ON FINITELY MEAN VALENT FUNCTIONS. II

BY

D. C. SPENCER

1. We suppose $f(z)$ is regular in $|z|<1$ and denote by W the Riemann domain which is the transform of $|z|<1$ by f. We shall say that $f(z)$ has valency p if $f(z)$ takes no value w more than p times. More generally, let $W(R)$ be the area (regions covered multiply being counted multiply) of that portion of W which lies in the circle $|w| \leqq R$; then, if

$$W(R) \leqq p\pi R^2 \tag{1.1}$$

for all $R>0$, where p is a positive number (not necessarily integral), we shall say that $f(z)$ is p mean valent (p.m.v.)(1). This paper is a sequel to one of the same title to appear shortly in the Proceedings of the London Mathematical Society(2) in which I have shown that many of the known theorems concerning p-valent functions may be extended to the wider class of p.m.v. functions. I discuss here the behavior of p.m.v. functions on paths tending to points on the circumference $|z|=1$.

The theorems which I discuss here remain true under hypotheses somewhat less restrictive than the one stated above. For example, the hypothesis that $W(R) \leqq p\pi R^2$ only for $R \geqq R_0 > 0$ would suffice (constants now depending on R_0 as well as p). Furthermore, slightly less precise versions of the theorems (with p replaced by $p+\epsilon$) could be stated subject to the still weaker condition that

$$\limsup_{R\to\infty} \frac{W(R)}{\pi R^2} \leqq p.$$

Certain theorems(3) proved elsewhere, however, require the full strength of (1.1) for all $R>0$, and for this reason I have not introduced a new definition here.

2. We begin by expressing the inequality (1.1) in a form more convenient for our purpose. Let $n(r, w)$ be the number of times (necessarily bounded by a constant depending on r) that $f(z)$ takes on the value w in $|z|<r$; and let us take

Presented to the Society, April 27, 1940; received by the editors October 13, 1939, and, in expanded form, April 4, 1940.

(1) This definition was suggested to me by Professor J. E. Littlewood, to whom I am also indebted for advice in the preparation of the paper.

(2) This paper will be referred to as V_1.

(3) For example, Theorem 1 of V_1. The complication of an additional parameter R_0 is avoided thereby as well.

$$p(r, R) = \frac{1}{2\pi}\int_{-\pi}^{\pi} n(r, Re^{i\Psi})d\Psi, \tag{2.1}$$

$$p(R) = p(1, R) = \lim_{r\to 1} p(r, R). \tag{2.2}$$

Since $p(r, R)$ is an increasing function of r, $p(R)$ exists (but may be infinite). We have

$$\begin{aligned} W(R) &= \lim_{r\to 1} \int_0^R \int_{-\pi}^{\pi} n(r, \mathrm{R}e^{i\Psi})\mathrm{R}d\mathrm{R}d\Psi \\ &= \int_0^R \left(\lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} n(r, \mathrm{R}e^{i\Psi})d\Psi \right) d(\pi \mathrm{R}^2) \\ &= \int_0^R p(\mathrm{R})d(\pi\mathrm{R}^2). \end{aligned}$$

Hence the hypothesis (1.1) may be expressed in the form

$$\int_0^R p(\mathrm{R})d(\pi\mathrm{R}^2) \leqq p\pi R^2 \qquad (R > 0). \tag{2.3}$$

3. We shall make frequent use of the following lemma:

LEMMA 1. *Suppose* $s_1 \geqq s_2$. *Then the hypothesis*

$$\int_0^{R_1} p(R)d(R^{s_1}) \leqq pR_1^{s_1} \qquad (R_1 > 0) \tag{3.1}$$

implies

$$\int_0^{R_1} p(R)d(R^{s_2}) \leqq pR_1^{s_2} \qquad (R_1 > 0), \tag{3.2}$$

but not conversely.

Making some trivial transformations of variable, we see it is enough to show that, if $s \geqq 1$,

$$\int_0^{R_1} p_1(R)d(R^{s}) \leqq pR_1^{s} \qquad (R_1 > 0) \tag{3.3}$$

implies

$$\int_0^{R_1} p_1(R)dR \leqq pR_1 \qquad (R_1 > 0), \tag{3.4}$$

where $p_1(R) = p(R^{1/s_2})$, but that (3.4) does not imply (3.3).

Integrating by parts, we have (dropping subscripts)

$$\begin{aligned}\int_0^{R_1} p(R)dR &= \int_0^{R_1} p(R)\cdot R^{s-1}\cdot R^{1-s}dR \\ &= \left[R^{1-s}\int_0^{R} p(\mathrm{R})\cdot \mathrm{R}^{s-1}d\mathrm{R}\right]_0^{R_1} \\ &\quad + (s-1)\int_0^{R_1}\left(\int_0^{R} p(\mathrm{R})\mathrm{R}^{s-1}d\mathrm{R}\right)\cdot R^{-s}dR \\ &\leqq \frac{1}{s}\,pR_1 + \frac{(s-1)}{s}\,pR_1 \\ &= p(R_1)\end{aligned}$$

by (3.3).

On the other hand, the converse implication is false. In fact, take

$$p(R) = \begin{cases} 1, & 2\mu - 1 \leqq R < 2\mu,\ \mu = 1, 2, \cdots, \\ 0, & \text{otherwise;}\end{cases} \tag{3.5}$$

and write $R_1 = n + \theta$, where n is an integer and $0 \leqq \theta < 1$. Then

$$\begin{aligned}\int_0^{R_1} p(R)dR &= \sum_{\mu=1}^{[n/2]}\left\{(2\mu - 1) - 2\mu\right\} + \begin{cases} 0, & n \text{ even,} \\ \theta, & n \text{ odd,}\end{cases} \\ &= \begin{cases}\frac{1}{2}n, & n \text{ even,} \\ \frac{1}{2}(n-1) + \theta, & n \text{ odd,}\end{cases} \\ &\leqq \tfrac{1}{2}R_1.\end{aligned}$$

