THE THEORY OF INTEGRATION*

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1. **Introduction.** There are several types of integration for numerically-valued functions, one of which is associated with the ideas and methods of Cauchy, Riemann, and Lebesgue, another of which is connected with the method of Perron, and so on. This paper will be concerned with the integration of functions with values in a Banach space by methods in the Cauchy-Riemann-Lebesgue tradition.

Numerous other studies of integration in abstract spaces have been made in recent times. Noteworthy among these are (a) papers by Graves [1],† Bochner [1], Dunford [1], and Gowurin [1]; (b) a paper by Birkhoff [1]; (c) and other papers by Birkhoff [2], Dunford [2], and Pettis [1]. In spite of the fact that the theories of integration in both (a) and (b) are developments of the ideas of Cauchy, Riemann, and Lebesgue, there is an underlying unity of method in the four papers in (a) which groups them and distinguishes them from the paper in (b). The Lebesgue integral for numerically-valued functions can be developed by either of these two general methods, but for functions with values in a Banach space they are not equivalent.

The purpose of the present paper is to study the two general methods used in (a) and (b) for integrating functions with values in a Banach space, to determine their interrelations and limitations, and to generalize and extend them.

We proceed to describe the extensions and generalizations obtained.

(1) *Generalized convex sets.* One of the important contributions made by Birkhoff [1] was in showing the fundamental importance of convex sets in the theory of integration. In §2 further properties of convex sets are established. In §3 generalized convex sets are defined and their properties established. These sets form an extensive class which includes convex sets; they have all the properties of convex sets needed in integration.

(2) *Generalized measure functions.* By using the generalized convex sets described in (1), it is shown in §§8–19 that the values of the measure function in both Bochner's and Birkhoff's integral, which they assumed to be numbers, may be bounded transformations with certain restrictions. The use of the gen-

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† Numbers in square brackets refer to the bibliography at the end.

eralized convex sets thus enables us to obtain an integral more general than that of Birkhoff.

(3) *Measurable functions.* A definition of measurable function is given in §14, and the properties of this class of functions are established in §15. It is shown in §19 that this class includes the measurable functions of Bochner as a proper subclass, and necessary and sufficient conditions that a measurable function be measurable in the sense of Bochner are established. In §16 a sufficient condition that a measurable function be integrable by the method of Birkhoff is established.

(4) *The Darboux theory of the upper and lower integral.* In Part V the properties of the Riemann-Stieltjes integral in abstract spaces are established. It is shown in §21 that there is a closed convex set which replaces the upper and lower integral for numerically-valued functions. In §25 it is shown that, at least in certain cases, the extreme points of this convex set (see Price [1]) are the exact analogues of the upper and lower integrals themselves.

(5) *A new class of Riemann-Stieltjes integrable functions.* In §§22–24 a new class of integrable functions and their integrals are studied. The values of these functions are sets instead of a single element in a Banach space; their integrals, of the Riemann-Stieltjes type, are closed convex sets. The integrals of these functions are additive in an extended sense (see Theorem 23.11).

(6) *A new class of measurable functions of the Bochner type.* It is shown in §26 that Bochner's method can be used to define measurable and summable functions whose values are sets in a Banach space; the integrals of these functions are closed convex sets and have the properties of the Lebesgue integral.

(7) *A new class of measurable functions of the Bochner-Dunford type.* It is shown in §27 that the method used by Bochner and Dunford can be extended still further to define measurable and summable functions whose values are sets in a Banach space; the integrals of these functions are closed convex sets and have the properties of the Lebesgue integral. This integral is the most general one obtained with the properties of the Lebesgue integral.

For numerically-valued functions there is only one Lebesgue integral—an integral which includes all others of the Cauchy-Riemann-Lebesgue type. There are many ways of developing the theory of this integral, but all methods lead to the same result. It is natural to ask whether the same is true for functions with values in a Banach space. The answer is in the negative. Birkhoff [1, p. 377] showed that his integral includes those of Graves and Bochner. But the integral of Bochner does not include that of Graves; an example given by Graves [1, p. 166] proves this fact. The integrals of Bochner and Dunford are known to be equivalent (see Dunford [3, p. 475]). The situation is changed radically, however, by the extensions given above. Neither the original Birkhoff integral nor the generalization of it described in (2) includes the generalized Bochner integral of the functions in (6); furthermore, neither the Bochner integral nor its extensions (see (2) and (6) above) include the generalized Bochner-Dunford integral of the functions described in (7). The generalization of the Bochner integral in (2) is the Gowurin integral; the generalized Birkhoff integral in (2) overlaps the Gowurin integral, but neither includes all cases of the other. It appears therefore that the extensions of the various methods associated with the Cauchy-Riemann-Lebesgue processes of integration lead not to one theory in abstract spaces, but to several.

As for methods and technique, there is one new element in the present case. The Hausdorff distance between sets in a Banach space is introduced (see §2); with this metric these sets become the elements of a complete metric space. In §4 the convergence of infinite series whose terms are elements of this metric space is studied; the types of convergence are numerous. Birkhoff also studied infinite series whose terms are sets in a Banach space, but without the Hausdorff metric. In §5 the interrelations of the various types of convergence introduced in §4 are examined, and in addition they are compared with Birkhoff's unconditional summation of complexes (see Birkhoff [1, pp. 361–364]). In §6 a theorem is proved which is fundamental in Birkhoff's method of integration; it is an extension of one given by Birkhoff [1, p. 364]. It is this theorem which requires the introduction of convex and generalized convex sets in Birkhoff's method of integration.

The use of the Hausdorff metric does not lead to a more general integral of the Birkhoff type (see Theorem 12.2), but it does make possible the generalizations listed under (4), (5), (6), (7) above. Furthermore, its use seems to simplify the details of some proofs, and to permit all of them to be stated in more familiar form and language.

The paper closes with applications of some of the results on integration to Fourier series and singular integrals. It is shown that the limit of the Fejér integral under very general conditions represents the value of the function—even when the values of the function are sets and it is impossible to establish the usual connection between the Fourier series and the Fejér integral. It is shown that the Fourier series of a function of bounded total variation converges to the usual value. It is necessary to construct a new proof of this theorem since the mean value theorems usually used are lacking.

Part I. Generalized convex sets

2. **The algebra of complexes.** We shall first explain the notation to be used. Let $\mathfrak{B}$ be a Banach space, that is, a space of type (B) (see Banach [1,

p. 53]), with elements $f, g, \cdots$ whose norms are $\|f\|, \|g\|, \cdots$. Let F denote a set or complex of elements f and aF the complex of elements af when a is any real number. If R is any set of real numbers r, let RF denote the set of elements rf, $r \varepsilon R, f \varepsilon F$. By the sum $F_1 + \cdots + F_n$, or $\sum_1^n F_i$, is meant the complex of elements $f_1 + \cdots + f_n$, $f_i \varepsilon F_i$. The closure of F is $\overline{F}$. The convex hull of F is $C[F]$, and the closed convex hull is $\overline{C}[F]$. The closure of $F+G$ will be represented also by $F+^*G$. Further details will be found in Birkhoff [1, pp. 358–365].

Throughout the paper a set $F \subset \mathfrak{B}$ is understood to be bounded unless there is a statement to the contrary.

(2.1) DEFINITION. *Let F_1, F_2 be two sets in $\mathfrak{B}$. Let d_1 be the upper bound of distances from points of F_1 to F_2 and d_2 the upper bound of distances from points of F_2 to F_1. The larger of the numbers d_1, d_2 is called the distance $D(F_1, F_2)$ between F_1 and F_2.*

This definition of distance between two sets in a metric space was introduced by Hausdorff [1, pp. 145–146]; it has been used by Blaschke [1, pp. 59–66].

The diameter $D(F-F, 0)$ of a set F is denoted by $\rho(F)$.

The following relations are consequences of the above definitions:

$$D(F_1, F_2) = D(F_2, F_1); \tag{2.2}$$

$$D(F_1, F_2) \geqq 0; \tag{2.3}$$

$$D(F_1, F_2) = 0 \text{ equivalent to } \overline{F}_1 = \overline{F}_2; \tag{2.4}$$

$$D(F_1, F_3) \leqq D(F_1, F_2) + D(F_2, F_3); \tag{2.5}$$

$$D(aF_1, aF_2) = |a| D(F_1, F_2) \qquad \text{for every real number } a; \tag{2.6}$$

$$D(F_1, F_2) = D(F_1, \overline{F}_2); \tag{2.7}$$

$$D(F_1 + \cdots + F_n, G_1 + \cdots + G_n) \leqq D(F_1, G_1) + \cdots + D(F_n, G_n); \tag{2.8}$$

$$D(C[F_1], C[F_2]) \leqq D(F_1, F_2); \tag{2.9}$$

$$D(F + G_1, F + G_2) \leqq D(G_1, G_2); \tag{2.10}$$

$$\rho(F_i) \leqq \rho(F_1 + F_2) \leqq \rho(F_1) + \rho(F_2) \qquad (i = 1, 2); \tag{2.11}$$

$$\rho(C[F]) = \rho(F), \qquad D(C[F], 0) = D(F, 0). \tag{2.12}$$

The first six of these relations follow at once from Definition 2.1; (2.11) and (2.12) were given by Birkhoff [1, pp. 368, 360]. Furthermore, (2.10) follows at once from (2.8) and (2.4). It can be shown by means of examples that the inequality may hold in (2.10). We shall give proofs of (2.8) and (2.9).

Proof of (2.8). Let $\epsilon > 0$ be given. Then it is possible to choose either a point $(f_1 + \cdots + f_n) \varepsilon (F_1 + \cdots + F_n)$ so that

$$d(f_1 + \cdots + f_n, G_1 + \cdots + G_n) \geqq D(F_1 + \cdots + F_n, G_1 + \cdots + G_n) - \epsilon$$

or a point $(g_1 + \cdots + g_n) \varepsilon (G_1 + \cdots + G_n)$ so that

$$d(g_1 + \cdots + g_n, F_1 + \cdots + F_n) \geqq D(F_1 + \cdots + F_n, G_1 + \cdots + G_n) - \epsilon,$$

where $d(f, F)$ represents the distance from f to the set F. It follows from Definition 2.1 that this choice is possible. Without loss of generality we may assume that the first case occurs. Then we can hold $f_1, \cdots, f_n$ fixed and choose points $g_1 \varepsilon G_1, \cdots, g_n \varepsilon G_n$ so that

$$\|f_1 - g_1\| < D(F_1, G_1) + \epsilon/n, \cdots, \|f_n - g_n\| < D(F_n, G_n) + \epsilon/n.$$

Then

$$\begin{aligned}\|(f_1 + \cdots + f_n) - (g_1 + \cdots + g_n)\| &\leqq \|f_1 - g_1\| + \cdots + \|f_n - g_n\| \\ &\leqq D(F_1, G_1) + \cdots + D(F_n, G_n) + \epsilon.\end{aligned}$$

But since

$$\begin{aligned}D(F_1 + \cdots + F_n, G_1 + \cdots + G_n) - \epsilon &\leqq d(f_1 + \cdots + f_n, G_1 + \cdots + G_n) \\ &\leqq \|(f_1 + \cdots + f_n) - (g_1 + \cdots + g_n)\|,\end{aligned}$$

we see that

$$D(F_1 + \cdots + F_n, G_1 + \cdots + G_n) - \epsilon \leqq D(F_1, G_1) + \cdots + D(F_n, G_n) + \epsilon.$$

Since ϵ is arbitrary, (2.8) follows and the proof is complete.

Proof of (2.9). The distance of a point of F_2 from $C[F_1]$ does not exceed its distance from F_1 since $F_1 \subseteqq C[F_1]$. Then the parallel set to $C[F_1]$ at the distance $D(F_1, F_2)$, that is, the set of all those points at distance from $C[F_1]$ equal to or less than $D(F_1, F_2)$, is a convex set which contains F_2. Then the distance from any point of $C[F_2]$ to $C[F_1]$ is equal to or less than $D(F_1, F_2)$. In the same way we show that the distance from any point of $C[F_1]$ to $C[F_2]$ is equal to or less than $D(F_1, F_2)$. Then (2.9) follows from Definition 2.1, and the proof is complete.

It is possible to give an example in which $D(C[F_1], C[F_2]) < D(F_1, F_2)$.

With the distance introduced in Definition 2.1, the complexes F are the elements of a complete metric space (see Price [3]). Two complexes F_1, F_2 are metrically identical when their closures coincide in the point-set sense.

(2.13) Theorem. *If $F_1, \cdots, F_n$ are convex sets and $a_1, \cdots, a_n$ are any real numbers, then $\sum_1^n a_i F_i$ is a convex set, and*

$$\sum_1^n a_i \overline{F}_i \subseteqq \overline{\sum_1^n a_i F_i}.$$

This theorem was given by Birkhoff [1, pp. 359–360].

(2.14) THEOREM. *If* $F_1, \cdots, F_n$, $F_i \subset \mathfrak{B}$, *are sets which are compact in* $\mathfrak{B}$, *and if* $a_1, \cdots, a_n$ *are any real numbers, then*

$$\sum_1^n a_i\overline{F}_i = \overline{\sum_1^n a_iF_i}.$$

The proof is omitted.

(2.15) THEOREM. *Let* $F_1, \cdots, F_n$, $F_i \subset \mathfrak{B}$, *be convex sets. If* g *is any extreme point of the convex set* $\sum_1^n a_iF_i$, *then* $g = \sum_1^n a_ig_i$, *and* $g_1, \cdots, g_n$ *are extreme points of* $F_1, \cdots, F_n$ *respectively.*

A point of a convex set is an extreme point if it is not an interior point of any segment which belongs to the set (see Price [1, p. 57]). Since g is a point of $\sum_1^n a_iF_i$, there exist points $g_1, \cdots, g_n$, $g_i \in F_i$, such that $g = \sum_1^n a_ig_i$. Suppose that one of the points $g_1, \cdots, g_n$, say g_1, is not an extreme point. Then there exist two points g_1', g_1'' in F_1 such that $g_1 = \theta g_1' + (1-\theta)g_1''$, $0 < \theta < 1$. Then $g = \theta g' + (1-\theta)g''$, where $g' = a_1g_1' + a_2g_2 + \cdots + a_ng_n$, $g'' = a_1g_1'' + a_2g_2 + \cdots + a_ng_n$, and g is not an extreme point. This contradiction establishes the theorem.

(2.16) THEOREM. *If* R *is a closed interval of real numbers, and if* $F \subset \mathfrak{B}$ *is closed, then* RF *is closed. If in addition* R *does not contain the number zero in its interior, and if* F *is convex, then* RF *is convex also.*

(2.17) COROLLARY. *If* R, R' *are arbitrary closed intervals of real numbers, and if* F *is the set* $R'f$, f *a fixed element in* $\mathfrak{B}$, *then* RF *is closed and convex.*

The conditions of neither the theorem nor the corollary are necessary.

(2.18) THEOREM. *If* R *is any set of numbers such that zero is not an interior point of its closed convex hull* $\overline{C}[R]$, *and if* F *is any set in* $\mathfrak{B}$, *then* $\overline{C}[RF] = \overline{C}[R]\overline{C}[F]$.

If $r_0 \in R$, then $r_0F \subseteq \overline{C}[RF]$ because of the convexity properties of $\overline{C}[RF]$. Again, if r_1 is a limit point of points of R, then $r_1F \subseteq r_1\overline{C}[F] \subseteq \overline{C}[RF]$ because $\overline{C}[RF]$ is closed. Finally, let $\overline{C}[R] \equiv (\alpha \leq r \leq \beta)$. Then by the statements just made, $\alpha\overline{C}[F] \subseteq \overline{C}[RF]$, $\beta\overline{C}[F] \subseteq C[RF]$; hence, because of the convexity once more, $\overline{C}[R]\overline{C}[F] \subseteq \overline{C}[RF]$. Also $RF \subseteq \overline{C}[R]\overline{C}[F]$ and, by Theorem 2.16, $\overline{C}[R]\overline{C}[F]$ is closed and convex. But $\overline{C}[RF]$ has no proper subset with these properties, and therefore $\overline{C}[RF] = \overline{C}[R]\overline{C}[F]$. The proof is complete.

(2.19) COROLLARY. *Let* R_1, R_2 *be two sets of numbers such that zero is not an interior point of* $\overline{C}[R_1]$, $\overline{C}[R_2]$, *and let* F_1, F_2 *be two arbitrary sets in* $\mathfrak{B}$. *Then*

$$\overline{C}[R_1F_1 + R_2F_2] = \overline{\overline{C}[R_1]C[F_1] + \overline{C}[R_2]C[F_2]}. \tag{2.20}$$

This corollary follows at once from the theorem and $C[R_1F_1+R_2F_2]=C[R_1F_1]+C[R_2F_2]$ (see Birkhoff [1, p. 360]).

(2.21) THEOREM. *Let R_1, R_2 be two sets of numbers such that zero is not an interior point of $C[R_1]$, $C[R_2]$, and let F be any set in $\mathfrak{B}$. Then*

$$\overline{C}[R_1]\overline{C}[F]+\overline{C}[R_2]\overline{C}[F]=\overline{C}[R_1+R_2]\overline{C}[F]; \tag{2.22}$$

$$D(\overline{C}[R_1]\overline{C}[F],\overline{C}[R_2]\overline{C}[F])\leqq D(\overline{C}[R_1],\overline{C}[R_2])D(\overline{C}[F],0). \tag{2.23}$$

Proof of (2.23). Consider the distance from any point $r_1f\in\overline{C}[R_1]\overline{C}[F]$ to $\overline{C}[R_2]\overline{C}[F]$. We can find a point $r_2\in\overline{C}[R_2]$ such that $|r_1-r_2|\leqq D(\overline{C}[R_1],\overline{C}[R_2])$. Then the distance from r_1f to $\overline{C}[R_2]\overline{C}[F]$ does not exceed

$$\|r_1f-r_2f\|\leqq|r_1-r_2|\cdot\|f\|\leqq D(\overline{C}[R_1],\overline{C}[R_2])D(\overline{C}[F],0).$$

In the same way it can be shown that the distance from points of $\overline{C}[R_2]\overline{C}[F]$ to $\overline{C}[R_1]\overline{C}[F]$ does not exceed the same number. The proof is then complete.

3. **Generalized convex sets.** In this section we shall establish the fundamental properties of a linear operator whose domain and range are sets $F\subset\mathfrak{B}$.

Let $\mathfrak{T}$ be the normed linear space of linear transformations T whose domain is $\mathfrak{B}$ and whose range is in $\mathfrak{B}$. Thus if $T\in\mathfrak{T}$ and a, b are real numbers, we have $Tf\in\mathfrak{B}$, $T(af+bg)=aTf+bTg$, $\|Tf\|\leqq\|T\|\cdot\|f\|$. If T_1, $T_2\in\mathfrak{T}$, then T_1T_2 and T_1+T_2 also belong to $\mathfrak{T}$. The identical transformation I belongs to $\mathfrak{T}$.

Let t denote a set of elements of $\mathfrak{T}$: $t\equiv(T_1,\cdots,T_r)$. Here $T_i\in\mathfrak{T}$, and $r=r(t)$ is a positive integer.

