GEOMETRIC ASPECTS OF RELATIVISTIC DYNAMICS*

BY

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Introduction

1. Kasner has studied the three-parameter families of trajectories of a particle moving in a plane under forces which are functions of position only, and has shown that all such families of curves, each particular family corresponding to a particular field of force, possess certain common geometrical properties which distinguish them from three-parameter families of curves defined in other ways.† He and his students have also studied a variety of other problems concerning families of trajectories of particles, but in all of this work it has been assumed that the particles obey the laws of Newtonian dynamics. So far there do not seem to have been any parallel investigations concerning the trajectories of particles obeying the laws of special relativistic dynamics.

For the sake of brevity, we shall call a particle obeying the laws of Newtonian dynamics a classical particle, and we shall call a particle obeying the laws of special relativistic dynamics a relativistic particle.

This article deals primarily with the problem of determining a set of geometrical properties which is characteristic of the families of trajectories of a relativistic particle moving in a plane under forces which are functions of position only. Whereas Kasner found that in the classical case the families of trajectories are characterized by a certain set of five properties, we find that in the relativistic case there are six characteristic properties.‡ Four of these correspond to four of the properties given by Kasner for the classical case, and resemble the latter in various degrees, while the remaining two properties have no classical analogues.

In the concluding sections of the article we deal with some other problems concerning trajectories of relativistic particles, most of the considerations being confined to the case of motion in a plane. In particular, we study the determination of the field of force by the properties of the family of trajectories,

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† These Transactions, vol. 7 (1906), pp. 401–424; also *Differential-Geometric Aspects of Dynamics*, American Mathematical Society Colloquium Publications, vol. 3_2, New York, 1913, pp. 9–17.

‡ When this paper was presented to the Society, on October 29, 1938 (Bulletin of the American Mathematical Society, abstract 44-9-397), it was announced that the families of trajectories can be characterized by a set of seven properties. It has since been found that one of those properties is a consequence of the others.

we investigate point transformations which transform families of trajectories into families of trajectories, and we consider the properties of certain special families of trajectories which are called natural families. (A natural family of trajectories is the family of possible trajectories of a particle moving in a conservative field of force with a prescribed value of the total energy.)

In many places the detailed proofs of the results will be omitted; for these proofs depend, for the most part, upon entirely elementary and straightforward, but tedious, calculations.

THE DIFFERENTIAL EQUATION DEFINING THE FAMILY OF TRAJECTORIES

2. We consider a relativistic particle, having rest-mass m_0, moving in a plane under a force which is a function of position only. If x and y are the rectangular coordinates of the particle with respect to a fixed set of axes, and if $X(x, y)$ and $Y(x, y)$ are, respectively, the x-component and the y-component of the force, the differential equations of motion of the particle can be written in the form

$$(1)\qquad \begin{aligned} \frac{d}{dt}\left[\dot{x}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}\right] &= \frac{1}{m_0}X(x, y) \equiv \phi(x, y), \\ \frac{d}{dt}\left[\dot{y}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}\right] &= \frac{1}{m_0}Y(x, y) \equiv \psi(x, y). \end{aligned}$$

Here, of course, c denotes the speed of light, and the dots indicate total differentiation with respect to the time t. If both ϕ and ψ are identically zero, the family of trajectories is merely the two-parameter family of straight lines in the plane. We explicitly exclude this degenerate case from all of our considerations. We shall assume that the functions ϕ and ψ are of class C^2, if not throughout the entire plane, at least throughout a certain open region to which our considerations are restricted.*

We first obtain the differential equation defining the family of possible trajectories, by eliminating the time from equations (1) in the usual way. The result is the equation

$$(2)\qquad y''' = -F + Gy'' + Hy''^2 + F(1 + Ky''^2)^{1/2},$$

where

$$(3)\qquad \begin{aligned} F &= \frac{1}{2c^4}(1+y'^2)(\psi-\phi y')(\phi+\psi y'), & H &= -\frac{3\phi}{\psi-\phi y'}, \\ G &= \frac{\psi_x+(\psi_y-\phi_x)y'-\phi_y y'^2}{\psi-\phi y'}, & K &= \frac{4c^4}{(1+y'^2)^2(\psi-\phi y')^2}. \end{aligned}$$

* Many of our results are valid under conditions which are slightly broader than these. The minimum conditions under which the conclusions hold cannot be stated in any simple form.

The primes indicate total differentiation with respect to x; and $\phi_x = \partial\phi/\partial x$, and so on. The positive value of the square root in the last term of (2) is the significant one; and wherever square roots appear in the following work it is to be understood, unless the contrary is explicitly indicated, that the positive values are intended. We note the identity

$$FK + 2H/3 = \frac{2y'}{1 + y'^2} \cdot \tag{4}$$

As may be seen by letting c tend to infinity, the equation which corresponds to (2) in the classical case is $y''' = Gy'' + Hy''^2$, G and H being given by the above formulas. We see that, for a given field of force, the family of trajectories is independent of the rest-mass of the particle in the classical case, but not in the relativistic case.

Equation (2) is not an arbitrary differential equation of the third order.* On the contrary, the equation is entirely special in respect to the way in which the derivatives are involved, and it is somewhat special in respect to the way in which x and y are involved. Hence, regardless of the forms of the functions ϕ and ψ, the family of curves defined by (2) must possess certain special geometrical properties, corresponding to the special features of the form of the equation. Our immediate problem is to discover these characteristic properties.

The characteristic properties of the family of trajectories

3. Following Kasner's procedure, we begin by considering the trajectories which pass through a fixed point O: (x, y) in the direction determined by a fixed value of y', the lineal element (x, y, y') being such that, for it, F, G, and H are all finite, and F and H are not zero.† These curves form a one-parameter family, the different curves having different curvatures at the point O. Considering each of the curves of this family, we construct the parabola which osculates the curve at the point O. Finally, we consider the locus Γ_1 of the foci of these parabolas.

