ON 0-REGULAR SURFACE TRANSFORMATIONS*

BY

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1. The continuous transformation $T(M)=M'$, where M is a compact metric space, is said to be 0-regular† provided that for each sequence of points $\{x_i'\}$ converging to x' in M', the sets $T^{-1}(x_i')$ converge 0-regularly‡ to $T^{-1}(x')$. This is equivalent to a continuous transformation sending open sets into open sets, while the inverse sets as a collection are uniformly locally connected (that is, for each $\epsilon>0$ a $\delta>0$ exists such that every two points x and y of any inverse set X whose distance apart is less than δ lie in a connected subset of X of diameter less than ϵ). This characterization suggests the projection of a convex euclidean set onto a plane. For example, the orthogonal projection of a solid circle onto a diameter is a 0-regular transformation. It is not 0-regular on the circumference, however, because of the folding about the diameter's end points. That there exist other types of 0-regular transformations is illustrated by the identification of diametrically opposite points of a 2-sphere to obtain a projective plane. A suggestive property of a 0-regular transformation is that the inverse sets must all contain the same number of components.†

In this paper a study is made of 0-regular transformations defined on 2-dimensional pseudo-manifolds. It is shown that if M is a 2-dimensional pseudo-manifold and $T(M)=M'$ is a monotone 0-regular transformation, then either T is topological, or M' is an arc or a simple closed curve. Moreover, it is shown that T must be topological or M' must be degenerate except in the following cases: (i) The sphere, 2-cell, and circular ring may be mapped onto an arc. (ii) The torus, Klein bottle, circular ring, Möbius band, pinched sphere, and 2-cell with two boundary points identified may be mapped onto a simple closed curve. In each of these cases the possible transformations are characterized. For example, it is shown that the only non-topological monotone 0-regular transformation of a sphere onto a nondegenerate image space is equivalent to an orthogonal projection onto a diameter.

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† See A. D. Wallace, *On 0-regular transformations*, to appear in the American Journal of Mathematics.

‡ A convergent sequence of closed sets $\{X_n\}$ is said to converge 0-regularly to X provided that for each $\epsilon>0$ there exist positive numbers δ and N such that if $n>N$, any pair of points x, y of X_n with $\rho(x, y)<\delta$ lie together in a continuum in X_n of diameter less than ϵ. See G. T. Whyburn, Fundamenta Mathematicae, vol. 25 (1935), pp. 408–426.

In §6 the above results are stated in terms of possible monotone 0-regular retracting transformations on pseudo-manifolds, while in §7 R. L. Moore's* self-compact equicontinuous collections of curves are used in stating the results. In the concluding section the possible images of pseudo-manifolds under general 0-regular transformations are considered.

2. Throughout this section the following notation will be used: Let M denote a 2-dimensional pseudo-manifold, that is, a 2-dimensional manifold or surface (with or without boundary) among q points of which identifications have been performed so as to produce r local separating points† of M. Let B be the boundary (that is, a finite number of simple closed curves) of M, and denote the finite set of local separating points of M by S. Finally, let $T(M)=M'$ be a *monotone* 0-*regular* transformation and assume M' *is non-degenerate.*

2.1. *If x is a point of S, then $T^{-1}T(x)=x$.*‡

Proof. Since M is a locally connected continuum and x is a local separating point of M, there exists a connected neighborhood $U(x)$ of x such that $U(x)-x=L_1+L_2+\cdots+L_\lambda$ $(\lambda\geqq 2)$, where the L_i are mutually separated open sets each having x as a limit point. Assume the assertion is false; then there exists an L_k such that $\overline{T^{-1}T(x)\cdot L_k}$ contains x. Let $\{x_i\}$ be a sequence of points in $L_n-T^{-1}T(x)$ (n not k) converging to x. Now $\{T^{-1}T(x_i)\}$ converges to $T^{-1}T(x)$. Hence for each sufficiently large i there exists a point y_i of L_k such that $T(y_i)=T(x_i)$ and $\{y_i\}$ converges to x, since $\overline{T^{-1}T(x)\cdot L_k}$ contains x. But any connected set in $T^{-1}T(x_i)$ containing x_i and y_i must extend outside $U(x)$. Hence $\{T^{-1}T(x_i)\}$ does not converge 0-regularly to $T^{-1}T(x)$ contrary to the hypothesis that T is a 0-regular transformation.

2.11. *If x is a point of S, then $T(x)$ is a local separating point, but not a cut point of M'.*

Proof. Since $T^{-1}T(x)=x$, $T^{-1}T(x)$ locally separates M. Hence by a known theorem§ on monotone transformations $T(x)$ locally separates M'. However, $T(x)$ cannot be a cut point of M' since $M-x$ is connected.

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* *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932, pp. 396–397. The essential definitions are given in §7 for completeness.

† The point x of a continuum M is called a local separating point of M provided there exists a neighborhood $U(x)$ of x in M such that x separates $\overline{U(x)}$ between some pair of points of the component of $\overline{(Ux)}$ containing x. See G. T. Whyburn, *Local separating points of continua*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 305–314.

‡ The theorem and proof given here are valid if M is any locally connected continuum and x is a local separating point of M.

§ See G. T. Whyburn, *Semi-closed sets and collections*, Duke Mathematical Journal, vol. 2 (1936), p. 686, (3.1).

2.2. *If x' is any point of M', then $T^{-1}(x')$ is either an arc or a simple closed curve.*

Proof. Since T is interior,* $T^{-1}(x')$ can contain no open set. Hence either $T^{-1}(x')$ is a single point (that is, a degenerate arc or simple closed curve) or $T^{-1}(x')$ is a 1-dimensional continuum. Moreover, $T^{-1}(x')$ is locally connected.† Therefore, in order to establish the assertion it must be shown that every point x of $X=T^{-1}(x')$ has an order not greater than 2 in X. Suppose x has an order greater than 2 in X; then there exist nondegenerate arcs α_1, α_2, α_3 in X which are disjoint except for $\alpha_1\cdot\alpha_2\cdot\alpha_3=x$, an end point of each α_i. Since, after 2.1, x cannot be a point of S, there exists a neighborhood $U(x)$ of x in M such that $\overline{U(x)}$ is a 2-cell. It may be assumed each α_i is disjoint‡ with $F(U(x))$ except for the other end point. Now $\overline{U(x)}-\sum\alpha_i$ must contain at least two components L_1, L_2 such that $\overline{L}_i$ contains x. (Observe that one cannot say three components here because x may be a point of B.) Moreover, it may be assumed $\alpha_1\cdot\overline{L}_1=x$, since $\overline{U(x)}$ is a 2-cell. There exists a sequence of points $\{y_i\}$ of $L_1\cdot(M-X)$ converging to x, since X contains no open subset of M. Now if $\{Y_i\}=\{T^{-1}T(y_i)\}$ converges to X, there must exist for all sufficiently large i points z_i of Y_i not contained in L_1, since $\alpha_1\cdot\overline{L}_1=x$. It may be assumed $\{z_i\}$ converges to x. Therefore $\{Y_i\}$ does not converge 0-regularly to X, since any connected subset of Y_i containing y_i and z_i must extend outside $\overline{U(x)}$. Thus the assumption that the order of x is greater than 2 has led to the contradiction that T is not 0-regular.