(3.1) DEFINITION. *The product t_1t_2 of t_1 and t_2, where $t_1\equiv(T_{1i})$, $i=1,\cdots,r_1$, and $t_2\equiv(T_{2j})$, $j=1,\cdots,r_2$, is $t\equiv(T_{1i}T_{2j})$, $i=1,\cdots,r_1$ and $j=1,\cdots,r_2$.*

Let C^* denote a set of elements t which satisfies the following hypotheses:

(3.2) HYPOTHESIS. *$t\in C^*$ implies $T_1+\cdots+T_r=I$.*

(3.3) HYPOTHESIS. *t_1, $t_2\in C^*$ implies $t_1t_2\in C^*$.*

(3.4) HYPOTHESIS. *There exists a constant W, depending only on C^*, such that for every $t\in C^*$ and any set of points $f_1,\cdots,f_r$ in $\mathfrak{B}$*

$$\left\|\sum_1^r T_if_i\right\|\leqq W\max_i\|f_i\|. \tag{3.5}$$

When Hypothesis 3.4 is satisfied, we shall assume that W represents the lower bound of the constants for which (3.5) holds. The W-property of a set of transformations t was first studied by Gowurin [1]. We shall say C^* is of bounded variation if there exists a constant V such that for every $t\in C^*$,

$\sum_1^r \|t_i\| \leqq V$. Hypothesis 3.5 is weaker than bounded variation and is implied by it. The convex operator C has $V=1$ and $W=1$.

It should be remarked that if t_1, t_2 satisfy Hypothesis 3.2, then t_1t_2 automatically satisfies the same hypothesis.

We shall use C^* also to denote an operator as explained in the following definition.

(3.6) DEFINITION. *The transform of a set $F \subset \mathfrak{B}$ by C^*, denoted by $C^*[F]$, is the set of all points $\sum_1^r T_i f_i$, where $f_i \varepsilon F$ and $t \equiv (T_1, \cdots, T_r) \varepsilon C^*$. If F is the null set, $C^*[F]$ is the null set.*

The fundamental properties of this operator are given in the following theorems and corollaries.

(3.7) THEOREM. $C^*[F+G]=C^*[F]+C^*[G]$.

(3.8) COROLLARY. *Let T_1, T_2 be any two numbers, or more generally, any two elements of $\mathfrak{T}$ which commute with all the transformations T_i in $t \varepsilon C^*$; then $C^*[T_1F+T_2G]=T_1C^*[F]+T_2C^*[G]$.*

(3.9) THEOREM. $F \subseteqq G$ *implies* $C^*[F] \subseteqq C^*[G]$.

(3.10) THEOREM. $F \subseteqq C^*[F]$.

(3.11) THEOREM. $C^*[\prod_{(\nu)} C^*[F_\nu]] = \prod_{(\nu)} C^*[F_\nu]$.

(3.12) COROLLARY. $C^*[C^*[F]]=C^*[F]$.

(3.13) THEOREM. $t \equiv (T_1, \cdots, T_r)$ ε C^* *implies* $C^*[F]=T_1C^*[F]+ \cdots + T_r C^*[F]$.

(3.14) THEOREM. $D(C^*[F_1], C^*[F_2]) \leqq WD(F_1, F_2)$.

(3.15) THEOREM. $D(F, 0) \leqq D(C^*[F], 0) \leqq WD(F, 0)$, *and also* $D(F, 0) \leqq D(C^*[F], 0) \leqq D(F, 0)+W\rho(F)$.

(3.16) THEOREM. $\rho(F) \leqq \rho(C^*[F]) \leqq W\rho(F)$.

From (3.8), (3.9), and (3.15) it follows that C^* is a homogeneous, additive, monotone, and bounded operator. It should be remarked that the proofs of (3.7)–(3.13) do not depend on Hypothesis 3.4; only the proofs of (3.14)–(3.16) depend on this third hypothesis. Theorems 3.14–3.16 are generalizations of properties of the convex operator C already given in (2.9) and (2.12); the latter can be obtained from the former by setting $W=1$.

Proof of (3.7). Since the transformations T_i in $t \varepsilon C^*$ are linear by hypothesis, it is clear that $C^*[F+G] \subseteqq C^*[F]+C^*[G]$. Thus we have only to show that $C^*[F]+C^*[G] \subseteqq C^*[F+G]$.

Let $\sum_1^{r_1} T_{1i} f_i$, $\sum_1^{r_2} T_{2j} g_j$ be any two points in $C^*[F]$, $C^*[G]$ respectively. The proof will be completed by showing that $\sum_1^{r_1} T_{1i}f_i+\sum_1^{r_2} T_{2j}g_j$, an arbitrary point of $C^*[F]+C^*[G]$, is identical with

$$\sum_{i=1}^{r_1}\sum_{j=1}^{r_2} T_{1i}T_{2j}(f_i+g_j), \tag{3.17}$$

which is a point of $C^*[F+G]$ by Hypothesis 3.3. The proof follows from the fact that

$$\begin{aligned}\sum_{i=1}^{r_1} T_{1i}\sum_{j=1}^{r_2}\,[T_{2j}f_i+T_{2j}g_j] &= \sum_{i=1}^{r_1} T_{1i}\left[\sum_{j=1}^{r_2} T_{2j}f_i+\sum_{j=1}^{r_2} T_{2j}g_j\right]\\ &= \sum_{i=1}^{r_1} T_{1i}\left[f_i+\sum_{j=1}^{r_2} T_{2j}g_j\right]\\ &= \sum_{i=1}^{r_1} T_{1i}f_i+\sum_{j=1}^{r_2} T_{2j}g_j,\end{aligned}$$

where (3.2) has been used in the transformations. The proof is complete.

The proofs of the remaining theorems and corollaries are left to the reader.

A trivial example of an operator C^* is formed by the set of elements t where t is a set of positive rational fractions with even denominators whose sum is 1. A more interesting example is the following one in the space of square matrices of order 2. The norm of a matrix is the square root of the sum of the squares of its elements. Let t be the r matrices

$$\begin{Vmatrix}\alpha_i & 0\\ 0 & \beta_i\end{Vmatrix} \qquad (i=1,\cdots,r),$$

where $0\leqq\alpha_i,\,\beta_i\leqq 1,\,\sum_i\alpha_i=\sum_i\beta_i=1$. A transformation T is here matrix multiplication. The set of all such elements t forms an operator C^* which satisfies Hypotheses 3.2, 3.3, 3.4. This operator is of bounded variation with $V=2$. It therefore has the W-property, and it can be shown that $2^{1/2}\leqq W\leqq 2$. A similar operator C^* exists in the space of square matrices of order n; in this case $V=n$ and $n^{1/2}\leqq W\leqq n$.

It is necessary to observe also that there are methods for constructing operators C^*. Let $\{\tau\}$ be a class of transformations with the following properties:

(3.18) Hypothesis. *$\tau\ \varepsilon\ \{\tau\}$ implies $\tau\ \varepsilon\ \mathfrak{T}$; $\tau\ \varepsilon\ \{\tau\}$ and $\tau\neq 0$ imply that τ has an inverse $\tau^{-1}\ \varepsilon\ \mathfrak{T}$; $\tau_1,\cdots,\tau_r\ \varepsilon\ \{\tau\}$ and $(\tau_1+\cdots+\tau_r)\neq 0$ imply that $(\tau_1+\cdots+\tau_r)$ has an inverse $(\tau_1+\cdots+\tau_r)^{-1}\ \varepsilon\ \mathfrak{T}$.*

Let $\tau_1,\cdots,\tau_r$ be any r elements in $\{\tau\}$; let t, t' be $[(\tau_1+\cdots+\tau_r)^{-1}\tau_1,\cdots,(\tau_1+\cdots+\tau_r)^{-1}\tau_r]$, $[\tau_1(\tau_1+\cdots+\tau_r)^{-1},\cdots,\tau_r(\tau_1+\cdots+\tau_r)^{-1}]$ respectively and C_0^* the set of all such t and t'. Then C_0^* satisfies Hypothesis 3.2.

In the usual way we can form the smallest set C^* which contains C_0^* and is closed with respect to the multiplication defined in (3.1). Then C^* satisfies Hypotheses 3.2, 3.3 and is a homogeneous, additive, and monotone operator.

(3.19) THEOREM. *Let C^* be the operator formed from $\{\tau\}$ as described, and let $\tau_1, \cdots, \tau_r$ be any elements in $\{\tau\}$. Then for any set $F \subset \mathfrak{B}$*

$$(\tau_1 + \cdots + \tau_r)C^*[F] = \tau_1 C^*[F] + \cdots + \tau_r C^*[F].$$

From Theorem 3.13 we have

$$C^*[F] = (\tau_1 + \cdots + \tau_r)^{-1}\tau_1 C^*[F] + \cdots + (\tau_1 + \cdots + \tau_r)^{-1}\tau_r C^*[F],$$

from which the theorem follows.

In the future C^* will be used to denote an operator which satisfies Hypotheses 3.2, 3.3, 3.4 and is therefore additive, homogeneous, monotone, and bounded.

(3.20) DEFINITION. *A set $F \subset \mathfrak{B}$ such that $F = C^*[F]$ will be called convex C^*.*

It follows from Corollary 3.12 that every set $C^*[F]$ is convex C^*.

PART II. INFINITE SERIES

4. **Convergence of infinite series.** We shall now consider various types of convergence of the infinite series

$$(4.1) \qquad \sum_{k=1}^{\infty} f_k, \qquad f_k \in \mathfrak{B};$$

$$(4.2) \qquad \sum_{k=1}^{\infty} F_k, \qquad F_k \subset \mathfrak{B}.$$

As stated in §2 the space with elements $F \subset \mathfrak{B}$ is a *complete* metric space.

(4.3) DEFINITION. *The series* (4.1) *converges unconditionally to $f \in \mathfrak{B}$ if and only if every rearrangement α of all the terms of the series gives a series $\sum_1^\infty f_{\alpha(k)}$ which converges to f.*

(4.4) DEFINITION. *The series* (4.2) *converges strongly unconditionally to the sum $F \subset \mathfrak{B}$ if and only if every series $\sum_1^\infty f_k$, $f_k \in F_k$, converges unconditionally and F is the locus of the sums of these series.*

These two definitions have been given by Birkhoff [1, pp. 361–362].

(4.5) DEFINITION. *The series* (4.2) *converges normally if and only if the series $\sum_1^\infty D(F_k, 0)$ converges.*

(4.6) DEFINITION. *The series* (4.2) *converges regularly if and only if for each $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that $D(\sum_{k=N}^{N+p} F_k, 0) < \epsilon$, $p \geqq 0$.*

It can be shown that this definition is equivalent to the following one.

(4.6′) DEFINITION. *The series* (4.2) *converges regularly if and only if for each* $\epsilon>0$ *there exists an* $M=M(\epsilon)$ *such that* $D(\sum_m^n F_k, 0)<\epsilon$, $n\geqq m\geqq M$.

(4.7) DEFINITION. *The series* $\sum_1^\infty F_k$ *converges and has the sum* $F\subset\mathfrak{B}$ *if and only if for each* $\epsilon>0$ *there exists an* $N=N(\epsilon)$ *such that* $D(\sum_1^n F_k, F)<\epsilon$, $n\geqq N$.

It follows from (2.7) that the sum of a convergent series is not uniquely determined. We agree, however, that the sum of a convergent series shall be taken as a closed set; with this convention the sum is unique.

Since we have associated two sums with the series $\sum_1^\infty F_k$, it will be convenient to introduce notation for them. If the series converges in the sense of Definition 4.7, its sum will be denoted by $\sum_1^\infty F_k$. If $\sum_1^\infty F_k$ converges strongly unconditionally in the sense of Definition 4.4, we indicate its (strong) sum by $\mathrm{S}\sum_1^\infty F_k$.

(4.8) DEFINITION. *The series* $\sum_1^\infty F_k$ *converges weakly unconditionally to* $F\subset\mathfrak{B}$ *if and only if every rearrangement* α *of all the terms of the series gives a series* $\sum_1^\infty F_{\alpha(k)}$ *which converges to* F.

(4.9) THEOREM. *A necessary and sufficient condition that* $\sum_1^\infty f_k$ *be unconditionally convergent to* f *is that to every* $\epsilon>0$ *there correspond an integer* N *such that the sum* η *of any finite set of terms of* $\sum_1^\infty f_k$ *including* $f_1, \cdots, f_N$ *satisfies* $\|\eta-f\|<\epsilon$.

(4.10) THEOREM. *A necessary and sufficient condition that* $\sum_1^\infty f_k$ *be unconditionally convergent to* f *is that for each* $\epsilon>0$ *there exist an* M *such that* $M<k(1)<k(2)<\cdots<k(r)$ *implies*

$$\|f_{k(1)}+f_{k(2)}+\cdots+f_{k(r)}\|<\epsilon.$$

(4.11) THEOREM. *A necessary and sufficient condition that* $\sum_1^\infty F_k$ *converge strongly unconditionally is that to each* $\epsilon>0$ *there correspond an* M *such that* $M<k(1)<k(2)<\cdots<k(r)$ *implies*

$$D(F_{k(1)}+F_{k(2)}+\cdots+F_{k(r)}, 0)<\epsilon.$$

These theorems have been given by Birkhoff [1, pp. 361–362].

(4.12) THEOREM. *A necessary and sufficient condition that* $\sum_1^\infty F_k$ *converge (see Definition* 4.7) *is that for each* $\epsilon>0$ *there exist an* N *such that* $D(\sum_{k=1}^{N+p} F_k, \sum_{k=1}^{N} F_k)<\epsilon$ *for all* $p\geqq 1$.

The proof follows easily from properties of the distance function given in §2. The proof that the condition is sufficient depends on the fact that the space with elements $F\subset\mathfrak{B}$ is a complete metric space.

(4.13) THEOREM. *A necessary and sufficient condition that* (4.2) *be weakly unconditionally convergent to F is that to every* $\epsilon>0$ *there correspond a number N such that the sum* Φ *of any finite set of terms of* (4.2) *including* $F_1, \cdots, F_N$ *satisfies* $D(\Phi, F)<\epsilon$.

This theorem is entirely similar to Theorem 4.9.

(4.14) THEOREM. *A sufficient condition that* $\sum_1^\infty F_k$ *be weakly unconditionally convergent to F is that for each* $\epsilon>0$ *there exist an M such that* $M<k(1)<k(2)<\cdots<k(r)$ *implies*

$$D(F_{k(1)}+F_{k(2)}+\cdots+F_{k(r)}, 0)<\epsilon.$$

We have

$$D\left(\sum_{k=1}^m F_k, \sum_{k=1}^n F_k\right)=D\left(\sum_{k=1}^n F_k+\sum_{k=n+1}^m F_k, \sum_{k=1}^n F_k\right)\leqq D\left(\sum_{k=n+1}^m F_k, 0\right)<\epsilon$$

provided only that $m\geqq n\geqq M$. From Theorem 4.12 it follows that $\sum_1^\infty F_k$ converges and has a sum F; then $D(\sum_1^m F_k, F)\leqq\epsilon$, $m\geqq M$. Now consider any rearrangement $\sum_1^\infty F_{\alpha(k)}$ of $\sum_1^\infty F_k$. We have

$$D\left(\sum_{k=1}^\nu F_{\alpha(k)}, F\right)\leqq D\left(\sum_{k=1}^\nu F_{\alpha(k)}, \sum_{k=1}^M F_k\right)+D\left(\sum_{k=1}^M F_k, F\right)<\epsilon+\epsilon$$

for all ν which are so large that $\sum_1^\nu F_{\alpha(k)}$ contains $F_1, \cdots, F_M$. Thus the series converges and has the sum F. The proof is complete.

If each set F_k in (4.2) consists of a single element f_k, weak unconditional convergence and strong unconditional convergence are identical and identical with unconditional convergence of (4.1) as defined in (4.3).

(4.15) THEOREM. *If* $\sum_1^\infty F_k$ *is strongly unconditionally convergent, and if* F_k *is convex* C^* *for all k, then the sum* $\mathrm{S}\sum_1^\infty F_k$ *is convex* C^*.

(4.16) THEOREM. *If* $\sum_1^\infty F_k$ *converges, and if* F_k *is convex* C^* *for all k, then the sum* $\sum_1^\infty F_k$ *is convex* C^*.

We have

$$\begin{aligned} D\left(\sum_1^n F_k, \overline{C}^*\left[\sum_1^\infty F_k\right]\right) &= D\left(\sum_1^n C^*[F_k], \overline{C}^*\left[\sum_1^\infty F_k\right]\right) \\ &= D\left(C^*\left[\sum_1^n F_k\right], \overline{C}^*\left[\sum_1^\infty F_k\right]\right) \\ &\leqq WD\left(\sum_1^n F_k, \sum_1^\infty F_k\right) < W\epsilon \end{aligned}$$

for all $n \geqq N$. Since the sum $\sum_1^\infty F_k$ is a closed set by agreement, we have $\sum_1^\infty F_k = \bar{C}^*[\sum_1^\infty F_k]$. The proof is complete.

5. **Relations between various types of convergence.** We shall now examine the relations between the various types of convergence defined in §4 and also the relations between the convergence of the two series

$$\sum_1^\infty F_k, \tag{5.1}$$

$$\sum_1^\infty C^*[F_k]. \tag{5.2}$$

(5.3) THEOREM. *The series* (5.2) *converges normally if and only if* (5.1) *converges normally.*

The proof follows from (3.15).

(5.4) THEOREM. *Normal convergence implies strong unconditional convergence.*

The proof follows from Theorem 4.11 and (2.8).

(5.5) THEOREM. *The series* $\sum_1^\infty C^*[F_k]$ *and* $\sum_1^\infty \bar{C}^*[F_k]$ *converge strongly unconditionally if and only if* $\sum_1^\infty F_k$ *converges strongly unconditionally.*

The proof follows from Theorems 4.11, 3.15 and

$$\begin{aligned} D(\bar{C}^*[F_{k(1)}] + \cdots + \bar{C}^*[F_{k(r)}], 0) &= D(C^*[F_{k(1)}] + \cdots + C^*[F_{k(r)}], 0) \\ &= D(C^*[F_{k(1)} + \cdots + F_{k(r)}], 0). \end{aligned}$$

This theorem was given by Birkhoff [1, p. 363].

(5.6) THEOREM. *Strong unconditional convergence implies weak unconditional convergence and regular convergence.*

The proof follows from Theorems 4.11, 4.14 and Definition 4.6.

(5.7) THEOREM. *The series* (5.2) *converges regularly if and only if* (5.1) *converges regularly.*

(5.8) THEOREM. *If a series converges regularly or weakly unconditionally, it converges.*

If a series converges regularly, the sequence of partial sums is a Cauchy sequence.

(5.9) THEOREM. *If* $\sum_1^\infty F_k$ *converges, then* $\sum_1^\infty C^*[F_k]$ *converges and* $\sum_1^\infty C^*[F_k] = \bar{C}^*[\sum_1^\infty F_k]$.

The proof follows from

$$D\left(\sum_1^n C^*[F_k],\ \bar{C}^*\left[\sum_1^\infty F_k\right]\right) = D\left(C^*\left[\sum_1^n F_k\right],\ \bar{C}^*\left[\sum_1^\infty F_k\right]\right) \leqq WD\left(\sum_1^n F_k,\ \sum_1^\infty F_k\right). \tag{5.10}$$

(5.11) THEOREM. *If $\sum_1^\infty F_k$ converges weakly unconditionally, $\sum_1^\infty C^*[F_k]$ converges weakly unconditionally.*

The proof follows from Theorem 4.13 and (5.10).