Hence (3.4) is satisfied with $p = \frac{1}{2}$. But

$$\begin{aligned}\int_0^{2\nu} p(R)d(R^s) &= \int_0^{(2\nu)^s} p(R^{1/s})dR = \sum_{\mu=1}^{\nu}\left\{(2\mu)^s - (2\mu - 1)^s\right\} \\ &= \tfrac{1}{2}(2\nu)^s + \tfrac{1}{4}s(2\nu)^{s-1} + O(\nu^{s-2}) > \tfrac{1}{2}(2\nu)^s,\end{aligned}$$

if $s > 1$ and $\nu > \nu_0(s)$. Thus, if $s > 1$, (3.3) is false for $R_1 = 2\nu$, $\nu > \nu_0$. We have shown that the converse of the lemma is false for some function $p(R)$, but not for a $p(R)$ corresponding to an actual Riemann domain. However, the $p(R)$ of the schlicht function which maps the unit circle on the domain shown in Fig. 1 differs as little as we please from the choice (3.5), and for it, therefore, the converse of the lemma is false.

4. Lemma 1 shows that the hypothesis

$$\int_0^{R_1} p(R)d(R^s) \leqq pR_1^s \qquad (R_1 > 0) \tag{s}$$

is the stronger the larger s is. For the sake of completeness I include the following two theorems (but they may be omitted by the reader if he so desires; they have no bearing on the rest of the theory).

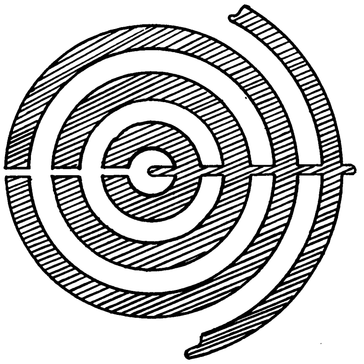

FIG. 1

THEOREM 1. *If* (s) *is true for all* $s>0$, *and* $p(R)$ *corresponds to a Riemann domain* W(4), *then* $p(R) \leqq p$.

THEOREM 2. *If*

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

is mean p-*valent, so is the (generally algebraic*(5)) *function* $\{f(z^k)\}^{1/k}$. *On the other hand, if* $k>1$, *the mean* p-*valency of the function*

$$f_k(z) = a_1 z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \cdots$$

does not imply that of the function (of form f_1)

$$\{f_k(z^{1/k})\}^k = a_1^k z + k a_1^{k-1} a_{k+1} z^2 + \cdots .$$

If $f_k(z)$ is p-valent, then so is $\{f_k(z^{1/k})\}^k$. This result and its converse are well known when the functions are p-valent(6).

We take Theorem 1 first, and note that if for a given value of R, R_0 say,

$$p(R_0) = p(1, R_0) > p,$$

then, since $p(r, R_0) \to p(R_0)$ as $r \to 1$, there exists a $\delta > 0$ and $r_0 = r_0(\delta) < 1$ such that, for $r > r_0$,

(4) The theorem is false if this clause is omitted (and is therefore not trivial).

(5) $\{f(z^k)\}^{1/k}$ has branch points at the zeros of f other than the origin. In the neighborhood of the origin, however,

$$\{f(z^k)\}^{1/k} = a_1^{1/k} z + (1/k) a_1^{1/k-1} a_2 z^{k+1} + \cdots .$$

If f is mean 1-valent, then f has at most one zero (by the definition of mean 1-valency), and in this case, therefore, $\{f(z^k)\}^{1/k}$ is regular in $|z| < 1$ (and so of the form f_k).

(6) See V_1.

$$(4.1) \qquad p(r, R_0) > p + \delta.$$

We show that this cannot happen.

Suppose it does. Then in the first place $p(r, R)$ is discontinuous at $R = R_0$, *qua* function of R, for each fixed $r > r_0$. For if it were continuous we should have(7)

$$\lim_{s\to\infty} \frac{1}{R_0^s}\int_0^{R_0} p(r, R)d(R^s) = \lim_{s\to\infty}\int_0^1 p(r, xR_0)d(x^s) = p(r, R_0),$$

which is incompatible with the combination (4.1) and (s) for $R = R_0$.

Now let $B(r)$ be the transform of the circumference $|z| = r$ by $f(z)$ ($B(r)$ is the boundary of $W(r)$). Then $B(r)$ is an analytic curve, and crosses the circumference $|w| = R_0$ a finite (even) number of times if it crosses it at all. If $B(r)$ does not meet $|w| = R_0$, or meets it only in points, then it is obvious that $p(r, R)$ is continuous at R_0. Hence the intersection of $B(r)$ with $|w| = R_0$ contains one or more intervals if $r > r_0$. These intervals depend upon r, but by (4.1) the intervals corresponding to any $r > r_0$ have positive total length. It follows that if $r > r_0$ the plane measure of $B(r)$ is positive, and so the length (or linear measure) of $B(r)$ is infinite. This is a contradiction of the regularity of $f(z)$ in $|z| < 1$, and proves Theorem 1.

Next, to prove the first part of Theorem 2, let $p(R)$, $p_k(R)$ correspond respectively to $f(z)$, $\{f(z^k)\}^{1/k}$. Then

$$p_k(R) = p(R^k).$$

In fact, $\{f(z)\}^{1/k}$ (or branch thereof) maps $|z| < 1$ cut along a radius from 0 to 1 on a surface S with function $(1/k)p(R^k)$; hence $\{f(z^k)\}^{1/k}$ (which maps $|z| < 1$ on S covered k-times) has for function

$$k\cdot(1/k)p(R^k) = p(R^k).$$

Finally

$$\int_0^{R_1} p_k(R)d(\pi R^2) = \int_0^{R_1} p(R^k)d(\pi R^2) = \int_0^{R_1^k} p(R)d(\pi R^{2/k}) \leqq p\pi R^2,$$

by the mean p-valency of f and Lemma 1. This proves the first half of the theorem.

