THE DIFFERENTIAL GEOMETRY OF SERIES OF LINEAL ELEMENTS*

BY
JOHN DE CICCO

1. **Introduction.** We shall begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A *turn* T_α converts each element into one having the same point and making a fixed angle α with the original direction. By a *slide* S_k, the line of the element remains the same and the point moves along the line a fixed distance k. These transformations together generate a continuous group of three parameters which we call the *whirl group* W_3. The group of whirls W_3 is isomorphic to the group of rigid motions M_3. These two three-parameter groups are commutative and together form a new group of six parameters which we term the *whirl-motion group* G_6. In preceding papers (see the bibliography at the end of this paper), Kasner and the author developed the geometry of this group G_6. In this paper, we wish to give the differential geometry of the series of lineal elements in the plane with respect to the whirl-motion group G_6.

A set of ∞^1 elements is called a *series*: this includes a union (curve or point) as a special case. A collection of ∞^2 elements is termed a *field*, which of course corresponds to a differential equation of first order, $F(x, y, y')=0$. The totality of ∞^3 elements of the plane is called the *opulence*.

A *turbine* is the series which is obtained by applying a turn T_α to each element of an oriented circle (the outer circle). It is said to be *nonlinear* or *linear* according as the base circle is not or is a straight line. A *nonlinear flat field* consists of the ∞^2 elements cocircular with a given element, called the *center* or *central element*. A *linear flat field* is the set of ∞^2 elements on the ∞^1 oriented lines, which are parallel and possess the same orientation.

In this paper, we shall consider the tangent turbines and the osculating flat fields of a series of lineal elements. We shall find the necessary and sufficient conditions that ∞^1 limaçon (circular) series be the osculating limaçon (circular) series of a general (equiparallel) series (Theorems 11 and 16). We shall define the *curvature* and *torsion* of any series (formulas (38), (39), and (47)). The curvature and torsion of a series $\bar{S}$ conjugate to a given series S will be obtained in terms of the curvature and torsion of the given series S. Finally we shall find that *any two general (equiparallel) series, which have their curvatures and torsions the same functions of the angle u (arc*

* Presented to the Society, March 26, 1937; received by the editors February 16, 1939.

length s), are equivalent under the whirl-motion group G_6 (Theorems 20 and 21). This then gives us the intrinsic equations of a series of lineal elements in the geometry of the whirl-motion group G_6.

For the analytic representation, it will be convenient to define an element by the hessian coordinates (u, v, w) where v is the length of the perpendicular from the origin, u is the angle between the perpendicular and the initial line, and w is the distance between the foot of the perpendicular and the point of the element.

2. **The tangent turbines of a general series.** Any series which consists of ∞^1 nonparallel elements is termed a *general series*, whereas an *equiparallel series* consists of ∞^1 parallel elements. Thus a general series is never contained in a linear flat field, while an equiparallel series always lies in a linear flat field.

Any general series is given by the equations

$$v = v(u), \qquad w = w(u), \tag{1}$$

while any equiparallel series is given by the equations

$$u = c, \qquad w = w(v), \tag{2}$$

where c is a constant.

The points of the elements of a series form a union which we call the *point-union* of the series. The lines of the elements of a general series are the tangent lines of a union which is called the *line-union* of the general series. For an equiparallel series, there is no line-union since the lines of the element all have a common direction. The point-union is called the *base curve* of the equiparallel series.

A nonlinear turbine is a general series. Its point-union is a circle, called the *outer circle*, and its line-union is also a circle, called the *inner circle*. These two circles are concentric, and their common center is called the center of the turbine. Of course, the inner circle is in the interior of the outer circle.

From the preceding remarks, we may have the following construction for a nonlinear turbine in addition to the one given in §1. A nonlinear turbine is the series which is obtained by applying a slide S_s to each element of an oriented circle (the inner circle). From this, we find that the equations of a nonlinear turbine are

$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u + s, \tag{3}$$

where (a, b) are the cartesian coordinates of the center, r is the radius of the inner circle, and s is the constant distance of the slide S_s. We call (a, b, r, s) a set of nonlinear turbine coordinates.