(5.12) THEOREM. *If $\sum_1^\infty F_k$ is strongly unconditionally convergent with the sum* $\mathrm{S}\sum_1^\infty F_k$, *then it is weakly unconditionally convergent and*

$$\sum_1^\infty F_k = \overline{\mathrm{S}\sum_1^\infty F_k}.$$

From Theorems 5.6 and 5.8 it follows that $\sum_1^\infty F_k$ converges; let its sum be F. Take any point $f \in F$. We shall show that

$$f \in \overline{\mathrm{S}\sum_1^\infty F_k}.$$

From $f \in F=\sum_1^\infty F_k$ it follows that corresponding to a given $\epsilon>0$ there exists an N_1 such that

$$d\left(f,\ \sum_1^n F_k\right) < \epsilon/2, \qquad n \geqq N_1. \tag{5.13}$$

Here $d(g, G)$ denotes the distance from g to G. Since $\sum_1^\infty F_k$ is strongly unconditionally convergent, it follows from Theorem 4.11 that there exists an integer N_2 such that $N_2<k(1)< \cdots <k(r)$ implies $D(F_{k(1)}+ \cdots +F_{k(r)},\ 0) <\epsilon/2$, and therefore

$$\left\| \sum_{i=N_2+1}^\infty f_i \right\| \leqq \epsilon/2, \qquad f_i \in F_i. \tag{5.14}$$

Let N be the larger of the numbers N_1, N_2. From (5.13) it follows that there exists a sum $\sum_1^N f_k$, $f_k \in F_k$, such that

$$\left\| f - \sum_1^N f_k \right\| < \epsilon/2. \tag{5.15}$$

The points $f_1, \cdots, f_N$ are chosen once for all and then held fixed. Then choose points $f_{N+1}, f_{N+2}, \cdots$ arbitrarily from $F_{N+1}, F_{N+2}, \cdots$ and consider the series $\sum_1^\infty f_k$. Since $\sum_1^\infty F_k$ is strongly unconditionally convergent, the series $\sum_1^\infty f_k$ converges and has a sum $g \in \mathrm{S}\sum_1^\infty F_k$. But

$$\|f - g\| \leqq \left\| f - \sum_{1}^{N} f_k \right\| + \left\| \sum_{N+1}^{\infty} f_k \right\| < \epsilon$$

by (5.15) and (5.14). Since $f \varepsilon F$, $g \varepsilon \mathrm{S}\sum_1^\infty F_k$ and ϵ is arbitrary, it follows that

$$f \varepsilon \overline{\mathrm{S}\sum_1^\infty F_k}, \qquad F \subseteqq \overline{\mathrm{S}\sum_1^\infty F_k}.$$

We prove next that

$$\overline{\mathrm{S}\sum_1^\infty F_k} \subseteqq F.$$

Let $g = \sum_1^\infty f_k$, $f_k \varepsilon F_k$, be an arbitrary point of $\mathrm{S}\sum_1^\infty F_k$. Corresponding to any given $\epsilon > 0$ we can find an M such that $D(\sum_1^n F_k, F) < \epsilon$ for $n \geqq M$. This fact and the definition of distance imply $d(\sum_1^n f_k, F) < \epsilon$, $n \geqq M$; hence, $d(\sum_1^\infty f_k, F) = d(g, F) \leqq \epsilon$. Since $\epsilon > 0$ was arbitrary, $d(g, F) = 0$; since F is closed, $g \varepsilon F$. Then

$$\mathrm{S}\sum_1^\infty F_k \subseteqq F, \qquad \overline{\mathrm{S}\sum_1^\infty F_k} \subseteqq F.$$

This inequality completes the proof that

$$\sum_1^\infty F_k = F = \overline{\mathrm{S}\sum_1^\infty F_k}$$

and the proof of the theorem.

(5.16) THEOREM. *If the series $\sum_1^\infty F_k$ converges strongly unconditionally, then*

$$C^*\left[\mathrm{S}\sum_1^\infty F_k\right] \subseteqq \mathrm{S}\sum_1^\infty C^*[F_k] \subseteqq \mathrm{S}\sum_1^\infty \overline{C}^*[F_k] \subseteqq \overline{C}^*\left[\mathrm{S}\sum_1^\infty F_k\right],$$

$$\overline{\mathrm{S}\sum_1^\infty C^*[F_k]} = \overline{\mathrm{S}\sum_1^\infty \overline{C}^*[F_k]} = \overline{C}^*\left[\mathrm{S}\sum_1^\infty F_k\right].$$

It follows from Theorems 5.5 and 4.15 that $\sum_1^\infty C^*[F_k]$ and $\sum_1^\infty \overline{C}^*[F_k]$ converge strongly unconditionally to sets convex C^*. It is obvious that

$$C^*\left[\mathrm{S}\sum_1^\infty F_k\right] \subseteqq \mathrm{S}\sum_1^\infty C^*[F_k] \subseteqq \mathrm{S}\sum_1^\infty \overline{C}^*[F_k]. \tag{5.17}$$

By Theorem 5.6, $\sum_1^\infty F_k$ converges weakly unconditionally, and by Theorem 5.12 it converges to the sum

$$\overline{\mathrm{S}\sum_1^\infty F_k};$$

hence, by Theorems 5.11 and 5.9, $\sum_1^\infty C^*[F_k]$ converges weakly unconditionally to the sum $\overline{C}^*[\mathrm{S}\sum_1^\infty F_k]$. Furthermore, $\sum_1^\infty C^*[F_k]$ and $\sum_1^\infty \overline{C}^*[F_k]$ converge weakly unconditionally to the same sum. Thus $\sum_1^\infty C^*[F_k]$ and $\sum_1^\infty \overline{C}^*[F_k]$ converge strongly unconditionally to $\mathrm{S}\sum_1^\infty C^*[F_k]$ and $\mathrm{S}\sum_1^\infty \overline{C}^*[F_k]$, and weakly unconditionally to $\overline{C}^*[\mathrm{S}\sum_1^\infty F_k]$. It follows from Theorem 5.12 that

$$\overline{\mathrm{S}\sum_1^\infty C^*[F_k]} = \overline{\mathrm{S}\sum_1^\infty \overline{C}^*[F_k]} = \overline{C}^*\left[\mathrm{S}\sum_1^\infty F_k\right].$$

Then $\mathrm{S}\sum_1^\infty \overline{C}^*[F_k] \subseteq \overline{C}^*[\mathrm{S}\sum_1^\infty F_k]$. The remainder of the proof follows at once from this fact and (5.17).

This theorem was given by Birkhoff [1, p. 363].

6. **An intersection theorem for series.** The principal object of this section is to establish Theorem 6.4 and two further theorems which give conditions under which its hypothesis are satisfied. This theorem is called an intersection theorem because of its application in the theory of integration. First, we shall state another theorem, whose proof we leave to the reader.

(6.1) THEOREM. *If the series* $\sum_1^\infty F_k$ *and* $\sum_1^\infty G_k$, $G_k \subseteq F_k$, *converge, then* $\sum_1^\infty G_k \subseteq \sum_1^\infty F_k$. *If* $\sum_1^\infty F_k$ *converges regularly, then* $\sum_1^\infty G_k$, $G_k \subseteq F_k$, *converges regularly, and* $\sum_1^\infty G_k \subseteq \sum_1^\infty F_k$.

Let $\{\tau\}$ be a class of transformations τ which satisfies Hypothesis 3.18 and has the additional property that the sum of any unconditionally convergent series of transformations in $\{\tau\}$ is a transformation in $\{\tau\}$. Let C^* be the operator formed from $\{\tau\}$ as explained at the end of §3, and let it satisfy (3.2), (3.3), (3.4).

Consider the infinite series

$$\sum_1^\infty \tau_i F_i, \tag{6.2}$$

$$\sum_{i,j=1}^\infty \tau_{ij} F_{ij}, \tag{6.3}$$

where the F_i and F_{ij} are sets in $\mathfrak{B}$ and the τ_i and τ_{ij} are elements in $\{\tau\}$.

(6.4) THEOREM. *Let the series* (6.2), (6.3) *satisfy the following hypotheses*:

(6.5) $\sum_1^\infty \tau_i F_i$ *converges strongly unconditionally*;

(6.6) $\sum_{j=1}^\infty \tau_{ij}$ *converges unconditionally*, $\sum_{j=1}^\infty \tau_{ij} = \tau_i$ *for all* i;

(6.7) F_i *and* F_{ij} *are convex* C^* *for all* i, j;

(6.8) $F_{ij} \subseteq F_i$ *for all* i, j; *not all* F_{ij} *are vacuous*; *each* F_i *is bounded.*

Then the following conclusions can be drawn:

(6.9) $\sum_{i,j=1}^\infty \tau_{ij} F_{ij}$ *converges strongly unconditionally*;

(6.10) *each row* $\sum_j \tau_{ij} F_{ij}$ *of* (6.3) *converges strongly unconditionally to a set contained in* $\tau_i F_i$;

(6.11) *the series* $\sum_{ij}\tau_{ij}F_{ij}$ *can be summed by rows*;

(6.12) $\sum_{ij}\tau_{ij}F_{ij} \subseteqq \sum_1^\infty \tau_i F_i$.

Let M be chosen so that $M<k(1)<\cdots<k(r)$ implies $D(\tau_{k(1)}F_{k(1)}+\cdots+\tau_{k(r)}F_{k(r)}, 0)<\epsilon/(2W)$, where W is the constant in (3.5). Such an M exists by the hypotheses of the theorem and Theorem 4.11. Let K denote the maximum of the M finite numbers $D(F_1, 0), \cdots, D(F_M, 0)$. Then choose N so that

$$\left\|\sum \tau_{ij}\right\| < \epsilon/(2MK) \qquad (i = 1, 2, \cdots, M), \tag{6.13}$$

where the summation is with respect to j and over any finite number of terms with $j>N$. Such an N exists by Theorem 4.10 and (6.6).

Now consider any finite sum of terms in $\sum_{ij}\tau_{ij}F_{ij}$ with $i\geqq M+1$ or $j\geqq N+1$. This sum can be broken into two parts, the first of which has $i\leqq M, j\geqq N+1$, and the second of which has $i\geqq M+1$. Consider the first one. We have

$$\begin{aligned} D\Big(\sum_i\sum_j \tau_{ij}F_{ij}, 0\Big) &\leqq \sum_i D\Big(\sum_j \tau_{ij}F_{ij}, 0\Big) \leqq \sum_i D\Big(\sum_j \tau_{ij}F_i, 0\Big) \\ &\leqq \sum_i D\Big(\Big(\sum_j \tau_{ij}\Big)F_i, 0\Big) \end{aligned}$$

by (2.8), (6.8), (6.7), and Theorem 3.19. Then from (6.13) we have

$$D\Big(\sum_i\sum_j \tau_{ij}F_{ij}, 0\Big) \leqq \sum_i \Big\|\sum_j \tau_{ij}\Big\| D(F_i, 0) < \sum_i \frac{\epsilon}{2MK}K \leqq \epsilon/2.$$

We proceed to show that the second part of the sum is also less than $\epsilon/2$. From (6.8) we have

$$D\Big(\sum_i\sum_j \tau_{ij}F_{ij}, 0\Big) \leqq D\Big(\sum_i\sum_j (\tau_{ij}\tau_i^{-1})\tau_i F_i, 0\Big).$$

From (6.6), the hypotheses on $\{\tau\}$ stated above, and the manner of constructing the operator C^*, it follows that $\sum_j(\tau_{ij}\tau_i^{-1})\tau_i F_i \subseteqq C^*[\tau_i F_i\ (+)\ 0]$, where $(+)$ denotes a logical sum rather than a vector sum. Then

$$\begin{aligned} D\Big(\sum_i\sum_j \tau_{ij}F_{ij}, 0\Big) &\leqq D\Big(\sum_i C^*[\tau_i F_i\ (+)\ 0], 0\Big) \\ &\leqq D\Big(C^*\Big[\sum_i (\tau_i F_i\ (+)\ 0)\Big], 0\Big) \\ &\leqq WD\Big(\sum_i (\tau_i F_i\ (+)\ 0), 0\Big) \\ &< \epsilon/2 \end{aligned}$$

by Theorems 3.7 and 3.15 and the original choice of M. From (2.8) it then follows that $D(\sum\sum\tau_{ij}F_{ij},\ 0)<\epsilon$, where the sum is any finite sum with $i\geqq M+1$ or $j\geqq N+1$. By Theorem 4.11 the series $\sum_{ij}\tau_{ij}F_{ij}$ is strongly unconditionally convergent. The proof of (6.9) is thus complete.

We turn now to (6.10). From Theorem 4.11 and the result just established, it follows that every row $\sum_{ij}\tau_{ij}F_{ij}$ converges strongly unconditionally. Furthermore, $\sum_{j=1}^{J}\tau_{ij}F_{ij}\subseteqq\sum_{j=1}^{J}\tau_{ij}F_i=(\sum_{j=1}^{J}\tau_{ij})F_i$ as a result of hypotheses (6.7), (6.8) and Theorem 3.19. Now by letting $J\to\infty$, we see from (6.6) that the second part of the statement in (6.10) follows.

We proceed to prove (6.11). Let F denote the weak sum of $\sum_{ij}\tau_{ij}F_{ij}$. Since this series converges unconditionally, we can find an m_1 and n_1 so large that

$$D\left(\sum_{i=1}^{r}\sum_{j=1}^{s}\tau_{ij}F_{ij},\ F\right)<\epsilon/2 \tag{6.14}$$

for $r\geqq m_1$, $s\geqq n_1$. Next, by Theorem 4.11 and the conclusion (6.9) that $\sum_{ij}\tau_{ij}F_{ij}$ converges strongly unconditionally, there exists an m_2, n_2 such that

$$D(\sum\sum\tau_{ij}F_{ij},\ 0)<\epsilon/2, \tag{6.15}$$

where the summation is extended over any set of terms outside of the rectangle $1\leqq i\leqq m_2$, $1\leqq j\leqq n_2$. Let M (N) be the larger of m_1, m_2 (n_1, n_2). Then

$$\begin{aligned} D\left(\sum_{i=1}^{m}\sum_{j=1}^{\infty}\tau_{ij}F_{ij},\ F\right) &\leqq D\left(\sum_{i=1}^{m}\sum_{j=1}^{\infty}\tau_{ij}F_{ij},\ \sum_{i=1}^{m}\sum_{j=1}^{n}\tau_{ij}F_{ij}\right)\\ &\quad + D\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\tau_{ij}F_{ij},\ F\right)\\ &< D\left(\sum_{i=1}^{m}\sum_{j>n}\tau_{ij}F_{ij},\ 0\right)+\epsilon/2\\ &< \epsilon/2+\epsilon/2=\epsilon \end{aligned}$$

provided $m\geqq M$, $n\geqq N$ by (2.5), (2.8), (2.4), (6.14), (6.15). This inequality establishes (6.11).

The final conclusion (6.12) now follows from (6.10), (6.11) and the first statement in Theorem 6.1. The proof of the entire theorem is thus complete.

This theorem is an extension of one first given by Birkhoff [1, p. 364].

(6.16) THEOREM. *If the series* $\sum_1^\infty\|\tau_i\|$ *converges, and if the sets* F_i *are uniformly bounded, then the series* $\sum_1^\infty\tau_iF_i$ *converges strongly unconditionally, that is, hypothesis* (6.5) *is satisfied.*

The proof follows from Theorem 4.11.

(6.17) THEOREM. *If* $\sum_i\tau_iF_i$ *is a finite sum, then hypothesis* (6.5) *is satisfied.*

7. **Summation of infinite series.** We shall now extend some well known theorems on the summation of infinite series to series of the form $\sum_1^\infty f_n$.

Let S denote the sequence $g_1, g_2, \cdots$ of elements in $\mathfrak{B}$. Let (A) denote the following infinite array of numbers:

$$\begin{array}{l} a_{11},\ a_{12},\ \cdots,\ a_{1k},\ \cdots, \\ a_{21},\ a_{22},\ \cdots,\ a_{2k},\ \cdots, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \\ a_{i1},\ a_{i2},\ \cdots,\ a_{ik},\ \cdots, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot. \end{array}$$

Set $A_i(S)=\sum_{k=1}^\infty a_{ik}g_k$. We say S is summable to the sum $A(S)$ when each of the series $A_i(S)$ is convergent and $A_i(S)\to A(S)$. The method A is called *regular* if every convergent sequence is summable by that method to its limit.

(7.1) THEOREM. *In order that the method A be regular, it is necessary and sufficient that the following conditions be satisfied simultaneously*:

$$\sum_{k=1}^{\infty} |a_{ik}| \leqq M, \qquad i = 1, 2, \cdots; \tag{7.2}$$

$$\lim_{i\to\infty} a_{ik} = 0, \qquad k = 1, 2, \cdots; \tag{7.3}$$

$$\lim_{i\to\infty} \sum_{k=1}^{\infty} a_{ik} = 1. \tag{7.4}$$

The statement of the theorem is identical with the well known theorem for sequences S whose elements are numbers (see Banach [1, pp. 90–91]). The reader will prove without difficulty that the conditions stated are sufficient. By considering sequences of the form $b_1f, b_2f, \cdots$, where f is a fixed element in $\mathfrak{B}$ and $b_1, b_2, \cdots$ is a sequence of numbers, and applying the known theorem for sequences whose elements are numbers, we see that the conditions stated are necessary in the present case also.

(7.5) THEOREM. *If the series $\sum_1^\infty f_n$ is summable C_k, that is, by Cesàro means of order k, and if $f_n=O(1/n)$, then $\sum_1^\infty f_n$ is convergent.*

The proof of this theorem is the same as for a series of numbers (see Knopp [1, pp. 486–487]).

PART III. A THEORY OF INTEGRATION

8. **Abstract space $\mathfrak{A}$.** We shall now consider a space which will be the domain of functions to be studied presently. We assume that $\mathfrak{A}$ is an abstract space with elements x and sets X. In addition we shall suppose that there is a

completely additive class of sets $\mathfrak{X}$ in $\mathfrak{A}$ with the following properties: (1) the empty class belongs to $\mathfrak{X}$; (2) if X belongs to $\mathfrak{X}$, the complement of X does also; (3) the sum of a sequence $\{X_n\}$ of sets selected from the class $\mathfrak{X}$ belongs also to the class $\mathfrak{X}$. We shall say that a set $(\mathfrak{X})$ is measurable $(\mathfrak{X})$. Finally, we shall suppose that there is a measure defined for the class of sets $\mathfrak{X}$. A function of a set $\tau(X)$ will be called a measure $(\mathfrak{X})$ if it has the following properties:

(8.1) $\tau(X)$ is defined for every set in $\mathfrak{X}$; $\tau(X) \varepsilon \mathfrak{T}$; if $\tau(X) \neq 0$, then $\tau^{-1}(X)$ exists and $\tau^{-1}(X) \varepsilon \mathfrak{T}$.

(8.2) $\tau(\sum_1^\infty X_n) = \sum_1^\infty \tau(X_n)$ for every sequence of sets $(\mathfrak{X})$ no two of which have points in common; the series $\sum_1^\infty \tau(X_n)$ converges unconditionally.

Then $\tau(X)$ is called, for every set X measurable $(\mathfrak{X})$, the measure of X.

