2^{a_\xi}$. If B is not a period block, a subblock $1^{a_1}2^{a_2}1^{a_3} \cdots 2^{a_j}$, $j < \xi$, of B is a period block of T. Then T_e is given by $\cdots B_e' B_e' B_e' \cdots$, where $B_e' = a_1a_2 \cdots a_j$. The trajectory T_e thus has a period less than ξ, which is impossible. It follows that (5) is valid.

It follows from the proof of Theorem 3 that if ξ is odd the period of T is not greater than $2(a_1 + a_2 + \cdots + a_\xi)$, and that T is given by (4). By the argument of the preceding paragraph the period of T cannot be less than $(a_1 + a_2 + \cdots + a_\xi)$. Hence T has the period block $1^{a_1}2^{a_2} \cdots 1^{a_\xi}2^{a_1}1^{a_2} \cdots 2^{a_j}$ or the equivalent block with the symbols 1 and 2 interchanged. Thus T_e is given by $\cdots B_e'' B_e'' B_e'' \cdots$, where $B_e'' = a_1a_2 \cdots a_\xi a_1 a_2 \cdots a_j$. Since ξ divides the length of B_e, we have $j = \xi$, whence (6) is valid.

From Theorem 4 it is evident that the number of periodic trajectories of period ω with two generating symbols is the number of solutions of

$$\sum_{j=1}^{2n} a_j = \omega, \qquad 2\left(\sum_{j=1}^{2n+1} a_j\right) = \omega,$$

where the a's and n are integers, and the blocks $a_1a_2 \cdots a_\xi$ $(\xi = 2n, 2n+1)$ are not of the form $DD \cdots D$, that is, formed by the repetition of a block.

LEMMA 1. *A recurrent trajectory T with two or more generating symbols is admissible.*

Since T contains a block ab, where a and b are distinct, and this block cannot be contained in a subray with one generating symbol, it follows that each exponent is finite, and the exponent trajectory T_e exists.

LEMMA 2. *If an admissible trajectory T is recurrent, its exponent trajectory contains a finite number of generating symbols.*

Consider again a subblock ab of T where $a \neq b$. If there exists in T a sequence of blocks $a_1^{n_1}, a_2^{n_2}, \cdots$, where the sequence $n_1, n_2, \cdots$ is unbounded, then there exists an arbitrarily long block which does not contain ab. Hence T is not recurrent. Thus Lemma 2 is proved.

THEOREM 5. *If an admissible trajectory T is recurrent, the exponent trajectory T_e of T is recurrent.*

Consider a block $B_e = ps \cdots q$ of T_e. There is a corresponding block $B = a^p b^s \cdots c^q$ of T bordered on the left and right by symbols g and h respectively, where $g \neq a$ and $h \neq c$. Since T is recurrent, the block gBh occurs in each block of T of length $R(n)$, where n is the length of gBh, and $R(n)$ is the recurrency function of T. Thus in each subblock B' of T of length $R(n)$ there occurs a block $g^\alpha B h^\beta$, where $\alpha \geqq 1$, and $\beta \geqq 1$. Each block B' is contained in a block B'', where B'' is preceded in T by a symbol distinct from the first symbol of B'', and followed by a symbol distinct from the last symbol of B'', and the exponent block of B'' has the same length as the exponent block of B'. Evidently, B_e is contained in the block of exponents of each block B''. Let t be the maximum length of the exponent blocks of the blocks of type B''. We denote the exponent block of a block B'' by B_e''. Each exponent block C_e in T_e of length t corresponds to a block C of T which contains a block B'' as subblock. It follows that each block of T_e of length t contains B_e. Let r denote the length of B_e. There are a finite number of blocks $B_{e1}, B_{e2}, \cdots, B_{e\rho}$ in T_e of length r. There exist numbers $t_1, t_2, \cdots, t_\rho$ such that for each i $(i=1, 2, \cdots, \rho)$ B_{ei} is contained in each t_i-block of T_e. Let $R_e(r)$ denote the maximum of the numbers $t_1, t_2, \cdots, t_\rho$. Then each r-block of T_e is contained in each $R_e(r)$-block. Thus T_e is recurrent.