As for the second half, let $p_1(R)$, $p_k(R)$ correspond respectively to $f_1 = \{f_k(z^{1/k})\}^k$, $f_k(z)$. Then

$$p_1(R) = p_k(R^{1/k}).$$

An argument similar to that given above to prove the negative part of Lemma 1 now shows that there exist mean p-valent functions f_k and arbitrarily large R_1 for which

(7) Since x^s increases practically from 0 to 1 in an arbitrarily small neighborhood of $x = 1$ when s is large.

$$\int_0^{R_1} p_1(R)d(\pi R^2) = \int_0^{R_1} p_k(R^{1/k})d(\pi R^2) = \int_0^{R_1^{1/k}} p_k(R)d(\pi R^{2k}) > p\pi R_1^2,$$

if $k>1$. If, however, $p_k(R) \leqq p$, then $p_1(R) \leqq p$, and in this case (in particular if f_k is p-valent) f_1 is mean p-valent.

5. After these preliminaries we now study the rate of growth of mean p-valent functions. The method depends on the distortion theory of Ahlfors(8), a theory which has already been applied by Cartwright(9) to obtain an upper bound of $M(r, f)$ (the maximum modulus of $f(z)$ on $|z| = r$) for p-valent functions. By $K(\alpha, \beta, \cdots)$ we denote a positive number depending on the parameters shown explicitly. If it is clear on what parameters K depends, as often happens, we simply write K. K's will not necessarily be the same in different contexts.

It is convenient to suppose first that $f(z)$ is regular for $|z| \leqq 1$. We write $w_0 = f(0)$. Let $C(R)$ be the circumference $|w| = R$ in the w-plane, and let $E(R) = W \times C(R)$, the set of points common to W and $C(R)$ (so that $mE(R) = 2\pi Rp(R)$). Two points of $E(R)$ are considered distinct if they correspond to distinct sheets of W, even though they have the same projection on the complex w-plane. $E(R)$ consists of a finite set of arcs $\{I_\nu(R)\}$(10), $(\nu = 1, 2, \cdots, N)$, where N depends on R (and f). For fixed R_1 let $r_\nu(R_1)$ be the value of r for which $B(r)$ (the transform of $|z| = r$ by f) just touches $I_\nu(R_1)$ (for the first time). If $|w_0| < R < R_1$, at least one arc of $E(R)$ separates $I_\nu(R_1)$ from w_0; if more than one, let $I_\nu(R)$ be the first which is met in describing a continuous curve lying in W and connecting w_0 with a point of $I_\nu(R_1)$. Let $mI_\nu(R) = \Theta_\nu(R)$.

THEOREM 3. *Suppose that* $0 < r < 1$, *and that* $R_1 > M(r_0, f)$. *Then*

$$(5.1) \quad 2\pi \int_{M(r_0,f)}^{R_1} \frac{dR}{\Theta_\nu(R)} \leqq \log \frac{1}{(1 - r_\nu(R_1))^2} + K(r_0), \quad (\nu = 1, 2, \cdots, N(R_1)).$$

Take $R_1 = M(r, f)$, and let $I_{\nu(r)}$ be any one of the intervals $\{I_\nu(R_1)\}$ which is touched by $B(r)$ (there is at least one). Then, if $r > r_0$, we have by Theorem 2 (with $R_1 = M(r, f)$, $\nu = \nu(r)$)

$$(5.2) \quad 2\pi \int_{M(r_0,f)}^{M(r,f)} \frac{dR}{\Theta_{\nu(r)}(R)} \leqq \log \frac{1}{(1 - r)^2} + K(r_0).$$

This formula has been proved in effect by Cartwright [3]. I omit the proof of the more general formula (5.1) since no essentially new ideas are involved.

6. Let

(8) Ahlfors [1].

(9) Cartwright [3].

(10) $C(R)$ may not cut B (the transform of $|z| = 1$ and the boundary of W), in which case each interval of $E(R)$ is the whole of $C(R)$, and the number of intervals is the number of sheets cut by $C(R)$ (zero for large R).

$$f_k(z) = a_1 z + a_{k+1} z^{k+1} + \cdots .$$

We deduce the following theorem from Theorem 3:

THEOREM 4. *If $f_k(z)$ is mean p-valent and $M(r, f_k)$ is the maximum modulus of f_k on the circle $|z| = r$, then*

$$M(r, f_k) \leqq K(p, k)\mu_{[p/k]}(1 - r)^{-2p/k}, \tag{6.1}$$

where

$$\mu_{[p/k]} = \max(|a_1|, \cdots, |a_{[p/k]}|).$$

Theorem 4 was stated without proof in V_1. It is known for p-valent functions, the case $k=1$ having been proved by Cartwright (loc. cit.); and the general case is an easy deduction from the case $k=1$ when f_k is p-valent(11). By combining Theorem 4 above with Theorem 3 of V_1 we obtain the following theorem (also stated without proof in V_1):

THEOREM 5. *If $f_k(z)$ is mean p-valent, then*

$$|a_n| \leqq K(p, k)\mu_{[p/k]} n^{2p/k-1}$$

provided $p > \frac{1}{4}k$.

Theorem 5 for p-valent functions was proved in V_1(12). The restriction that $p > \frac{1}{4}k$ is necessary. In fact, if $n \geqq 1$, take

$$a_{nk+1} = \begin{cases} \dfrac{1}{\nu(\lambda_\nu)^{1/2}} z^{nk+1}, & \text{if } |(nk+1) - \lambda_\nu| \leqq k/2 \text{ and } \lambda_\nu \geqq nk+1, \\ 0, & \text{otherwise,} \end{cases}$$

where (λ_ν) is a rapidly increasing sequence, and take $a_1 = 1$. We suppose that the λ_ν satisfy the inequality

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu(\lambda_\nu)^{1/2}} \leqq 1 - (\pi/6)^{1/2}.$$

Then $f_k(z)$ is zero only at the origin, and each point of the circle $|w| < (\pi/6)^{1/2}$ is covered by W_k (the transform of $|z| < 1$ by f_k) once and only once. Since the area of W_k is less than or equal to $\pi \sum_{\nu=1}^{\infty} 1/\nu^2 = \pi^3/6$, we see that $W(R) \leqq \pi^2 R^2$ for $R > 0$, so that f_k is mean π-valent. On the other hand, given any function $\psi(n)$ tending steadily to 0 as $n \to \infty$, we can choose the λ_ν such that $|a_n| > \psi(n)/n^{1/2}$ for an infinity of n. This *Gegenbeispiel* in modified form was suggested to me by Professor J. E. Littlewood.