We shall not require that the measure of the entire space be finite. We shall admit the case in which the measure $\tau(X)$ of certain sets X in $\mathfrak{X}$ is "infinite" and not a proper transformation in $\mathfrak{T}$. In particular, let $\{X_n\}$ be a denumerable disjoint decomposition of $\mathfrak{A}$ into sets X_n of $\mathfrak{X}$ with measure $\tau(X_n)$. If

$$\lim_{k\to\infty} \left\| \left(\sum_{n=1}^{k} \tau(X_{\alpha(n)}) \right)^{-1} \right\| = 0$$

for every arrangement α of the sets in $\{X_n\}$ and for every such decomposition $\{X_n\}$ of $\mathfrak{A}$, we shall say that the measure of $\mathfrak{A}$ is "infinite," and we shall consider that (8.2) is satisfied.

We shall now describe the construction of the operator C^* to be used in the future. Let $X_1, \cdots, X_r$ be any finite number of disjoint sets in $\mathfrak{X}$ with measures $\tau(X_1), \cdots, \tau(X_r)$ or $\tau_1, \cdots, \tau_r$ for short. Let t, t' be $[(\tau_1 + \cdots + \tau_r)^{-1}\tau_1, \cdots, (\tau_1 + \cdots + \tau_r)^{-1}\tau_r]$, $[\tau_1(\tau_1 + \cdots + \tau_r)^{-1}, \cdots, \tau_r(\tau_1 + \cdots + \tau_r)^{-1}]$ and C_0^* the set of all such t, t'. Then C^* is formed from C_0^* as indicated in §3. We see that C^* automatically satisfies Hypotheses 3.2 and 3.3, and we now make the following further assumption, which is an additional hypothesis concerning the measure function $\tau(X)$:

(8.3) C^* is a bounded operator, that is, C^* satisfies Hypothesis 3.4.

We observe that if the measure function $\tau(X)$ satisfies (8.1), (8.2), (8.3), then the operator C^*, formed as just described, has all the properties assumed in any of the theorems given above. In particular, the assumption was made in §6 that the sum of any unconditionally convergent series of transformations in $\{\tau\}$ is a transformation in $\{\tau\}$; this property is a consequence of (8.2). Hypothesis 3.18 follows from (8.1), (8.2).

Finally, if the measure $\tau(X)$ of a set X in $\mathfrak{X}$ is a positive number $m(X)$, the ordinary convex operator C can be substituted in all later developments for the operator C^*. In this special case it is clear that for any set F, $C^*[F] \subseteqq C[F]$, and that C has all the desired properties.

We shall associate a second measure function $\nu(X)$ with the class of sets $\mathfrak{X}$. More precisely, let X be any set in $\mathfrak{X}$ and $\{Y_n\}$, $Y_n \varepsilon \mathfrak{X}$, any disjoint decomposition of X. Then

$$\nu(X) = \sup_{\{Y_n\}} \sum_{n=1}^{\infty} \|\tau(Y_n)\|. \tag{8.4}$$

If $\tau(X)$ is a positive numerically-valued measure function $m(X)$, then $\nu(X) = \tau(X)$ for every X in $\mathfrak{X}$.

It can be shown that $\nu(X)$ is a measure function with the properties specified above. In particular, $\nu(X)$ has the following properties: (a) $\nu(X)$ is defined for every set $X \varepsilon \mathfrak{X}$ and $\nu(X) \geqq 0$; (b) if $\{X_n\}$ is any denumerable set of disjoint sets of $\mathfrak{X}$, then $\nu(\sum_1^\infty X_n) = \sum_1^\infty \nu(X_n)$.

The total variation $\nu(\mathfrak{A})$ of $\tau(X)$ over $\mathfrak{A}$ may be finite. On the other hand it may happen that $\nu(\mathfrak{A})$ is infinite, but that there is a decomposition of $\mathfrak{A}$ into a denumerable set $\{X_n\}$ of disjoint sets of $\mathfrak{X}$ such that $\nu(X_n)$ is finite, $n = 1, 2, \cdots$. These are the only two cases which we shall consider in the future.

We shall say a certain property holds almost everywhere $(\mathfrak{X}, \tau)$ [or $(\mathfrak{X}, \nu)$] and mean thereby that the property in question is satisfied in $\mathfrak{A}$ except at the points of a set $X \varepsilon \mathfrak{X}$ with $\tau(X) = 0$ [or $\nu(X) = 0$]. It is clear that almost everywhere $(\mathfrak{X}, \nu)$ implies almost everywhere $(\mathfrak{X}, \tau)$ but not conversely.

9. **Functions.** We shall now describe the class of functions whose integrals we shall consider.

Let $\mathfrak{A}$ be the space described in §8. For each x in $\mathfrak{A}$ let $F(x)$ be a bounded set in $\mathfrak{B}$. Then $F(x)$ is a function whose integral we shall consider in §§10–13 and 20–27. In §§14–19 we shall consider a class of functions $f(x)$ which will be defined as measurable. Here for each x in $\mathfrak{A}$, $f(x)$ is a single element in $\mathfrak{B}$. For the present, however, it is not necessary to restrict ourselves either to measurable functions or to single-valued functions.

10. **Integral ranges.** Let a function $F(x)$ with domain $\mathfrak{A}$ be given. Let Δ_k denote a decomposition of $\mathfrak{A}$ into a denumerably infinite sequence $\{X_i^k\}$ of disjoint sets X_i^k each of which is measurable $(\mathfrak{X})$ and has a measure $\tau(X_i^k)$. The product $\Delta_1\Delta_2$ of two decompositions Δ_1, Δ_2 of $\mathfrak{A}$ is the decomposition of $\mathfrak{A}$ into the sets $X_i^1 X_j^2$ which are nonvacuous.

Consider the series

$$\sum_{i=1}^{\infty} \tau(X_i^k) C^*[F(X_i^k)], \tag{10.1}$$

corresponding to the decomposition Δ_k, where $F(X_i^k)$ is the point-set sum of the sets $F(x)$, $x \varepsilon X_i^k$.

(10.2) DEFINITION. *If each set* $F(X_i^k)$ *is bounded, and if the series* (10.1) *is strongly unconditionally convergent, the sum* (10.1) *will be called the integral range* $I(F, \mathfrak{A}, \Delta_k)$ *of* $F(x)$ *with respect to the decomposition* Δ_k.

(10.3) THEOREM. *If the integral ranges* $I(F, \mathfrak{A}, \Delta_1)$, $I(F, \mathfrak{A}, \Delta_2)$ *exist, then* $I(F, \mathfrak{A}, \Delta_1\Delta_2)$ *also exists.*

The proof follows from Definition 10.2, Property 8.2, and (6.9) in Theorem 6.4.

(10.4) THEOREM. *Any two integral ranges of* $F(x)$ *overlap. More precisely, if* $I(F, \mathfrak{A}, \Delta_1)$, $I(F, \mathfrak{A}, \Delta_2)$, $I(F, \mathfrak{A}, \Delta_1\Delta_2)$ *are integral ranges of* $F(x)$ *with respect to* Δ_1, Δ_2, $\Delta_1\Delta_2$, *then*

$$I(F, \mathfrak{A}, \Delta_1\Delta_2) \subseteqq I(F, \mathfrak{A}, \Delta_1)I(F, \mathfrak{A}, \Delta_2).$$

Let the sets in Δ_1, Δ_2 be $\{X_i^1\}$, $\{X_j^2\}$. Then the sets in $\Delta_1\Delta_2$ are $\{X_i^1X_j^2\}$. Let τ_i and F_i of Theorem 6.4 be identified with $\tau(X_i^1)$ and $C^*[F(X_i^1)]$, and τ_{ij} and F_{ij} with $\tau(X_i^1X_j^2)$ and $C^*[F(X_i^1X_j^2)]$. We verify that all the hypotheses of Theorem 6.4 are satisfied. Then from (6.12) we have at once that $I(F, \mathfrak{A}, \Delta_1\Delta_2) \subseteqq I(F, \mathfrak{A}, \Delta_1)$. The remainder of the proof follows by symmetry.

This section follows closely results given by Birkhoff [1, pp. 366–367].

11. **Integrable functions and their integrals.** We proceed now to the definition of integrable function and integral of an integrable function.

(11.1) DEFINITION. *A function* $F(x)$ *will be called integrable if and only if the inferior limit of the diameters of its integral ranges is zero.*

(11.2) THEOREM. *If* $F(x)$ *is integrable, then the intersection of the integral ranges of* $F(x)$ *is a single element* $(\mathfrak{X})\int_{\mathfrak{A}}F(x)d\tau(x)$ *of* $\mathfrak{B}$.

(11.3) DEFINITION. *The element* $(\mathfrak{X})\int_{\mathfrak{A}}F(x)d\tau(x)$ *of Theorem* 11.2 *is called the integral of* $F(x)$ *over* $\mathfrak{A}$.

(11.4) THEOREM. $F(x)$ *is integrable if and only if to every* $\epsilon > 0$ *there corresponds a decomposition* Δ *under which the series* $\sum_i \tau(X_i)C^*[F(X_i)]$ *is strongly unconditionally convergent and its sum has a diameter less than* ϵ.

These definitions and theorems are similar to ones given by Birkhoff [1, p. 367].

12. **Relation to an integral of Birkhoff.** In the special case in which the measure function $\tau(X)$ is a positive numerically-valued function $m(X)$, it is possible to compare the integral $(\mathfrak{X})\int_{\mathfrak{A}}F(x)d\tau(x)$ which we have just defined with an integral $J(F)$ given by Birkhoff [1, p. 367]. In the case now under con-

sideration, therefore, the operator C^* reduces to, or may be replaced by, the ordinary convex operator C.

Birkhoff has defined the integral range $J_\Delta(F)$ of $F(x)$ relative to Δ to be

$$J_\Delta(F) = \overline{C}\left[\mathrm{S}\sum_i m(X_i)F(X_i)\right]. \tag{12.1}$$

His definition of the integral $J(F)$ is identical with that given in Definition 11.3.

(12.2) THEOREM. *If the measure function $\tau(X)$ is a numerically-valued function $m(X)$ and if C^* is replaced by C, then $I(F, \mathfrak{A}, \Delta) = J_\Delta(F)$ and $(\mathfrak{X})\int_{\mathfrak{A}} F(x)dm(x) = J(F)$.*

By Definition 10.2

$$I(F, \mathfrak{A}, \Delta) = \sum_i m(X_i)C[F(X_i)]. \tag{12.3}$$

We observe first from Theorem 5.5 that $\sum_i C[m(X_i)F(X_i)]$, which is the same as (12.3), converges strongly unconditionally if and only if $\sum_i m(X_i)F(X_i)$ converges strongly unconditionally. Hence, the existence of either of the integral ranges implies that of the other.

Next, by Theorem 5.12 we have

$$\sum_i m(X_i)F(X_i) = \overline{\mathrm{S}\sum_i m(X_i)F(X_i)}. \tag{12.4}$$

Again, by Theorem 5.6, we see that the existence of $I(F, \mathfrak{A}, \Delta)$, $J_\Delta(F)$ implies that $\sum_i m(X_i)C[F(X_i)]$ and $\sum_i m(X_i)F(X_i)$ are weakly unconditionally convergent. Finally, from Theorem 5.9 we have

$$\sum_i m(X_i)C[F(X_i)] = \overline{C}\left[\sum_i m(X_i)F(X_i)\right], \tag{12.5}$$

whence, substituting from (12.4) we obtain

$$\begin{aligned} \sum_i m(X_i)C[F(X_i)] &= \overline{C}\overline{[\mathrm{S}\textstyle\sum_i m(X_i)F(X_i)]} \\ &= \overline{C}\left[\mathrm{S}\sum_i m(X_i)F(X_i)\right]. \end{aligned} \tag{12.6}$$

Then (12.1), (12.3), (12.6) show that $I(F, \mathfrak{A}, \Delta) = J_\Delta(F)$. From the identity of the integral ranges there follows at once the identity of the integrals. The proof is complete.

It follows from this theorem that the integral defined in §11 genuinely includes that of Birkhoff and reduces to his in a special case.

13. **Properties of the integral.** We shall now give the principal properties of the integral defined in §11.

(13.1) THEOREM. (First theorem on additivity.) *If X is in $\mathfrak{X}$, and if $(\mathfrak{X})\int_{\mathfrak{A}}F(x)d\tau(x)$ exists, then $(\mathfrak{X})\int_X F(x)d\tau(x)$ exists. If Δ is a decomposition of $\mathfrak{A}$, and if $(\mathfrak{X})\int_{\mathfrak{A}}F(x)d\tau(x)$ exists, then $\sum_i(\mathfrak{X})\int_{X_i}F(x)d\tau(x)$ converges unconditionally and*

$$\sum_i (\mathfrak{X}) \int_{X_i} F(x)d\tau(x) = (\mathfrak{X}) \int_{\mathfrak{A}} F(x)d\tau(x). \tag{13.2}$$

(13.3) THEOREM. (Second theorem on additivity.) *If $(\mathfrak{X})\int_{X_i}F(x)d\tau(x)$ exists for all sets X_i of a decomposition Δ of $\mathfrak{A}$, and if $\sum_i(\mathfrak{X})\int_{X_i}F(x)d\tau(x)$ converges unconditionally, then $(\mathfrak{X})\int_{\mathfrak{A}}F(x)d\tau(x)$ exists, and the relation* (13.2) *holds.*

(13.4) THEOREM. (Theorem on distributivity.) *If $F_1(x)$, $F_2(x)$ are integrable over $\mathfrak{A}$ and T_1, T_2 are transformations in $\mathfrak{T}$ which commute with all elements of $\mathfrak{T}$, then $T_1F_1(x)+T_2F_2(x)$ is integrable, and*

$$(\mathfrak{X}) \int_{\mathfrak{A}} \left[\sum_{i=1}^{2} T_iF_i(x) \right] d\tau(x) = \sum_{i=1}^{2} T_i (\mathfrak{X}) \int_{\mathfrak{A}} F_i(x)d\tau(x). \tag{13.5}$$

(13.6) THEOREM. (Theorem on integration of sequences.) *If the functions $F_n(x)$ are defined and integrable over $\mathfrak{A}$, if the $F_n(x)$ tend uniformly to a limit function $F(x)$, and if $\tau(X)$ has bounded total variation $\nu(\mathfrak{A})$ over $\mathfrak{A}$, then $F(x)$ is integrable and*

$$\lim_{n\to\infty} (\mathfrak{X}) \int_{\mathfrak{A}} F_n(x)d\tau(x) = (\mathfrak{X}) \int_{\mathfrak{A}} F(x)d\tau(x). \tag{13.7}$$

(13.8) THEOREM. (Law of the mean.) *If the measure $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is finite, then*

$$(\mathfrak{X}) \int_{\mathfrak{A}} F(x)d\tau(x) = \tau(\mathfrak{A})f, \qquad f \in \overline{C}^*[F(\mathfrak{A})]. \tag{13.9}$$

(13.10) COROLLARY. *If $F(x)$ is bounded in $\mathfrak{A}$ and $\tau(\mathfrak{A})$ is finite, then*

$$\left\| (\mathfrak{X}) \int_{\mathfrak{A}} F(x)d\tau(x) \right\| \leqq \|\tau(\mathfrak{A})\| D(C^*[F(\mathfrak{A})], 0). \tag{13.11}$$

The proofs of the first four of the above theorems can be supplied from proofs of similar theorems given by Birkhoff [1, pp. 367–372]. The proof of Theorem 13.8 follows from (10.1) and Theorem 3.19. Each integral range is contained in $\tau(\mathfrak{A})\overline{C}^*[F(\mathfrak{A})]$. Then the intersection of all the integral ranges is a point $\tau(\mathfrak{A})f$ in $\tau(\mathfrak{A})\overline{C}^*[F(\mathfrak{A})]$ and (13.9) follows. Corollary 13.10 follows from (13.9).

(13.12) THEOREM. *Let $F(x)$ be defined in $\mathfrak{A}$, bounded, and equal to the zero element in $\mathfrak{B}$ almost everywhere $(\mathfrak{X}, \tau)$. Then $(\mathfrak{X})\int_{\mathfrak{A}} F(x)d\tau(x)$ exists and equals the zero element.*

PART IV. MEASURABLE FUNCTIONS AND THEIR INTEGRALS

14. **Measurable functions.** We turn now to a consideration of a class of functions to be known as measurable.

(14.1) DEFINITION. *Let $f(x)$ be a single-valued function whose domain is $\mathfrak{A}$ and range is in $\mathfrak{B}$. We shall say that $f(x)$ is measurable $(\mathfrak{X})$ if and only if for every $f_0 \in \mathfrak{B}$ and $r>0$ the set $E_x[\|f(x)-f_0\| \leqq r]$ is measurable $(\mathfrak{X})$.*

We shall consider only functions which, although they may be unbounded, have only finite values.

The definition of measurability in (14.1) can be extended to functions whose ranges are in metric spaces.

(14.2) THEOREM. *If $f(x)$ is measurable, then for all $f_0 \in \mathfrak{B}$ and $r>0$ the following sets are measurable $(\mathfrak{X})$:*

$$E_x[\|f(x) - f_0\| > r], \qquad E_x[\|f(x) - f_0\| = r], \qquad E_x[\|f(x) - f_0\| \geqq r],$$
$$E_x[\|f(x) - f_0\| < r], \qquad E_x[f(x) = f_0].$$

(14.3) THEOREM. *If for all $f_0 \in \mathfrak{B}$ and $r>0$ the set $E_x[\|f(x)-f_0\|<r]$ (or $E_x[\|f(x)-f_0\|\geqq r]$, or $E_x[\|f(x)-f_0\|>r]$) is measurable $(\mathfrak{X})$, then $f(x)$ is measurable $(\mathfrak{X})$.*

We shall have occasion to consider functions of another kind. Let T be a bounded linear transformation with domain $\mathfrak{B}$ and range in $\mathfrak{B}$ and norm $\|T\|$. The space $\mathfrak{T}$ with elements T is a normed vector space. Let $T(x)$ be a function such that $T(x) \in \mathfrak{T}$ for $x \in \mathfrak{A}$. The definition of a function $T(x)$ measurable $(\mathfrak{X})$ can be obtained by replacing $f(x)$ and f_0 in Definition (14.1) by $T(x)$ and T_0. Then f can be replaced by T in Theorems 14.2 and 14.3. A function whose values are real numbers and which is measurable $(\mathfrak{X})$ is a special case both of functions $f(x)$ and $T(x)$ which are measurable $(\mathfrak{X})$.

(14.4) DEFINITION. *Let $f(x)$ be a function with domain $\mathfrak{A}$ and range in $\mathfrak{B}$. We say $f(x)$ is separable if and only if the set $f(\mathfrak{A})$ is separable. The term almost separable $(\mathfrak{X}, \tau)$ [or $(\mathfrak{X}, \nu)$] will have its usual meaning.*

15. **Elementary properties of measurable functions.** In this section we shall establish the fundamental properties of measurable functions.

(15.1) THEOREM. *If $f(x)$ is measurable $(\mathfrak{X})$, then the real-valued function $\|f(x)\|$ is measurable $(\mathfrak{X})$.*

Since $f(x)$ is measurable, the set $E_x[\|f(x)-0\| \leqq r]$ or $E_x[\|f(x)\| \leqq r]$ is measurable.

(15.2) THEOREM. *Let $f(x)$ be measurable $(\mathfrak{X})$. Let N be any positive number, and let a function $f_N(x)$ be defined as follows: $f_N(x)=f(x)$ when $\|f(x)\| \leqq N$ and $f_N(x)=0$ when $\|f(x)\|>N$. Then $f_N(x)$ is measurable $(\mathfrak{X})$.*

Let X_0 be the set on which $f_N(x)=0$. It is measurable $(\mathfrak{X})$ by Theorem 14.2. Then the proof follows from the identities

$$E_x[\|f_N(x) - f_0\| \leqq r] = E_x[\|f(x) - f_0\| \leqq r]\cdot E_x[\|f(x)\| \leqq N] + X_0,$$
$$\text{or } E_x[\|f(x) - f_0\| \leqq r]\cdot E_x[\|f(x)\| \leqq N].$$

The first statement holds in case the sphere $\|f-f_0\| \leqq r$ contains the zero element, the second in case it does not.