COROLLARY 1. *If T is a recurrent nonperiodic trajectory in two generating symbols, the exponent trajectory T_e of T is a recurrent nonperiodic trajectory.*

It is obvious that a non-recurrent trajectory T may have a recurrent exponent trajectory T_e. It is necessary even in the case of two generating symbols to impose an additional restriction on T_e to insure the recurrence of T. We shall give the additional restriction for the case of two generating symbols.

We say that a trajectory T in two generating symbols is *strongly recurrent* if for each n and n-block B in T there exists an integer $R(n)$ such that if B_1, B_2 are any nonoverlapping blocks of length $R(n)$, the block B_1 contains a block B whose first symbol is separated from the first symbol of a block B in B_2 by an odd number of symbols. An immediate result is the following theorem.

THEOREM 6. *An admissible trajectory T in two generating symbols is recurrent if and only if its exponent trajectory is strongly recurrent.*

Certain inequality relations exist between the recurrency function of a recurrent trajectory T and the recurrency function of the exponent trajectory T_e of T. For the sake of brevity these relations will be omitted.

That the following theorem is true appears from the definition of transitivity.

THEOREM 7. *A transitive ray in two or more generating symbols is admissible.*

THEOREM 8. *The exponent ray R_e of a transitive ray R in two or more generating symbols is transitive.*

It is evident that R_e has the infinite set $1, 2, 3, \cdots$ of generating symbols. We denote this set by S. Let $l, m, n, \cdots, p$ be an arbitrary subset of S containing μ symbols not necessarily distinct, and let $q, r, s, \cdots, t$ be a second subset of μ symbols in S not necessarily distinct. By assumption R contains at least two distinct generating symbols a, b. Since R is transitive, R contains the block $\alpha B \gamma$, where

$$B = \overset{l}{a_1}\overset{l}{a_2} \cdots \overset{l}{a_q}\overset{m}{a_{q+1}} \cdots \overset{m}{a_{q+r}}\overset{n}{a_{q+r+1}} \cdots \overset{n}{a_{q+r+s}} \cdots \overset{p}{a_{q+r+s+\cdots+u+1}} \cdots \overset{p}{a_{q+r+s+\cdots+u+t}},$$

$\alpha \neq a_1$, $\gamma \neq a_{q+r+s+\cdots+u+t}$, the exponent block of B is $B_e = l^q m^r n^s \cdots p^t$, and the a's are alternately equal to a and b so that $a_1 = a$, $a_2 = b$, $a_3 = a, \cdots$. Thus R_e contains each block B_e that can be formed from the symbols in S, whence R_e is transitive.

Theorem 8 can be extended to "transitive trajectories" with no subray generated by one symbol only.

4. **A trajectory identical with its exponent trajectory.** In Theorem 2 we noted that a periodic trajectory is distinct from its exponent trajectory. That this is not true for trajectories in general is a consequence of the theorem which follows.

THEOREM 9. *There exists a trajectory identical with its exponent trajectory.*

We let B_0 denote the block 212, and let $B_1 = 2$. We form the trajectory

(7) $$\cdots B_3^{-1}B_2^{-1}B_1^{-1}B_0B_1B_2B_3\cdots,$$

where B_i is the exponent block of B_{i+1} for each $i>0$, the last symbol of B_i is distinct from the first symbol of B_{i+1} for each $i>0$, and B_i^{-1} denotes the block obtained from B_i by reversing the symbols in B_i. We illustrate by giving some of the blocks B_i explicitly:

$$B_2 = 11,\ B_3 = 21,\ B_4 = 221,\ B_5 = 22112,\ B_6 = 11221211.$$

We note that for $i>0$ the block B_i^{-1} is the exponent block of B_{i+1}^{-1}. Thus (7) is the sequence

(8) $$\cdots, 21122, 122, 12, 11, 2, 212, 2, 11, 21, 221, 22112, \cdots,$$

where we have separated the blocks B_i and B_i^{-1} by commas. The exponent block of $B_1^{-1}B_0B_1$ is B_0. From this statement and the definition of (7), it appears that the exponent block of $B_r^{-1}\cdots B_2^{-1}B_1^{-1}B_0B_1B_2\cdots B_r$ is the block $B_{r-1}^{-1}\cdots B_1^{-1}B_0B_1\cdots B_{r-1}$. Thus (7) has an exponent trajectory and is identical with it.