From (3) or by synthetic reasoning, it may be shown that (1) two elements which are not simultaneously parallel and of the same orientation determine a unique nonlinear turbine, and (2) two nonlinear turbines possess either one common element or no common elements.

If a one-parameter family of series has the property that consecutive series have a common element, then the family is called a *set of enveloping series*. The locus of intersection of consecutive series of the family is called the *envelope* of the family. Thus the one-parameter family of general series $v=v(u, t)$, $w=w(u, t)$ is a set of enveloping series if and only if the equations $v_t=0$ and $w_t=0$ have a common solution in u. The envelope is then given by the two eliminants with respect to t of these four equations.

Let two series S_1 and S_2 possess a common element E_0. These two series are said to be *tangent* (or to have contact of first order) at E_0 if and only if they have two consecutive elements in common at E_0. Thus the two general series S_1: $v=v_1(u)$, $w=w_1(u)$, and S_2: $v=v_2(u)$, $w=w_2(u)$ are tangent at the common element $E_0(u_0, v_0, w_0)$ if and only if

$$\begin{aligned} v_0 &= v_1(u_0) = v_2(u_0), & w_0 &= w_1(u_0) = w_2(u_0), \\ v_1'(u_0) &= v_2'(u_0), & w_1'(u_0) &= w_2'(u_0). \end{aligned} \tag{4}$$

Let S_t: $v=v(u, t)$, $w=w(u, t)$ denote a one-parameter family of enveloping series, and let S denote the envelope of this family. From the equations of the envelope S and by (4), it easily follows that *any series S_t of the one-parameter family of enveloping series is tangent to the envelope S at any one of their common elements.*

If a one-parameter family of turbines is an enveloping set of turbines, then we shall say that the turbines are the *tangent turbines* of the envelope.

THEOREM 1. *The ∞^1 nonlinear turbines*

$$v = a(t) \cos u + b(t) \sin u + r(t), \quad w = -a(t) \sin u + b(t) \cos u + s(t) \tag{5}$$

constitute a set of tangent turbines if and only if

$$a'^2 + b'^2 = r'^2 + s'^2. \tag{6}$$

For, this is the condition that the equations

$$a' \cos u + b' \sin u + r' = 0, \qquad -a' \sin u + b' \cos u + s' = 0 \tag{7}$$

be compatible in u.

The envelope of the ∞^1 nonlinear turbines is given by the equations (5) and (7). Solving (7) for $\cos u$ and $\sin u$, we obtain the

COROLLARY. *The series to which the nonlinear turbines of Theorem* 1 *are the* *tangent turbines either consists of one element or is a general series. It is given by the equations*

$$\cos u = \frac{-a'r' - b's'}{a'^2 + b'^2}, \qquad \sin u = \frac{a's' - b'r'}{a'^2 + b'^2}, \tag{8}$$
$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u + s.$$

The envelope (8) of the tangent turbines is given by the equations (5), where the value of t in terms of u is defined by the equations (7). If equations (5), subject to the conditions (7), are differentiated totally with respect to u, the resulting equations are

$$v' = -a \sin u + b \cos u, \qquad w' = -a \cos u - b \sin u, \tag{9}$$

where the accent denotes total differentiation with respect to u. But these equations and (5) may be solved for a, b, r, s. Thus, we have established the following result.

THEOREM 2. *The tangent turbines of the general series* (1) *are the nonlinear turbines whose parameter values are*

$$a = -v' \sin u - w' \cos u, \qquad b = v' \cos u - w' \sin u, \tag{10}$$
$$r = v + w', \qquad s = -v' + w,$$

where the accent denotes differentiation with respect to u.

It is noted that, *if a general series is a curve, then the tangent turbines are the osculating circles of the curve.*

A tangent turbine of a general series S at an element E may be defined as the unique limiting turbine of the set of nonlinear turbines such that any nonlinear turbine of this set contains the element E and any other nearby element of S.