For convenience in discussing the curve Γ_1 and certain other curves, we introduce two auxiliary systems of rectangular coordinates with their origins at the point O. The one, (ξ, η), system is such that the ξ-axis and η-axis are

* By an arbitrary differential equation of the third order we mean an equation of the form $y''' = f(x, y, y', y'')$, where the right-hand member is an arbitrary function of the four arguments indicated.

† In order to satisfy the condition $H \neq 0$, it may be necessary to make an adjustment of the coordinate system. We may as well assume that the adjustment of the coordinate system is such that ψ also does not vanish at the point O.

parallel to the x-axis and y-axis, respectively. The other, (u, v), system is such that the u-axis is the common tangent, at O, of the ∞^1 trajectories we are considering. The orientations of both of these sets of axes are the same as that of the (x, y) set. The relation between the auxiliary coordinate systems is represented by the equations

$$\xi + y'\eta = (1 + y'^2)^{1/2}u, \qquad - y'\xi + \eta = (1 + y'^2)^{1/2}v.$$

The focus of the parabola determined by the differential element of the third order (x, y, y', y'', y''') has the coordinates

$$\xi = \frac{3}{2}y''\frac{(1 + y'^2)y''' - 6y'y''^2}{(1 + y'^2)y'''^2 - 6y'y''^2y''' + 9y''^4},$$

$$\eta = \frac{3}{2}y''\frac{(1 + y'^2)y'y''' + 3(1 - y'^2)y''^2}{(1 + y'^2)y'''^2 - 6y'y''^2y''' + 9y''^4}.$$

The equation of the curve Γ_1 is obtained by eliminating y'' and y''' from these equations and equation (2). We find that the resulting equation, written in terms of the coordinates (u, v), is

$$(5)\qquad \begin{aligned} &(u^2 + v^2)^2[G(u^2 + v^2) - 2u_0u - 4v_0v/3] \\ &\qquad - \frac{1}{4F}(1 + y'^2)^{3/2}v[G(u^2 + v^2) - 2u_0u - 2v_0v]^2 = 0, \end{aligned}$$

where

$$(6)\qquad u_0 = (3/4)(1 + y'^2)^{1/2}, \qquad v_0 = (1/4)(1 + y'^2)^{1/2}[3y' - (1 + y'^2)H].$$

The curve Γ_1 is a quintic or a sextic according as G is, or is not, zero.

Let a be an arbitrarily chosen positive constant. The inverse of the curve Γ_1 with respect to the circle $u^2+v^2=a^2$ is the cubic Γ_1' represented by the equation

$$a^4[Ga^2 - 2u_0u - 4v_0v/3] - \frac{1}{4F}(1 + y'^2)^{3/2}v[Ga^2 - 2u_0u - 2v_0v]^2 = 0.$$

This cubic can be obtained from the particular cubic Γ_0 represented by the equation

$$(7)\qquad a^2(u + 2v/3) + v(u + v)^2 = 0$$

by means of the affine transformation

$$(8)\qquad u \longrightarrow \frac{A}{a}\left(u - \frac{Ga^2}{2u_0}\right), \qquad v \longrightarrow \frac{Av_0}{au_0}v,$$

where

$$A = \frac{3^{1/2}c^2(1 + y'^2)^{1/2}}{2(\phi + \psi y')} \cdot \tag{9}$$

We observe that the cubic Γ_1' passes through the point O when, and only when, G is zero. This is the case in which Γ_1 reduces to a quintic. If, and only if, the field of force is given by the equations

$$\phi = a_1 + a_2 x, \qquad \psi = a_3 + a_2 y,$$

where the a's are constants, the cubic Γ_1' always passes through the corresponding point O. Various physically important fields of force satisfy this condition.

The cubic Γ_1' has the three asymptotes represented by the equations

$$v = 0, \qquad u + \frac{v_0}{u_0} v = \frac{Ga^2}{2u_0} \pm 3^{-1/2} \frac{a^2}{A} \cdot$$

The curve has three real branches, one, and only one, of which is asymptotic to both of the parallel asymptotes and is not asymptotic to the third asymptote, $v=0$. Let us call this particular branch the transverse branch of Γ_1'.

Because the square root in the last term of equation (2) is positive, and because the significant values of y'' are all of one sign, as y'' varies the focus of the osculating parabola does not describe the entire curve represented by equation (5), but only a certain arc of the curve. When y'' approaches zero, the coordinates of the inverse of the focus of the osculating parabola approach the values

$$u = \frac{Ga^2}{2u_0}, \qquad v = 0.$$

It follows from this fact and some simple continuity considerations that the foci of the osculating parabolas lie on an arc of Γ_1 which is the inverse of a part of the transverse branch of Γ_1'.

Hence we can state the first property of the family of trajectories in the following form:

Property I. (1) *If, for each of the* ∞^1 *trajectories passing through a given point in a given direction, we construct the parabola which osculates the trajectory at the given point, the locus* Γ_1 *of the foci of these parabolas is the inverse of a cubic* Γ_1' *with respect to the circle* $u^2+v^2=a^2$, *where* a *is an arbitrary positive constant. The cubic* Γ_1' *can be obtained from the particular cubic* Γ_0 *represented by equation* (7), *by means of an affine transformation of the form* (8), *where* u_0 *is given by the first of equations* (6), *and where* A, G, *and* v_0 *are functions of* x, y, *and y', and are independent of a.* (2) *More particularly, the foci of the osculating parabolas lie on an arc of* Γ_1 *which is the inverse of a part of the transverse branch of* Γ_1'.

The calculations which establish Property I can be reversed unambiguously, and in this way we get a converse of the property. We find that if a three-parameter family of plane curves possesses Property I, the defining differential equation is of the form (2), where now F, G, and H are some functions of x, y, and y', K is defined by equation (4), and the square root in the final term has its positive value. If a family of curves has the first part of Property I, but not necessarily the second part, the defining differential equation is as just described, except that the sign of the square root is not determined.

We see that Property I is characteristic of families of curves defined by differential equations which have the structure of equation (2) as regards y'' and y''', the functions (of x, y, and y') F, H, and K being subject to the restriction (4).