(11) For then the function $f_1(z) = \{f_k(z^{1/k})\}^k$ is p-valent. This line of argument is not possible here (see Theorem 2).

(12) The theorem for p-valent functions was known subject to certain restrictions on f_k; in V_1 these restrictions were removed.

7. We now prove Theorem 4. We define $\Theta_\nu(\rho, R)$, the function of Theorem 3, in terms of $f_k(\rho, z)$, where $\rho<1$. Then we define

$$\Theta_\nu(R) = \lim_{\rho\to 1} \Theta_\nu(\rho, R) \leqq \lim_{\rho\to 1} 2\pi R p(\rho, R) = 2\pi R p(R),$$

since f_k is mean p-valent. $\Theta_\nu(R)$ is thus an integrable function of R (over any finite interval). Now let W_k be the transform of $|z|<1$ by f_k. Since rotation of W_k about the origin through an angle $2\pi/k$ transforms W_k into itself, we see that if T_0 is any "tube" of W_k extending to ∞, there are $(k-1)$ other tubes T_ν, $(\nu=1, 2, \cdots, (k-1))$, each identical to T_0. If, therefore, $\Theta_\nu(R)$ is the width of T_ν measured on $C(R)$, we have

$$\sum_{\nu=0}^{k} \Theta_\nu(R) = k\Theta_0(R) \leqq mE(R) = 2\pi p(R),$$

and so

$$\Theta_0(R) \leqq \frac{2\pi}{k} R p(R).$$

Hence

$$\begin{aligned} 2\pi \int_{M(r_0)}^{M(r)} \frac{dR}{\Theta_{\nu(r)}(R)} &\geqq k \int_{M(r_0)}^{M(r)} \frac{1}{p(R)} \frac{dR}{R} = k \int_{\log M(r_0)}^{\log M(r)} \frac{dR}{p(e^R)} \\ &\geqq k \frac{(\log M(r) - \log M(r_0))^2}{\int_{\log M(r_0)}^{\log M(r)} p(e^R)dR}, \end{aligned} \tag{7.1}$$

since (writing $\psi(R) = 1/p(e^R)$)

$$(b-a)^2 = \left(\int_a^b \psi^{1/2}\psi^{-1/2} dR\right)^2 \leqq \int_a^b \psi dR \int_a^b \psi^{-1} dR$$

by Schwarz's inequality. But

$$\begin{aligned} \int_{R_1}^{R_2} p(e^R)dR &= \int_{e^{R_1}}^{e^{R_2}} p(R) \frac{dR}{R} \\ &= \left[\frac{1}{R}\int_{e^{R_1}}^{R} p(\mathrm{R})d\mathrm{R}\right]_{e^{R_1}}^{e^{R_2}} + \int_{e^{R_1}}^{e^{R_2}} \left(\int_0^R p(\mathrm{R})d\mathrm{R}\right)\frac{1}{R^2} dR \\ &\leqq p + p(R_2 - R_1) \end{aligned} \tag{7.2}$$

by the hypothesis of mean valency p and Lemma 1 (with $s_1=2$, $s_2=1$). Substituting from (7.2) (with $R_1=\log M(r_0)$, $R_2=\log M(r)$) in (7.1) and using (5.2), we have

$$\log M(r) \leqq \frac{p}{k} \log \frac{1}{(1-r)^2} + K(p, k, r_0) + \log M(r_0). \tag{7.3}$$

Theorem 4 will follow at once from (7.3) (with $r_0=\frac{1}{2}$, say) if

$$M(r_0, f) < K(p, k, r_0)\mu_{[p/k]}. \tag{7.4}$$

To prove (7.4) it is sufficient to show that the family of mean p-valent functions f_k is quasi-normal of order $[p/k]$ at most, and this follows from the definition of mean valency p and the form of f_k[13]. This completes the proof of Theorem 4.

8. The full strength of the hypothesis of mean valency p is not used in Theorem 4; all that is used is (7.2), and this in the form

$$\int_{R_1}^{R_2} p(e^R)dR \leqq p/s + p(R_2 - R_1), \tag{l}$$

where $s>0$, is implied by (s). The hypothesis (l) is, in fact, sufficient for the truth of all theorems proved in this paper. Furthermore, only the properties of W in the neighborhood of ∞ are relevant. For example, if $W(R) \leqq p\pi R^2$ only for $R>R_0>0$, then

$$M(r, f) = O(1 - r)^{-2p},$$

where the constant implied in the O depends on R_0, f, and p. More generally if

$$\limsup_{R\to\infty} \frac{W(R)}{\pi R^2} \leqq p, \tag{8.1}$$

then, for every $\epsilon>0$,

$$M(r, f) = O(1 - r)^{-2p-\epsilon}.$$

Moreover, if (8.1) is satisfied with $p=0$, then

$$M(r, f) = O(1 - r)^{-\epsilon}.$$

We thus obtain, in particular, the striking result that a schlicht function which fills only an infinitesimal part of the w-plane is of infinitesimal order.

9. We shall say that a set of points in a domain D is a path P if it is a Jordan curve. If the equation of P is

$$P(t) = x(t) + iy(t),$$

where t varies from 0 to 1, and if, given ϵ,

$$|P(t) - a| < \epsilon$$

for $t_0(\epsilon)<t<1$, then we say a is an end of P, or that P converges to the point a. A path in $|z|<1$ with end $e^{i\theta}$ will be denoted by $P(\theta)$.

(13) See Montel [5, p. 73]. The test given there for quasi-normality $[p/k]$ is satisfied if applied to the functions $f_k(z^{1/k})$, which are regular in the unit circle slit along a radius, and this implies that the family $f_k(z)$ is quasi-normal of order $[p/k]$.