Employing the same technique as that used in constructing (7) and using more than two symbols, one can construct an unlimited number of trajectories identical with their exponent trajectories. We shall prove later the uniqueness of (7) for the class of trajectories in two generating symbols 1, 2.

5. **Proper exponent blocks and join-blocks in trajectories with generating symbols** 1, 2. Consider an arbitrary subblock B of a trajectory T in generating symbols 1, 2 where the exponent trajectory T_e of T contains the same generating symbols. The block B has an exponent block B_e which does not necessarily occur as a subblock of the exponent trajectory T_e of T since B may be preceded by or followed by a symbol identical with the first or last symbol of B respectively. For this reason we associate with B a new type of exponent block. Consider the block B_e of exponents of B *which occur in* T_e and can be determined without reference to T from B alone and the fact that the exponents equal 1 or 2. We term B_e the *proper exponent block of* B. We similarly speak of a *proper exponent ray*. We let C_1, C_2 be consecutive subblocks of the trajectory T so that C_1C_2 is a subblock of T. We denote the proper exponent blocks of C_1 and C_2 by D_1 and D_2 respectively. The proper exponent block of C_1C_2 is a block D_1JD_2. We shall say that J is the *exponent block due to the join of* C_1 *and* C_2. Obviously, J is either vacuous, or is one of the blocks 1, 2, or 11.

THEOREM 10. *Let T_e be the exponent trajectory of a trajectory T, and suppose that T and T_e have the same generating symbols* 1, 2. *The length of the proper exponent block B_e of a block B in T satisfies the formula*

(9) $$L(B_e) \leqq L(B) - 2$$

if $B \neq \alpha, \alpha^2$ $(\alpha = 1, 2)$. *If B has an intermediate block* 1^2 *or* 2^2, *then*

(10) $$L(B_e) \leqq L(B) - 3.$$

In any case $L(B_e) \leqq L(B) - 1$. We write $B = a_1 a_2^r a_3^s \cdots a_{n-1}^t a_n$, $n \geqq 2$, where the a's are distinct and alternate between 1 and 2. Obviously, $L(B_e) = n - 2$, and $L(B) \geqq n$. If $B = a_1^2 a_2^r a_3^s \cdots a_{n-1}^t a_n$, then $L(B_e) = n - 1$, $L(B) \geqq n + 1$. If finally $B = a_1^2 a_2^r a_3^s \cdots a_{n-1}^t a_n^2$, then $L(B_e) = n$, $L(B) \geqq n + 2$. Thus (9) is valid. The validity of (10) is obvious.

Theorems 11–13 to follow will be needed in a later section.

THEOREM 11. *Let* T_{ee} *and* T_e *be the exponent trajectories of trajectories* T_e *and* T *respectively, and suppose that* T, T_e, *and* T_{ee} *have the generating symbols* 1, 2. *Let JED be a subblock of T, and suppose that the blocks J, E, and D are so related that E is the proper exponent block of D, while J is the exponent block due to the to the join of E and D. If* $L(D) \geqq 4$, *then*

(11) $$L(JE) < L(D).$$

We write $D = GH$, where G is a block of length 4. We let J_e denote the exponent block due to the join of G and H, and let G_e, H_e denote the proper exponent blocks of G and H respectively. We have the following relations:

(12) $$L(JE) = L(J) + L(E),$$

(13) $$L(E) = L(G_e) + L(J_e) + L(H_e).$$

We consider first the case where G begins with the block α^2. By the assumption $L(G) = 4$ we have $G \neq \alpha, \alpha^2$, whence by Theorem 10 the relation $L(E) \leqq L(D) - 2$ follows. Since D begins with α^2, the block J contains no exponent arising from D. Hence J is vacuous or 1, whence $L(J) \leqq 1$. It follows by (12) that (11) is valid.