It is of interest to consider the relations between these results and the corresponding results for the classical case given by Kasner.* The curve which corresponds in the classical case to our curve Γ_1 is a circle, or a straight line, according as G is not, or is, zero. The inverse of this curve with respect to the circle $u^2+v^2=a^2$ is the straight line represented by the equation

$$2u_0u + 2v_0v = Ga^2.$$

This line is parallel to the parallel asymptotes of Γ_1', and is midway between them.

It will be observed that if we let c tend to infinity, the curve Γ_1 degenerates, not into the classical circle, but into that circle taken twice, together with the line $v=0$. We can see without difficulty that the second circle and the line $v=0$ constitute the degenerate form of the nonsignificant part of Γ_1 (the part of Γ_1 formed by the inverses of the points of Γ_1' which do not lie on the transverse branch).

4. The tangent at the point O to the line of force passing through that point is represented by the equation

$$v = \frac{\psi - \phi y'}{\phi + \psi y'} u.$$

As we have seen, the slope of the parallel asymptotes of Γ_1' is $-u_0/v_0$ in the

* It is understood, of course, that here and elsewhere we are comparing the properties of the family of relativistic trajectories with the properties of the family of classical trajectories in the same field of force.

uv-coordinate system. It readily follows from equations (3) and (6) that

$$-\frac{u_0}{v_0} = -\frac{\psi - \phi y'}{\phi + \psi y'}.$$

Hence we have a second property of the family of trajectories, which can be stated as follows:

PROPERTY II. *The cubic* Γ_1' *which corresponds, according to Property* I, *to a lineal element* (x, y, y') *is such that the lineal element bisects the angle between the direction of the parallel asymptotes of* Γ_1' *and a certain direction which is fixed for the given point O (the direction of the force acting at O).*

Conversely, it is easily shown that if a family of curves possessing Property I also possesses Property II, the function $H(x, y, y')$ in the defining differential equation must be of the form

$$H = \frac{3}{y' - \omega(x, y)}, \tag{10}$$

where $\omega(x, y)$ is the slope of the direction, associated with the point (x, y), which is referred to in the statement of the property.

Property II is very closely related to the second property in Kasner's set. The remarks previously made concerning the relation between Property I and Kasner's corresponding property will suffice to make this connection clear.

5. The point P on the u-axis midway between the parallel asymptotes of the curve Γ_1' has the coordinates $u = Ga^2/2u_0$, $v = 0$. If, at the point O, we have the relations $\psi_x = \psi_y - \phi_x = \phi_y = 0$, the point P coincides with O for all values of y'; otherwise, as y' varies the point P describes a certain curve Γ_2. We readily find that Γ_2 is represented by the equation

$$(\xi^2 + \eta^2)(\psi\xi - \phi\eta) = (2a^2/3)\,[\psi_x\xi^2 + (\psi_y - \phi_x)\xi\eta - \phi_y\eta^2].$$

The inverse of Γ_2 with respect to the circle $\xi^2 + \eta^2 = a^2$, a being the constant used in defining Γ_1', is represented by the equation

$$\psi\xi - \phi\eta = (2/3)\,[\psi_x\xi^2 + (\psi_y - \phi_x)\xi\eta - \phi_y\eta^2].$$

Thus we have

PROPERTY III. (1) *Either the point P on the u-axis midway between the parallel asymptotes of* Γ_1' *coincides with O for all values of* y', *or, as* y' *varies, P describes a curve* Γ_2, *which is the inverse, with respect to the circle* $\xi^2 + \eta^2 = a^2$, *of a conic* Γ_2' *passing through the point O.* (2) *If the conic* Γ_2' *exists,* its tangent at O has the direction, fixed for O, referred to in the statement of Property* II.

* We consider that the conic does not exist if P coincides with O for all values of y'.

Conversely, if a family of curves possessing Property I also possesses the first part of Property III, the function $G(x, y, y')$ in the defining differential equation has the form

$$G = \frac{\mu_1 + \mu_2 y' + \mu_3 y'^2}{\omega_1 - y'}, \tag{11}$$

where μ_1, μ_2, μ_3, and ω_1 are functions of x and y. If a family of curves possessing Properties I and II also possesses both parts of Property III, we have the relation $\omega_1 = \omega$, where ω is the function introduced in connection with the converse of Property II.

If we have the relations $\phi = \Phi_x$, $\psi = \Phi_y$, where Φ is some function of x and y, we say that the field of force is conservative. We note that if, and only if, the field of force is conservative, the conic Γ_2', when it exists, is always either a rectangular hyperbola or a pair of perpendicular straight lines. The only conservative fields of force for which the conic never exists are those derived from functions Φ of the form

$$\Phi = a_1 + a_2 x + a_3 y + a_4(x^2 + y^2),$$

where the a's are constants.

If ϕ and ψ are, respectively, the real and the imaginary parts of an analytic function of $x+iy$, we have what Lecornu has called an analytic field of force. We see that if, and only if, the field of force is analytic, the conic Γ_2' when it exists is always a circle. The only analytic fields of force for which the conic never exists are those for which the expressions $\phi + i\psi$ are linear functions of $x+iy$.

Our remarks concerning the relations between Property III and the corresponding classical property will be postponed until after we have given IV.

6. If the conic Γ_2' corresponding to the point O exists, its curvature at the point O is

$$-\frac{4}{3}\,\frac{\psi_x\phi^2 + (\psi_y - \phi_x)\phi\psi - \phi_y\psi^2}{\phi^3[1 + (\psi^2/\phi^2)]^{3/2}}.$$

The curvature at O of the line of force through that point is

$$\frac{\psi_x\phi^2 + (\psi_y - \phi_x)\phi\psi - \phi_y\psi^2}{\phi^3[1 + (\psi^2/\phi^2)]^{3/2}}.$$

Hence we can state

PROPERTY IV. *If the conic Γ_2' corresponding to O exists, the ratio of its curvature at O to the curvature (at O) of the line of force through that point is $-4/3$. If the conic does not exist, the curvature of the line of force at O is zero.*

In connection with the converse of this property, we observe that the lines of force are defined geometrically by the property that the tangent at any point has the direction, associated with that point, referred to in the statement of Property II.