THEOREM 6. *Suppose that $f(z)$ is p.m.v. and that E_θ is a set of distinct points. If to each point θ of E_θ there corresponds at least one path $P(\theta)$ for which*

$$\liminf_{P_\theta} (1 - r)^{\alpha(\theta)} |f(z)| > 0, \tag{9.1}$$

then

$$\sum_{E_\theta} \alpha(\theta) \leqq 2p. \tag{9.2}$$

It is sufficient to prove the theorem for an enumerable set (θ_ν)(14). It is then enough to show that

$$\sum_{\nu=1}^{n} \alpha_\nu \leqq 2p, \tag{9.3}$$

where $\alpha_\nu = \alpha(\theta_\nu) > 0$. Under these circumstances there correspond to $R > R_0(n, f)$, n arcs $I_\nu(R)$, $(1 \leqq \nu \leqq n)$, such that the transform of $I_\nu(R)$ by $z = f^{-1}(w)$ is a cross section(15) $\gamma_\nu(R)$ of the unit circle separating the point $e^{i\theta_\nu}$ from the origin and converging to $e^{i\theta_\nu}$ as $R \to \infty$ (16). Let $R_\nu(r)$ be the largest R for which $\gamma_\nu(R)$ has points in common with the circle $|z| = r$, and write

$$mI_\nu(R) = \Theta_\nu(R) = 2\pi R \Xi_\nu(R).$$

Then

$$2\pi \int_K^{R_\nu(r)} \frac{dR}{\Theta_\nu(R)} = \int_{K_1}^{\log R_\nu(r)} \frac{dR}{\Xi_\nu(e^R)} \geqq \frac{(\log R_\nu(r) - R_1)^2}{\int_{R_1}^{\log R_\nu(r)} \Xi_\nu(e^R) dR},$$

as in the proof of Theorem 4. That is to say,

$$\int_{K_1}^{\log R_\nu(r)} \Xi_\nu(e^R) dR \geqq \frac{(\log R_\nu(r) - R_1)^2}{2\pi \int_K^{R_\nu(r)} dR/\Theta_\nu(R)} \geqq \frac{(\log R_\nu(r) - K_1)^2}{\log 1/(1 - r)^2 + K}$$

by Theorem 3,

$$\geqq \tfrac{1}{2}\alpha_\nu \log R_\nu(r) + o(\log R_\nu(r)),$$

by the hypothesis (9.1). This inequality may be written in the form

$$\tfrac{1}{2} R \alpha_\nu \leqq \int_K^R \Xi_\nu(e^R) dR + o(R).$$

Summing over ν from 1 to n, we obtain

(14) But even a schlicht function may tend to ∞ at a non-enumerable set of discrete points $e^{i\theta}$.

(15) By a cross section of a domain D we mean a path lying in D (except for its end-points) and connecting two distinct boundary points of D.

(16) That is, given ϵ, $\gamma_\nu(R)$ lies in a circle of radius ϵ and center $e^{i\theta_\nu}$ if $R > R_0(\epsilon)$. The statement is intuitive, and in any case is covered by familiar arguments.

$$\tfrac{1}{2}R\sum_{\nu=1}^{n}\alpha_\nu \leqq \int_K^R \sum_{\nu=1}^{n} \Xi_\nu(e^R)dR + o(R) \leqq \int_K^R p(e^R)dR + o(R),$$

since $\sum_{\nu=1}^{n}\Xi_\nu(R) \leqq p(R)$,

$$\leqq pR + o(R),$$

by the hypothesis of mean valency p (see (7.2)). Dividing by R and letting $R\to\infty$, we obtain (9.3), and (since n is arbitrary) this proves the theorem.

10. THEOREM 7. *Suppose $f(z)$ is p.m.v. and that*

$$f(z) = O(1) \tag{10.1}$$

on some path $P_1(\theta_0)$. Then on any path $P(\theta_0)$

$$\limsup\, (1 - r)^{2p} \,|\, f(z) \,| = 0. \tag{10.2}$$

We suppose there is an infinite sequence of points, (z_n) say, tending to $e^{i\theta_0}$ and a number $K>0$ such that $|f(z_n)| > |f(z_{n-1})|$, and

$$|\, f(z_n) \,| > K(1 - r_n)^{-2p}, \qquad |\, z_n \,| = r_n; \tag{10.3}$$

we argue by *reductio ad absurdum.*

Suppose first that there exists an arbitrarily large R such that the transform of $E(R)$ by $z=f^{-1}(w)$ contains an infinity of nonoverlapping cross sections $\gamma_\nu(R)$ of $|z|<1$ converging to $e^{i\theta_0}$ as $\nu\to\infty$ ([17]), and that each γ_ν separates at least one point z_{n_ν} from the origin. Changing the numeration (if necessary) we may suppose that $\gamma_\nu(R)$ separates z_ν from $z=0$. Let $I_\nu(R)$ be the transform by $f(z)$ of $\gamma_\nu(R)$, and write $mI_\nu(R)=\Theta_\nu(R)=2\pi R\Xi_\nu(R)$, $R_\nu=|f(z_\nu)|$. Then

$$p \log R_\nu \leqq \frac{(\log R_\nu - K)^2}{2\pi\int_K^{R_\nu} dR/\Theta_\nu(R)} + O(1). \tag{10.4}$$

Otherwise

$$\log R_\nu < 2\pi p \int_K^{R_\nu} \frac{dR}{\Theta_\nu(R)} + O(1) < \log \frac{1}{(1 - r_\nu)^2} + O(1)$$

by Theorem 3, and this contradicts the hypothesis (10.3). But (as in the proof of Theorem 6)

$$\frac{(\log R_\nu - K)^2}{2\pi\int_K^{R_\nu} dR/\Theta_\nu(R)} \leqq \int_K^{\log R_\nu} \Xi_\nu(e^R)dR,$$

and so, substituting in (10.4),

([17]) But no γ_ν separates $e^{i\theta_0}$ from the origin.

$$(10.5)\qquad p\log R_\nu \leqq \int_K^{\log R_\nu} \Xi_\nu(e^R)dR + O(1).$$

Finally, since $\gamma_{\nu-1}$ and γ_ν are nonoverlapping, we see that $\Xi_{\nu-1}(R)$ and $\Xi_\nu(R)$ are distinct for all (sufficiently) large R, and so

$$(10.6)\qquad \begin{aligned}\int_K^{\log R_\nu} \Xi_\nu(e^R)dR &\leqq \int_K^{\log R_\nu} p(e^R)dR - \int_K^{\log R_\nu} \Xi_{\nu-1}(e^R)dR \\ &\leqq p\log R_\nu - p\log R_{\nu-1} + O(1)\end{aligned}$$

by the hypothesis of mean valency p and (10.5) for $\nu-1$. (10.5) and (10.6) give a contradiction if $\nu>\nu_0$, and so the infinity of nonoverlapping cross sections with the properties stated cannot exist. The alternative is that for $R>R_0$ one cross section, $\gamma(R)$ say, separates all but a finite number of (z_ν) from $z=0$.