3. **The tangent turbines of an equiparallel series.** A linear turbine is the series which is obtained by applying a turn T_ω to each element of an oriented straight line. Thus a linear turbine is an equiparallel series whose base curve is a straight line. The equations of a linear turbine are

$$u = U - \omega, \qquad v \cos \omega + w \sin \omega = V, \tag{11}$$

where (U, V) are the hessian coordinates of the base line and ω is the constant angle of the turn T_ω.

By the same process of reasoning as that used in the preceding section, we obtain the following results.

THEOREM 3. *The* ∞^1 *linear turbines*

$$u = U(t) - \omega(t), \qquad v \cos \omega(t) + w \sin \omega(t) = V(t) \tag{12}$$

constitute a set of tangent turbines if and only if

$$U' = \omega' \neq 0. \tag{13}$$

COROLLARY. *The series to which the linear turbines of Theorem 3 are the tangent turbines either consists of one element or is an equiparallel series. It is given by the equations*

$$u = U - \omega = \text{const.}, \quad v = V \cos \omega - \frac{V'}{\omega'} \sin \omega, \quad w = V \sin \omega + \frac{V'}{\omega'} \cos \omega. \tag{14}$$

THEOREM 4. *The tangent turbines of the equiparallel series* (2) *are the linear turbines whose parameter values are*

$$U = c - \text{arc tan} \frac{1}{w'} + n\pi, \quad V = \pm \frac{vw' - w}{(1 + w'^2)^{1/2}}, \quad \omega = - \text{arc tan} \frac{1}{w'} + n\pi, \tag{15}$$

where the accent denotes differentiation with respect to v.

A tangent turbine of an equiparallel series S at an element E may be defined as the unique limiting turbine of the set of linear turbines such that any linear turbine of the set contains the element E and any other nearby element of S.

It may be now observed that two series at a common element E are tangent at E if and only if they have the same tangent turbine at E.

4. **Conjugate series of elements.** Two turbines T and $\overline{T}$ are said to be *conjugate* if they have the same circle as point locus and the elements of the two turbines are symmetrically related to the elements of the circle.

Two series S and $\overline{S}$ are said to be conjugate if there exists a one-to-one correspondence between their elements in such a way that the tangent turbines of the two series at the corresponding elements are conjugate turbines.

By Theorem 1, we find

THEOREM 5. *For any general series S, there always exists one and only one conjugate series $\overline{S}$ which either consists of one element or is a general series. This series $\overline{S}$ is given by the equations*

$$\begin{aligned} \cos \bar{u} &= \frac{-a'r' + b's'}{a'^2 + b'^2}, & \sin \bar{u} &= \frac{-a's' - b'r'}{a'^2 + b'^2}, \\ \bar{v} &= a \cos \bar{u} + b \sin \bar{u} + r, & \bar{w} &= -a \sin \bar{u} + b \cos \bar{u} - s, \end{aligned} \tag{16}$$

where (a, b, r, s) are the parameter values of the tangent turbines of S.

It is noted that the only self-conjugate series are the unions.

It may be observed that, if the conjugate series of a general series consists of only one element $\bar{E}$, then S is contained in the nonlinear flat field whose central element is $\bar{E}$. In this case, we shall say that S is a *co-flat series.*

Obviously if an equiparallel series is not a turbine, then there is no series which is conjugate to it.

5. **The osculating flat fields of a series of elements.** The flat field which has three consecutive elements in common with a series S at an element E of S is called the *osculating flat field* of the series S at the element E.

THEOREM 6. *The osculating flat fields of a general series S are the nonlincar flat fields whose central elements are the elements of the series $\bar{S}$ conjugate to S.*

If S is a co-flat series, then S has one and only one osculating flat field, namely, the nonlinear flat field in which it is contained.

THEOREM 7. *Any equiparallel series has one and only one osculating flat field, namely, the linear flat field in which it is contained.*

Of course, the tangent turbine of a series S at an element E of S is contained in the osculating flat field of S at E.