The converse of Property IV can be expressed as follows. If a family of curves possessing Properties I to III inclusive also possesses Property IV, the functions ω, μ_1, μ_2, and μ_3, which have appeared above, must satisfy the relation

$$\mu_1 + \mu_2\omega + \mu_3\omega^2 - \omega_x - \omega\omega_y = 0. \tag{12}$$

The relation of Properties III and IV to the classical theory is very much he same as that of Property II. The properties could be taken over, with slight changes of wording, into the classical theory as alternatives to the third and fourth properties in Kasner's set. The properties are not very directly connected with the two given by Kasner, although their converses have the same effect as the converses of his properties in restricting the form of the function $G(x, y, y')$.

7. The parallel asymptotes of the curve Γ_1' intersect the u-axis in points P_1 and P_2 having the abscissae

$$u = \frac{Ga^2}{2u_0} - 3^{-1/2}\frac{a^2}{A}, \qquad u = \frac{Ga^2}{2u_0} + 3^{-1/2}\frac{a^2}{A},$$

respectively. From the point O, as initial point, we draw a vector $\overrightarrow{OQ}$ equal to the vector $\overrightarrow{P_1P_2}$. Then we study the curve Γ_3 described by the terminus Q of this vector as y' varies. The result can be stated as follows:

Property V. *The curve* Γ_3 *is a circle which passes through the point* O; *and the tangent to the circle at* O *is perpendicular to the direction, fixed for* O, *referred to in the statement of Property* II.

For the sake of future use, we note that the equation of the circle Γ_3 is

$$\xi^2 + \eta^2 = \frac{4a^2}{3c^2}(\phi\xi + \psi\eta). \tag{13}$$

Now let us consider the converse of Property V.

If a family of curves possesses Properties I and II, the curve Γ_3 described by the point Q is represented by the equation

$$\xi^2 + \eta^2 = \pm\frac{2^{5/2}}{3}a^2\xi\left[F\frac{1 + (\eta/\xi)\omega}{(1 + (\eta/\xi)^2)(\omega - (\eta/\xi))}\right]^{1/2}, \tag{14}$$

where the symbol F is to be interpreted as $F(x, y, \eta/\xi)$. If the family of curves has also Property V, equation (14) must be of the form

$$\xi^2 + \eta^2 = \frac{2^{5/2}}{3} a^2\lambda(\xi + \omega\eta), \tag{15}$$

where λ is some function of x and y; and hence we must have

$$F(x, y, y') = -\lambda^2(1 + y'^2)(1 + \omega y')(y' - \omega). \tag{16}$$

Property V has no analogue in the Newtonian case. This is natural; for we see that the property is connected essentially with the occurrence of the terms $-F$ and $F(1+Ky''^2)^{1/2}$ in the right-hand member of equation (2), and no such terms exist in the corresponding classical equation.

8. The five properties which we have obtained may be looked upon as the geometrical meaning of the special way in which the derivatives enter into equation (2). They even go somewhat beyond this, in that their converses restrict to some extent the way in which the variables x and y occur in the defining differential equation of a family of curves possessing the properties. However, the most general differential equation defining a family of curves possessing the five properties contains four arbitrary functions of x and y, namely, λ, ω, μ_1, and μ_3, whereas equation (2) depends on only two such functions, namely, ϕ and ψ. We must, therefore, proceed to find one or more additional properties to complete the characterization of the families of dynamical trajectories.

9. Referring to equation (13), we see that the ξ-axis intersects the circle Γ_3 in the point M having the coordinates $\xi=(4a^2/3c^2)\phi$, $\eta=0$. The line through the point O and the center of the circle intersects the circle again in the point M' having the coordinates $\xi=(4a^2/3c^2)\phi$, $\eta=(4a^2/3c^2)\psi$. The distance from the point O to the point M is $OM=(4a^2/3c^2)\phi$, and the distance from the point M to the point M' is $MM'=(4a^2/3c^2)\psi$.

If the conic Γ_2' corresponding to O exists, the ξ-axis intersects it in the point A having the coordinates $\xi=(3/2)\psi/\psi_x$, $\eta=0$, and the η-axis intersects it in the point B having the coordinates $\xi=0$, $\eta=(3/2)\phi/\phi_y$. We let OA and OB, respectively, denote the distances from the point O to the points A and B. Then $OA=(3/2)\psi/\psi_x$, $OB=(3/2)\phi/\phi_y$.

We have immediately

PROPERTY VI. *When the initial point O is changed, the associated circle Γ_3 changes in the manner described by the following equations*:

$$\frac{\partial}{\partial x} MM' = \frac{3}{2}\,\frac{MM'}{OA} \quad \textit{or} \quad 0$$

according as the conic Γ_2' *corresponding to O exists or does not exist*;

$$\frac{\partial}{\partial y} OM = \frac{3}{2} \frac{OM}{OB} \quad or \quad 0$$

according as the conic exists or does not exist.

Conversely, if we take the equation of Γ_3 in the form (15), and the equation of Γ_2' in the form

$$\omega\xi - \eta = (2/3)(\mu_1\xi^2 + \mu_2\xi\eta + \mu_3\eta^2),$$

and proceed to define distances OA, OB, OM, and MM' as above, we get the results

$$OA = \frac{3}{2} \frac{\omega}{\mu_1}, \qquad OB = -\frac{3}{2\mu_3}, \qquad OM = \frac{2^{5/2}}{3} a^2\lambda, \qquad MM' = \frac{2^{5/2}}{3} a^2\lambda\omega.$$

Hence, if a family of curves possessing Properties I, II, III, and V, also possesses Property VI, we have the relations

$$(17) \qquad (\lambda\omega)_x = \lambda\mu_1, \qquad \lambda_y = -\lambda\mu_3.$$

Since Property VI relates to the circle Γ_3, it, like Property V, has no analogue in the Newtonian case. On the other hand, having Property VI, we have no need of an analogue of the complicated fifth property in Kasner's set.