Next, we suppose that G begins with $\alpha\beta$ $(\alpha \neq \beta)$. If $G = \alpha\beta\alpha\alpha$, then $G_e = 12$. If H is vacuous, $E = 12$ and $ED = 12\alpha\beta\alpha\alpha$. If $\alpha = 1$, the proper exponent block of $12\alpha\beta\alpha\alpha$ contains a subblock 1^3, which is impossible in view of the fact that T_{ee} contains only the symbols 1, 2. Hence $\alpha = 2$, and $J = 2$. Thus $L(JE) = 3$, and (11) holds. If H is not vacuous, the block GH begins with $\alpha\beta\alpha\alpha\beta$ since T_e contains only the generating symbols 1, 2. By Theorem 10, $L(E) \leqq L(D) - 3$. Since $L(J) \leqq 2$, formula (11) is valid. We now let $G = \alpha\beta\beta\alpha$. Since we have an intermediate block β^2, by Theorem 10 we have $L(E) \leqq L(D) - 3$, whence (11) holds. If finally $G = \alpha\beta\alpha\beta$, G is preceded in T by α since we cannot have a block 1^3 in T_e. Then $J = 2$, and $L(J) = 1$. By Theorem 10 we have $L(E) \leqq L(D) - 2$, whence (11) holds. Thus in any case (11) is valid.

THEOREM 12. *Let T_e be the exponent trajectory of a trajectory T, and let T and T_e be trajectories in the generating symbols* 1, 2. *Let JED be a subblock of T, where J, E, and D are related as in Theorem* 11. *If $L(D) \geqq 4$, then $L(JE) \geqq 2$.*

If the leading 4-block of D is of the form $\alpha\alpha\beta\beta$, $\alpha\alpha\beta\alpha$, $\alpha\beta\alpha\alpha$, or $\alpha\beta\alpha\beta$ $(\alpha \neq \beta)$, the proper exponent block of this block is of length 2, whence $L(JE) \geqq 2$. If the leading 4-block of D is of the form $\alpha\beta\beta\alpha$, this block has the proper exponent block 2, whence $L(E) \geqq 1$. The leading symbol α of D will yield an exponent in J. Thus in any case $L(JE) \geqq 2$.

THEOREM 13. *Let T, T_e, J, E, and D be defined as in Theorem* 12. *If J is non-vacuous, then E is non-vacuous.*

6. **Subrays of a trajectory identical with its exponent trajectory.** The theorem which follows is valid for trajectories based on an arbitrarily given set of generating symbols, and is not restricted to the 1, 2 case.

THEOREM 14. *If a trajectory T is identical with its exponent trajectory T_e, the trajectory T does not contain two identical subrays R_1, R_2 with initial elements in different positions in T.*

Suppose that the rays R_1, R_2 are directed to the right in the sense that $R_1 = R_2 = abc \cdots$. The rays R_1 and R_2 overlap, whence it is no restriction to suppose that R_1 overlaps R_2. Let the subblock of R_1 which precedes R_2 in R_1 be denoted by B. Since $R_1 = R_2$, the ray R_2 contains a subray R_3 identical with R_2 and preceded in R_2 by the block B. Thus T contains the subray $N = BBB \cdots$. Since $T = T_e$, the trajectory T_e contains a subray N_1 identical with the ray N. Let N_e denote the proper exponent ray of N. The rays N_1 and N_e overlap in T_e. Therefore the ray N_e contains a subray N_2 identical with N. Clearly, N_2 is the exponent ray of a subray $N_3 = B_1B_1B_1 \cdots$ of N where B is the exponent block of B_1. Since the ray N_3 is a subray of the ray N, and $l(B_1) \geqq l(B)$, we can write B_1 as $B_{11}B^rB_{12}$, where $B_{12}B_{11} = B$, and $r \geqq 0$. If $r = 0$ it is understood that the block B^r is vacuous. Thus the trajectory $T_1 = \cdots B_1B_1B_1 \cdots$ obtained by continuing N_3 to the left is identical with the trajectory $T_2 = \cdots BBB \cdots$. But T_2 is the exponent trajectory of T_1, whence by Theorem 2 we have arrived at a contradiction.