6. **The limaçon series.** Let T be a nonlinear turbine (not a point-turbine), let $\bar{E}$ be a fixed element on the conjugate turbine $\bar{T}$ of T, and let γ be a real number. Let O be the point of $\bar{E}$, and let P be the point of any element E of the turbine T. On the line (OP), let us select the points P_i, $(i=1, 2)$, such that $d(P, P_i)=2\gamma$. Let E_i be the element whose point is P_i and whose direction is that of E. By this construction, to each element E of T, there are associated two elements E_1 and E_2. The totality of elements E_1, E_2 is called a *limaçon series* with *central turbine* T and *radius* γ.

Let T be a point-turbine (point, or star), let $\bar{E}$ be a fixed element of T, and let γ be a real number. Let L be the angle bisector of the angle whose initial and terminal sides are the lines of $\bar{E}$ and of any element E of T respectively. On L, let us select the points P_i, $(i=1, 2)$, such that $d(O, P_i)=2\gamma$, where O is the point of T. Let E_i be the element whose point is P_i and whose direction is that of E. By this construction, to each element E of T, there are associated two elements E_1 and E_2. The totality of elements E_1, E_2 is called a *limaçon series* with *central turbine* T and *radius* γ.

The equations of any limaçon series are

$$\begin{aligned} v &= A\cos u + B\sin u + 2\gamma\sin(u-\bar{u})/2 + R, \\ w &= -A\sin u + B\cos u + 2\gamma\cos(u-\bar{u})/2 + S, \end{aligned} \tag{17}$$

where (A, B, R, S) are the parameters of the central turbine T, $\bar{u}$ is the normal angle of the fixed element $\bar{E}$, and γ is the radius of the limaçon series.

Upon setting

$$C = -2\gamma\sin\bar{u}/2, \qquad D = 2\gamma\cos\bar{u}/2, \tag{18}$$

the equations (17) of the limaçon series take the form

$$
\begin{aligned}
v &= A \cos u + B \sin u + C \cos u/2 + D \sin u/2 + R, \\
w &= - A \sin u + B \cos u - C \sin u/2 + D \cos u/2 + S.
\end{aligned} \tag{19}
$$

We call $L(A, B, C, D, R, S)$ a set of limaçon series coordinates. Obviously,

$$L(A, B, C, D, R, S) = L(A, B, -C, -D, R, S).$$

The point-union of the limaçon series (19) is the limaçon

$$X + iY = (A + iB) + (C + iD)e^{iu/2} + (R + iS)e^{iu}, \tag{20}$$

while the line-union is

$$X + iY = (A + iB) + (1/4)(C - iD)e^{3iu/2} + (3/4)(C + iD)e^{iu/2} + Re^{iu}. \tag{21}$$

From (10) and (19), we obtain

THEOREM 8. *A limaçon series is a co-flat series. The tangent turbines of a limaçon series are such that their conjugate turbines contain the element* $\overline{E}$ *and such that their centers are on the circle with center* (A, B) *and radius* γ.

From this theorem, we derive

THEOREM 9. *Three co-flat nonlinear turbines which do not all contain one element determine a unique limaçon series. Three elements, no two of which are parallel, and which do not all lie on one turbine, determine four limaçon series. Three elements, two of which are parallel without all being parallel, determine two limaçon series.*

7. **The circular series.** The equiparallel series whose point-union is a circle with center (A, B) and radius γ is called the *circular series* with center (A, B) and radius γ.

The equations of a circular series are

$$u = c, \qquad (v - \alpha)^2 + (w - \beta)^2 = \gamma^2, \tag{22}$$

where (c, α, β) are the hessian coordinates of the element whose point is (A, B) and whose inclination is $c+\pi/2$. Thus we must have the relations

$$A = \alpha \cos c - \beta \sin c, \qquad B = \alpha \sin c + \beta \cos c. \tag{23}$$

THEOREM 10. *Three parallel elements which are not all on one turbine determine a unique circular series. Three linear turbines which all possess the same common direction and no two of whose base lines are parallel determine four circular series. Three linear turbines which all possess the same common direction and only two of whose base lines are parallel determine two circular series.*