10. Now we proceed to show that the six properties which we have obtained are in fact characteristic of the family of relativistic trajectories.

Suppose that a certain three-parameter family of plane curves possesses all six of the properties. Then, as we have seen, the family is defined by a differential equation of the form (2), where the square root has its positive value, and where F, G, H, and K, are given by the formulas (16), (11), (10), and (4), respectively, λ and ω being some functions of x and y, and μ_1, μ_2, μ_3, and ω_1 being defined by the equations (12), (17), and $\omega_1=\omega$.

Let us define two new functions $\phi(x, y)$ and $\psi(x, y)$ as follows:

$$\lambda = 2^{-1/2}c^{-2}\phi, \qquad \omega = \psi/\phi.$$

Then, by (12) and (17), we have the relations

$$\mu_1 = \psi_x/\phi, \qquad \mu_2 = (\psi_y - \phi_x)/\phi, \qquad \mu_3 = -\phi_y/\phi.$$

When, in the formulas for F, G, and H, we replace λ, ω, μ_1, μ_2, and μ_3 by these expressions in terms of ϕ and ψ, we obtain the formulas (3).

Thus, not only does every family of curves defined by a system of equations such as (2) and (3), with the square root positive, possess the six properties given, but also if a three-parameter family of plane curves possesses the six properties, it is defined by such a system of equations, with suitably chosen functions $\phi(x, y)$ and $\psi(x, y)$. Moreover, if a family of curves is defined by such a system of equations, it is the family of trajectories of a particle moving according to the differential equations of motion (1). Hence, if a family of curves possesses the six properties, it is the family of trajectories of a relativistic particle moving in a suitably chosen positional field of force. Therefore, the set of six properties is characteristic of the families of trajectories of a relativistic particle moving in a plane under forces which are functions (not identically zero) of position only.

It will be observed that the six properties are ordinally independent, that is, no one of them can be derived from those which precede it.

THE DETERMINATION OF THE FIELD OF FORCE BY THE GEOMETRICAL PROPERTIES OF THE FAMILY OF TRAJECTORIES

11. It is of interest to discuss the way in which a field of force is determined by the geometrical properties of the family of trajectories of a particle moving in the field. In the Newtonian case the geometry of the family of trajectories is incapable of determining more than the direction of the force acting at any point and the ratio of the magnitudes of the forces acting at any two points. On the other hand, in the relativistic case, if the rest-mass of the particle is given,* the geometry of the family of trajectories determines the field of force completely. This is because the right-hand member of equation (2) is not homogeneous and of degree zero in ϕ, ψ, and their partial derivatives, as is the right-hand member of the corresponding classical equation.

When the complete three-parameter family of relativistic trajectories of a particle in a positional field of force is given, we can determine the circle Γ_3 corresponding to any point (x, y), and, by equation (13), this determines the values of the functions ϕ and ψ at (x, y). The components of the force acting at the point are $m_0\phi(x, y)$ and $m_0\psi(x, y)$. Thus, when the complete three-parameter family of trajectories is given, the field of force is fully determined. However, we are mainly interested in showing that we can determine the force acting at a particular point, or the field of force, without making use of the complete family of trajectories.

12. We shall first show that the force acting at a particular point is determined when three trajectories, passing through that point in the same direction, are given.†

* Throughout this section we suppose that m_0 is given.

† The proof given is based on the assumption that the trajectories are such that two constants, v_2 and v_3^{-1}, are sufficiently small in absolute value. The extent to which this restriction can be removed by the use of continuity considerations has not been investigated.

If the cubic Γ_1' corresponding to a lineal element (x, y, y') is known, the circle Γ_3 corresponding to the point (x, y) can be constructed immediately;* and then, as has been said above, the force acting at (x, y) is determined. Hence, it will suffice to show that Γ_1' is determined when three trajectories, passing through (x, y) in the direction determined by y', are given.

Let T_1, T_2, and T_3 be three such trajectories. We construct the corresponding three osculating parabolas, determine their foci, and then obtain the inverses of these points with respect to the circle $u^2+v^2=a^2$. The coordinates (in the uv-coordinate system) of the last three points will be denoted by (u_1, v_1), (u_2, v_2), (u_3, v_3). We have the equations

$$(18)\quad Ga^2 - 2u_0u_n - 4v_0v_n/3 - \frac{(1+y'^2)^{3/2}}{4a^4F} v_n[Ga^2 - 2u_0u_n - 2v_0v_n]^2 = 0, \qquad n = 1, 2, 3,$$

which we have to solve for F, G, and v_0, in order to determine Γ_1'.

From equations (18) we obtain the equations

$$(19)\quad \begin{aligned} &v_1[Ga^2 - 2u_0u_1 - 2v_0v_1]^2[Ga^2 - 2u_0u_n - 4v_0v_n/3] \\ &\quad - v_n[Ga^2 - 2u_0u_1 - 4v_0v_1/3][Ga^2 - 2u_0u_n - 2v_0v_n]^2 = 0, \quad n = 2, 3, \end{aligned}$$

which we have to solve for G and v_0. To each solution (G, v_0) of equations (19) there corresponds a unique value of F which is given by any one of equations (18). Now equations (19) have a finite set of solutions. Our problem is to show that only one of these solutions is significant, and to show how the significant solution can be distinguished.

For the time being, let us regard u_1 and v_1 as constants, u_2, v_2, u_3, and v_3 as variables, and the solutions of equations (19) as pairs of functions of these variables.

It follows from the second part of Property I and the elementary properties of the curve Γ_1' that the significant solutions of equations (19) are such that as v_2 approaches zero $2u_0u_2$ approaches Ga^2, and as v_3^{-1} approaches zero u_0r approaches $-v_0$, where $r=u_3/v_3$. Now, for $v_2=v_3^{-1}=0$, equations (19) reduce to

$$v_1[Ga^2 - 2u_0u_1 - 2v_0v_1]^2[Ga^2 - 2u_0u_2] = 0,$$
$$[Ga^2 - 2u_0u_1 - 4v_0v_1/3][2u_0r + 2v_0]^2 = 0.$$

Hence, two of the solutions of (19) satisfy the above elementary criteria for

* The construction is an easy consequence of Properties II and V and the definition of Γ_3. There is an ambiguity in the construction, arising from the two possible ways of drawing a vector from one of the intersections of asymptotes of Γ_1' to the other. However, this ambiguity is removed when we take account of the fact that a trajectory lies on that side of its tangent toward which the force is directed.

significance, and we must seek an additional criterion to distinguish between them.