Now we can find a number R_1 such that, for $R>R_1$, $\gamma(R)$ does not separate $e^{i\theta_0}$ from $z=0$. Otherwise there would exist no path $P_1(\theta_0)$ on which $f=O(1)$, contrary to the hypothesis of the theorem. Since, on the other hand, $\gamma(R)$ separates all but a finite number of the (z_ν) from $z=0$, we see that, for $R>R_1$, $\gamma(R)$ has $e^{i\theta_0}$ as one end-point. This, I say, is impossible[18]. In fact, suppose $R_1<R_2<R_3$, and connect $\gamma(R_2)$ with $\gamma(R_3)$ by a simple analytic curve lying in $|z|<1$. Let q_1 be the last intersection of this curve with $\gamma(R_2)$, q_2 the first intersection with $\gamma(R_3)$. Then the portion P of the curve connecting q_1 with q_2 lies in a sub-domain D of $|z|<1$ (bounded by $\gamma(R_2)$, $\gamma(R_3)$, and points of $|z|=1$), and divides D into two domains. Let D_1 be the domain bounded by $\gamma(R_2)$, $\gamma(R_3)$, P, and $e^{i\theta_0}$; and let W_1 be the transform of D_1, Π the transform of P, by f. Suppose $R_2<R<R_3$, and let $I(R)$ be the first cross section of W_1 on $C(R)$ which is met in describing a continuous curve from $E(R_2)$ to $E(R_3)$ in W_1. We write $\Theta(R)=mI(R)$. Then, by the hypothesis of mean valency p,

$$\int_{R_1}^{R_2} mI(R)dR \leqq p\pi R_2^2.$$

Hence, if $K=2p\pi R_2^2/(R_2-R_1)$, and E is the set of values of R in the interval $R_1<R<R_2$ for which

$$mI(R) > K,$$

then

$$(10.7)\qquad mE < \tfrac{1}{2}(R_2 - R_1).$$

Next, we define

$$J(R) = \begin{cases} I(R), \text{ if } mI(R) \leqq K, \\ \text{a portion of } I(R) \text{ of length } K \text{ measured from } \Pi \text{ if } mI(R) > K. \end{cases}$$

[18] For finitely mean valent functions, but not for infinitely mean valent functions.

Let W_2 be one of the sub-domains of W_1 swept out by $J(R)$ as R varies from R_2 to R_3, which contains, as part of its boundary, a set A of boundary points of W of positive measure. Such a sub-domain exists by (10.7). Further, W_2 is plainly a finitely valent domain; and every point of its boundary is accessible (by the definition of accessibility). We map W_2 on a sub-domain D_2 of $|z|<1$ by f^{-1}, the set A corresponding to the boundary point $e^{i\theta_0}$. This contradicts well known theorems on the correspondence of boundaries(19) and proves our statement.

11. The conclusion (9.2) of Theorem 6 is a best possible one when p is integral, as shown by the p-valent function

$$f_n(z) = \frac{z^p}{(1 + z^n)^{2p/n}} \cdot$$

On the other hand, the hypothesis (9.1) cannot be relaxed to the extent of replacing lim inf by lim sup. We have in fact

THEOREM 8. *If $\psi(r)$ is any real function of r satisfying*

$$(1 - r)^2 = o(\psi(r)), \tag{11.1}$$

then there is a function $f(z)$ regular and schlicht in $|z|<1$ such that, for at least one path $P(\theta_\nu)$,

$$\limsup_{P(\theta_\nu)} \psi(r) \,|\, f(z) \,|\, > 0 \tag{11.2}$$

at an enumerable infinity of discrete points (θ_ν).

The following theorem shows that Theorem 7 is best possible.

THEOREM 9. *Suppose $\psi(r)$ satisfies* (11.1). *Then there is a schlicht function $f(z)$ such that the radial limit,* $\lim_{r\to 1} f(re^{i\theta})$, *exists everywhere and is finite, but*

$$\limsup \psi(r) \,|\, f(z) \,|\, > 0 \tag{11.3}$$

on at least one path $P(\theta_0)$.

The function whose existence is asserted in Theorem 9 is simpler and we discuss it first. We take $f(z)$ to be the function which maps $|z|<1$ on the simply-connected domain W shown in Fig. 2, with $f(0)=0$ and $f'(0)$ real and positive (so that f is uniquely defined by W). W consists of the whole w-plane slit along an infinity of concentric circles of radii R_ν, $(\nu=1, 2, \cdots)$, each annular region $(R_\nu, R_{\nu+1})$ being connected by a "thin tube" to the interior of the circle of radius R_1. Every point of the boundary B of W is accessible except points on the line extending from ω to ∞. The line from ω to ∞ is an infinite prime-end with the single accessible nuclear point (Hauptpunkt)

(19) See, for example, Carathéodory [2].