2.3. *If x' is a point of M' such that $T^{-1}(x')$ is nondegenerate and not contained in B, then x' is a local separating point of M'.*

Proof. Since $X=T^{-1}(x')$ is nondegenerate and not contained in B, it follows from 2.2 and 2.1 that there exists a subarc α of X which is disjoint with $B+S$. Let x be an interior point of α and let α_1, α_2 be arcs such that $\alpha=\alpha_1+\alpha_2$ while $\alpha_1\cdot\alpha_2=x$. There exists a neighborhood $U(x)$ of x which is an open 2-cell and does not contain either of the α_i. Let $x_1xx_2=\beta$ be the subarc of α containing x such that $\beta\cdot F(U(x))=x_1+x_2$; then $U(x)-\beta=L_1+L_2$ is a separation such that $\overline{L}_1\cdot\overline{L}_2=\beta$. Now let $\{U_i(x)\}$ be a sequence of connected neighborhoods contained in $U(x)$ and closing down on x. Then for each i, $U_i(x)-X=L_{i1}+L_{i2}$, where $L_{ij}=L_j\cdot(U_i(x)-X)$, is a separation, since X contains no open subset of M. Suppose x' is not a local separating point of M';

* That is, open sets go into open sets. That a 0-regular transformation is interior follows from the Eilenberg characterization of an interior transformation. See Fundamenta Mathematicae, vol. 24 (1935), p. 174.

† See A. D. Wallace, loc. cit.

‡ For any open set U, $F(U)$ denotes the set-theoretic boundary of U, that is, the set $\overline{U}-U$.

then for each i, $T(U_i(x)-X)=T(U_i(x))-x'=0'$ must be a connected set.* Hence there must exist a point y_i' of $0'$ for each i such that $T^{-1}(y_i')\cdot L_{ij}$ contains a point y_{ij} for $j=1, 2$. Now $\{y_i'\}$ converges to x' and $\rho(y_{i1}, y_{i2})$ converges to zero, since the $U_i(x)$ close down on x. But from the definition of the L_{ij} it follows that any connected set in $T^{-1}(y_i')$ containing y_{i1} and y_{i2} must go outside $U(x)$. Therefore $\{T^{-1}(y_i')\}$ does not converge 0-regularly to X, contrary to the hypothesis that T is a 0-regular transformation.

2.31. *Under the conditions of* 2.3, *x' locally separates M' into exactly two components.*

Proof. Suppose x' locally separates M' into more than two components; then there exists a connected neighborhood $V(x')$ such that $V(x')-x'=L_1'+L_2'+L_3'+\cdots$, where the L_j are mutually separated connected open sets with x' a point of $F(L_j')$ for each j. For each j, $T^{-1}(L_j')$ is a connected set, since connectedness is invariant under the inverse of a monotone transformation. Thus, since T is interior, $T^{-1}(x')$ is on the boundary of at least three mutually separated connected open sets in M. This is impossible since $T^{-1}(x')$ is a nondegenerate arc or simple closed curve.

2.32. *If x' is an end point of $T^{-1}(x')$ for some point x' of M', then x belongs to B.*

2.33. *If x' is not a local separating point of M', then $T^{-1}(x')$ is a single point or is contained in B.*

2.4. *If x' is a point of M' such that $B\cdot T^{-1}(x')$ contains a nondegenerate continuum K, then $T^{-1}(x')$ is contained in B and x' is an end point of M'.*

Proof. If $T^{-1}(x')$ is not contained in B, it follows from 2.31 that x' locally separates M' into exactly two components. Thus there exists a connected neighborhood $V(x')$ of x' such that $V(x')-x'=L_1'+L_2'$, where L_1' and L_2' are mutually separated connected open sets with x' a point of $\overline{L_1'}\cdot\overline{L_2'}$. Hence $T^{-1}(L_1')$ and $T^{-1}(L_2')$ are disjoint connected open sets in M whose boundaries have $T^{-1}(x')$ in common, consequently have K in common. This is impossible since $T^{-1}(x')$ is locally connected and K is contained in B.

Let x' be a point of M' such that $T^{-1}(x')$ is nondegenerate and contained in B. Now $T^{-1}(x')$ can contain no point of S. Hence for any point x of $T^{-1}(x')$ which is not an end point there exists a neighborhood $U(x)$ such that $\overline{U(x)}$ is a 2-cell. Moreover, $U(x)$ may be taken so small that x is an interior point of an arc lying in $T^{-1}(x')$ with its end points not in $\overline{U(x)}$.

* See G. T. Whyburn, *Concerning points of a continuous curve defined by certain im kleinen properties*, Mathematische Annalen, vol. 102 (1929), pp. 313–336, Theorem 1.

Thus there exists a neighborhood $V(x')$ such that for each point y_i' of $V(x')-x'$ the set $T^{-1}(y_i')$ contains an arc Y_i, with its end points only in $F(U(x))$, which separates $\overline{U(x)}$ into exactly two components, since T is monotone 0-regular. In case $\{y_i'\}$ is any sequence of points converging to x', the corresponding Y_i may be so selected that if L_i denotes the component of $U(x)-Y_i$ containing x, then x is a point of $L=\prod L_i$ which is contained in $T^{-1}(x')$. Finally, every sufficiently small neighborhood $W(x)$ must have the property that $T^{-1}(x')\cdot W(x)$ is contained in L and for every point y_i' of $T(W(x))-x'$ the product $L_i\cdot W(x)\cdot T^{-1}(y_i')$ is empty. Suppose x' is not an end point of M'. Then it is an interior point of an arc $z_1'x'z_2'$. Now $T^{-1}(z_i'x')$ $(i=1, 2)$ is locally connected.* Hence there exist arcs y_ix_i in $W(x)$ such that $T(y_ix_i)=y_i'x'$ is contained in $z_i'x'$, where it is assumed y_i is the last point of y_ix_i in $T^{-1}(y_i')$ and x_i is the first point in $T^{-1}(x')$. From the choice of $W(x)$ it follows that y_i is contained in Y_i and x_i in L. Since $\overline{U(x)}$ is a 2-cell it may be assumed Y_2 separates Y_1 and L in $U(x)$ and consequently in $W(x)$. Thus y_1x_1 contains a point of Y_2 which contradicts the fact that $T(y_1x_1)$ is contained in $z_1'x'$.