A little consideration of the properties of the curve Γ_1' suffices to show that if the absolute value of v_3 is large, we have a relation of the form

$$v_0 = -u_0 r + (Ga^2 + C)/(2v_3) + O(v_3^{-2}),$$

where C is a constant which has the same sign as the product $v_0 v_3$. On the other hand, if we regard the second of equations (19) as a relation between an independent variable v_3 and a dependent variable v_0, we readily find that the two roots which reduce to $-u_0 r$ for $v_3^{-1}=0$ are given, for small values of v_3^{-1}, by the expansions

$$\begin{aligned} v_0 = & - u_0 r + \frac{Ga^2}{2v_3} \pm \frac{1}{2v_3}(Ga^2 - 2u_0 u_1 + 2u_0 v_1 r) \\ & \cdot \left[\frac{-2u_0 v_1 r/3}{Ga^2 - 2u_0 u_1 + 4u_0 v_1 r/3}\right]^{1/2} + O(v_3^{-2}). \end{aligned} \tag{20}$$

Hence, only that solution of equations (19) is significant which, for small values of v_3^{-1}, gives the third term in the right-hand member of (20) the same sign as $-u_0 r$. (It is easily shown that the square root in (20) is real when v_3^{-1} is small.)

To summarize: When the three trajectories T_1, T_2, and T_3, are given, equations (18) determine a finite set of solutions (F, G, v_0). Only one of these solutions is significant, namely, the one which behaves as described above when v_2 and v_3^{-1}, regarded momentarily as variables, approach zero. When the significant solution has been obtained, the curve Γ_1' corresponding to the lineal element (x, y, y') is determined, and the force acting at (x, y) can be calculated.

13. We shall show that, subject to certain restrictions of an analytical character, a positional field of force is determined throughout a neighborhood of a point when the force acting at the point and four one-parameter families of trajectories, each of which covers the neighborhood simply,* are given.

Let us suppose that in a neighborhood of a point P we have four one-parameter families of curves, of the type just described, which are known to be trajectories of a particle of rest-mass m_0 in an unknown positional field of force. We also suppose that the force acting at the point P is known.

The equations of the curves of the four given families will be written

$$f_n(x, y) = a_n, \qquad n = 1, 2, 3, 4, \tag{21}$$

* That is, so that through each point of the neighborhood there passes just one curve of each family.

where the a's are the parameters of the families. We assume that the left-hand members of these equations are analytic functions of their arguments, and that none of the partial derivatives $\partial f_n/\partial y$ vanish in the neighborhood of P.

The functions $y_n = y_n(x, a_n)$ defined by equations (21) satisfy the relations

$$
\begin{aligned}
(\psi - \phi y_n')y_n''' = &-\frac{1}{2c^4}(1 + y_n'^2)(\psi - \phi y_n')^2(\phi + \psi y_n') \\
&+ [\psi_x + (\psi_y - \phi_x)y_n' - \phi_y y_n'^2]y_n'' - 3\phi y_n''^2 \\
&+ \frac{1}{2c^4}(1 + y_n'^2)(\psi - \phi y_n')^2(\phi + \psi y_n') \\
&\cdot \left[1 + \frac{4c^4}{(1 + y_n'^2)^2(\psi - \phi y_n')^2} y_n''^2\right]^{1/2},
\end{aligned} \tag{22}
$$

where $m_0\phi(x, y)$ and $m_0\psi(x, y)$ are the (unknown) components of the force acting at (x, y), and where y_n', y_n'', and y_n''' are to be interpreted, in an obvious way, as definite analytic functions of x, y determined by equations (21).

We assume that the determinant

$$
\begin{vmatrix}
y_1'' & y_1' y_1'' & y_1'^2 y_1'' & y_1''^2 \\
y_2'' & y_2' y_2'' & y_2'^2 y_2'' & y_2''^2 \\
y_3'' & y_3' y_3'' & y_3'^2 y_3'' & y_3''^2 \\
y_4'' & y_4' y_4'' & y_4'^2 y_4'' & y_4''^2
\end{vmatrix}
$$

does not vanish at the point P. (This determinant cannot vanish at every point of a neighborhood of P, for all choices of the one-parameter families of trajectories, unless the force vanishes throughout the neighborhood. Otherwise, the family of all trajectories in a nonzero field of force would consist merely of a finite set of two-parameter families.) Consequently, we can solve equations (22) algebraically for ψ_x, $(\psi_y - \phi_x)$, ϕ_y, and the ϕ which appears in the third terms of the right-hand members, obtaining a set of relations which we shall write schematically as follows:

$$
\begin{aligned}
\psi_x &= f(x, y, \phi, \psi), & \psi_y - \phi_x &= g(x, y, \phi, \psi), \\
\phi_y &= h(x, y, \phi, \psi), & \phi &= k(x, y, \phi, \psi).
\end{aligned} \tag{23}
$$

It is to be emphasized that, in virtue of the given equations (21), the right-hand members of equations (23) are entirely definite analytic functions of the arguments indicated.

It follows from equations (23) that we have the system of partial differential equations

$$\begin{aligned} \phi_x &= (k_x + k_\psi f)/(1 - k_\phi), & \phi_y &= h, \\ \psi_x &= f, & \psi_y &= g + (k_x + k_\psi f)/(1 - k_\phi). \end{aligned} \tag{24}$$

By our assumption that the given curves (21) are known to be trajectories in an unknown positional field of force, and that the force acting at P is given, the system of equations (24) is satisfied by a pair of functions $\phi(x, y)$, $\psi(x, y)$, which have given values at the point P.