7. **The uniqueness of a trajectory identical with its exponent trajectory in the case of generating symbols** 1, 2. We shall prove in this section that the trajectory (7) is the only one of its kind for trajectories in generating symbols 1, 2. We let T^{-1} denote the trajectory obtained from a trajectory T by reversing the order of the symbols in T.

LEMMA 3. *If a trajectory T is identical with its exponent trajectory T_e, and T*

contains the generating symbols 1, 2 *only, the trajectory* T *or* T^{-1} *contains a subray*

(14) $$R = B_1' B_2' B_3' \cdots,$$

where B_i' *is the exponent block of* B_{i+1}', *and the last symbol of* B_i' *is different from the first symbol of* B_{i+1}' *for each* i.

Let a denote a symbol of T in a *fixed position* in T. The corresponding symbol a of the exponent trajectory T_e is the exponent of a symbol b in T so that the block b^a occurs in T. It is no restriction to assume that the block b^a is not to the left of the symbol a in T. We suppose first that the block b^a of T does not contain the symbol a, so that b^a is to the right of a in T.

We let B_1' denote the block of symbols in T starting with a and ending with the symbol preceding the block b^a in T. Since $T = T_e$, the symbol a in T_e is the initial symbol of a block B_1' in T_e. The block of T starting with b^a and having B_1' as exponent block is unique since consecutive exponents in T are exponents on distinct bases alternating between the symbols 1, 2. We emphasize that T is of the form

$$\cdots 1^{a_{-2}} 2^{a_{-1}} 1^{a_0} 2^{a_1} 1^{a_2} \cdots.$$

We denote the block of T starting with b^a and having exponent block B_1' by B_2'. Thus T contains the block $B_1' B_2'$. We assume now that T contains the block $B_1' B_2' \cdots B_r'$ where $B_1' B_2' \cdots B_{r-1}'$ is the exponent block of $B_2' B_3' \cdots B_r'$. Since $T = T_e$, the block $B_1' B_2' \cdots B_{r-1}'$ in T_e is followed by B_r', whence $B_1' B_2' \cdots B_r'$ in T is followed by a block B_{r+1}' whose exponent block is B_r', and the first symbol of B_{r+1}' is distinct from the last symbol in B_r'. Thus T contains the subray R.

Finally, we suppose that the block b^a of T contains the symbol a of T. If $a = 1$, then b^a is the block 1^1. Since the bases alternate between 1 and 2, the block b^a is preceded and followed in T by the base 2. Thus T contains the block $B_0 = 2a2 = 212$, where B_0 is the block B_0 occurring in (7). Since $T = T_e$, the symbol a in T_e is preceded and followed by 2 in T_e, whence a is the central symbol in a block B_0 of T_e. It follows that B_0 is the exponent block of a block $B_1^{-1} B_0 B_1$ in T with central symbol a and $B_1 = 2$. Making use of the equality $T = T_e$ and developing T to the right and left of $B_1^{-1} B_0 B_1$ as in §4 we obtain (7). The subray

(15) $$B_1 B_2 B_3 \cdots$$

of (7) is clearly a subray of the type (14). If now $a = 2$, the symbol a in T is either the leading or final symbol in the block b^a, so that b^a is either $a2$ or $2a$. If $b^a = a2$, then since $a = 2$, the block b^a is preceded in T by the symbol 1, and thus this block is preceded in T_e by the symbol 1. Thus the block 1^1 precedes b^a in T, and since both base and exponent in 1^1 are followed by a in T and T_e respectively, the base and exponent in the power 1^1 are corresponding symbols. The argument thus reduces to the preceding case where $a=1$. If now $a=2$, while $b^a=2a$, the block b^a is followed in T_e by the symbol 1, so that the block b^a in T is followed by the power 1^1. Clearly the base and exponent in this power are corresponding symbols, whence we have again reduced the argument to the case where $a=1$. Thus T contains the subray (14), and Lemma 3 is proved, for if b^a is to the left of a in T, then b^a is to the right of a in T^{-1}.