ω[20]. Let $e^{i\theta_0}$ be the point corresponding to this prime-end by f^{-1}. The function $f(z)$ tends to a finite limit on every radius, the limit being, however, unbounded in the neighborhood of $e^{i\theta_0}$. We choose (successively) the radii

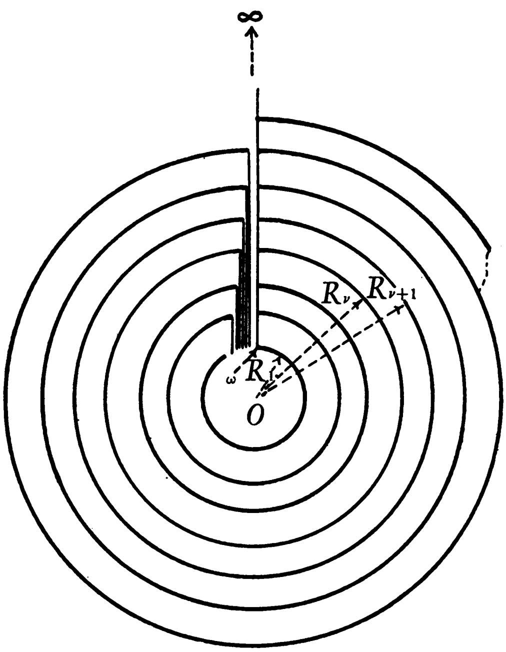

FIG. 2

(R_ν) of the figure and construct a path $P(\theta_0)$ whose transform by $f(z)$ approximates to every point of the infinite prime-end and on which (11.3) is satisfied.

We show that the radii (R_ν) can be so chosen that

$$\frac{R_\nu + R_{\nu+1}}{2} > \frac{K}{\psi(r_\nu)} \qquad (\nu = 1, 2, \cdots) \tag{11.4}$$

where r_ν is the value of r for which $B(r)$ first touches the circumference $C(\frac{1}{2}(R_\nu+R_{\nu+1}))$. Then if z_ν satisfies

$$f(z_\nu) = \frac{R_\nu + R_{\nu+1}}{2}, \qquad |z| = r_\nu,$$

we have only to connect the z_ν to obtain the desired path $P(\theta_0)$. For if $R > R_1$, the set $E(R)$ transforms by $z = f^{-1}(w)$ into a sequence of nonoverlapping cross sections $(\gamma_\nu(R))$, where $\gamma_\nu(R)$ separates z_ν from the origin if $\nu > \gamma_0(R)$. Since

(20) See Carathéodory [2].

$\gamma_\nu(R)$ converges to $e^{i\theta_0}$ as $\nu\to\infty$, z_ν tends to the same point (as $\nu\to\infty$) and

$$\psi(r_\nu)\,|\,f(z_\nu)\,| = \psi(r_\nu)\frac{R_\nu + R_{\nu+1}}{2} > K$$

by (11.4).

If, having chosen R_ν, we can choose $R_{\nu+1}$ such that (11.4) is satisfied by the function $f_\nu(z)$ which maps (with $f_\nu(0)=0$, $f_\nu'(0)>0$) the circle $|z|<1$ on the sub-domain W_ν of W shown in Fig. 3, it will follow by the subordination principle(21) that (11.4) is *a fortiori* satisfied by f uniformly in ν.

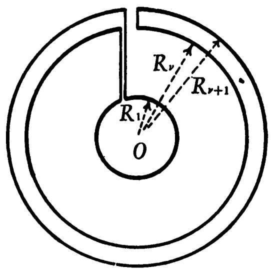

FIG. 3

To show that, for suitable choice of $R_{\nu+1}$, f_ν satisfies (11.4), we cut W_ν along a radius from 0 to a point w of its boundary on $|w|=R_1$, and map the resulting domain by means of $s_\nu(z)=\sigma+i\tau=\log w$ on a strip S_ν. Now for a parallel strip U defined by

$$\zeta(z) = \xi + i\eta, \qquad \xi_0 < \xi < \xi_1, \ |\,\eta\,| < a\pi,$$

we have

$$M(r,\zeta) \geqq \log\frac{1}{(1-r)^{2a}} + K(\xi_0) \tag{11.5}$$

if $\xi_0+A<M(r,\zeta)<\xi_1-A$. But for suitable z_0 (depending on R), the function $s_\nu(h(z))$, where

$$h(z) = \frac{z-z_0}{z\bar{z}_0-1},$$

is superordinate to a U with $\xi_0=\log R_\nu$, $\xi_1=\log R_{\nu+1}$ and $a=1-1/\xi_1$ (since we may make the angular spread of the annular region of W_ν as near to 2π as we please). Hence

$$M(r,s_\nu) \geqq M(r,s_\nu(h)) - K(R_\nu) \geqq M(r,\zeta) - K(R_\nu),$$

by subordination. If $K_1(R_\nu)<M(r,\zeta)<\log R_{\nu+1}-K_1$, this is not less than

(21) See Littlewood [6].

$$\log \frac{1}{(1-r)^2} - \frac{1}{\xi_1} \log \frac{1}{1-r} - K(R_\nu),$$

by (11.5); and is greater than or equal to

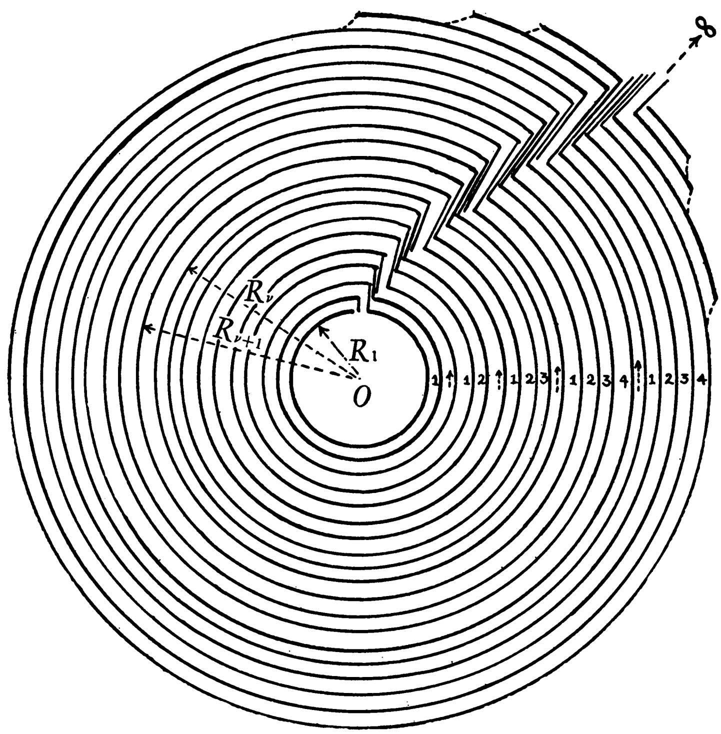

FIG. 4

$$\log \frac{1}{(1-r)^2} - K(R_\nu) \geqq \log \frac{K}{\psi(r)}$$

if r (and so $R_{\nu+1}$) is large enough. In particular,

$$\log \frac{R_\nu + R_{\nu+1}}{2} = M(r_\nu, s_\nu) \geqq \log \frac{K}{\psi(r_\nu)}$$

if $R_{\nu+1} > R_0(R_\nu)$. We can thus choose $R_{\nu+1}$ so that f_ν satisfies (11.4), and this completes the proof of Theorem 7.