2.5. *If for a point x' of M' the set $T^{-1}(x')$ is not contained in B, then $B\cdot T^{-1}(x')$ can contain only end points of $T^{-1}(x')$.*

Proof. Suppose the assertion is not true. Then there exists a point x of $B\cdot T^{-1}(x')$ which is not an end point of $T^{-1}(x')$. Now x is not a point of S, since $T^{-1}(x')$ must be nondegenerate. Thus there exists a neighborhood $U(x)$ such that $\overline{U(x)}$ is a 2-cell. It may be assumed there exist points of $T^{-1}(x')$ which are not in $\overline{U(x)}$. Let x_1, x_2 be points of $T^{-1}(x')$ different from x such that the arc x_1xx_2 is contained in $U(x)\cdot T^{-1}(x')$, and furthermore, let y, z be points of B different from x such that the arc yxz is contained in $B\cdot U(x)$. Then $(x_1xx_2)\cdot(yxz)=x$; for suppose the product set contained another point x_3. Then x_1xx_2+yxz contains a simple closed curve J, since no subcontinuum of $T^{-1}(x')$ can be contained in B because of 2.4. Thus J separates $\overline{U(x)}$ into exactly two components and there is one component C such that $\overline{C}\cdot[\overline{M-U(x)}]=0$, since J is contained in $U(x)$. Let $\{w_i\}$ be a sequence of points in C converging to a point x_0 of $J\cdot T^{-1}(x')$; then $T^{-1}T(w_i)$ is contained in $\overline{C}=C+J$ for each i. Hence $\{T^{-1}T(w_i)\}$ cannot converge to $T^{-1}T(x_0)$ $=T^{-1}(x')$, since the latter set contains points outside $\overline{U(x)}$. Thus the four arcs x_1x, x_2x, yx, zx are contained in the 2-cell $\overline{U(x)}$ and have by pairs only x in common. Hence there exists a neighborhood $W(x)$ in $U(x)$ which is separated into three components by x_1xx_2 such that x is on the boundary of

* See W. T. Puckett, *Concerning local connectedness* . . . , American Journal of Mathematics, vol. 61 (1939), p. 752, (3.1).

each, and only one can have both x_1x and x_2x on its boundary. Let L_i be the component such that $\overline{L}_i$ contains x_ix but not x_jx, and let $\{w_n\}$ be a sequence of points in L_1 converging to x. Then $\{T^{-1}T(w_n)\}$ cannot converge 0-regularly to $T^{-1}T(x)=T^{-1}(x')$, since they must go outside $W(x)$ to converge to the arc x_2x.

2.6. *If the sequence $\{x_n'\}$ of local separating points of M' converges to a point x' of $M'-T(S)$ and $T^{-1}(x')$ is degenerate, then x' is an end point of M'.*

Proof. There exists a neighborhood $U(x)$ of $x=T^{-1}(x')$ such that $\overline{U(x)}$ is a 2-cell, since x is not a point of S. Moreover, it may be assumed that each $T^{-1}(x_n')$ is contained in $U(x)$, since $\{T^{-1}(x_n')\}$ converges to x. Since x_n' is a local separating point of M', $T^{-1}(x_n')$ locally separates* M and, consequently, separates $\overline{U(x)}$ because it is a 2-cell. But since $T^{-1}(x_n')$ is contained in $U(x)$ it follows that $T^{-1}(x_n')$ separates M; that is, $M-T^{-1}(x_n')=L_n+N_n$ is a separation and it is assumed x is a point of L_n. Now $F(L_n)$ is contained in $T^{-1}(x_n')$, whence $F[T(L_n)]$ contains at most the single point x_n'. Thus $T(L_n)$ is an open set containing x' whose boundary consists of at most a single point. Thus in order to complete the proof it remains to be shown that $T(L_n)$ is a sequence of sets closing down on x'. It may be assumed that for each $n>k$, $T^{-1}(x_n')$ is contained in L_k. Moreover, because the transformation is interior, $F(L_n)=F(N_n)=T^{-1}(x_n')$. Hence x is a point of L_{n+1} which is contained in L_n, and consequently x is a point of $L=\prod L_n$. But $N=\sum N_n$ is open and $F(N)=x$, since for $n<k$, N_n is contained in N_k, $F(N_n)=T^{-1}(x_n')$, and $\{T^{-1}(x_n')\}$ converges to x. Thus if L_n does not close down on x, that is, x is not L, then $M-x=(L-x)+N$ is a separation. But this is impossible, since M is a 2-dimensional pseudo-manifold. Therefore $x=L$, and the proof is complete.

3. We next prove the following theorem.

THEOREM. *If M is a 2-dimensional pseudo-manifold and $T(M)=M'$ is a monotone 0-regular transformation, then T is topological, or M' is either an arc or a simple closed curve.*

Proof. In case M' is degenerate there is nothing to prove. Thus assume M' is nondegenerate and let K be the set of all points of M on which T is one-to-one. Then K is closed, since T is interior. Let $G=M-K$; then by 2.3 and 2.4, $T(G)$ consists of local separating points and end points of M'. Suppose G is not empty, and let x be a point of $F(G)$. If x is not a point of S, then it follows from 2.6 that $T(x)$ is an end point of M'. If x is a point of S, then it follows from 2.11 that $T(x)$ is a local separating point of M'. Hence $T(\overline{G})$ consists of local separating points and end points of M'. Suppose $M-\overline{G}$ is not

* See G. T. Whyburn, *Semi-closed sets and collections*, loc. cit.

empty; then there exist points y of $(M-S)\cdot(M-\overline{G})$ and z of $(M-S)\cdot G$, since $M-\overline{G}$ and G are both open and S is finite. Now $M-S$ is a region in the locally connected continuum M. Hence there exists an arc yz in $M-S$ which must intersect $F(G)$. Let x be the first such point from y to z. Now the arc yx is contained in K, and consequently $T(yx)$ is topological. Therefore no point x_1' of $T(yx)$ can be a local separating point of M', since no point of $M-S$ locally separates M. But $T(x)$ is an end point of M', which is a contradiction. Thus either G is empty or $M-\overline{G}$ is empty. In the first case T is topological, while in the second M' consists of end points and local separating points and consequently is a 1-dimensional continuum. But T is interior and for each point x' of M', $T^{-1}(x')$ is locally connected. Hence it follows from a known theorem* that M' is either an arc or a simple closed curve.

4. It is proposed in this section to show that a monotone 0-regular transformation on a 2-dimensional pseudo-manifold must be topological except in a few specific cases. The notation is that used in §2.

4.1. *In order that T be topological it is sufficient that either* (a) *S contains a point which locally separates M into at least three components, or* (b) *S contains more than one point.*

Proof. Suppose T is not topological; then M' is either an arc or a simple closed curve. Now after 2.11 the image of a point x of S cannot be an end point of M'. Hence $x'=T(x)$ must locally separate M' into exactly two components; that is, there exists a connected neighborhood $V(x')$ in M' such that $V(x')-x'=L'+N'$, where L', N' are open arcs with $x'=\overline{L}'\cdot\overline{N}'$. Suppose x locally separates M into more than two components; then there exists a connected neighborhood $U(x)$ such that $T(U(x))$ is contained in $V(x')$ and $U(x)-x=M_1+M_2+\cdots+M_k$ $(k\geqq 3)$, where, for each integer $i<k$, M_i is a component and x is a point of $\overline{M}_i$. Thus, since $T^{-1}(x')=x$, it may be assumed $L'\cdot T(M_1)\cdot T(M_2)$ contains a sequence of points $\{x_n'\}$ converging to x'. But $\{T^{-1}(x_n')\}$ does not converge 0-regularly to x, since for each n, $T^{-1}(x_n')\cdot M_1$ is not empty and $T^{-1}(x_n')\cdot M_2$ is not empty. Thus for (a) T must be topological. Now under the assumption that T is not topological it follows that if y, z are points of S, then $T(y+z)$ separates M'. Hence $y+z$ separates M, which is impossible. Thus for (b), T is also topological.