(15.3) THEOREM. *If $f_1(x)$ and $f_2(x)$ are measurable $(\mathfrak{X})$, and if one of the functions, say $f_1(x)$, is separable, then $f_1(x)+f_2(x)$ is measurable $(\mathfrak{X})$.*

Let S denote the open sphere $\|f-f_0\|<r$, where $f_0 \in \mathfrak{B}$ and $r>0$ are arbitrary but fixed. To prove that $f_1(x)+f_2(x)$ is measurable, it is sufficient to prove that $E_x[f_1(x)+f_2(x) \in S]$ is measurable (see Theorem 14.3).

From the hypothesis of the theorem it follows that there is a denumerable set of points $g_1, g_2, \cdots$ which is dense in the range of $f_1(x)$. Let $S_i(\xi_k)$ denote the open sphere with center g_i and radius ξ_k, where ξ_k is a rational number such that $0 \leqq \xi_k < r$. Let $S_i'(r-\xi_k)$ denote the open sphere with center f_0-g_i and radius $r-\xi_k$.

The proof of the theorem will be complete when we have established the following identity

$$(15.4)\quad E_x[f_1(x) + f_2(x) \in S] = \sum_{i=1}^{\infty}\sum_{\xi_k} E_x[f_1(x) \in S_i(\xi_k)]\cdot E_x[f_2(x) \in S_i'(r - \xi_k)].$$

The proof of this identity will depend on the following lemma, whose proof we leave to the reader.

(15.5) LEMMA. *The vector sum $S_i(\xi_k)+S_i'(r-\xi_k)$ of the open spheres S_i and S_i' with centers g_i and f_0-g_i and radii ξ_k and $r-\xi_k$ is the open sphere S with center f_0 and radius r.*

From this lemma it is immediately obvious that the set on the left in (15.4) contains the set on the right, and we have to establish only the opposite inequality. Let x_0 be any point of the set on the left in (15.4), that is, a point such that $f_1(x_0)+f_2(x_0) \in S$. We shall show that there is an i and a ξ_k such that

$$x_0 \in E_x[f_1(x) \in S_i(\xi_k)]\cdot E_x[f_2(x) \in S_i'(r - \xi_k)].$$

Since S is open,

$$\|f_1(x_0) + f_2(x_0) - f_0\| = \rho < r. \tag{15.6}$$

Choose ξ_{k_0} as a rational number such that

$$0 < \xi_{k_0} < (r - \rho)/2. \tag{15.7}$$

Next, since the set $g_1, g_2, \cdots$ is dense in the range of $f_1(x)$, there exists a point g_{i_0} such that

$$\|f_1(x_0) - g_{i_0}\| < \xi_{k_0}. \tag{15.8}$$

Thus $S_{i_0}(\xi_{k_0})$ contains $f_1(x_0)$. Then $S'_{i_0}(r-\xi_{k_0})$ contains $f_2(x_0)$, for the contrary assumption gives $\|f_2(x_0)-(f_0-g_{i_0})\| \geqq r-\xi_{k_0}$ and

$$\begin{aligned}\rho = \|f_1(x_0) + f_2(x_0) - f_0\| &\geqq \big|\, \|f_2(x_0) - (f_0 - g_{i_0})\| - \|f_1(x_0) - g_{i_0}\| \,\big| \\ &\geqq \|f_2(x_0) - (f_0 - g_{i_0})\| - \|f_1(x_0) - g_{i_0}\| \\ &> (r - \xi_{k_0}) - \xi_{k_0} = r - 2\xi_{k_0} \\ &> r - (r - \rho) = \rho.\end{aligned}$$

We are able to drop the absolute value signs because $\|f_2(x_0)-(f_0-g_{i_0})\| > r/2$ and $\|f_1(x_0)-g_{i_0}\| < r/2$ by (15.7), (15.8). The assumption has led to a contradiction; hence, $f_2(x_0) \in S'_{i_0}(r-\xi_{k_0})$ as well as $f_1(x_0) \in S_{i_0}(\xi_{k_0})$. Therefore

$$x_0 \in E_x[f_1(x) \in S_{i_0}(\xi_{k_0})]\cdot E_x[f_2(x) \in S'_{i_0}(r - \xi_{k_0})],$$

and any point contained in the set on the left in (15.4) is also contained in the set on the right. This statement completes the proof of the identity (15.4) and the proof of the theorem.

(15.9) THEOREM. *Let $f_1(x), f_2(x), \cdots$ be a sequence of functions measurable $(\mathfrak{X})$ defined on $\mathfrak{A}$ such that* $\lim f_n(x)$ *exists and equals $f(x)$ for $x \in \mathfrak{A}$. Then $f(x)$ is measurable $(\mathfrak{X})$. If the functions $f_1(x), f_2(x), \cdots$ are separable, then $f(x)$ is separable.*

To prove the theorem we must show that $E_x[\|f(x)-f_0\| \leqq r]$, for arbitrary $f_0 \in \mathfrak{B}$ and $r>0$, is a set $(\mathfrak{X})$. Let $\epsilon_1, \epsilon_2, \cdots$ be a monotone decreasing sequence of positive numbers whose limit is zero. Set

$$X_n(\epsilon) = E_x[\|f_n(x) - f_0\| \leqq r + \epsilon]\cdot E_x[\|f_{n+1}(x) - f_0\| \leqq r + \epsilon]\cdots,$$

$$X(\epsilon) = \sum_1^\infty X_n(\epsilon), \qquad X = \prod_n X(\epsilon_n).$$

It is clear that X is measurable, and that we have to prove only that $E_x[\,\|f(x)-f_0\| \leqq r] = X$.

Let x_0 be any point of $E_x[\|f(x)-f_0\| \leqq r]$. Then to each ϵ of $\epsilon_1, \epsilon_2, \cdots$ there corresponds an $N=N(f_0, r, x_0, \epsilon)$ such that $\|f_n(x_0)-f_0\| \leqq r+\epsilon$ for $n \geqq N$. Then $x_0 \in X_N(\epsilon)$ and therefore to each $X(\epsilon)$ and finally to X. Conversely, suppose $x_0 \in X$. Then x_0 belongs to each $X(\epsilon)$, and there exists an $N=N(f_0, r, x_0, \epsilon)$ so that $x_0 \in X_N(\epsilon)$. Thus $\|f_n(x_0)-f_0\| \leqq r+\epsilon$ for $n \geqq N$. Then $\|f(x_0)-f_0\| \leqq r+\epsilon$, and since this is true for each ϵ, we have $\|f(x_0)-f_0\| \leqq r$. Therefore $x_0 \in E_x[\|f(x)-f_0\| \leqq r]$.

It is obvious that $f(x)$ is separable under the conditions stated. The proof is complete.

It was pointed out in §14 that the definition of measurable function which we have given applies equally well to functions with values in a metric space. It may be added now that Theorem 15.9 holds also for functions of this kind.

At this point we shall establish the relation between functions which are measurable $(\mathfrak{X})$ and almost separable and two other classes of measurable functions. A function $f(x)$ on $\mathfrak{A}$ to $\mathfrak{B}$ is called a *step-function* if and only if $f(x)$ is constant on each of a finite number of disjoint sets of $\mathfrak{X}$. Then $f(x)$ is measurable (Bochner) if and only if it is almost everywhere the strong limit of a sequence of step-functions (see Bochner [1]). A measurable function which has only a countable number of distinct values will be said to be *countably-valued*. A function $f(x)$ is *weakly measurable* if and only if $u(f(x))$ is measurable for every linear functional $u(f)$ (an additive, continuous, real-valued function) defined in $\mathfrak{B}$ (see Pettis [1, p. 278]). A necessary and sufficient condition that $f(x)$ be measurable (Bochner) is that it be weakly measurable and almost separable (Pettis [1, p. 278]).

(15.10) THEOREM. *A function $f(x)$ is measurable and almost separable if and only if it is the limit almost everywhere of a sequence of step-functions, that is, if and only if it is measurable* (*Bochner*).

First, a necessary and sufficient condition that an almost separable function $f(x)$ be measurable is that the set $E_x[f(x) \in G]$ be measurable for every open set G in $\mathfrak{B}$. The condition is sufficient by Theorem 14.3, and we shall prove that it is necessary. Without loss of generality, we may assume that $f(x)$ is separable; then $f(\mathfrak{A})G$ contains a denumerable dense set $g_1, g_2, \cdots$. By considering the set of open spheres which are contained in G, have rational radii, and have centers at the points g_i, we see that $E_x[f(x) \in G]$ is the sum of a denumerable number of measurable sets and is therefore measurable. Second, the class of measurable and almost separable functions contains the class of functions which are measurable (Bochner) and is contained in the class of weakly measurable functions. It is clear that a step-function is measurable and separable; then by Theorem 15.9 a function which is measurable (Bochner) is measurable and almost separable. Next, the set $E_f[u(f)<u_0]$, where $u(f)$ is a linear functional defined in $\mathfrak{B}$, is an open set G in $\mathfrak{B}$. Since $E_x[u(f(x))<u_0]=E_x[f(x) \varepsilon G]$, which is a measurable set by the first step in the proof, it follows that an almost separable, measurable function $f(x)$ is weakly measurable. Third, it is known that for almost separable functions the class of functions which are measurable (Bochner) is the same as the class of functions which are weakly measurable (see Pettis [1, p. 278]). Then the three classes of functions considered in the second step of the proof coincide. The proof is complete.

(15.11) Corollary. *An almost separable function is measurable if and only if it is the uniform limit, except possibly on a set of measure zero, of measurable countably-valued functions.*

This corollary follows from the last theorem and a result of Pettis [1, p. 279, Corollary 1.12].

The following corollary is a generalization of the familiar theorem that the product of two measurable functions whose values are numbers is a measurable function.

(15.12) Corollary. *If $T(x)$ and $f(x)$ are defined on $\mathfrak{A}$, $T(x) \varepsilon \mathfrak{T}, f(x) \varepsilon \mathfrak{B}$, are measurable $(\mathfrak{X})$, and have separable ranges, then $T(x)f(x)$ is measurable $(\mathfrak{X})$ and separable.*

(15.13) Lemma. *If $\mathfrak{A}$ has finite measure $\tau(\mathfrak{A})$ and if $\{f_n(x)\}$ is a sequence of finite measurable functions on $\mathfrak{A}$, converging on this set to a finite measurable function $f(x)$, there exists, for each pair of positive numbers ϵ, η, an integer N and a measurable subset X of $\mathfrak{A}$ such that $\|\tau(X)\|<\eta$ and $\|f_n(x)-f(x)\|<\epsilon$ for every $n>N$ and $x \varepsilon \mathfrak{A}-X$.*

(15.14) Egoroff's theorem. *If $\mathfrak{A}$ has finite measure $\tau(\mathfrak{A})$ and if $\{f_n(x)\}$ is a sequence of finite measurable functions defined on $\mathfrak{A}$ that converges almost everywhere $(\mathfrak{X}, \tau)$ on this set to a finite measurable function $f(x)$, then there exists, for each $\epsilon>0$, a subset Y of $\mathfrak{A}$ such that $\|\tau(\mathfrak{A}-Y)\|<\epsilon$, and such that the convergence of $\{f_n(x)\}$ to $f(x)$ is uniform on Y.*

The proofs of Lemma 15.13 and of Egoroff's theorem are similar to those given for functions whose values are numbers (see Saks [1, pp. 17–18]).

16. **Integrals of bounded, measurable, and almost separable functions; an existence theorem.** We shall now prove a theorem which states sufficient conditions for the existence of the integral which was defined in §11. This theorem is a generalization of the familiar one that a function which is bounded and measurable on a set with finite measure is integrable.

(16.1) EXISTENCE THEOREM. *Let $f(x)$ be defined in $\mathfrak{A}$, measurable $(\mathfrak{X})$, almost separable $(\mathfrak{X}, \tau)$, and bounded:*

$$\|f(x)\| < K, \qquad x \varepsilon \mathfrak{A}. \tag{16.2}$$

Let the measure function $\tau(X)$ satisfy (8.1), (8.2), (8.3), *and let the measure $\tau(\mathfrak{A})$ of $\mathfrak{A}$ be finite. Finally, let $\tau(X)$ have the following property: if $X_1, \cdots, X_r$ are any r disjoint sets of $\mathfrak{X}$, and if $f_1, \cdots, f_r$ are any r points in $\mathfrak{B}$, there exists a constant W' such that*

$$\left\| \sum_1^r \tau(X_i) f_i \right\| \leqq W' \max_i \|f_i\|. \tag{16.3}$$

Then the integral $(\mathfrak{X})\int_{\mathfrak{A}} f(x) d\tau(x)$ exists.

Let $X_0 \varepsilon \mathfrak{X}$ be the set such that $\tau(X_0)=0$ and $f(\mathfrak{A}-X_0)$ is separable; let $g_1, g_2, \cdots$ be the denumerable dense set in $f(\mathfrak{A}-X_0)$. Let S_i^k be the closed sphere $\|f-g_i\| \leqq 1/k$. For a fixed k let sets X_i^k be defined as follows:

$$X_1^k = (\mathfrak{A} - X_0)\cdot E_x[f(x) \varepsilon S_1^k],$$

$$X_i^k = (\mathfrak{A} - X_0)\cdot E_x[f(x) \varepsilon S_i^k] - \sum_{j=1}^{i-1} X_j{}^k,$$

for $i=2, 3, \cdots$. Then for a fixed k the sets $X_0, X_1^k, X_2^k, \cdots$ form a decomposition Δ_k of $\mathfrak{A}$, and Δ_k, $k=1, 2, \cdots$, is a sequence of decompositions of $\mathfrak{A}$.

Corresponding to Δ_k we have the series

$$\sum_{i=1}^{\infty} \tau_i^k C^*[f_i^k], \tag{16.4}$$

where τ_i^k, f_i^k have been written for the longer expressions $\tau(X_i^k), f(X_i^k)$. Since $\tau(\mathfrak{A})$ is finite by hypothesis, and since $\sum_{i=1}^{\infty}\tau_i^k$ converges unconditionally by (8.2), for any $\epsilon>0$, there exists an M such that $M<i(1)< \cdots <i(r)$ implies $\|\tau_{i(1)}^k + \cdots + \tau_{i(r)}^k\| < \epsilon/(WK)$. Let S denote the sphere $\|f\| \leqq K$. Then by (16.2) and Theorems 3.9, 3.19 we have

$$\begin{aligned} \tau_{i(1)}^k C^*[f_{i(1)}^k] + \cdots + \tau_{i(r)}^k C^*[f_{i(r)}^k] &\subseteqq \tau_{i(1)}^k C^*[S] + \cdots + \tau_{i(r)}^k C^*[S] \\ &\subseteqq (\tau_{i(1)}^k + \cdots + \tau_{i(r)}^k) C^*[S]. \end{aligned}$$

From this result and Theorem 3.15 we have

$$\begin{aligned} D(\tau_{i(1)}^k C^*[f_{i(1)}^k] + \cdots + \tau_{i(r)}^k C^*[f_{i(r)}^k], 0) &\leqq \|\tau_{i(1)}^k + \cdots + \tau_{i(r)}^k\| D(C^*[S], 0) \\ &< \epsilon WK/WK = \epsilon. \end{aligned}$$

By Theorem 4.11 the series (16.4) therefore converges strongly unconditionally. Then by Definition 10.2 the sum of (16.4) is an integral range $I(f, \mathfrak{A}, \Delta_k)$.

To complete the proof, it is sufficient, by Theorem 11.4, to show that for each $\epsilon>0$ there exists a decomposition Δ_k of $\mathfrak{A}$ corresponding to which the diameter of the integral range $I(f, \mathfrak{A}, \Delta_k)$ is less than ϵ.

Let I_k denote $I(f, \mathfrak{A}, \Delta_k)$. Then from the definition of the diameter of a set and (2.5), we have

$$\rho(I_k) = D(I_k - I_k, 0) \leqq D\left(I_k - I_k, \sum_{i=1}^{n} \tau_i^k C^*[f_i^k] - \sum_{i=1}^{n} \tau_i^k C^*[f_i^k]\right) + D\left(\sum_{i=1}^{n} \tau_i^k C^*[f_i^k] - \sum_{i=1}^{n} \tau_i^k C^*[f_i^k], 0\right).$$

Next, by (2.8) and (2.6) we have

$$\rho(I_k) \leqq 2D\left(I_k, \sum_{i=1}^{n} \tau_i^k C^*[f_i^k]\right) + D\left(\sum_{i=1}^{n} \tau_i^k \{C^*[f_i^k] - C^*[f_i^k]\}, 0\right). \tag{16.5}$$

First choose k so that $k>4WW'/\epsilon$, and let it be held fixed thereafter. Then since f_i^k is contained in the sphere S_i^k with diameter $2/k$, we see that the second term in (16.5) does not exceed

$$W'D(C^*[f_i^k - f_i^k], 0) \leqq 2WW'/k < \epsilon/2. \tag{16.6}$$

But for a fixed k the first term in (16.5) can be made less than $\epsilon/2$ by taking n sufficiently large. We observe that (16.6) is independent of n. We have thus shown that it is possible to choose k so that $\rho(I_k)<\epsilon$. The proof is complete.

(16.7) Corollary. *Let $f(x)$ be defined in $\mathfrak{A}$, measurable, almost separable $(\mathfrak{X}, \tau)$ and bounded. Let the measure function $\tau(X)$ satisfy* (8.1), (8.2), (8.3), *and let the measure $\tau(\mathfrak{A})$ of $\mathfrak{A}$ be finite. Finally, let $\tau(X)$ have bounded total variation $v(\mathfrak{X})$ over $\mathfrak{A}$, Then the integral $(\mathfrak{X})\int_{\mathfrak{A}} f(x)d\tau(x)$ exists.*

17. **Properties of integrals of bounded, measurable functions.** In §13 we have given a number of properties of the integrals of integrable functions. In this section we shall give certain additional properties possessed by the integrals of bounded measurable functions but not by those of integrable functions in general.

(17.1) Theorem. (Lebesgue's convergence theorem.) *Let the measure $\tau(\mathfrak{A})$ be finite. Let $f_1(x), f_2(x), \cdots$ be a bounded sequence of functions, defined in $\mathfrak{A}$, which are measurable $(\mathfrak{X})$ and integrable, and which approach almost everywhere $(\mathfrak{X}, \tau)$ an integrable limit function $f(x)$. Then*

$$\lim_{n\to\infty} (\mathfrak{X}) \int_{\mathfrak{A}} f_n(x)d\tau(x) = (\mathfrak{X}) \int_{\mathfrak{A}} f(x)d\tau(x). \tag{17.2}$$

The proof can be given in the usual way by means of Theorems 15.9, 13.4, and 13.1, Lemma 15.13, and Corollary 13.10.