8. **The osculating limaçon series of a general series.** Let two series S_1 and S_2 possess a common element E_0. These two series are said to be *osculating* (or to have contact of the second order) at E_0 if and only if they have three consecutive elements in common at E_0. Thus the two general series S_1: $v=v_1(u)$, $w=w_1(u)$, and S_2: $v=v_2(u)$, $w=w_2(u)$ are said to be osculating at the common element $E_0(u_0, v_0, w_0)$ if and only if

$$
\begin{aligned}
v_0 &= v_1(u_0) = v_2(u_0), & w_0 &= w_1(u_0) = w_2(u_0), \\
v_1'(u_0) &= v_2'(u_0), & w_1'(u_0) &= w_2'(u_0), \\
v_1''(u_0) &= v_2''(u_0), & w_1''(u_0) &= w_2''(u_0).
\end{aligned} \tag{24}
$$

Let us consider the one-parameter family of enveloping series S_t: $v=v(u, t)$, $w=w(u, t)$. Every series S_t of the family is tangent to the envelope S. If every series S_t of the family is also an osculating series of the envelope S, then the family is called a *set of osculating series.* Our given one-parameter family of series is a set of osculating series if and only if the four equations

$$
v_t = 0, \qquad w_t = 0, \qquad v_{ut} = 0, \qquad w_{ut} = 0 \tag{25}
$$

have a common solution in u. The series S_t of the family are then the osculating series of the envelope S.

THEOREM 11. *The ∞^1 limaçon series*

$$
\begin{aligned}
v &= A(t) \cos u + B(t) \sin u + C(t) \cos u/2 + D(t) \sin u/2 + R(t), \\
w &= -A(t) \sin u + B(t) \cos u - C(t) \sin u/2 + D(t) \cos u/2 + S(t)
\end{aligned} \tag{26}
$$

constitute a set of osculating limaçon series if and only if

$$
\begin{gathered}
4(A'R' - B'S') = C'^2 - D'^2, \qquad 2(A'S' + B'R') = C'D', \\
A'^2 + B'^2 = R'^2 + S'^2.
\end{gathered} \tag{27}
$$

For, these are the conditions that the equations

$$
\begin{aligned}
A' \cos u + B' \sin u - R' &= 0, & -A' \sin u + B' \cos u - S' &= 0, \\
C' \cos u/2 + D' \sin u/2 + 2R' &= 0, & -C' \sin u/2 + D' \cos u/2 + 2S' &= 0,
\end{aligned} \tag{28}
$$

which are equivalent to the equations (25) for the ∞^1 limaçon series (26), be compatible in u.

The envelope of the ∞^1 limaçon series is given by the equations (26) and (28). Solving (28) for $\cos u$ and $\sin u$, we obtain the

COROLLARY. *The series to which the ∞^1 limaçon series of Theorem* 11 *are the osculating limaçon series either consists of one element or is a general series. It is given by the equations*

$$
(29)\quad \begin{aligned}
&\cos u = \frac{A'R' + B'S'}{A'^2 + B'^2}, \qquad \sin u = \frac{-A'S' + B'R'}{A'^2 + B'^2},\\
&v = A\cos u + B\sin u + C\cos u/2 + D\sin u/2 + R,\\
&w = -A\sin u + B\cos u - C\sin u/2 + D\cos u/2 + S.
\end{aligned}
$$

The series S of (29) to which the limaçon series are the osculating limaçon series is given by the equations (26), where the value of t in terms of u is defined by equations (28). If equations (26), subject to the conditions (28), are differentiated totally with respect to u and if the results are again differentiated totally with respect to u, we find that these are equivalent to

$$
(30)\quad \begin{aligned}
&C\cos u/2 + D\sin u/2 = -4s', && -C\sin u/2 + D\cos u/2 = 4r',\\
&A\cos u + B\sin u = -w' + 2s', && -A\sin u + B\cos u = v' - 2r',
\end{aligned}
$$

where r and s are the last two parameters of the tangent turbines to the series S. Solving (26) and (30) for A, B, C, D, R, S, we obtain the

THEOREM 12. *The osculating limaçon series of the general series S of* (1) *are those whose parameter values are*

$$
(31)\quad \begin{aligned}
&A = a + 2r'\sin u + 2s'\cos u, && B = b - 2r'\cos u + 2s'\sin u,\\
&C = -4r'\sin u/2 - 4s'\cos u/2, && D = 4r'\cos u/2 - 4s'\sin u/2,\\
&R = r + 2s', && S = s - 2r',
\end{aligned}
$$

where (a, b, r, s) are the parameters of the tangent turbines of S and the accent denotes total differentiation with respect to u.