The following statement follows immediately from 2.11:

4.11. *In order that M' be an arc it is necessary that S be empty.*

4.2. *If T is not topological, then either $T^{-1}(x')$ is an arc for every point x'*

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* See G. T. Whyburn, *Interior transformation on certain curves*, Duke Mathematical Journal, vol. 4 (1938), p. 612.

of M' or every $T^{-1}(x')$ is a simple closed curve. In neither case can more than two $T^{-1}(x')$ be degenerate.

Proof. Let L' be the set of all points x' of M' such that $T^{-1}(x')$ is degenerate. Since T is not topological either x' must be an end point of M' or $x=T^{-1}(x')$ must be a point of S. After 4.11 it follows that S is empty when M' has end points. Thus, since M' is either an arc or simple closed curve and consequently S consists of not more than one point, L' can contain at most two points and $M'-L'$ is connected. Let N_1' be the set of all points x' of M' such that $T^{-1}(x')$ is an arc, and N_2' the set of all points such that $T^{-1}(x')$ is a simple closed curve. From 2.2 it follows that $M'-L'=N_1'+N_2'$. But $\overline{N}_1' \cdot N_2' = N_1' \cdot \overline{N}_2' = 0$ since the 0-regular limit of a sequence of arcs (simple closed curves) is an arc (simple closed curve).* Hence either $N_1'=0$ or $N_2'=0$, since $M'-L'$ is connected.

4.21. *For each point x' of M' let $T^{-1}(x')$ be a simple closed curve, at least one of which is nondegenerate. If J is any simple closed curve of B, then some $T^{-1}(x')=J$.*

Proof. From 2.5 it follows that if $B \cdot T^{-1}(x')$ is not empty then $T^{-1}(x')$ is contained in B. Moreover, it follows from 4.2 that every $T^{-1}(x')$, except possibly two, is a nondegenerate simple closed curve. Thus there exists a nondegenerate simple closed curve $T^{-1}(x')$ contained in B such that $J \cdot T^{-1}(x')$ is not empty. But $J \cdot (\overline{B-J})$ is contained in S (of course, may be empty), while $S \cdot T^{-1}(x')=0$ by 2.1. It follows that $T^{-1}(x')$ is contained in J and is therefore J.

The following assertion also comes out of the above proof:

4.22. *Under the hypotheses of* 4.21, $S \cdot B=0$.

Let $p_m^1(M)$ denote the first Betti number (mod m) of M, for $m \geqq 0$. Also, if N is a closed subset of M, let $p_m^1(N, M)$ denote the first Betti number (mod m) of N relative to M, that is, the number of independent cycles in N relative to homologies (mod m) in M. In another paper† the writer has shown that

(i) *If x', y' are any two points of M', then*

$$p_m^1[T^{-1}(x'), M] = p_m^1[T^{-1}(y'), M]. \tag{1}$$

(ii) *If x' is any point of M', then*

$$p_m^1(M) = p^1(M') + p_m^1[T^{-1}(x'), M], \tag{2}$$

* See G. T. Whyburn, *On sequences and limiting sets*, Fundamenta Mathematicae, vol. 25 (1935), pp. 409–426, particularly (3.1) and (3.2), p. 416.

† *On regular transformations* (offered for publication to Duke Mathematical Journal).

which in the case considered here may be written

$$(2') \qquad 0 \leqq p_m^1(M) - p_m^1(M') = p_m^1[T^{-1}(x'), M],$$

since all the numbers involved are finite.

The two relations above, along with 2.2, give the following assertions:

4.31. *In any case* $0 \leqq p_m^1(M) - p_m^1(M') \leqq 1$ $(m \geqq 0)$.

4.32. *If, for any* $m \geqq 0$, $p_m^1(M) > 2$, *then* T *is topological.*

4.33. *If, for some point* x' *of* M', $T^{-1}(x')$ *is degenerate or an arc, then* $p_m^1(M) = p_m^1(M')$ $(m \geqq 0)$.

4.34. *In order that* M' *be an arc it is necessary that* $p_m^1(M) \leqq 1$ $(m \geqq 0)$.

4.35. *In order that* M' *be a simple closed curve it is necessary that* $1 \leqq p_m^1(M) \leqq 2$ $(m \geqq 0)$.

The assertions 4.1 and 4.11 may be obtained from 4.33, 2.1, and 2.2 as follows: Let M be a pseudo-manifold with $S = y_1 + y_2 + \cdots + y_\lambda$; then it may be assumed M was obtained from a manifold L, which contains no identifications, by identifying μ_i points to obtain y_i. Thus it follows from the Euler-Poincaré formula that

$$(3) \qquad p_m^1(M) = p_m^1(L) + \sum_{i=1}^{\lambda} (\mu_i - 1) \qquad (m \text{ prime}).$$

Therefore, if $\lambda > 1$ or some $\mu_i > 2$ it follows that $p_m^1(M) \geqq 2$. Thus it follows from 2.1 and 4.33 that $p_m^1(M) = p_m^1(M') \geqq 2$, since S is not empty. Hence T must be topological, since M' cannot be an arc of a simple closed curve. By the same reasoning it follows that if S contains a single point then $p_m^1(M') = p_m^1(M) \geqq 1$, and therefore M' cannot be an arc.

A 2-dimensional closed surface M (that is, $S=0$, $B=0$) can possess no 0- or 2-dimensional torsion,* and if M is orientable (that is, a sphere, or torus, and so on) it can possess no 1-dimensional torsion. However, if M is not orientable (that is, projective plane, Klein bottle, and so on) its 1-dimensional torsion group is cyclic of order 2.† Hence it follows from a known theorem‡ that if M is a 2-dimensional closed surface, then

(4a) $p_2^1(M) = p_0^1(M)$ when M is orientable, and

(4b) $p_2^1(M) = p_0^1(M) + 1$ when M is non-orientable.

Therefore it follows† that

(5a) $p_0^1(M) = p_2^1(M) = 0$, if M is a sphere;

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* See Alexandroff-Hopf, *Topologie* I, Berlin, 1935, Theorems I and II′, p. 212.

† See *Topologie* I, paragraph 10, pp. 266–269.

‡ See *Topologie* I, Theorem VIII′, p. 227.

(5b) $p_0^1(M)=0$, $p_2^1(M)=1$, if M is a projective plane;

(5c) $p_0^1(M)=1$, $p_2^1(M)=2$, if M is a Klein bottle;

(5d) $p_0^1(M)=p_2^1(M)=2$, if M is a torus; and

(5e) $p_2^1(M)>2$, if M is any other 2-dimensional closed surface.

Now let M be a 2-dimensional surface with boundary (that is, let $S=0$, $B=J_1+J_2+\cdots+J_\beta$, where the J_i are disjoint simple closed curves); then M may be thought of as a closed 2-dimensional surface L with β open 2-cells cut out. Thus if β is not 0, then*

(6a) M possesses no torsion;

(6b) $p_2^1(M)=p_0^1(M)=p_0^1(L)+\beta-1$ when L is orientable; and

(6c) $p_2^1(M)=p_0^1(M)=p_0^1(L)+\beta$ when L is non-orientable.