We remark that the exponent ray R_e of $B_2' B_3' B_4' \cdots$ in (14) is the ray R.

We consider now a trajectory T with $T=T_e$, whence by the lemma just proved T contains the subray R of (14). We let E_0 be a block such that the ray

$$E_0 B_1' B_2' \cdots \tag{16}$$

is the proper exponent ray of R in (14). The block E_0 may be vacuous. Since $T=T_e$, the trajectory T contains (16) as a subray. We let J_0 denote the exponent block due to the join of E_0 and B_1' in E_0B_1'. It is clear that T contains the subray $J_0E_0B_1'B_2' \cdots$. For $i>0$, we let E_i denote the proper exponent block of a block $J_{i-1}E_{i-1}$, and J_i the exponent block due to the join of E_i and $J_{i-1}E_{i-1}$. In the following lemma we use G_i to denote the block $J_iE_i \cdots J_1E_1J_0E_0$, and R as in (14). Here $G_{i+1}=G_i$ if $J_{i+1}E_{i+1}$ is vacuous.

Lemma 4. *If a trajectory T is equal to its exponent trajectory T_e, and T contains the subray*

$$G_iR, \tag{17}$$

the trajectory T contains the subray

$$G_{i+1}R, \tag{18}$$

where $J_{i+1}E_{i+1}$ in G_{i+1} may be vacuous.

We assume that T contains the subray (17). The proper exponent ray of (17) is the ray

$$E_{i+1}G_iR. \tag{19}$$

Since $T=T_e$, the trajectory T contains the subray (19). Evidently the proper exponent ray of (19) contains the subray (18).

Theorems 11 and 12 yield at once the following lemma.

Lemma 5. *If the subray (17) in a trajectory T with $T=T_e$ is continued to the left, one arrives at a block $J_\sigma E_\sigma$ of length 2 or 3, provided the trajectory T contains a subblock J_jE_j of length at least 2.*

LEMMA 6. *If a trajectory T in generating symbols 1, 2 is identical with its exponent trajectory T_e, the trajectory T or T^{-1} contains a subray identical with the subray* (15) *of* (7).

By Lemma 3, the trajectory T or T^{-1} contains a subray (14). Suppose that T contains (14). We continue the subray (14) of T to the left to obtain a subray (17) of T where E_{i+1} is vacuous. We suppose first that the subray (17), which is explicitly the ray

$$J_iE_i \cdots J_1E_1J_0E_0B_1'B_2'B_3' \cdots$$

contains a block J_jE_j of length at least 2, whence by Lemma 5 the ray (17) contains a subblock $J_\sigma E_\sigma$ of length 2 or 3.