From Theorem 11 and the Corollary to Theorem 11, we obtain

THEOREM 13. *The necessary and sufficient conditions that ∞^1 limaçon series be a set of osculating limaçon series are that they be a set of enveloping limaçon series and their central turbines be a set of tangent turbines in such a way that the element E of the envelope of the limaçon series on any particular limaçon series L is antiparallel (parallel but of opposite orientation) to the element E' of the envelope of the central turbines which is on the central turbine of L.*

THEOREM 14. *The tangent turbines and the central turbines of any general series have in common the envelope of the central turbines.*

The envelope of the central turbines is called the *series of curvature* of the given series. It is given by the equations

$$
(32)\qquad U = u + \pi, \qquad V = v + 2w', \qquad W = -2v' + w.
$$

THEOREM 15. *There is one and only one general series which contains a given element E_0 and which possesses a given series as series of curvature.*

An osculating limaçon series of a general series S at an element E may be defined as the unique limiting limaçon series of the set of limaçon series such that any limaçon series of this set contains the element E and any other two nearby elements of S.

9. **The osculating circular series of an equiparallel series.** By a process of reasoning similar to that used in the preceding section, we obtain the following results.

THEOREM 16. *The* ∞^1 *circular series*

$$u = c(t), \qquad [v - \alpha(t)]^2 + [w - \beta(t)]^2 = \gamma(t)^2 \tag{33}$$

constitute a set of osculating circular series if and only if

$$c' = 0, \qquad \alpha'^2 + \beta'^2 = \gamma'^2. \tag{34}$$

COROLLARY. *The series to which the* ∞^1 *circular series of Theorem* 16 *are the osculating circular series either consists of one element or is an equiparallel series. It is given by the equations*

$$u = c, \qquad v = \alpha - \frac{\alpha'\gamma}{\gamma'}, \qquad w = \beta - \frac{\beta'\gamma}{\gamma'}\cdot \tag{35}$$

THEOREM 17. *The osculating circular series of the equiparallel series S of* (2) *are those whose parameter values are*

$$\alpha = v - \frac{w'(1 + w'^2)}{w''}, \qquad \beta = w + \frac{1 + w'^2}{w''}, \qquad \gamma = \frac{(1 + w'^2)^{3/2}}{w''}, \tag{36}$$

where the accent denotes differentiation with respect to v.

From Theorem 16 and the Corollary to Theorem 16, we obtain

THEOREM 18. *The necessary and sufficient conditions that* ∞^1 *circular series be an osculating set of circular series are that they all have a common direction and that the circles of the circular series be a set of osculating circles.*

The equiparallel series which has the common direction of the given equiparallel series S and whose point-union is the curve of centers of the osculating circular series of S is called the *series of curvature*. It is given by the equations

$$U = c, \qquad V = v - \frac{w'(1 + w'^2)}{w''}, \qquad W = w + \frac{1 + w'^2}{w''}\cdot \tag{37}$$

An osculating circular series of an equiparallel series S at the element E may be defined as the unique limiting circular series of the set of circular series such that any circular series of the set contains the element E and any other two nearby elements of S.

At this point, we note that two series S_1 and S_2 are osculating at a common element E, if and only if they have the same osculating limaçon (or circular) series at E.

10. **The curvature and torsion of a general series.** The *curvature* κ at an element E of a general series S is defined by the formula

$$\kappa = (r'^2 + s'^2)^{1/2}, \tag{38}$$

where (a, b, r, s) are the parameters of the tangent turbine at E and the accent denotes differentiation with respect to u.