The relations (3), (5), and (6) are enough to determine the first Betti number (mod 0 or 2) of any 2-dimensional pseudo-manifold with which this paper hereafter is concerned.

4.4. *If M' is a nondegenerate arc, then M must be either a sphere, a 2-cell, or a circular ring.*

Proof. Since M' is an arc, it follows from 4.11 that $S=0$, and from 4.34 that $p_2^1(M)\leqq 1$. Hence, besides those surfaces given in the theorem, M may be either a projective plane or a Möbius band, both of which have $p_2^1(M)=1$. Suppose M to be either one of these and M' to be an arc; then it follows from 4.33 that $T^{-1}(x')$ is a nondegenerate simple closed curve for each point x' of M'. But let y', z' be the end points of M'; then it follows from 2.33 that $T^{-1}(y')+T^{-1}(z')$ is contained in B. This is impossible, since in the first case $B=0$ while in the second B consists of a single simple closed curve.

4.5. *If M' is a nondegenerate simple closed curve, then M is either a torus, a Klein bottle, a circular ring, a Möbius band, a pinched sphere (that is, a sphere with two points identified), or a 2-cell with two boundary points identified.*

Proof. Since M' is a simple closed curve it follows from 4.35 that $1\leqq p_m^1(M)\leqq 2$ for $m=0$ or 2. Thus the torus and Klein bottle are the only closed surfaces which can possibly transform into a simple closed curve. However, if $S=0$ and M has boundary, it is possible, besides the circular ring and Möbius band, that M may be either a 2-cell with two holes, a Möbius band with a hole, a torus with a hole, or a Klein bottle with a hole. In each of these cases $p_2^1(M)=2$ and $\beta\geqq 1$. Suppose M is one of these; then from 4.33 it follows that $T^{-1}(x')$ is a nondegenerate simple closed curve for each point x' of M'. Hence after 4.21 it follows that there exists a $T^{-1}(x')$ contained in B. Thus this x' is an end point of M' because of 2.4. This is impossible under

* See *Topologie* I, paragraph 11, pp. 269–270.

the assumption that M' is a simple closed curve. If S is not empty, then it follows from 2.1 and 4.33 that $p_m^1(M)=p_m^1(M')=1$, for $m=0$ or 2. Hence the only possibilities here are the pinched sphere and 2-cell with two points identified. It remains to show that in the case of the 2-cell the two points which are identified must be on the boundary. Let x_1, x_2 be the two points of a 2-cell, at least one of which is not on the boundary, which are identified to give the point $x=S$. Then there exists a neighborhood $U(x)$ such that $U(x)-x$ has two components, one of which, say C, is such that $\overline{C}$ is a 2-cell. Let $\{y_i\}$, contained in C, converge to x. Then for sufficiently large i, $T^{-1}T(y_i)$ is contained in C and is disjoint with $F(C)$. Now each of these $T^{-1}T(y_i)$ must locally separate M and consequently locally separate $\overline{C}$. Thus each must be a nondegenerate simple closed curve. Therefore, $T^{-1}(x')$ is a simple closed curve for each x' of M' after 4.2, and consequently some $T^{-1}(x')$ which is nondegenerate is contained in B. But this x' would be an end point of M' by 2.4, which is impossible since by assumption M' is a simple closed curve.

5. In 4.4 and 4.5 it is shown that only a few of the 2-dimensional pseudomanifolds can possibly be transformed into a nondegenerate arc or simple closed curve by a monotone 0-regular transformation. In this section it is shown that each of these transformations is possible. Moreover, the transformations are completely characterized.

A transformation $T(M)=M'$ is said to be *topologically equivalent** or simply *equivalent* to a transformation $W(N)=N'$ provided one can write $T(M)\equiv hWH(M)=M'$, where $H(M)=N$, $h(N')=M'$ are homeomorphisms.

5.1. *If M is a 2-cell and M' is a nondegenerate arc, then $T(M)=M'$ is equivalent to one of the transformations $W(N)=N'$, where N' is the interval $0\leqq\xi'\leqq 1$ and either* (a) *N is the square*

$$0 \leqq \xi \leqq 1, \qquad 0 \leqq \eta \leqq 1$$

with $\xi'=\xi$, (b) *N is the triangle*

$$0 \leqq \xi \leqq 1, \qquad 0 \leqq \eta \leqq \xi$$

with $\xi'=\xi$, (c) *N is the triangle*

$$0 \leqq \xi \leqq 1/2, \qquad 0 \leqq \eta \leqq \xi; \qquad\qquad 1/2 \leqq \xi \leqq 1, \qquad 0 \leqq \eta \leqq 1-\xi$$

with $\xi'=\xi$, or (d) *N is the solid circle*

$$0 \leqq \xi^2+\eta^2 \leqq 1$$

with $\xi'=(\xi^2+\eta^2)^{1/2}$.

* See G. T. Whyburn, *Completely alternating transformations*, Fundamenta Mathematicae, vol. 27 (1936), p. 140.

Proof. Suppose, after 4.2, that the inverse of every point of $M'=x_0'x_1'$ is an arc. Then there are three possibilities: (i) neither of the continua $X_i=T^{-1}(x_i')$ ($i=0$ or 1) is degenerate, (ii) one, say $X_0=x_0$, is degenerate, and (iii) both $X_0=x_0$ and $X_1=x_1$ are degenerate. It will be shown that the transformations arising from these possibilities are equivalent to (a), (b), and (c) respectively.

Since in (i) X_0 and X_1 are nondegenerate arcs, it follows from 2.33 that they lie in B. Hence $B=\alpha_0+X_0+\alpha_1+X_1$, where each α_i is an arc disjoint with X_0 and X_1 except for one end point in each. Now if x' is an interior point of M', the arc $X=T^{-1}(x')$ must separate X_0 and X_1 in M. Moreover, after 2.5, B can contain only end points of X. Hence $\phi=[X]$, where $X=T^{-1}(x')$ for some x' of M', is an equicontinuous (since it is 0-regular) collection* of arcs satisfying a theorem of R. L. Moore.† Thus there exists a self-compact* collection $G=[g]$ of mutually disjoint arcs such that $\sum g=M$ and for each X of ϕ and g of G the product $X\cdot g$ is a single point. Let $h(N')=M'$ be a topological transformation, and for N of (a) in the theorem let $H_0(M)=N$ be a topological transformation such that $H_0(X_i)=W^{-1}h^{-1}(x_i')$ ($i=0$, or 1). Now $\phi_0=[H_0(X)]$ and $G_0=[H_0(g)]$ are self-compact collections of arcs filling up N. Let $\phi_0'=[X_j^0]$ and $G_0'=[g_j^0]$ be countable subcollections of ϕ_0 and G_0 respectively such that

$$\overline{\sum_{j=1}^{\infty} X_j^0} = N = \overline{\sum_{j=1}^{\infty} g_j^0}.$$

These subcollections may be used in order to set up a sequence $H_j(N)=N$ of topological transformations, each of which is the identity on $\xi=0$, $0\leqq\eta\leqq 1$, whose limit is the homeomorphism $H_\infty(N)=N$ with the property $H_\infty H_0(X)=W^{-1}h^{-1}(x')$ for every point x' of M'. Now define $H\equiv H_\infty H_0$; then $T(M)\equiv hWH(M)$.