We assume that $L(J_\sigma E_\sigma)=2$. If $J_\sigma E_\sigma=22$, then $E_{\sigma+1}=2$, and $E_{\sigma+1}J_\sigma E_\sigma$ contains the subblock 2^3. Hence $J_\sigma E_\sigma\neq 22$. Suppose that $J_\sigma E_\sigma=12$. We cannot have $J_\sigma=12$, since by Theorem 13 the block E_σ is then not vacuous. Also, we cannot have $E_\sigma=12$ since the symbol 2 in E_σ yields an exponent in J_σ due to the join of E_σ with the block following E_σ in T, whence J_σ is not vacuous. Thus $J_\sigma=1$, and $E_\sigma=2$. Then $J_\sigma E_\sigma$ is followed by a block 122 in T. Now $J_\sigma E_\sigma 122$ has the proper exponent block $112=1J_\sigma E_\sigma$. Since $T=T_e$, the trajectory T contains the subray $R_a=112G_{\sigma-1}R$, where if $\sigma=0$, we understand that $G_{\sigma-1}=G_{-1}$ is vacuous. The leading block 11 in R_a has exponent 2, whence T contains the subray $R_b=KG_\sigma R$, where $K=21$. The proper exponent ray of R_b is R_b itself. We write B_1 for the symbol 2 in K, and B_2 for the block 11 in KJ_σ. The exponent of B_2 in R_b is the initial symbol of the proper exponent ray of R_b. If we define $B_3, B_4, \cdots$ as in §4, it is clear that R_b is identical with the ray (15). If $J_\sigma E_\sigma=11$, then $E_{\sigma+1}=2$. Writing $B_1=B_{\sigma+1}$, $B_2=J_\sigma E_\sigma$, defining B_i $(i>2)$ as in §4, and using the fact that (19) is the proper exponent ray of (17), we find that in this case (17) with $i=\sigma+1$ is identical with the subray (15) of (7). Finally, we write $J_\sigma E_\sigma=21$. If $E_\sigma=21$, the symbol 1 in E_σ yields an exponent so that J_σ is not vacuous. Hence $J_\sigma=2$, $E_\sigma=1$. Now $J_\sigma E_\sigma$ is followed in T by the block 121. Writing B_1 for J_σ, and B_2 for the block 11 which follows J_σ in $J_\sigma E_\sigma 121$, and defining B_i $(i>2)$ as in §4, we find that in this case (17) with $i=\sigma$ is identical with (15).

If $J_\sigma E_\sigma=221$, then $E_{\sigma+1}=2$ and $E_{\sigma+1}J_\sigma E_\sigma=2^31$ which is impossible. If $J_\sigma E_\sigma=211$, then $E_{\sigma+1}=2$, $J_{\sigma+1}=2$, which in the paragraph above was proved impossible. If $J_\sigma E_\sigma=212$, then $E_{\sigma+1}=J_{\sigma+1}=1$, which case was treated above. If $J_\sigma E_\sigma=121$, then $E_{\sigma+1}=1$, $J_{\sigma+1}=2$, which was also treated above. If $J_\sigma E_\sigma=112$, then $E_{\sigma+1}=2$, and $J_{\sigma+1}$ is vacuous. Writing $B_1=E_{\sigma+1}$, and B_2 for the leading block 11 of $J_\sigma E_\sigma$, and defining B_i $(i>2)$ as in §4, it is clear that (17) with $i=\sigma+1$ is in this case identical with (15). If $J_\sigma E_\sigma=122$, then

$J_{\sigma+1}E_{\sigma+1}=12$, which case was treated above. This completes the cases where $L(J_\sigma E_\sigma)=3$, since the blocks 111 and 222 cannot occur in T.

Suppose now that T contains no block $J_\sigma E_\sigma$ with $L(J_\sigma E_\sigma)\geqq 2$. Assume that T contains a subblock $J_\sigma E_\sigma$ with $L(J_\sigma E_\sigma)=1$. We cannot have $J_\sigma E_\sigma=1$, whence $E_\sigma=1$, since E_σ is the proper exponent block of 121 or 212, and the block $E_\sigma 121$ or $E_\sigma 212$ yields a non-vacuous block J_σ due to the join of E_σ with 121 or 212. If $J_\sigma E_\sigma=2$, then $E_\sigma=2$, and E_σ is followed by 11 in T. Writing $B_1=J_\sigma E_\sigma$, $B_2=11$, and defining B_i $(i>2)$ as in §4, we find that T contains the subray (15).

We suppose, finally, that T contains no subray (17) with a block $J_\sigma E_\sigma$ for which $L(J_\sigma E_\sigma)\geqq 1$. Thus the subray (14) of T cannot be continued to the left. The block B_1' cannot be of length greater than or equal to 3 since then B_1' would yield a non-vacuous block E_0. In the same way $B_1'\neq 11$, 22. If $B_1'=12$, then $B_2'=122$, and $E_0=1$, whereas if $B_1'=21$, then $B_2'=221$, and $E_0=1$. Thus $L(B_1')=1$. If $B_1'=1$, then $B_2'=2$, $B_3'=11$. In this case dropping the first block in (14), we obtain the subray (15) of (7) as a subray by writing $B_i=B_{i+1}'$, $i\geqq 1$. If $B_1'=2$, writing $B_i=B_i'$ we obtain (15) from (14).