The quantity κ is one-half of the radius of the osculating limaçon series L of S at E; and also it is one-half of the distance between the centers of the tangent and central turbines of S at E. When the direction is from the center of the tangent turbine to the center of the central turbine, we regard κ as positive. Otherwise, we take it to be negative.

The *torsion* τ at an element E of a general series S is defined by the formula

$$\tau = \frac{d\bar{u}}{du}, \tag{39}$$

where u and $\bar{u}$ are the normal angles of the element E of S and the element $\bar{E}$, which is the central element of the osculating flat field of S at E, respectively.

It is seen that the torsion τ at an element E of a general series S is the rate of change of the angle of the central element of the osculating flat field per unit radian measure of the angle of the element E.

It is observed that a series is a *whirl series* if and only if its torsion is unity.

From (38) and (39), we find

THEOREM 19. *The curvature ¯ of the conjugate series $\bar{S}$ of the general series S is equal to the quotient of the curvature κ and torsion τ of the series S. The torsion $\bar{\tau}$ of the conjugate series $\bar{S}$ of the general series S is the reciprocal of the torsion τ of the series S. That is,*

$$\bar{\kappa} = \frac{\kappa}{\tau}, \qquad \bar{\tau} = \frac{1}{\tau} \cdot \tag{40}$$

THEOREM 20. *Two general series which have their curvatures and torsions the same functions of u, the angle between the initial element and any element, are equivalent under the whirl-motion group G_6.*

Theorem 20 proves that the intrinsic equations of any general series in the geometry of the whirl-motion group G_6 are

$$(41)\qquad \kappa = \kappa(u), \qquad \tau = \tau(u),$$

where κ is the curvature, τ is the torsion, and u is the angle between the initial element and any element.

It is seen that the necessary and sufficient condition that a general series be co-flat is that its torsion be zero.

Before beginning the proof of Theorem 20, let us consider briefly the feuillets of the plane. Any *feuillet* consists of a lineal element E, a turbine T passing through E, and a flat field F containing both E and T. We recognize three distinct types of feuillets: (1) a *general* feuillet is one where both the turbine T and the flat field F are nonlinear, (2) an *intermediate* feuillet is one where the turbine T is linear and the flat field F is nonlinear, and (3) an *equiparallel* feuillet is one where both the turbine T and the flat field F are linear. The number of general (or intermediate, or equiparallel) feuillets in the plane is ∞^6 (or ∞^4, or ∞^4).

Under the whirl-motion group G_6, any two general (*or intermediate, or equiparallel*) *feuillets are equivalent.* In particular, under G_6, any general feuillet can be carried into the general feuillet such that the point and direction of its element E_0 are the origin and the positive direction of the y-axis respectively, its nonlinear turbine T_0 consists of all the lineal elements through the origin (the point-union or the star at the origin), and its nonlinear flat field F_0 is the one whose central element is E_0. We shall call this the normal feuillet. This result is very important in the proof of our fundamental Theorem 20.

Any feuillet of a general (equiparallel) series S is a general (equiparallel) feuillet which consists of an element E of S, the tangent nonlinear (linear) turbine T to S at E, and the osculating nonlinear (linear) flat field F to S at E. Obviously a general (equiparallel) series S possesses ∞^1 general (equiparallel) feuillets.

We shall now begin the proof of Theorem 20. First, we shall show that *there are only two general series S_1 and S_2 with the curvature and torsion given functions of the angle u and with the normal feuillet as initial feuillet.* (The angle u is the angle between any element E of S_1 or S_2 and the element E_0 of the normal feuillet.) By (39) and (41), we find

$$(42)\qquad \bar{u} = \int_0^u \tau du.$$

By equations (6), (8), (16), (38), and (41), we obtain

$$(43)\quad e^{iu} = -\frac{r' - is'}{a' - ib'},\ e^{i\bar{u}} = -\frac{r' + is'}{a' - ib'},\ \kappa = (r'^2 + s'^2)^{1/2} = (a'^2 + b'^2)^{1/2},$$

where the accent denotes differentiation with respect to u. Solving these equations for $a'+ib'$ and $r'+is'$, and integrating these results with respect to u, we find