In possibility (ii) $X_0=x_0$ is a single point and X_1 is a nondegenerate arc. Just as above X_1 must be contained in B. Moreover, x_0 is contained in B, for suppose it were not. Then there exists a neighborhood $U(x_0)$ disjoint with B and a point x' interior to $x_0'x_1'$ such that $T^{-1}(x')$ is contained in $U(x_0)$ and separates X_1 and x_0 in M. This is impossible, since in the case considered $T^{-1}(x')$ is an arc. Now let $\{y_i'\}$ be a sequence of points converging to x_0'. Then on $M_1=T^{-1}(x_1'y_1')$, and generally on $M_i=T^{-1}(y_{i-1}'y_i')$ the transformation behaves as in (1). Hence just as for (1) there exists for each i a self-compact collection $G^i=[g^i]$ of mutually disjoint arcs such that $\sum g^i=M_i$ and for each $X=T^{-1}(x')$ of M_i, $g^i\cdot X$ is a single point. Now it may be assumed that

* *Foundations of Point Set Theory*, pp. 396–397.

† *Foundations of Point Set Theory*, Theorem 1, p. 397.

for each i, y'_{i+1} precedes y'_i in $x'_0x'_1$. For any point y_1 of $T^{-1}(y'_1)$ let g^1 be the arc of G^1 which has y_1 for an end point, and let g^2 be the arc of G^2 which has y_1 for an end point. Then the other end point of g^2 is a point y_2 of $T^{-1}(y'_2)$. Step by step for each i let g^i be the arc of G^i having y_{i-1} of $T^{-1}(y'_{i-1})$ for one end point and denote the other, which must be in $T^{-1}(y'_i)$, by y_i. Define

$$g = \overline{\sum_{i=1}^{\infty} g^i};$$

then g is an arc. Thus as y_1 ranges over $T^{-1}(y'_1)$ it generates a self-compact collection $G=[g]$ of arcs such that $\sum g=M$ and each g intersects any $X=T^{-1}(x')$ of M in a single point. As for the previous case, this collection along with $\phi=[X]$ may be used in connection with an arbitrary homeomorphism $h(N')=M'$, where N is given by (b), to obtain a topological transformation $H(M)=N$ such that $T(M)\equiv hWH(M)$.

By the argument used above it follows that in possibility (iii) both the points $x_0=X_0$ and $x_1=X_1$ lie in B. Moreover, if x' is an interior point of $M'=x'_0x'_1$, then T behaves on $M_i=T^{-1}(x'_ix')$ as in (ii). Hence if $h(N')=M'$ is an arbitrary homeomorphism, where N and W are given by (c), it follows from (b) that there exist homeomorphisms $H_i(M_i)=W^{-1}h^{-1}(x'_ix')$ such that $T(M_i)\equiv hWH_i(M_i)$. Moreover, the $H_i(M_i)$ may be so defined that $H_0T^{-1}(x')\equiv H_1T^{-1}(x')$. Define $H(x)=H_i(x)$ for x a point of M_i; then $T(M)\equiv hWH(M)$. Thus all three possibilities for $T^{-1}(x')$ an arc are characterized.

If for each x' of M', $T^{-1}(x')$ is a simple closed curve, then it follows from 4.21 and 2.33 that the inverse of one end point of $M'=x'_0x'_1$, say $T^{-1}(x'_1)$, is the whole of B while the inverse of the other end point $T^{-1}(x'_0)=x_0$ is a single point in the interior of M. Since no point separates M, it therefore follows that the collection $\phi=[T^{-1}(x')]$ for all points x' of M' satisfies a theorem of Kerékjártó.* Hence the collection ϕ is homeomorphic with a collection of concentric circles filling a circle. Thus for N and W of (d) there exists a topological transformation $H(M)=N$ such that $WHT^{-1}(x')$ is a point of N' for every point x' of M' and conversely. Thus for each point a' of N' define $h(a')=x'$, where $WHT^{-1}(x')=a'$. Then $h(N')=M'$ is topological and such that $T(M)\equiv hWH(M)$.

5.2. *If M is a sphere and M' is a nondegenerate arc, then $T(M)=M'$ is equivalent to $W(N)=N'$, where N is the sphere $\xi^2+\eta^2+\zeta^2=1$, N is the interval $-1\leqq\xi'\leqq 1$, and W is the transformation $\xi'=\xi$.*

Proof. Since no arc separates the sphere, it follows from 4.2 and 2.33

* See Kerékjártó, *Topologie* I, Berlin, 1933, p. 246. See also H. Whitney, *Regular families of curves*, Annals of Mathematics, (2), vol. 34 (1933), example, p. 260.

that the inverse of every point x' of $M'=x_0'x_1'$ is a nondegenerate simple closed curve except for the end points x_i', which must have degenerate inverses. Let y' be any interior point of M', $M_i'=x_i'y'$, $M_i=T^{-1}(M_i')$, N_i' the interval $[0, (-1)^i]$, and $N_i=W^{-1}(N_i')$ for $i=0$ and 1. Then it follows from 5.1 (d) that there exist homeomorphisms $h_i(N_i')=M_i'$ and $H_i(M_i)=N_i$ such that $T(M_i)\equiv h_iWH_i(M_i)$. It may be assumed $h_0(0)=h_1(0)$ and $H_0T^{-1}(y')\equiv H_1T^{-1}(y')$. Define $H(x)=H_i(x)$ for x a point of M_i and $h(N')=M'$ accordingly. Then $T(M)\equiv hWH(M)$.

5.3. *If M is a circular ring and M' is a nondegenerate arc, then $T(M)=M'$ is equivalent to $W(N)=N'$ where N is the ring $1\leqq\xi^2+\eta^2\leqq 2$, N' is the interval $0\leqq\xi'\leqq 1$, and W is the transformation $\xi'=(\xi^2+\eta^2)^{1/2}-1$.*

Proof. Since $p_2^1(M)$ and $p_2^1(M')$ are not equal, it follows from 4.33 that the inverse of every point x' of M' is a nondegenerate simple closed curve. Let $B=J_0+J_1$, where J_i is a simple closed curve. Then it follows from 4.21 and 2.33 that $J_i=T^{-1}(x_i')$, where $M'=x_0'x_1'$. Let $H_0(M)=N$ be topological and suppose N^* to be the solid circle $0\leqq\xi^2+\eta^2\leqq 1$. Let this be filled with the family of concentric circles $G=[g]$. Then the families G and $\phi=[H_0T^{-1}(x')]$ for all x' of M' satisfy Kerékjártó's condition† that there exists a homeomorphism $H_1(N+N^*)=N+N^*$ such that $[H_1(g)]$ and $[H_1H_0T^{-1}(x')]$ together form a family of concentric circles filling $N+N^*$. Moreover, it may be assumed that for the circle X_0 (that is, for $\xi^2+\eta^2=1$) $H_1(X_0)=X_0$. Define $H\equiv H_1H_0$; then for each x' of M', $WHT^{-1}(x')=a'$, a point of N', and conversely. Now for each point a' of N' define $h(a')=x'$, where $WHT^{-1}(x')=a'$; then $h(N')=M'$ is topological and $T(M)\equiv hWH(M)$.