Thus in any case the trajectory T or T^{-1} contains the subray (15) of (7).

THEOREM 15. *There is one and only one trajectory T in generating symbols* 1, 2 *identical with its exponent trajectory.*

By Theorem 14 and Lemma 6 the trajectory T or T^{-1} contains the subray (15) of the trajectory (7) exactly once. Suppose that (15) is a subray of T. If (15) is preceded by the symbol 2 in T, the subray $\rho_1=2B_1B_2\cdots$ of T must be preceded by the symbol 1, since no block 2^3 can occur in T. Thus T contains the subray $\rho_2=12B_1B_2\cdots$. The proper exponent ray of ρ_1 is ρ_1 itself. In particular since the proper exponent ray ρ_1 of ρ_2 is preceded by the symbol 1, the ray ρ_2 is preceded in T by the symbol 2, so that $212B_1B_2\cdots$ occurs in T. We write $B_0=212$, whence the ray $\rho_3=B_0B_1B_2\cdots$ occurs in T. Since ρ_3 occurs in T_e, the ray $\rho_4=2212B_1B_2\cdots$ occurs in T, whence $B_1^{-1}B_0B_1B_2\cdots$ occurs in T. By induction, since

$$B_{r-1}^{-1}\cdots B_1^{-1}B_0B_1B_2\cdots$$

occurs in T, the same ray occurs in T_e, and is the exponent ray of the ray

$$B_r^{-1}\cdots B_1^{-1}B_0B_1B_2\cdots.$$

Thus T is identical with (7).

If, on the other hand, the subray (15) is preceded by 1 in T, the trajectory T contains either the subray $R_1=21B_1B_2\cdots$ or the subray $R_2=11B_1B_2\cdots$. The proper exponent rays of R_1 and R_2 are respectively R_2 and R_1. Since

$T=T_e$, the trajectory T then contains both R_1 and R_2. Since $R_1\neq R_2$, the subray (15) in R_1 and R_2 occurs twice in T, contradicting Theorem 14.

If T is the trajectory (7), then $T=T^{-1}$. Thus Theorem 15 is proved.

THEOREM 16. *The trajectory* (7) *is the exponent trajectory of two distinct trajectories in generating symbols* 1, 2.

Theorem 16 states that (7) is not symmetric in the symbols 1, 2. Suppose, on the contrary, that (7) is unchanged when we interchange the symbols 1 and 2. Let C_i be the block obtained from B_i by interchanging 1 and 2 in B_i. Then we have the trajectory

$$\cdots C_2^{-1}C_1^{-1}C_0C_1C_2\cdots . \tag{20}$$

We remark that the block C_i, $i\geqq 2$, has the exponent block B_{i-1}, whence C_i^{-1} has the exponent block B_{i-1}^{-1}. The block $C_1^{-1}C_0C_1$ has the exponent block B_0. We note that the symbol 2 in $C_1^{-1}C_0C_1=11211$ yields the exponent 1 in B_0. Now the ray $C_1^{-1}C_0C_1C_2\cdots$ has the proper exponent ray $\sigma=B_0B_1B_2\cdots$. If the trajectory (20) is identical with the trajectory (7), the trajectory (20) contains a subray $\sigma'=B_1^{-1}\sigma$ with proper exponent ray σ, where the symbol 1 in the subblock B_0 of the exponent ray σ is the exponent of the symbol 1 in the subblock B_0 of the ray σ'. Thus the exponent trajectory (7) of (20) contains the subray σ twice with initial symbols of each σ in different positions in (7). By Theorem 14 we have arrived at a contradiction.

Although (7) is the exponent trajectory of two distinct trajectories T_1 and T_2 in generating symbols 1, 2, the trajectories T_1 and T_2 are equivalent in the sense that these trajectories differ only in the notation used for the generating symbols.

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