$$(44)\qquad a+ib=\mp\int_0^u \kappa e^{i(\bar{u}+u)/2}du, \qquad r+is=\pm\int_0^u \kappa e^{i(\bar{u}-u)/2}du,$$

where the upper (or lower) signs are taken simultaneously and where $\bar{u}$ is defined by the equation (42). Since these are the parameters of the tangent turbines of our required series, we find that our two general series S_1 and S_2 are given by

$$(45)\qquad v+iw=\mp\left[e^{-iu}\int_0^u \kappa e^{i(\bar{u}+u)/2}du-\int_0^u \kappa e^{i(\bar{u}-u)/2}du\right],$$

where $\bar{u}$ is defined by the equation (42). This establishes our assertion.

Next we shall show that *the two series* S_1 *and* S_2 *as given by* (45) *are equivalent under the whirl-motion group* G_6. For the transformation of G_6

$$(46)\qquad U=u, \qquad V=-v, \qquad W=-w,$$

which is the product of a rotation R_π through π radians about the origin by the turn T_π through π radians, converts either one of the two series S_1 and S_2 into the other.

Let S' be any other general series with the curvature and torsion the same functions of the angle u. (The angle u is the angle between any element E and the initial element E' of S'.) Since any two general feuillets are equivalent under the whirl-motion group G_6, we can carry the initial general feuillet of S' (determined by the initial element E') into the normal feuillet. Under any such transformation of G_6, the general series S' is converted into a general series S''. Since S'' and either one of our original general series S_1 or S_2 possess the same initial feuillet (the normal feuillet) and since their curvatures and torsions are the same functions of the angle u, it follows by what we have proved above that S'' must coincide with either S_1 or S_2. Hence the three series S', S_1, and S_2 are all equivalent to each other under the whirl-motion group G_6. The proof of Theorem 20 is therefore complete.

11. **The curvature of an equiparallel series.** The *curvature* $\kappa=1/\gamma$ at an element E of an equiparallel series S is defined by the formula

$$(47)\qquad \kappa=\frac{1}{\gamma}=\frac{w''}{(1+w'^2)^{3/2}},$$

where the accent denotes differentiation with respect to v.

The quantity γ is the radius of the osculating circular series C of S at E. When it is the distance from the point of E to the center of the osculating circular series C, we regard $\kappa=1/\gamma$ as positive. Otherwise, we take it to be negative.

The *torsion* of an equiparallel series is taken to be zero.

THEOREM 21. *Two equiparallel series which have their curvatures the same functions of s, the arc length of the point-union from the initial element to any element, are equivalent under the whirl-motion group G_6.*

Theorem 21 shows that the intrinsic equations of any equiparallel series in the geometry of the whirl-motion group G_6 are

$$\kappa = \kappa(s), \qquad \tau = 0, \tag{48}$$

where κ is the curvature, τ is the torsion, and s is the arc length of the point-union between the initial element and any element.

Theorem 21 is a consequence of the fact that the whirl-motion group G_6 induces the group of rigid motions between the linear flat fields of the plane.

Now we may observe that the curvature of *any* series is the rate of change of the tangent turbine per unit measure of the elements of the series; and the torsion of any series is the rate of change of the osculating flat field per unit measure of the elements of the series.

BIBLIOGRAPHY

1. Kasner, *The group of turns and slides, and the geometry of turbines*, American Journal of Mathematics, vol. 33 (1911), pp. 193–202.
2. Kasner and De Cicco, *The geometry of turbines, flat fields, and differential equations*, American Journal of Mathematics, vol. 59 (1937), pp. 545–563.
3. De Cicco, *The geometry of whirl series*, these Transactions, vol. 43 (1938), pp. 344–358.
4. Kasner and De Cicco, *The geometry of the whirl-motion group G_6: elementary invariants*, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 399–403.
5. Kasner and De Cicco, *Quadric fields in the geometry of the whirl-motion group G_6*, American Journal of Mathematics, vol. 61 (1939), pp. 131–142.

BROOKLYN COLLEGE,
BROOKLYN, N. Y.