5.4. *If M is a circular ring and M' is a nondegenerate simple closed curve, then $T(M)=M'$ is equivalent to the transformation $\xi'=\cos\theta$, $\eta=\sin\theta$ on the circular ring N:*

$$\xi = r\cos\theta, \qquad \eta = r\sin\theta \quad (0\leqq\theta\leqq 2\pi,\ 1\leqq r\leqq 2).$$

Proof. Since $S=0$ and M' has no end points, $T^{-1}(x')$ is nondegenerate for each x' of M' and is not contained in $B=J_0+J_1$. Thus, after 4.21, each $T^{-1}(x')$ is a nondegenerate arc, and $B\cdot T^{-1}(x')$ consists of the end points of $T^{-1}(x')$ because of 2.32 and 2.5. Moreover, the end points of $T^{-1}(x')$ must lie one in J_0 and one in J_1, for if both were contained in J_i, $T^{-1}(x')$ would separate M and consequently x' would separate M'. Let y_1' and y_2' be any two points of M'; then $M'=\alpha_1'+\alpha_2'$, where α_i' is an arc and $\alpha_1'\cdot\alpha_2'=y_1'+y_2'$. Define $M_i=T^{-1}(\alpha_i')$. Let $W(N)=N'$ designate the transformation of the theorem and let a_1', a_2' be any two points of N'. Express N' as the sum of

† *Topologie* I, p. 246.

two arcs, $N'=\beta_1'+\beta_2'$ where $\beta_1'\cdot\beta_2'=a_1'+a_2'$. Define $N_i'=W^{-1}(\beta_i')$; then after 5.1 (a) there exist homeomorphisms $h_i(\beta_i')=\alpha_i'$ and $H_i(M_i)=N_i$ such that $T(M_i)\equiv h_iWH_i(M_i)$. Moreover, the homeomorphisms may be so defined that $h_1(a_i')=h_2(a_i')$ and $H_1T^{-1}(y_i')\equiv H_2T^{-1}(y_i')$. Let $H(x)=H_i(x)$ on M_i and $h(a')=h_i(a')$ on β_i; then $T(M)\equiv hWH(M)$.

5.5. *If M is a Möbius band and M' is a nondegenerate simple closed curve, then $T(M)=M'$ is equivalent to the transformation W: $\xi'=\cos\theta$, $\eta'=\sin\theta$, $\zeta'=0$, on the Möbius band N:*

$$\xi = (2 + r\cos\theta/2)\cos\theta,\quad \eta = (2 + r\cos\theta/2)\sin\theta,\quad \zeta = r\sin\theta/2$$
$$(0 \leqq \theta < 2\pi,\ -1 \leqq r \leqq 1).$$

Proof. Just as in 5.4 it follows that the inverse of each point x' of M' must be a nondegenerate arc with its end points only in B. Again just as in 5.4, 5.1 (a) may be used to define the homeomorphisms h and H such that $T(M)\equiv hWH(M)$, where $W(N)=N'$ is the analytical transformation of the theorem.

5.6. *If M is a 2-cell with two boundary points identified and M' is a nondegenerate simple closed curve, then $T(M)=M'$ is equivalent to the transformation $W(N)=N'$: $\xi'=\cos\theta$, $\eta'=\sin\theta$, where N is defined by*

$$\xi = r\cos\theta,\qquad \eta = r\sin\theta,\qquad 0 \leqq \theta \leqq 2\pi,\ 1 \leqq r \leqq (1/2)(3-\cos\theta).$$

Proof. Here $S=y_1$ is a single point. Thus it follows from an argument similar to that used in 5.4 that except for $T^{-1}T(y_1)$ the inverse of every point x' of M' is a nondegenerate arc with its end points in B and separated in B by y_1. Express M' as the sum of two nondegenerate arcs, that is, as $M'=\alpha_1'+\alpha_2'$, where $\alpha_1'\cdot\alpha_2'=y_1'+y_2'$. Then 5.1 (b) may be applied here as 5.1 (a) was in 5.4 to give homeomorphisms such that $T(M)\equiv hWH(M)$.

5.7. *If M is a pinched sphere and M' is a nondegenerate simple closed curve, then $T(M)=M'$ is equivalent to the transformation $W(N)=N'$: $\xi'=\cos\theta$, $\eta'=\sin\theta$, $\zeta'=0$, where N is defined by*

$$\xi = (2+\sin^2(\theta/2)\cos\phi)\cos\theta,\ \eta = (2+\sin^2(\theta/2)\cos\phi)\sin\theta,$$
$$\zeta = \sin^2(\theta/2)\sin\phi \qquad (0 \leqq \theta, \phi \leqq 2\pi).$$

Proof. Here again $S=y_1$ is a single point, Just as in 5.2 it follows that the inverse of every point of M', except $T(y_1)$, is a nondegenerate simple closed curve. Express M' as the sum of two nondegenerate arcs one common end point of which is $T(y_1)$. Then 5.1 (d) may be used to define homeomorphisms such that $T(M)\equiv hWH(M)$.

5.8. *If M is a torus and M' is a nondegenerate simple closed curve, then $T(M)=M'$ is equivalent to the transformation $W(N)=N'$: $\xi'=\cos\theta$, $\eta'=\sin\theta$, $\zeta'=0$, where N is defined by*

$$\xi = (2+\cos\phi)\cos\theta, \qquad \eta = (2+\cos\phi)\sin\theta, \qquad \zeta = \sin\phi \qquad (0 \leqq \theta, \phi \leqq 2\pi).$$

Proof. Since $p_2^1(M) > p_2^1(M')$ it follows from 4.32 and 4.2 that the inverse of every point of M' is a nondegenerate simple closed curve. Let $M'=\alpha_1'+\alpha_2'$, where α_1' and α_2' are nondegenerate arcs with common end points (that is, $\alpha_1'\cdot\alpha_2'=y_1'+y_2'$). Now $M_i=T^{-1}(\alpha_i')$ is a 2-dimensional manifold with $B=T^{-1}(y_1')+T^{-1}(y_2')$. Moreover, $T(M_i)=\alpha_i'$, an arc. Thus M_i must be a circular ring, since this is the only 2-dimensional surface with B consisting of two simple closed curves, which maps into an arc by a monotone 0-regular transformation. Express $N'=\beta_1'+\beta_2'$, where β_1' and β_2' are nondegenerate arcs with end points only in common. Define $N_i=W^{-1}(\beta_i')$. Then 5.3 gives homeomorphisms $h_i(\beta_i')=\alpha_i'$, $H_i(M_i)=N_i$ such that $T(M_i)\equiv h_iWH_i(M_i)$. Moreover, the h_i may be so chosen that $h_1(\beta_1'\cdot\beta_2')\equiv h_2(\beta_1'\cdot\beta_2')$ and the H_i so chosen that $H_1(M_1\cdot M_2)\equiv H_2(M_1\cdot M_2)$. As several times before define $H(x)=H_i(x)$ for x a point of M_i and $h(a')=h_i(a')$ for a' a point of β_i'; then $T(M)\equiv hWH(M)$.