(17.3) COROLLARY. *Let $\tau(\mathfrak{A})$ be finite. Let $f_1(x), f_2(x), \cdots$ be a bounded sequence of functions, defined in $\mathfrak{A}$, which are measurable $(\mathfrak{X})$ and almost separable $(\mathfrak{X}, \tau)$ and approach almost everywhere $(\mathfrak{X}, \tau)$ a limit function $f(x)$. Then $f(x)$ is integrable and the relation* (17.2) *holds.*

From Theorem 15.9 it follows that $f(x)$ is measurable $(\mathfrak{X})$, and it can be shown without difficulty that it is amost separable $(\mathfrak{X}, \tau)$. The remainder of the proof follows from Theorems 16.1 and 17.1.

(17.4) THEOREM. *If $f(x)$ is defined in $\mathfrak{A}$, bounded, measurable $(\mathfrak{X})$, and integrable, and if $\tau(X)$ has bounded total variation $\nu(X)$ over $\mathfrak{A}$, then $(\mathfrak{X})\int_{\mathfrak{A}}\|f(x)\|d\nu(x)$ exists and*

$$\left\| (\mathfrak{X}) \int_{\mathfrak{A}} f(x) d\tau(x) \right\| \leqq (\mathfrak{X}) \int_{\mathfrak{A}} \|f(x)\| d\nu(x). \tag{17.5}$$

By Theorem 15.1 the function $\|f(x)\|$ is measurable $(\mathfrak{X})$. The existence of $(\mathfrak{X})\int_{\mathfrak{A}}\|f(x)\|d\nu(x)$ then follows from Theorem 16.1. Next, consider the integral range (16.4). We have

$$\begin{aligned} D\left(\sum_{i=1}^{\infty} \tau(X_i^k) C^*[f(X_i^k)], 0 \right) &\leqq \sum_{i=1}^{\infty} \|\tau(X_i^k)\| D(C^*[f(X_i^k)], 0) \\ &\leqq \sum_{i=1}^{\infty} \nu(X_i^k)[D(f(X_i^k), 0) + W\rho(f(X_i^k))] \\ &\leqq \sum_{i=1}^{\infty} \nu(X_i^k) \sup_{x \in X_i^k} \|f(x)\| + 2W\nu(\mathfrak{A})/k. \end{aligned}$$

By taking the limit as $k \to \infty$, we obtain (17.5). The proof is complete.

(17.6) COROLLARY. *If $f(x)$ is defined in $\mathfrak{A}$, bounded, measurable $(\mathfrak{X})$, and almost separable $(\mathfrak{X}, \tau)$, and if $(\mathfrak{X})\int_{\mathfrak{A}}\|f(x)\|d\nu(x)$ exists, then* (17.5) *holds.*

There exists a denumerable decomposition $\{X_n\}$ of $\mathfrak{A}$ into sets of $\mathfrak{X}$ such that $\nu(X_n)$ is finite for all n (see §8). Then by the theorem, (17.5) holds for each set X_n. The remainder of the proof follows from Theorem 13.1.

18. **Summable functions and their integrals.** In the preceding two sections we have treated the integrals of bounded, measurable $(\mathfrak{X})$, and almost separable $(\mathfrak{X}, \tau)$ functions. We shall now indicate briefly how these results can be extended to unbounded functions.

(18.1) DEFINITION. *A function $f(x)$, defined in $\mathfrak{A}$, is said to be summable if and only if it is measurable $(\mathfrak{X})$ and almost separable $(\mathfrak{X}, \tau)$ and $(\mathfrak{X})\int_{\mathfrak{A}}\|f(x)\|d\nu(x)$ exists.*

Let $u(x)$ be a real-valued and positive function defined on $\mathfrak{A}$ which is summable (ν). Let $f_1(x), f_2(x), \cdots$ be a sequence of functions defined on $\mathfrak{A}$ each of which is bounded and measurable $(\mathfrak{X})$, almost separable $(\mathfrak{X}, \tau)$ and which approach $f(x)$ almost everywhere $(\mathfrak{X}, \nu)$ and satisfy $\|f_n(x)\| \leqq u(x)$, $x \in \mathfrak{A}$. Then $(\mathfrak{X})\int_{\mathfrak{A}} f_n(x)d\tau(x)$ exists for every n by Theorem 16.1. Furthermore $\lim_{n\to\infty}(\mathfrak{X})\int_{\mathfrak{A}} f_n(x)d\tau(x)$ exists, for we have

$$\left\| (\mathfrak{X})\int_{\mathfrak{A}} f_m(x)d\tau(x) - (\mathfrak{X})\int_{\mathfrak{A}} f_n(x)d\tau(x) \right\| \leqq (\mathfrak{X})\int_{\mathfrak{A}} \|f_m(x) - f_n(x)\| d\nu(x)$$

$$\leqq (\mathfrak{X})\int_{\mathfrak{A}} 2u(x)d\nu(x).$$

There exists a set A, measurable $(\mathfrak{X})$, such that $\mathfrak{A}-A$ has finite measure (ν), and such that $(\mathfrak{X})\int_A 2u(x)d\nu(x) < \epsilon/2$. Also, by Egoroff's theorem and the known properties of summable functions, $(\mathfrak{X})\int_{\mathfrak{A}-A}\|f_m(x) - f_n(x)\|d\nu(x) < \epsilon/2$ for all m, n sufficiently large. Thus for all m, n sufficiently large $\|(\mathfrak{X})\int_{\mathfrak{A}} f_m(x)d\tau(x) - (\mathfrak{X})\int_{\mathfrak{A}} f_n(x)d\tau(x)\| < \epsilon$, and the limit exists as stated. Finally, an argument used by Bochner [1, pp. 266–267] shows that this limit is unique. More precisely, let $v(x)$ and $g_1(x), g_2(x), \cdots$ have the properties specified above for $u(x)$ and $f_1(x), f_2(x), \cdots$. Then

$$\lim_{n\to\infty} (\mathfrak{X})\int_{\mathfrak{A}} g_n(x)d\tau(x) = \lim_{n\to\infty} (\mathfrak{X})\int_{\mathfrak{A}} f_n(x)d\tau(x).$$

Let $f(x)$ be a summable function, and let $f_N(x)$ be the truncated function of $f(x)$ as in Theorem 15.2. Then $f_N(x)$, $N=1, 2, \cdots$, has all the properties of the sequences just considered. It follows therefore that $\lim_{N\to\infty} (\mathfrak{X})\int_{\mathfrak{A}} f_N(x)d\tau(x)$ exists and is unique.

(18.2) Definition. *Let $f(x)$, $x \in \mathfrak{A}$, be summable, and let $\{f_n(x)\}$, $\|f_n(x)\| \leqq u(x)$, be any sequence of functions, each of which is defined in $\mathfrak{A}$, bounded and measurable $(\mathfrak{X})$, almost separable $(\mathfrak{X}, \tau)$, and which approach $f(x)$ almost everywhere $(\mathfrak{X}, \nu)$. Then the integral of $f(x)$ over $\mathfrak{A}$, denoted by $(\mathfrak{X})\int_{\mathfrak{A}} f(x)d\tau(x)$, is defined by*

$$(\mathfrak{X})\int_{\mathfrak{A}} f(x)d\tau(x) = \lim_{n\to\infty} (\mathfrak{X})\int_{\mathfrak{A}} f_n(x)d\tau(x). \tag{18.3}$$

The integral of a summable function is effectively defined since the sequence of truncated functions forms one sequence of the needed type.

We shall omit the proofs of the following two theorems.

(18.4) Theorem. *If $f(x)$ and $g(x)$ are summable, then $f(x)+g(x)$ is summable, and*

$$(\mathfrak{X})\int_{\mathfrak{A}}[f(x)+g(x)]d\tau(x)=(\mathfrak{X})\int_{\mathfrak{A}}f(x)d\tau(x)+(\mathfrak{X})\int_{\mathfrak{A}}g(x)d\tau(x).$$

(18.5) CONVERGENCE THEOREM. *If $f_1(x), f_2(x), \cdots$ is a sequence of summable functions on $\mathfrak{A}$ which approaches almost everywhere $(\mathfrak{X}, \nu)$ a function $f(x)$, and if there exists a summable function $u(x)$ such that $\|f_n(x)\| \leqq u(x)$ for $n=1, 2, \cdots$ and $x \varepsilon \mathfrak{A}$, then $f(x)$ is summable and*

$$\lim_{n\to\infty}(\mathfrak{X})\int_{\mathfrak{A}}f_n(x)d\tau(x)=(\mathfrak{X})\int_{\mathfrak{A}}f(x)d\tau(x). \tag{18.6}$$

19. **Comparison with the measurable and summable functions of Bochner and Gowurin.** Other measurable and summable functions have been treated by Bochner [1] and Gowurin [1]. Bochner, however, considered only the case in which the measure function is numerically-valued. In Theorem 15.10 we have shown that a function is measurable and almost separable if and only if it is measurable (Bochner). Furthermore, it follows from the results in §18 that the classes of functions which are summable and summable (Bochner) are one, and that their integrals are the same in the two cases. Other results on the relations of the various integrals are given by Pettis [1, p. 292].

There are functions which are measurable but not measurable (Bochner); one simple example is the everywhere discontinuous, Riemann integrable function given by Graves [1, p. 166] (see Price [2, §4]).

Gowurin's results are patterned after those of Bochner. In his treatment of the Radon integral (see Gowurin [1, pp. 264–265]), which is the part of his paper most closely related to §§14–18, his results are more general than those obtained above in some respects, and less general in others. In particular, his treatment includes only totally measurable functions $f(x)$, where a totally measurable function is bounded and the limit of a uniformly convergent series of functions $g_1(x), g_2(x), \cdots$. On the other hand, some of the restrictions placed on the measure function $\tau(X)$ in §8 are not needed for Gowurin's results. The limit theorems given above (Theorems 17.1, 18.5) are better, disregarding the stronger hypotheses on $\tau(X)$, than that obtained by Gowurin [1, p. 265].

PART V. RIEMANN-STIELTJES INTEGRALS

20. **Definitions and hypotheses.** We proceed to develop the theory of the Riemann-Stieltjes integral. Much of it can be obtained by specializing the general results in Part III. For the sake of simplicity, the exposition will be given for functions defined on a linear interval and a numerically-valued

measure function; from the preceding part of the paper obvious extensions will occur to the reader.

We collect our hypotheses here.

(20.1) HYPOTHESIS. *$F(x)$ is defined on $\mathfrak{A}$: $a \leqq x \leqq b$; $F(x) \subset \mathfrak{B}$.*

(20.2) HYPOTHESIS. *$D(C[F(x)], 0) \leqq K$, $x \in \mathfrak{A}$.*

The measure function $m(I)$ is a numerically-valued, additive, non-decreasing function of intervals $I \subset \mathfrak{A}$:

(20.3) HYPOTHESIS. *$m(I) \geqq 0$, $I \subseteqq \mathfrak{A}$; $0 < m(\mathfrak{A}) < M$, a constant.*

(20.4) HYPOTHESIS. *$m(I_1 + I_2) = m(I_1) + m(I_2)$ for any two nonoverlapping intervals I_1, I_2 on $\mathfrak{A}$.*

We denote by I both an interval and its length. Occasionally it will be assumed that $m(I)$ is continuous:

(20.5) HYPOTHESIS. *For every $\epsilon > 0$ there exists an η such that $m(I) < \epsilon$ for every $I \subseteqq \mathfrak{A}$ for which $I < \eta$.*

Since the measure function is numerically-valued, we shall replace the operator C^* by the ordinary convex operator C as explained in §8. Furthermore, Δ_k will now denote a decomposition of $\mathfrak{A}$ into a finite number of nonoverlapping intervals $\{I_i^k\}$, $i = 1, \cdots, n_k$. The integral range of $F(x)$ over $\mathfrak{A}$ with respect to the decomposition Δ_k is

$$I(F, \mathfrak{A}, \Delta_k) = \sum_{i=1}^{n_k}{}^* C[F(I_i^k)] m(I_i^k) = \sum_{i=1}^{n_k}{}^* \overline{C}[F(I_i^k)] m(I_i^k),$$

where $\sum^*$ denotes the closure of the sum (see Definition 10.2). Theorem 6.17 shows that Theorem 10.4 still holds. The definitions and theorems of §11 apply to the present integral $\int_a^b F(x) dm(x)$, which corresponds to one called the generalized Riemann-Stieltjes integral by Hobson [1, vol. 1, p. 547]. We shall now define a set in $\mathfrak{B}$ which generalizes the upper and lower generalized Riemann-Stieltjes integrals (see Hobson [1, vol. 1, p. 546]).

(20.6) DEFINITION. *A set $\mathrm{RS}[\int_a^b F(x) dm(x)]$ such that*

$$D\left(\mathrm{RS}\left[\int_a^b F(x) dm(x)\right], I(F, \mathfrak{A}, \Delta_k)\right) \to 0$$

as $k \to \infty$ for every sequence of decompositions Δ_k, $k = 1, 2, \cdots$, for which as $k \to \infty$ $\max_i I_i^k \to 0$ will be called the Riemann-Stieltjes integral set of $F(x)$ on $\mathfrak{A}$.

In the remainder of the paper we propose to study some of the fundamental properties and applications of both $\int_a^b F(x) dm(x)$ and $\mathrm{RS}[\int_a^b F(x) dm(x)]$.

21. **Darboux's theorem and other results.** We shall give first a theorem which generalizes for RS $[\int_a^b F(x)dm(x)]$ the well known theorems of Darboux for upper and lower Riemann integrals (see Hobson [1, vol. 1, pp. 462–463]).

As a matter of notation, let $\Delta_k' : \{I_i'^k\}$ $(i=1, \cdots, n_k)$, $k=1, 2, \cdots$, denote a sequence of consecutive decompositions of $\mathfrak{A}$, that is, a sequence in which Δ_{k+1}' can be obtained from Δ_k' by subdividing one or more intervals. It will be assumed for all sequences Δ_k, Δ_k' that $\max_i I_i^k$, $\max_i I_i'^k \to 0$ as $k \to \infty$.

(21.1) THEOREM. *Let $F(x)$ and $m(I)$ satisfy Hypotheses* 20.1–20.5. *If there exists a set $G \subset \mathfrak{B}$ such that $D(G, I(F, \mathfrak{A}, \Delta_k')) \to 0$ as $k \to \infty$, then $D(G, I(F, \mathfrak{A}, \Delta_k)) \to 0$ as $k \to \infty$ for every sequence of decompositions $\Delta_1, \Delta_2, \cdots$, and G is the Riemann-Stieltjes integral set* RS $[\int_a^b F(x)dm(x)]$.

Although not difficult, the details of the proof are long and will be omitted.

(21.2) THEOREM. *Let $F(x)$ and $m(I)$ satisfy Hypotheses* 20.1–20.5. *If $F(\mathfrak{A})$ is compact in $\mathfrak{B}$, then* RS $[\int_a^b F(x)dm(x)]$ *exists.*

To prove the theorem we shall show that there exists a set G such that $D(G, I(F, \mathfrak{A}, \Delta_k')) \to 0$ as $k \to \infty$. It will then follow from Theorem 21.1 that RS $[\int_a^b F(x)dm(x)]$ exists and is identical with G.

From Theorem 10.4 and the fact that the decompositions are consecutive,

$$I(F, \mathfrak{A}, \Delta_1') \supseteq I(F, \mathfrak{A}, \Delta_2') \supseteq \cdots . \tag{21.3}$$

Since $F(\mathfrak{A})$ is compact in $\mathfrak{B}$ by hypothesis, it follows that $C[F(\mathfrak{A})]$ is compact (see Mazur [1] or Price [1, §7]). From this fact it follows that each set in (21.3) is compact. Under these conditions Cantor's theorem states that the product G of the sets in (21.3) is not null (see Sierpiński [1, p. 30] or Hausdorff [1, p. 129]). Finally, since each set of (21.3) is closed and convex, G is closed and convex.

We shall show next that $D(G, I(F, \mathfrak{A}, \Delta_k')) \to 0$ as $k \to \infty$. Suppose that this limit is not zero. Since $G \subseteq I(F, \mathfrak{A}, \Delta_k')$, $k=1, 2, \cdots$, the distance from any point of G to $I(F, \mathfrak{A}, \Delta_k')$ is zero. Then for some $\epsilon > 0$ there exist sets $G_k \subset I(F, \mathfrak{A}, \Delta_k')$, $k=1, 2, \cdots$, such that the distance from each point of G_k to G is ϵ or greater, for otherwise the limit would be zero. For each k the set G_k is closed, compact, and non-null, and $G_1 \supseteq G_2 \supseteq \cdots$. Then by Cantor's theorem the product G' of the sets G_k is non-null. Finally, the distance from any point of G' to G is at least ϵ. But since $G' \subseteq G$, this is impossible, and therefore $D(G, I(F, \mathfrak{A}, \Delta_k')) \to 0$ as $k \to \infty$. The proof is complete.

(21.4) COROLLARY. *The Riemann-Stieltjes integral set is closed and convex.*

(21.5) THEOREM. *Let $F(x)$ and $m(I)$ satisfy Hypotheses* 20.1–20.4. *A necessary and sufficient condition that* RS $[\int_a^b F(x)dm(x)]$ *exist is that for every se-*

quence of subdivisions Δ_k, $k=1, 2, \cdots$, *of* $\mathfrak{A}$ *the sets* $I(F, \mathfrak{A}, \Delta_k)$ *form a Cauchy sequence.*

A necessary condition that a sequence converge is that it be a Cauchy sequence; the necessity of the condition stated in the theorem then follows from Definition 20.6. Also the condition is sufficient. Consider the two sequences of subdivisions Δ_k, Δ_k^*. Since the space whose elements are sets in $\mathfrak{B}$ is complete in the Hausdorff metric (see §2), each of the Cauchy sequences $I(F, \mathfrak{A}, \Delta_k)$, $I(F, \mathfrak{A}, \Delta_k^*)$ has a limit. If these limits are not the same, the sequence of decompositions $\Delta_1, \Delta_1^*, \Delta_2, \Delta_2^*, \cdots$ leads to a sequence of integral ranges which is not a Cauchy sequence. This fact contradicts the hypothesis. The proof is complete.

22. **Existence theorems.** We shall now give two theorems which state sufficient conditions of another type for the existence of $\int_a^b F(x)dm(x)$ and RS $[\int_a^b F(x)dm(x)]$.

First, we shall give a definition of continuity. Let I be a closed interval on $\mathfrak{A}$. Then the fluctuation $\Phi(F, I)$ of $F(x)$ on I is the diameter $\rho(F(I)) = \rho(C[F(I)])$ of $F(I)$. In terms of the fluctuation we can define the saltus $\omega(F, x_0)$ at a point $x_0 \varepsilon \mathfrak{A}$. Let I_k be a closed interval of length $2k$ whose midpoint is x_0, the proper modification being made in case x_0 is an end point of $\mathfrak{A}$. The limit of $\Phi(F, I_k)$ as $k \to 0$ exists and is unique, and is $\omega(F, x_0)$ by definition. We shall say that $F(x)$ is continuous at x_0 if and only if $\omega(F, x_0)=0$. Thus if $F(x)$ is continuous at x_0, $F(x_0)$ denotes a single element in $\mathfrak{B}$.

With this definition of continuity, we have the following theorem.