If the two equations forming the conditions for integrability of the system (24) are satisfied identically, the system is completely integrable. In this case the field of force is determined throughout a neighborhood of P by the differential equations (24) and the given value of the force acting at P, at least if the coordinates of P and the values of ϕ and ψ at P form a system of values in the neighborhood of which the right-hand members of the equations are holomorphic. If the conditions for integrability are not satisfied identically, and are two independent equations, these equations determine implicitly a certain finite number of distinct pairs of functions $\phi(x, y)$ and $\psi(x, y)$. Then, if the point P is one at which the distinct pairs of functions have distinct pairs of values, the field of force is determined throughout a neighborhood of P by the conditions for integrability and the given values of ϕ and ψ at P. There is a third conceivable case, namely, that in which just one of the two conditions for integrability is not satisfied identically, or in which, while neither condition is satisfied identically, the two conditions are equivalent to a single equation. In this case we have in effect to deal with a completely integrable system of partial differential equations in one of the unknown functions (say ϕ) and an equation which determines the other unknown function (say ψ) implicitly in terms of x, y, and ϕ. Again we see that if the point P is such that certain conditions of analyticity are satisfied, and certain distinct pairs of functions have distinct pairs of values, the field of force is determined throughout a neighborhood of P by the equations (24), the conditions for integrability, and the given value of the force acting at P.

The point transformations which convert every family of dynamical trajectories into a family of dynamical trajectories

14. Kasner has shown that, in the Newtonian case, collineations are the only point transformations of the plane which convert every three-parameter family of trajectories (belonging to a positional field of force) into such a family of curves.* We proceed now to obtain the corresponding result for the relativistic case.

* In general, the fields of force corresponding to the original family and the transformed family, respectively, are different.

Suppose that we have a family of dynamical trajectories, defined by a system of equations such as (2), (3). We apply a point transformation

$$x = x(\bar{x}, \bar{y}), \qquad y = y(\bar{x}, \bar{y}), \tag{25}$$

where the functions $x(\bar{x}, \bar{y})$, $y(\bar{x}, \bar{y})$ are of class C^3, and the Jacobian $x_{\bar{x}}y_{\bar{y}} - x_{\bar{y}}y_{\bar{x}}$ does not vanish in the region under consideration; and we require that the transformation be specialized so that the transformed family of curves shall be defined by a system of equations of the form

$$\bar{y}''' = -\bar{F} + \bar{G}\bar{y}'' + \bar{H}\bar{y}''^2 + \bar{F}(1 + \bar{K}\bar{y}''^2)^{1/2},$$

$$\bar{F} = \frac{1}{2c^4}(1 + \bar{y}'^2)(\bar{\psi} - \bar{y}'\bar{\phi})(\bar{\phi} + \bar{y}'\bar{\psi}), \qquad \bar{H} = -\frac{3\bar{\phi}}{\bar{\psi} - \bar{\phi}\bar{y}'},$$

$$\bar{G} = \frac{\bar{\psi}_{\bar{x}} + (\bar{\psi}_{\bar{y}} - \bar{\phi}_{\bar{x}})\bar{y}' - \bar{\phi}_{\bar{y}}\bar{y}'^2}{\bar{\psi} - \bar{\phi}\bar{y}'}, \qquad \bar{K} = 4c^4(1 + \bar{y}'^2)^{-2}(\bar{\psi} - \bar{\phi}\bar{y}')^{-2}.$$

Here $\bar{\phi}$ and $\bar{\psi}$ denote functions of $\bar{x}$ and $\bar{y}$, and the primes denote differentiation with respect to $\bar{x}$.

On transforming equations (2), (3) by means of (25) and its extensions, we obtain an equation of the form

$$\begin{aligned} \bar{y}''' &= R_1 + R_2\bar{y}'' + R_3\bar{y}''^2 \\ &+ R_4\left[1 + \frac{4c^4(\alpha + \beta\bar{y}' + \gamma\bar{y}'^2 + \delta\bar{y}'^3 + \epsilon\bar{y}'')^2}{[(x_{\bar{x}} + y_{\bar{y}}\bar{y}')^2 + (y_{\bar{x}} + y_{\bar{y}}\bar{y}')^2]^2[A(x_{\bar{x}} + x_{\bar{y}}\bar{y}') - B(y_{\bar{x}} + y_{\bar{y}}\bar{y}')]^2}\right]^{1/2}, \end{aligned} \tag{26}$$

where A and B are functions of $\bar{x}$ and $\bar{y}$, the R's are functions of $\bar{x}$, $\bar{y}$, and $\bar{y}'$, which are rational in $\bar{y}'$, and where

$$\begin{aligned} &\alpha = x_{\bar{x}}y_{\bar{x}\bar{x}} - x_{\bar{x}\bar{x}}y_{\bar{x}}, \qquad \beta = 2(x_{\bar{x}}y_{\bar{x}\bar{y}} - x_{\bar{x}\bar{y}}y_{\bar{x}}) + (x_{\bar{y}}y_{\bar{x}\bar{x}} - x_{\bar{x}\bar{x}}y_{\bar{y}}), \\ &\gamma = 2(x_{\bar{y}}y_{\bar{x}\bar{y}} - x_{\bar{x}\bar{y}}y_{\bar{y}}) + (x_{\bar{x}}y_{\bar{y}\bar{y}} - x_{\bar{y}\bar{y}}y_{\bar{x}}), \qquad \delta = x_{\bar{y}}y_{\bar{y}\bar{y}} - x_{\bar{y}\bar{y}}y_{\bar{y}}, \\ &\epsilon = x_{\bar{x}}y_{\bar{y}} - x_{\bar{y}}y_{\bar{x}}. \end{aligned} \tag{27}$$

In order that equation (26) shall reduce to the required form, for all choices of the functions $\phi(x, y)$, $\psi(x, y)$ in the original equations, it is obviously necessary that we have the relations

$$\alpha = \beta = \gamma = \delta = 0, \qquad x_{\bar{x}}^2 + y_{\bar{x}}^2 = x_{\bar{y}}^2 + y_{\bar{y}}^2, \qquad x_{\bar{x}}x_{\bar{y}} + y_{\bar{x}}y_{\bar{y}} = 0. \tag{28}$$