Observation. Let Z_1 and Z_2 be simple closed curves on M which, when oriented, may be considered as generators of the Betti group $B_0^1(M)$. For each pair of positive integers k_1, k_2 there exists a monotone 0-regular transformation of M into a nondegenerate simple closed curve such that for each x' of M' the simple closed curve $T^{-1}(x')$ can be so oriented as to carry a cycle which is homologous to $k_1Z_1+k_2Z_2$. In case $k_i=0$, however, one must choose $k_j=1$.

5.9. While the Klein bottle can be mapped onto a nondegenerate simple closed curve by a monotone 0-regular transformation, $T(M)=M'$, there is not a convenient analytical description as in the previous cases. However, the possible transformations can be characterized after a fashion. In the first place $T^{-1}(x')$, for every point x' of M', must be a nondegenerate simple closed curve, since $p_2^1(M)>p_2^1(M')$. Let Z_1, Z_2 be simple closed curves on M which, when oriented, may be considered as generators of the Betti group $B_0^1(M)$. Since M is a Klein bottle, it may be assumed that $2kZ_2\sim 0$ for all integers k. There exist integers k_1 and k_2 such that

$$T^{-1}(x') \sim k_1Z_1 + k_2Z_2,$$

after $T^{-1}(x')$ is oriented. However k_1 must be zero, for if it were not, then $p_0^1[T^{-1}(x'), M]=1$. Consequently

$$p_0^1(M') = p_0^1(M) - p_0^1[T^{-1}(x'), M] = 0,$$

contrary to the fact that M' is a nondegenerate simple closed curve. Thus for every x' of M', $T^{-1}(x') \sim k_2 Z_2$, when oriented. Moreover, k_2 must be odd, since $p_2^1[T^{-1}(x'), M]$ cannot be zero.

In order to demonstrate such a mapping suppose M to arise from the oriented square $ABCD$ by identifying the oriented sides, AB with DC and BC with DA.* Let $T(M) = M'$ be such that the collection $[T^{-1}(x')]$ in $ABCD$ is a collection of straight lines parallel to BC.

6. The continuous transformation $T(M) = N$, a subset of M, is said to be *retracting*† provided that for each point x of N, $T(x) = x$. The following statements are immediate consequences of the results in the preceding sections:

6.1. *In order that there exist a monotone* 0*-regular retracting transformation of the* 2*-dimensional pseudo-manifold M onto a nondegenerate arc, it is necessary and sufficient that M be a* 2*-cell, a circular ring, or a sphere.*

6.2. *In order that there exist a monotone* 0*-regular retracting transformation of the* 2*-dimensional pseudo-manifold M onto a nondegenerate simple closed curve, it is necessary and sufficient that M be a circular ring, a Möbius band, a torus, a Klein bottle, a* 2*-cell with two boundary points identified, or a pinched sphere.*

6.3. *There exist no monotone* 0*-regular retracting transformations of* 2*-dimensional pseudo-manifolds onto nondegenerate sets except those given by* 6.1 *and* 6.2.

7. A collection G of continua is said to be *equicontinuous*‡ with respect to a given set M if for every collection H of open sets covering M there exists a finite collection H' of open sets covering M such that if x_1 and x_2 are two points of M lying in some one set of H' and belonging to a continuum X of G, then there exists an arc x_1x_2 lying both in X and in some set of the collection H. The collection G is said to be *self-compact*‡ if every infinite sequence of continua of the collection G contains an infinite subsequence which converges to some set of the collection G.

Let M be any compact space and $T(M) = M'$ be a monotone 0-regular transformation. Then, obviously, the collection $G = [T^{-1}(x')]$ for all points x' of M' is an equicontinuous self-compact collection of mutually disjoint continua filling M. Moreover, it is easily seen that an equicontinuous self-compact collection $G = [X]$ of mutually disjoint continua filling M gives rise

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* See Alexandroff-Hopf, p. 207.

† See Borsuk, Fundamenta Mathematicae, vol. 18 (1932), p. 204.

‡ *Foundations of Point Set Theory*, pp. 396–397.

to a monotone 0-regular transformation. Thus the following assertions are immediate consequences of the results of §§4 and 5:

7.1. *Let M be a 2-dimensional pseudo-manifold and $G=[X]$ be an equicontinuous self-compact collection of mutually disjoint continua filling M. If G contains more than one element, then each X is an arc or each X is a simple closed curve.*

7.2. *In order that a 2-dimensional pseudo-manifold M may be decomposed into an equicontinuous self-compact collection of mutually disjoint arcs, at least one of which is nondegenerate, it is necessary and sufficient that M be a 2-cell, a 2-cell with two boundary points identified, a circular ring, or a Möbius band.*

7.3. *In order that a 2-dimensional pseudo-manifold M may be decomposed into an equicontinuous self-compact collection of mutually disjoint simple closed curves, at least one of which is nondegenerate, it is necessary and sufficient that M be a 2-cell, a circular ring, a sphere, a pinched sphere, a torus, or a Klein bottle.*

8. A. D. Wallace* has shown that if $T(M)=T_2T_1(M)$ is the usual monotone-light factoring of any 0-regular transformation, then T_1 is a monotone 0-regular transformation and T_2 is a local homeomorphism.† Moreover, he has shown that when the image is nondegenerate any 0-regular transformation on an arc or on a simple closed curve is a homeomorphism or a local homeomorphism respectively. Thus the following assertions are consequences of the results of §§3, 4, and 5:

8.1. *If M is a 2-dimensional pseudo-manifold and $T(M)=M'$ is a 0-regular transformation, then M' is a 2-dimensional pseudo-manifold, an arc, or a simple closed curve.*

8.2. *Let M be a 2-dimensional pseudo-manifold and $T(M)=M'$ a 0-regular transformation. If M' is a nondegenerate arc, then T is monotone and, consequently, is equivalent to one of the transformations in* 5.1, 5.2, *or* 5.3.

Since a local homeomorphism on a simple closed curve is equivalent to the transformation W': $\xi'=\cos k\theta$, $\eta'=\sin k\theta$ (k an integer) on the circle $\xi=\cos\theta$, $\eta=\sin\theta$ $(0\leqq\theta\leqq 2\pi)$,‡ the following assertion is immediate:

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* *On 0-regular transformations*, loc. cit.

† The transformation $T(M)=M'$ is said to be a local homeomorphism if for each point x of M there exists a neighborhood $U(x)$ on which T is topological. See S. Eilenberg, Fundamenta Mathematicae, vol. 24 (1935), p. 35.

‡ See G. T. Whyburn, *Interior transformations on compact sets*, Duke Mathematical Journal, vol. 3 (1937), p. 374, (3.2).

8.3. *Let M be a 2-dimensional pseudo-manifold and $T(M)=M'$ a* 0*-regular transformation. If M' is a nondegenerate simple closed curve, then T is equivalent to the transformation $W'W$, where W' is given above and W is one of the transformations in* 5.4, 5.5, 5.6, 5.7, 5.8, *or* 5.9.

The following assertion results from 6.3, 8.2, and 8.3:

8.4. *If M is a 2-dimensional pseudo-manifold and $T(M)=N$ is a* 0*-regular retracting transformation, then T must be monotone.*

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