(22.1) Theorem. *Let* $F(x)$ *and* $m(I)$ *satisfy Hypotheses* 20.1–20.4. *If the variation of* $m(I)$ *over the points of discontinuity of* $F(x)$ *is zero, then the Riemann-Stieltjes integral* $\int_a^b F(x)dm(x)$ *exists.*

The proof of this theorem can be patterned very closely after a well known existence theorem for the Riemann-Stieltjes integral of a numerically-valued function (see Hobson [1, pp. 542–544]). In the present case the proof is somewhat simpler than that given by Hobson as a result of the general theorems proved above (see §§10, 11). We show that $\rho(I(F, \mathfrak{A}, \Delta_k)) \to 0$ as $k \to \infty$ for an arbitrary sequence Δ_k, $k=1, 2, \cdots$. It follows that $\int_a^b F(x)dm(x)$ exists, and that $I(F, \mathfrak{A}, \Delta_k) \to \int_a^b F(x)dm(x)$ for an arbitrary sequence Δ_k.

The corresponding theorem for the Riemann integral in abstract spaces was first proved by Graves [1].

We shall now give a second definition of continuity and from it obtain another existence theorem. The two definitions of continuity are distinct, and they lead to different existence theorems.

The fluctuation $\Phi(F, I)$ of $F(x)$ on the closed interval I is defined by

$$(22.2) \qquad \Phi(F, I) = \sup_{x_1, x_2 \in I} D(C[F(x_1)], C[F(x_2)]).$$

The saltus $\omega(F, x_0)$ of $F(x)$ at $x_0 \in \mathfrak{A}$ is defined in terms of this fluctuation in the usual way. Finally, $F(x)$ is continuous at $x_0 \in \mathfrak{A}$ if and only if $\omega(F, x_0) = 0$. A point of continuity of $F(x)$ according to this definition may be a point of discontinuity according to the first definition.

(22.3) THEOREM. *Let* $F(x)$ *and* $m(I)$ *satisfy Hypotheses* 20.1–20.4. *If the variation of* $m(I)$ *over the points of discontinuity (in the second sense) of* $F(x)$ *is zero, then the Riemann-Stieltjes integral set* $\mathrm{RS}[\int_a^b F(x)dm(x)]$ *exists.*

As a result of Theorem 21.5, this theorem will be established if we show that $I(F, \mathfrak{A}, \Delta_k)$, $k = 1, 2, \cdots$, is a Cauchy sequence for every sequence of decompositions Δ_k. The analysis used in establishing this fact is similar to that in the proof of Theorem 22.1 and is omitted.

The conditions stated in Theorems 22.1, 22.3 are sufficient but are not necessary (see Graves [1]).

23. **Properties of the Riemann-Stieltjes integral set.** We shall first investigate the additive properties of $\mathrm{RS}[\int_a^b F(x)dm(x)]$ with respect to intervals and functions.

It will be convenient to use $F +^* G$ to denote the closure of $F + G$.

(23.1) THEOREM. *If* $\mathrm{RS}[\int_a^c F(x)dm(x)]$ *and* $\mathrm{RS}[\int_c^b F(x)dm(x)]$ *exist,* $a < c < b$, *then* $\mathrm{RS}[\int_a^b F(x)dm(x)]$ *exists and* $\mathrm{RS}[\int_a^b] = \mathrm{RS}[\int_a^c] + \mathrm{RS}[\int_c^b]$.

Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ denote the intervals $a \leqq x \leqq c$, $c \leqq x \leqq b$. By (2.7), (2.8) we have

$$D\left(\mathrm{RS}\left[\int_a^c F(x)dm(x)\right] +^* \mathrm{RS}\left[\int_a^b F(x)dm(x)\right],\ I(F, \mathfrak{A}_1, \Delta_k') +^* I(F, \mathfrak{A}_2, \Delta_k'')\right) \leqq D\left(\mathrm{RS}\left[\int_a^c\right],\ I(F, \mathfrak{A}_1, \Delta_k')\right) + D\left(\mathrm{RS}\left[\int_c^b\right],\ I(F, \mathfrak{A}_2, \Delta_k'')\right),$$

from which the proof follows.

(23.2) THEOREM. *If integrals* $\mathrm{RS}[\int_a^b F_1(x)dm(x)]$, $\mathrm{RS}[\int_a^b F_2(x)dm(x)]$, *and* $\mathrm{RS}[\int_a^b \{F_1(x) + F_2(x)\}dm(x)]$ *exist, then*

$$\mathrm{RS}\left[\int_a^b \{F_1(x) + F_2(x)\}dm(x)\right] \leqq \mathrm{RS}\left[\int_a^b F_1(x)dm(x)\right] +^* \mathrm{RS}\left[\int_a^b F_2(x)dm(x)\right].$$

Let $G(x)=F_1(x)+F_2(x)$. Then $G(I)\subseteqq F_1(I)+F_2(I)$ for any interval $I\subseteqq\mathfrak{A}$. The proof follows from this fact and the properties of the convex operator C.

(23.3) THEOREM. *Let $u(x)$, $v(x)$ be numerically-valued functions such that $u(x)\geqq 0$, $v(x)\geqq 0$. For each $x \in \mathfrak{A}$ let $F(x)$ be a convex set in $\mathfrak{B}$. If* $\mathrm{RS}[\int_a^b u(x)F(x)dm(x)]$ *and* $\mathrm{RS}[\int_a^b v(x)F(x)dm(x)]$ *exist, then* $\mathrm{RS}[\int_a^b \{u(x)+v(x)\}F(x)dm(x)]$ *also exists, and*

$$\mathrm{RS}\left[\int_a^b \{u(x)+v(x)\}F(x)dm(x)\right] = \mathrm{RS}\left[\int_a^b \{u(x)F(x)+v(x)F(x)\}dm(x)\right]. \tag{23.4}$$

(23.5) THEOREM. *Let $u(x)$, $v(x)$ be functions defined on $\mathfrak{A}$ whose values are numbers but otherwise arbitrary. For each $x \in \mathfrak{A}$ let $F(x)$ be an element in $\mathfrak{B}$. If* $\mathrm{RS}[\int_a^b u(x)F(x)dm(x)]$, $\mathrm{RS}[\int_a^b v(x)F(x)dm(x)]$ *exist, then* $\mathrm{RS}[\int_a^b \{u(x)+v(x)\}F(x)dm(x)]$ *exists and* (23.4) *holds.*

The proofs of these two theorems are omitted.

We shall now examine analogues of removing a constant from under the sign of integration.

(23.6) THEOREM. *Let T be a continuous linear transformation with domain and range $\mathfrak{B}$ which has an inverse T^{-1} with the same properties. If* $\mathrm{RS}[\int_a^b F(x)dm(x)]$ *exists, then* $\mathrm{RS}[\int_a^b TF(x)dm(x)]$ *exists and equals* $T\{\mathrm{RS}[\int_a^b F(x)dm(x)]\}$.

The corresponding theorem for the Lebesgue integral was proved byBirkhoff [1, pp. 370–371].

(23.7) THEOREM. *Let $u(x)\geqq 0$ be a bounded, numerically-valued function defined on $\mathfrak{A}$, and F an arbitrary bounded set in $\mathfrak{B}$. Let $m(I)$ satisfy Hypotheses* 20.3, 20.4. *If* $\mathrm{RS}[\int_a^b u(x)dm(x)]$ *exists, then* $\mathrm{RS}[\int_a^b u(x)Fdm(x)]$ *exists and equals* $\mathrm{RS}[\int_a^b u(x)dm(x)]\overline{C}[F]$.

From the definition of the integral range in §20 and Theorems 2.18, 2.21 we have

$$\begin{aligned} I(u(x)F, \mathfrak{A}, \Delta_k) &= \sum_i{}^*\overline{C}[u(I_i^k)F]m(I_i^k) = \sum_i{}^*\overline{C}[u(I_i^k)]\overline{C}[F]m(I_i^k) \\ &= \overline{C}\left\{\sum_i{}^*\overline{C}[u(I_i^k)]m(I_i^k)\right\}\overline{C}[F] \\ &= \left\{\sum_i{}^*\overline{C}[u(I_i)]m(I_i^k)\right\}\overline{C}[F] \\ &= I(u(x), \mathfrak{A}, \Delta_k)\overline{C}[F]. \end{aligned}$$

Next, by Theorem 2.21 we have

$$D\left(\mathrm{RS}\left[\int_a^b u(x)dm(x)\right]\overline{C}[F],\ I(u(x)F,\ \mathfrak{A},\ \Delta_k)\right)$$
$$\leqq D\left(\mathrm{RS}\left[\int_a^b u(x)dm(x)\right],\ I(u(x),\ \mathfrak{A},\ \Delta_k)\right)D(\overline{C}[F],\ 0).$$

Since $I(u(x),\ \mathfrak{A},\ \Delta_k)\rightarrow\mathrm{RS}\left[\int_a^b u(x)dm(x)\right]$ as $k\rightarrow\infty$ and F is bounded, the proof follows.

(23.8) THEOREM. *Let $u(x)$ be a bounded, numerically-valued function defined on $\mathfrak{A}$; let $m(I)$ satisfy Hypotheses* 20.3, 20.4; *and let f be an arbitrary element of $\mathfrak{B}$. If* $\mathrm{RS}\left[\int_a^b u(x)dm(x)\right]$ *exists, then* $\mathrm{RS}\left[\int_a^b u(x)fdm(x)\right]$ *exists and equals* $\mathrm{RS}\left[\int_a^b u(x)dm(x)\right]f$.

The proof follows from Corollary 2.17.

We shall now prove the analogue of the inequality $\left|\int_a^b u(x)dx-\int_a^b v(x)dx\right| \leqq\int_a^b \left|u(x)-v(x)\right|dx$ for numerically-valued functions $u(x)$, $v(x)$.

(23.9) THEOREM. *Let $F_1(x)$, $F_2(x)$ and $m(I)$ satisfy Hypotheses* 20.1–20.4. If $\mathrm{RS}\left[\int_a^b F_i(x)dm(x)\right]$, $i=1,\ 2$, *exist, then*

$$(23.10)\qquad \begin{aligned} &D\left(\mathrm{RS}\left[\int_a^b F_1(x)dm(x)\right],\ \mathrm{RS}\left[\int_a^b F_2(x)dm(x)\right]\right)\\ &\qquad\leqq \overline{\int_a^b} D(C[F_1(x)],\ C[F_2(x)])dm(x).\end{aligned}$$

The integral on the right in (23.10) is the upper generalized Riemann-Stieltjes integral of a numerically-valued function (see Hobson [1, p. 546]).

To prove the theorem, we observe first that

$$\begin{aligned} &D\left(\mathrm{RS}\left[\int_a^b F_1(x)dm(x)\right],\ \mathrm{RS}\left[\int_a^b F_2(x)dm(x)\right]\right)\\ &\qquad\leqq D\left(\mathrm{RS}\left[\int_a^b F_1(x)dm(x)\right],\ I(F_1,\ \mathfrak{A},\ \Delta_k)\right)\\ &\qquad\quad+D(I(F_1,\ \mathfrak{A},\ \Delta_k),\ I(F_2,\ \mathfrak{A},\ \Delta_k))\\ &\qquad\quad+D\left(I(F_2,\ \mathfrak{A},\ \Delta_k),\ \mathrm{RS}\left[\int_a^b F_2(x)dm(x)\right]\right).\end{aligned}$$

Then by (2.6), (2.7), (2.8), and the definition of $I(F,\ \mathfrak{A},\ \Delta_k)$ we have

$$D(I(F_1,\ \mathfrak{A},\ \Delta_k),\ I(F_2,\ \mathfrak{A},\ \Delta_k))\leqq\sum_i D(C[F_1(I_i^k)],\ C[F_2(I_i^k)])m(I_i^k).$$

Finally, we observe that

$$D(C[F_1(I_i^k)],\ C[F_2(I_i^k)]) \leqq \sup_{x \in I_i^k} D(C[F_1(x)],\ C[F_2(x)]).$$

From these facts the proof follows.

The next theorem states a sufficient condition that the Riemann-Stieltjes integral set be additive (see Theorem 23.2). In it and in its proof, we shall understand fluctuation, saltus, and continuity in the second of the two senses of §22 (see (22.2)).

(23.11) THEOREM. *Let $F_1(x)$, $F_2(x)$, and $m(I)$ satisfy Hypotheses* 20.1–20.4, *and let the variation of $m(I)$ over the points of discontinuity of $F_1(x)$ and $F_2(x)$ be zero. Then*

$$\text{(23.12)}\quad \begin{aligned} &\mathrm{RS}\left[\int_a^b \{F_1(x) + F_2(x)\}\, dm(x)\right] \\ &\qquad = \mathrm{RS}\left[\int_a^b F_1(x)dm(x)\right] +^* \mathrm{RS}\left[\int_a^b F_2(x)dm(x)\right], \end{aligned}$$

the existence of these integrals following from Theorem 22.3.

The proof of this theorem depends on the following two lemmas, whose proof we leave to the reader.

(23.13) LEMMA. *If $\Phi(F, I)$ is defined by* (22.2), *then* $\Phi(F_1+F_2, I) \leqq \Phi(F_1, I) + \Phi(F_2, I)$.

(23.14) LEMMA. *Let $F_1(x)$, $F_2(x)$ be constant, that is, let $F_1(x) = F_1$, $F_2(x) = F_2$ for all $x \in \mathfrak{A}$. Then the relation* (23.12) *holds.*

We proceed to the proof of the theorem. Let ζ and ϵ be any positive numbers. Then with each point ξ of $\mathfrak{A}$ we can associate an interval I of length not greater than ζ such that $\Phi(F_i, I) \leqq \omega(F_i, \xi) + \epsilon/2$, $i = 1, 2$ (see Hobson [1, pp. 466–467]). Then by Lemma 23.13, $\Phi(F_1+F_2, I) \leqq \omega(F_1, \xi) + \omega(F_2, \xi) + \epsilon$. By the Heine-Borel theorem a finite number of these intervals cover $\mathfrak{A}$. Finally, we can replace these overlapping intervals by a finite set of non-overlapping intervals I such that in each of them $\Phi(F_1+F_2, I) \leqq \omega(F_1, \xi) + \omega(F_2, \xi) + \epsilon$, where ξ is either an interior or end point of I (see Hobson [1, pp. 466–467]). Let $\eta \leqq \zeta$ be the length of the shortest of these intervals, and let their end points be $a = x_0 < x_1 < \cdots < x_n = b$.

Let two functions $G_1(x)$, $G_2(x)$ be defined on $\mathfrak{A}$ as follows: $G_i(x) = F_i(x_{k-1})$, $x_{k-1} \leqq x < x_k$, for $k = 1, 2, \cdots, n$ and $i = 1, 2$. Set $G(x) = G_1(x) + G_2(x)$.

Let $A(\epsilon)$ be the closed set of points of $\mathfrak{A}$ at which the saltus of either $F_1(x)$ or $F_2(x)$ is greater than or equal to ϵ. With each point $x \in A(\epsilon)$ associate the interval $x-\rho\leqq x\leqq x+\rho$. These intervals coalesce to form a finite number of nonoverlapping intervals contained in $\mathfrak{A}$; denote them by $A(\epsilon, \rho)$. The set $A(\epsilon)$ is the inner limiting set of the sets $A(\epsilon, \rho)$ as $\rho\rightarrow 0$. Since the variation of $m(I)$ over $A(\epsilon)$ is zero by hypothesis, it is possible to choose ρ so that the variation of $m(I)$ over $A(\epsilon, \rho)$ is less than an arbitrary positive number ζ'. Choose ζ so that $\zeta<\rho$; then also $\eta<\rho$. Then if one of the intervals I contains a point of $A(\epsilon)$, it is contained entirely in $A(\epsilon, \rho)$. Let $\mathfrak{A}_1$ denote the sum of the intervals I which contain no point of $A(\epsilon)$ and $\mathfrak{A}_2$ the sum of the remaining ones. Then $m(\mathfrak{A}_2)<\zeta'$.

We can now establish (23.12) by showing that the distance between the sets on the two sides of the equation is zero. By (2.5) we have

$$
\begin{aligned}
&D\left(\mathrm{RS}\left[\int_a^b \{F_1(x)+F_2(x)\}dm(x)\right], \mathrm{RS}\left[\int_a^b F_1(x)dm(x)\right]\right.\\
&\qquad\left. +^* \mathrm{RS}\left[\int_a^b F_2(x)dm(x)\right]\right)\\
&\leqq D\left(\mathrm{RS}\left[\int_a^b \{F_1(x)+F_2(x)\}dm(x)\right], \mathrm{RS}\left[\int_a^b G(x)dm(x)\right]\right) \qquad (23.15)\\
&\qquad + D\left(\mathrm{RS}\left[\int_a^b G(x)dm(x)\right], \mathrm{RS}\left[\int_a^b F_1(x)dm(x)\right]\right.\\
&\qquad\qquad\left. +^* \mathrm{RS}\left[\int_a^b F_2(x)dm(x)\right]\right).
\end{aligned}
$$

By Theorems 23.1, 22.3, and (2.8) the first term in the right-hand member of (23.15) does not exceed

$$
\begin{aligned}
&D\left(\mathrm{RS}\left[\int_{\mathfrak{A}_1} \{F_1(x)+F_2(x)\}dm(x)\right], \mathrm{RS}\left[\int_{\mathfrak{A}_1} G(x)dm(x)\right]\right)\\
&\qquad + D\left(\mathrm{RS}\left[\int_{\mathfrak{A}_2} \{F_1(x)+F_2(x)\}dm(x)\right], D\mathrm{S}\left[\int_{\mathfrak{A}_2} G(x)dm(x)\right]\right).
\end{aligned} \tag{23.16}
$$

By Theorem 23.9 and the results above, the first term here does not exceed

$$
\begin{aligned}
\int_{\mathfrak{A}_1} D(C[F_1(x)+F_2(x)], C[G(x)])dm(x) &\leqq \int_{\mathfrak{A}_1} [\omega(F_1, \xi)+\omega(F_2, \xi)+\epsilon]dm(x)\\
&\leqq 3\epsilon m(\mathfrak{A}_1) \leqq 3\epsilon m(\mathfrak{A}).
\end{aligned}
$$

The upper integral of Theorem 23.9 is an ordinary integral here by Theorem 22.1 and the hypotheses of the present theorem. Also by Theorem 23.9, (2.5), Hypothesis 20.2, and the results above, the second term in (23.16) does not exceed

$$\int_{\mathfrak{A}_2} D(C[F_1(x) + F_2(x)], C[G(x)])dm(x) \leqq \int_{\mathfrak{A}_2} 4Kdm(x) < 4K\zeta'.$$

Thus the first term in the right-hand member of (23.15) is less than $3\epsilon m(\mathfrak{A})+4K\zeta'$.

Now consider the second term in the right member of (23.15). By Lemma 23.14 and the definition of $G(x)$ we have $\text{RS}[\int_a^b G(x)dm(x)] = \text{RS}[\int_a^b G_1(x)dm(x)] +^* \text{RS}[\int_a^b G_2(x)dm(x)]$. Then by (2.7) and (2.8), the second term in the right-hand member of (23.15) does not exceed

$$D\left(\text{RS}\left[\int_a^b G_1(x)dm(x)\right], \text{RS}\left[\int_a^b F_1(x)dm(x)\right]\right) + D\left(\text{RS}\left[\int_a^b G_2(x)dm(x)\right], \text{RS}\left[\int_a^b F_2(x)dm(x)\right]\right);$$

as before we can show that this expression does not exceed $3\epsilon m(\mathfrak{A})+4K\zeta'$.

Collecting results, we see that the left-hand member of (23.15) is less than $6\epsilon m(\mathfrak{A})+8K\zeta'$; since $m(\mathfrak{A})$, K are constants and ϵ, ζ' are arbitrary, it is therefore zero. The proof is complete.