It readily follows from equations (27) and (28) that $x_{\bar{x}}$, $x_{\bar{y}}$, $y_{\bar{x}}$, and $y_{\bar{y}}$ must be constants, and that we must have the relations

$$y_{\bar{x}} = \pm x_{\bar{y}}, \qquad y_{\bar{y}} = \mp x_{\bar{x}}.$$

Hence it is necessary, in order that a point transformation shall convert every three-parameter family of relativistic trajectories into a family of relativistic trajectories, that the transformation be a rigid motion, a magnification, a reflection with respect to a straight line, or a combination of such transformations. Also, we easily see that this condition is sufficient to insure that the transformation has the required property.

NATURAL FAMILIES OF RELATIVISTIC TRAJECTORIES

15. So far we have been considering the family of all possible trajectories of a particle in a positional field of force. Now we wish to study certain important subfamilies of trajectories, which we call natural families. Since, in the study of natural families, we can easily deal with the case of a particle moving in three-dimensional space, we shall do so. The results for the case in which the particle moves in a fixed plane can be obtained by a simple specialization.

Let us consider a relativistic particle, of rest-mass m_0, moving in three-dimensional space under a force which is derived from a potential energy function $V(x, y, z)$. Here x, y, and z are the rectangular coordinates of the particle with respect to a fixed set of axes. The differential equations of motion are the following:

$$(29)\qquad \begin{aligned} \frac{d}{dt}\left[\dot{x}\left(1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}\right)^{-1/2}\right] &= -\phi_x, \\ \frac{d}{dt}\left[\dot{y}\left(1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}\right)^{-1/2}\right] &= -\phi_y, \\ \frac{d}{dt}\left[\dot{z}\left(1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}\right)^{-1/2}\right] &= -\phi_z, \end{aligned}$$

where $\phi = V/m_0$.

Equations (29) possess the integral

$$(30)\qquad c^2\left(1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}\right)^{-1/2} = h - \phi,$$

where h is a constant of integration. Hence, the five-parameter family of trajectories defined by equations (29) consists of ∞^1 four-parameter families, each particular one of which corresponds to a particular value of h. Each of these four-parameter families will be called a natural family of trajectories.*

For the sake of convenience, we shall write $h-\phi=\Phi$.

It is easily shown that the defining differential equations of the natural family of trajectories corresponding to h can be written in the form

* Natural families of trajectories of classical particles are defined in an analogous way. See Kasner, *Differential-Geometric Aspects of Dynamics*, p. 34.

$$(31)\quad \begin{aligned} &(1+y'^2+z'^2)^{-1/2}\frac{d}{dx}\left[y'\left(\frac{\Phi^2-c^4}{1+y'^2+z'^2}\right)^{1/2}\right]=\frac{\partial}{\partial y}(\Phi^2-c^4)^{1/2},\\ &(1+y'^2+z'^2)^{-1/2}\frac{d}{dx}\left[z'\left(\frac{\Phi^2-c^4}{1+y'^2+z'^2}\right)^{1/2}\right]=\frac{\partial}{\partial z}(\Phi^2-c^4)^{1/2}, \end{aligned}$$

where $y'=dy/dx$, $z'=dz/dx$.

Now in Newtonian dynamics the equations corresponding to (29) are $\ddot{x}=-\phi_x, \ddot{y}=-\phi_y, \ddot{z}=-\phi_z$; the equation corresponding to (30) is $(\dot{x}^2+\dot{y}^2+\dot{z}^2)/2 =h-\phi$; and hence the differential equations defining the natural family of trajectories corresponding to h are

$$(32)\quad \begin{aligned} &(1+y'^2+z'^2)^{-1/2}\frac{d}{dx}\left[y'\left(\frac{2\Phi}{1+y'^2+z'^2}\right)^{1/2}\right]=\frac{\partial}{\partial y}(2\Phi)^{1/2},\\ &(1+y'^2+z'^2)^{-1/2}\frac{d}{dx}\left[z'\left(\frac{2\Phi}{1+y'^2+z'^2}\right)^{1/2}\right]=\frac{\partial}{\partial z}(2\Phi)^{1/2}. \end{aligned}$$

On comparing the systems of equations (31) and (32), we get the

THEOREM. *If the constants E_1, E_2, A, and m_0, and the functions $V_1(x, y, z)$ and $V_2(x, y, z)$ are such that we have identically*

$$A[E_1 - V_1(x, y, z)] = [E_2 - V_2(x, y, z)]^2 - m_0^2c^4,$$

the natural family of trajectories of a classical particle moving with (classical) total energy E_1 in the field of force derived from the potential energy function $V_1(x, y, z)$ is identical with the natural family of trajectories of a relativistic particle, of rest-mass m_0, moving with (relativistic) total energy E_2 in the field of force derived from the potential energy function $V_2(x, y, z)$.

Undoubtedly, the content of this theorem is more or less familiar, since it is an immediate consequence of the well known fact that, whereas the classical trajectories are defined by the principle of least action

$$\delta\int (E_1 - V_1)^{1/2}ds = 0,$$

the relativistic trajectories are defined by the principle

$$\delta\int [(E_2 - V_2)^2 - m_0^2c^4]^{1/2}ds = 0.$$

However, the theorem does not seem to be stated explicitly in any of the readily accessible literature.

We have seen in the preceding sections that the sets of properties which characterize the families of all trajectories of a particle in an arbitrary field of force are very different in the classical and relativistic cases, respectively. The present theorem shows that if we consider not the families of all trajectories but only natural families of trajectories, the characteristic properties are the same in the two cases. The characteristic properties of a natural family of trajectories have been given by Kasner.*

* *Differential-Geometric Aspects of Dynamics*, pp. 37–42. See also J. Lipka, these Transactions vol. 13 (1912), pp. 77–95.

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