We turn next to a theorem of a different type. Let T denote an element of $\mathfrak{T}$ (see §3). We shall consider a transformation $T(x)$ which depends on the real parameter x. Then $\|[T(x_1)-T(x_2)]f\| \leqq C(x_1, x_2)\|f\|$. We shall say that $T(x)$ is continuous in x at x_0 if and only if for each $\epsilon>0$ there exists a $\delta(x_0, \epsilon)$ such that $C(x, x_0)<\epsilon$ for $|x-x_0|<\delta(x_0, \epsilon)$.

(23.17) THEOREM. *Let $F(x)$ and $m(I)$ satisfy Hypotheses* 20.1–20.4, *and let $T(x)$ be continuous in x on the closed interval $\mathfrak{A}$. If* $\text{RS}[\int_c^d F(x)dm(x)]$ *exists for every interval $c \leqq x \leqq d$ on $\mathfrak{A}$, then* $\text{RS}[\int_a^b T(x)F(x)dm(x)]$ *also exists.*

First we approximate to $T(x)$ by a sequence $T_n(x)$, $n=1, 2, \cdots$, such that $T_n(x)$ is constant on subintervals of $\mathfrak{A}$. Then the proof is completed by means of Theorem 23.1, a slight extension of Theorem 23.6, and Theorem 13.6.

24. **Improper integrals.** Let $F(x)$ and $m(I)$ satisfy Hypotheses 20.1, 20.3, 20.4, and let $F(x)$ be bounded except at $x=c$, $a<c<b$. If $\lim \text{RS}[\int_a^{c-\epsilon_1} F(x)dm(x)]$, $\lim \text{RS}[\int_{c+\epsilon_2}^b F(x)dm(x)]$ exist as $\epsilon_1, \epsilon_2 \to 0$ independently, then we say that the improper integral $\text{RS}[\int_a^b F(x)dm(x)]$ exists and that its value is the closure of the sum of these limits. If

$$\lim \overline{\int_a^{c-\epsilon_1}} D(C[F(x)], 0)dm(x), \quad \lim \overline{\int_{c+\epsilon_2}^b} D(C[F(x)], 0)dm(x)$$

exist as ϵ_1, $\epsilon_2 \to 0$ independently, we shall say that the improper integral $\mathrm{RS}[\int_a^b F(x)dm(x)]$ converges absolutely. Similar considerations apply if there are any finite number of points of $\mathfrak{A}$ at which $F(x)$ is unbounded.

(24.1) THEOREM. *Let $T(x)$ satisfy the hypotheses of Theorem 23.17. If $\mathrm{RS}[\int_a^b F(x)dm(x)]$ is an improper integral which converges absolutely, then $\mathrm{RS}[\int_a^b T(x)F(x)dm(x)]$ exists and converges absolutely.*

25. **Extreme integrals.** We shall show that there are certain points of the Riemann-Stieltjes set $\mathrm{RS}[\int_a^b F(x)dm(x)]$ which are exact analogues of the upper and lower Riemann-Stieltjes integrals of a numerically-valued function.

Let $F(x)$ be a function defined and bounded on $\mathfrak{A}$ which has its values in n-dimensional euclidean space but is otherwise arbitrary. By Theorem 2.14 and §20, we have $I(F, \mathfrak{A}, \Delta_k) = \sum_i \overline{C}[F(I_i^k)]m(I_i^k)$. Assume that $\mathrm{RS}[\int_a^b F(x)dm(x)]$ exists. Then the closed convex set $\mathrm{RS}[\int_a^b F(x)dm(x)]$ has at least two extreme points (unless it is a single point), and they are limit points of extreme points of $I(F, \mathfrak{A}, \Delta_k)$ (see Price [1, §9]). Furthermore, if g is an extreme point of $I(F, \mathfrak{A}, \Delta_k)$, then $g = \sum_i g_i m(I_i^k)$, and by Theorem 2.15 g_i is an extreme point of $\overline{C}[F(I_i^k)]$, $i = 1, \cdots, n_k$. Moreover, the extreme points g_i are contained in the closure of the sets $F(I_i^k)$. Finally, the set $\mathrm{RS}[\int_a^b F(x)dm(x)]$ is the closed convex hull of its extreme points.

For a numerically-valued function, the set

$$I(u, \mathfrak{A}, \Delta_k) = \sum_i \overline{C}[u(I_i^k)]m(I_i^k) \tag{25.1}$$

has exactly two extreme points, and they are

$$\sum_i \left[\sup_{x \in I_i^k} u(x) \right] m(I_i^k), \qquad \sum_i \left[\inf_{x \in I_i^k} u(x) \right] m(I_i^k). \tag{25.2}$$

Also, $\sup_{x \in I_i^k} u(x)$ and $\inf_{x \in I_i^k} u(x)$ are extreme points of the set $\overline{C}[u(I_i^k)]$ and belong to the closure of $u(I_i^k)$. Moreover, the limits of the two extreme points (25.2) of (25.1) are

$$\overline{\int_a^b} u(x)dm(x), \qquad \underline{\int_a^b} u(x)dm(x),$$

the two extreme integrals of $\mathrm{RS}[\int^b u(x)dm(x)]$. Finally, the set $\mathrm{RS}[\int_a^b u(x)dm(x)]$ is the interval bounded by—that is, the closed convex hull of—the points

$$\overline{\int_a^b} u(x)dm(x), \qquad \underline{\int_a^b} u(x)dm(x).$$

These results hold at least in n-dimensional euclidean space and possibly in more general spaces (see Price [1, §9] and Theorems 2.14, 2.15 above).

It was in connection with the results of this paragraph that the author rediscovered extreme points of convex sets.

Part VI. Other measurable functions

26. **An extension of Bochner's results.** We shall now show how the program of Bochner can be carried through in a more general case.

Let $F(x)$, $G(x)$, $\cdots$ denote functions defined for x in $\mathfrak{A}$ whose values are sets in $\mathfrak{B}$. Let $m(X)$ be a numerically-valued, nonnegative measure function defined for sets X in $\mathfrak{X}$. Let $G(x)$ denote a function which takes on only a finite number of values, that is, a function for which there exists a finite decomposition of $\mathfrak{A}$ into sets $X_1, \cdots, X_r$ of $\mathfrak{X}$ with $G(x) = G_i \subset \mathfrak{B}$ for $x \in X_i$, $i = 1, \cdots, r$. A function $F(x)$ will be called measurable if there exists a sequence $G_n(x)$ of these functions such that $\lim_{n\to\infty} D(F(x), G_n(x)) = 0$ almost everywhere $(\mathfrak{X}, m)$.

(26.1) Definition. $\int_{\mathfrak{A}} G(x)dm(x) = \sum_i^* C[G_i]m(X_i)$.

With this definition, we see that the integral of $G_1(x)+G_2(x)$ exists, and that $\int_{\mathfrak{A}}\{G_1(x)+G_2(x)\}dm(x) = \int_{\mathfrak{A}} G_1(x)dm(x) +^* \int_{\mathfrak{A}} G_2(x)dm(x)$. The real-valued function $D(G(x), 0)$ is measurable. From (2.5) we have $|D(F(x), 0) - D(G_n(x), 0)| \leqq D(F(x), G_n(x))$, from which it follows that $D(F(x), 0)$ also is measurable. We see then that $\int_{\mathfrak{A}} D(G(x), 0)dm(x)$ exists, and from (2.8) it follows that $D(\int_{\mathfrak{A}} G(x)dm(x), 0) \leqq \int_{\mathfrak{A}} D(G(x), 0)dm(x)$.

We shall say that the measurable function $F(x)$ is summable if $D(F(x), 0)$ is summable. Let $u(x)$ be a positive, real-valued, summable function; and let $\{F_n(x)\}$ be a sequence of bounded, measurable functions such that $D(F_n(x), 0) \leqq u(x)$ for all n. By arguments similar to those in §18, we can show that if $F(x)$ is summable and $F_n(x) \to F(x)$ almost everywhere $(\mathfrak{X}, m)$, then $\lim \int_{\mathfrak{A}} F_n(x)dm(x)$ exists and is unique. We define $\int_{\mathfrak{A}} F(x)dm(x)$ to be the closure of $\lim \int_{\mathfrak{A}} F_n(x)dm(x)$; it is therefore a closed convex set. Summable functions and their integrals have the following properties:

(26.2) $D(\int_{\mathfrak{A}} F(x)dm(x), 0) \leqq \int_{\mathfrak{A}} D(F(x), 0)dm(x)$;

(26.3) if $F(x)$ is constant and equal to F, then $\int_{\mathfrak{A}} F(x)dm(x) = \overline{C}[F]m(\mathfrak{A})$;

(26.4) if c_1, c_2 are constants and $F_1(x)$, $F_2(x)$ are summable functions, then

$$\int_{\mathfrak{A}} \{c_1F_1(x) + c_2F_2(x)\}dm(x) = c_1\int_{\mathfrak{A}} F_1(x)dm(x) +^* c_2\int_{\mathfrak{A}} F_2(x)dm(x);$$

(26.5) if $\mathfrak{A} = X_1 + X_2 + \cdots$ and $X_mX_n = 0$ for $m \neq n$, then

$$\int_{\mathfrak{A}} F(x)dm(x) = \sum_1^\infty \int_{X_n} F(x)dm(x);$$

(26.6) if the summable functions $F_1(x), F_2(x), \cdots$ on $\mathfrak{A}$ have a summable real-valued bound $u(x)$, that is, if $D(F_n(x), 0) \leqq u(x)$ for all n, and if $\lim F_n(x)$ exists for almost all x, then $\lim \int_{\mathfrak{A}} F_n(x)dm(x) = \int_{\mathfrak{A}} \lim F_n(x)dm(x)$.

27. **A more general class of measurable functions.** Bochner [1] and Dunford [1] have obtained extensive classes of measurable functions from classes of finitely-valued functions and continuous functions respectively. It is possible to show by means of examples that a more general class of measurable functions can be obtained by starting from a more general class. We shall describe such a class of functions.

Let $G(x), G_1(x), \cdots$ be functions defined on $\mathfrak{A}$: $a \leqq x \leqq b$ whose values are sets in $\mathfrak{B}$. Let Φ be a class of these functions with the following properties:

(27.1) $G(x) \in \Phi$ implies $D(G(x), 0)$ is bounded and continuous except possibly on a set of measure zero.

(27.2) $G(x) \in \Phi$ implies $\mathrm{RS}[\int^b G(x)dx]$ exists;

(27.3) $G_1(x), G_2(x) \in \Phi$ implies $G_1(x) + G_2(x) \in \Phi$.

(27.4) $G_1(x), G_2(x) \in \Phi$ implies the points of discontinuity of $D(G_1(x), G_2(x))$ are a set of measure zero at most.

(27.5) $G_1(x), G_2(x) \in \Phi$ implies $\mathrm{RS}[\int_a^b \{G_1(x) + G_2(x)\} dx] = \mathrm{RS}[\int_a^b G_1(x)dx] +^* \mathrm{RS}[\int_a^b G_2(x)dx]$.

The extension of the class Φ and the determination of the properties of the integrals of functions in the extended class are left to the reader.

Part VII. Fourier series

28. **Fejér's theorem.** For functions $f(x)$ which are measurable and almost separable, the fundamental results for singular integrals hold as for numerically-valued functions. The proofs are so similar that they need not be repeated here (see Hobson [1, vol. 2, chap. 7] and Bochner [1, §7]). We shall give certain results for the Fejér integral of functions $F(x)$, however.

Let $F(x)$ be defined for $-\pi \leqq x < \pi$ and elsewhere by $F(x+2\pi) = F(x)$; the values of $F(x)$ are sets in $\mathfrak{B}$. Let $\mathrm{RS}[\int_I F(t)dt]$ exist for every interval I contained in $-\pi \leqq t \leqq \pi$, and if it is an improper integral, let it be absolutely convergent. The Fejér integral corresponding to this function is

$$(28.1) \qquad \frac{1}{2\pi m} \mathrm{RS}\left[\int_{-\pi}^{\pi} \frac{\sin^2 [m(x-t)/2]}{\sin^2 [(x-t)/2]} F(t)dt\right].$$

The existence of this integral, not only for the interval indicated but also for every interval contained in it, for all m and x follows from the hypotheses concerning $F(x)$ and Theorem 24.1.

(28.2) Theorem. *The limit as $m \to \infty$ of the Fejér integral* (28.1) *exists for* *each value of x for which $C[F(x\pm 0)]$ exist, and the value of this limit is* $(1/2)\{C[F(x+0)]+{}^*C[F(x-0)]\}$.

The proof of this theorem follows the general outline of the proof of the corresponding theorem for numerically-valued functions. In the course of the proof it is necessary to use (2.5), (2.6), (2.7), the second definition of continuity in §22, and Theorems 23.1, 23.7, 23.9. The details are somewhat long and will be omitted.

Let $f(x)$ satisfy all the hypotheses imposed on $F(x)$ above; in addition let $f(x)$ for each x be an element in $\mathfrak{B}$, and let its points of discontinuity form a set of measure zero. Then $\int_{-\pi}^{\pi} f(t)dt$ exists, and Theorem 28.2 holds for $f(x)$. Let elements a_n, b_n in $\mathfrak{B}$ be defined by

$$(28.3) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt$$

for $n=0, 1, 2, \cdots$. These integrals exist by Theorem 24.1. With these coefficients we form the Fourier series

$$(28.4) \qquad a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

The investigation of the summation by Cesàro means of order one (see Knopp [1, chap. 13]) of (28.4) leads in the usual way to Fejér's integral (28.1). Theorems 23.5, 23.6, and 23.11 enable us to make the necessary transformations. Then from Theorem 28.2 we have the following theorem.

(28.5) THEOREM. *Let $f(x)$ have the properties just specified. Then the Fourier series* (28.4) *associated with $f(x)$ is summable C_1 at all points x at which the two limits $f(x\pm 0)$ exist, and its sum C_1 is* $(1/2)[f(x+0)+f(x-0)]$.

Bochner [1, §7] has obtained a more general result by using the Lebesgue integral rather than the Riemann integral. In the present case a theorem more general than (28.5) might be expected, in particular, a theorem for functions $F(x)$ rather than $f(x)$, the series (28.4) having sets rather than elements as coefficients, corresponding to the greater generality in Theorem 28.2. The more general result is lacking because of the restrictive hypotheses in Theorems 23.3, 23.5, and 23.11. It seems therefore that there may be functions which can be represented by the limits of their Fejér integrals but which cannot be represented by their Fourier series.

29. **A theorem on the Fourier coefficients.** Bochner [1, §7] has shown that the coefficients a_n, b_n in (28.3) tend to zero as $n\to\infty$ for any function which is measurable and almost separable. The following generalization of a

well known theorem has a certain amount of interest because it is necessary to construct a new proof.

(29.1) THEOREM. *Let $f(x)$ be a function defined on $\mathfrak{A}$ whose values are elements in $\mathfrak{B}$; let $f(x)$ have bounded total variation on $\mathfrak{A}$. Then as $\lambda \to \infty$*

$$(29.2) \qquad \int_a^b f(x) \sin \lambda x \, dx, \qquad \int_a^b f(x) \cos \lambda x \, dx$$

are $O(1/\lambda)$.

Since $f(x)$ has bounded total variation on $\mathfrak{A}$, its points of discontinuity are denumerable and of the first kind only; hence the integrals in (29.2) exist by Theorem 22.1. We shall set up a sequence of subdivisions of the interval $\mathfrak{A}$ which can be used in constructing the integral $\int_a^b f(x) \sin \lambda x \, dx$. On the x axis mark the points $2\pi m/(2^k\lambda)$, $m=0, \pm 1, \pm 2, \cdots$. The intervals or parts of intervals of this subdivision of the x axis which lie on $\mathfrak{A}$ will be taken as the intervals of the decomposition Δ_k of $\mathfrak{A}$, $k=1, 2, \cdots$. The decompositions of $\mathfrak{A}$ have been chosen in such a way that the zeros of $\sin \lambda x$ on $\mathfrak{A}$ coincide with end points of intervals. Let the end points of intervals of Δ_k taken from left to right be x_i^k, $i=0, 1, \cdots, n_k$, and let Δx_i^k denote the length of the interval $x_{i-1}^k \leqq x \leqq x_i^k$.

It is clear that

$$(29.3) \qquad \begin{aligned} \left\| \int_a^b f(x) \sin \lambda x \, dx \right\| &\leqq \left\| \sum_i f(x_i^k) \sin \lambda x_i^k \, \Delta x_i^k \right\| \\ &+ \left\| \sum_i f(x_i^k) \sin \lambda x_i^k \, \Delta x_i^k - \int_a^b f(x) \sin \lambda x \, dx \right\|. \end{aligned}$$

Since $\int_a^b f(x) \sin \lambda x \, dx$ exists, for each $\epsilon > 0$ it is possible to find a k_0 such that

$$(29.4) \qquad \left\| \sum_i f(x_i^k) \sin \lambda x_i^k \Delta x_i^k - \int_a^b f(x) \sin \lambda x \, dx \right\| < \epsilon$$

for $k \geqq k_0$ (see Definition 11.3).

Next consider the first term in the right-hand member of (29.3). Set $s_i = \sin \lambda x_1^k \, \Delta x_1^k + \cdots + \sin \lambda x_i^k \, \Delta x_i^k$ for $i=1, 2, \cdots, n_k$. Then

$$\begin{aligned} \sum_i f(x_i^k) \sin \lambda x_i^k \Delta x_i^k &= f(x_1^k) s_1 + \sum_{i=2}^{n_k} f(x_i^k)(s_i - s_{i-1}) \\ &= s_1[f(x_1^k) - f(x_2^k)] + s_2[f(x_2^k) - f(x_3^k)] \\ &\quad + \cdots + s_{n_k} f(x_{n_k}^k), \end{aligned}$$

from which it follows that

$$\left\|\sum_i f(x_i^k)\sin\lambda x_i^k\,\Delta x_i^k\right\| \leqq |s_1|\,\|f(x_1^k)-f(x_2^k)\| + |s_2|\,\|f(x_2^k)-f(x_3^k)\| + \cdots + |s_{n_k}|\,\|f(x_{n_k}^k)\|.$$

From the definition of s_i it follows that $|s_i| < \pi/\lambda$ for $i=1, 2, \cdots, n_k$. Then since $f(x)$ has bounded total variation on $\mathfrak{A}$, there is a constant V such that

$$(29.5) \qquad \left\|\sum_i f(x_i^k)\sin\lambda x_i^k\,\Delta x_i^k\right\| < \pi V/\lambda.$$

Collecting results from relations (29.3), (29.4), and (29.5) we have $\|\int_a^b f(x)\sin\lambda x\,dx\| < \pi V/\lambda+\epsilon$. Since ϵ is an arbitrary positive number, it follows that $\|\int_a^b f(x)\sin\lambda x\,dx\| \leqq \pi V/\lambda$. These results and similar considerations for the other integral in (29.2) complete the proof.

30. **Fourier's theorem.** From Theorems 7.5, 28.5, and 29.1 we have the following theorem.

(30.1) FOURIER'S THEOREM. *Let $f(x)$ be defined on $-\pi \leqq x < \pi$ and elsewhere by $f(x+2\pi)=f(x)$; let the values of $f(x)$ be elements in $\mathfrak{B}$. If $f(x)$ has bounded total variation on $-\pi \leqq x \leqq \pi$, then the Fourier series* (28.4) *associated with $f(x)$ converges for each value of x to* $(1/2)[f(x+0)+f(x-0)]$.

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