12. In Theorem 8 let $f(z)$ be the function which maps $|z| < 1$ on the w-plane slit as shown in Fig. 4. The domain consists of an infinity of "tubes" (numbered as shown) connecting the circle $|w| \leqq R_1$ with ∞. If we write

$$N(n) = \sum_{\mu=1}^{n} \mu,$$

then the νth tube has an angular spread of nearly 2π over the "long" intervals (of R):

$$(12.1) \qquad R_{N(n)+\nu-1} < R < R_{N(n)+\nu} \qquad (n \geqq \nu).$$

Using the same notation as in Theorem 9, we see it is enough to show that the radii may be chosen successively in such a way that

$$\frac{R_\nu + R_{\nu+1}}{2} > \frac{K}{\psi(r_\nu)} \qquad (\nu = 1, 2, \cdots).$$

The proof is, however, now similar to that given already in the preceding section for the corresponding inequality (11.4), and I omit it.

13. I add finally a theorem of a somewhat different sort:

THEOREM 10. *Suppose that $f(z)$ is regular in $|z|<1$, and satisfies the condition that $W(R)<\infty$, $0\leqq R<\infty$. Let $E_1(\theta)$, $E_2(\theta)$ be the sets of limit points as $f(z)$ tends to $e^{i\theta}$ along two paths $P_1(\theta)$, $P_2(\theta)$ respectively. Then $E_1(\theta)\times E_2(\theta)\neq 0$*(22).

This theorem is related to a well known theorem of Lindelöf(23) which states that, if f is bounded in $|z|<1$ and tends to limits l_1, l_2, along two paths $P_1(\theta)$, $P_2(\theta)$, then $l_1=l_2$. Theorem 10 is false for bounded functions; there exist (infinitely mean valent) bounded functions such that, for at least one point $e^{i\theta}$, $E_1(\theta)\times E_2(\theta)=0$(24). On the other hand, if $F(\theta)=E_1(\theta)\times E_2(\theta)$, the hypothesis "$F(\theta)\neq 0$ for all θ" does not imply the finiteness of $W(R)$(25), so that the conditions $F(\theta)\neq 0$, $W(R)<\infty$ are not equivalent.

In proving Theorem 10 we may plainly suppose that $|f|$ is bounded on $P_1(\theta)$, $P_2(\theta)$ and that P_1, P_2 do not intersect. Then, joining P_1 to P_2 by a path Q lying inside $|z|<1$, we can map the sub-domain of $|z|<1$ bounded by P_1, P_2, and Q, onto the unit circle, the paths P_1, P_2 being transformed into two arcs, Π_1, Π_2, abutting at a point $e^{i\theta}$. Let L_1, L_2 be the transforms of Π_1, Π_2 by f, and let Λ_1, Λ_2 be the projections of L_1, L_2 on the w-plane. We suppose $E_1\times E_2=0$, and argue by *reductio ad absurdum*.

If $E_1\times E_2=0$, there exist two positive numbers δ and r_1 such that the portions of Λ_1 and Λ_2 corresponding to the arc of $|z|=1$ which lies inside a circle of radius r_1 and center $e^{i\theta}$ are separated by a distance δ. Let $c(r)$ be that arc of the circle of radius r and center $e^{i\theta}$ which lies in $|z|<1$, and let $\Gamma(r)$ be

(22) That is, E_1 and E_2 contain a common point (which may be ∞).

(23) Lindelöf [4].

(24) An example is the function f which maps the unit circle on the circle $|w|\leqq R$ covered infinitely many times, with winding point at $w=0$. There is then one point $e^{i\theta}$, and two paths P_1, P_2 converging to it, such that the transforms of P_1 and P_2 by f are concentric circles.

(25) In fact, if f maps the unit circle on a Riemann domain bounded by a "spiral" with asymptotic point $w=0$, then f tends to a limit on every path $P(\theta)$. By coiling the spiral sufficiently loosely, the sum of the areas bounded by successive loops can be made infinite.

that portion of the transform of $c(r)$ which connects Λ_1 to Λ_2. Let W_{r_1} be the simply connected domain bounded by $\Gamma(r_1), \cdots, \Gamma(r_1)$, and subsets $B_1(r_1)$, $B_2(r_1)$ of the boundary continua Λ_1 and Λ_2. Now I say no point w is covered by W_{r_1} more than a finite number of times. For, by the construction of W_{r_1}, the boundary curves $B_1(r_1)$, $B_2(r_1)$ are both simple, and $\Gamma(r_1)$ (the transform of a portion of $c(r_1)$) is analytic. Hence, if a point w were covered an infinity of times, some neighborhood of w would be covered an infinity of times, and the area of W_{r_1} would therefore be infinite, contradicting the hypothesis that $W(R) < \infty$, for finite R (since W_{r_1} is a finite domain). Similarly, if W_{r_1}' is an interior domain, the boundary of which is at distance $\delta/4$, say, from the boundary of W_r, then the valency of points of W_r' is uniformly bounded by a number K.

Finally, as $r \to 0$, $\Gamma(r)$ converges to an "end" ξ of $W(r_1)$ in the sense of Carathéodory(26). Since $W(r_1)$ is bounded and $\Lambda(r_1)$, $\Lambda(r_2)$ are separated by a distance $\delta/2$, $\Gamma(r)$ cannot converge to a point or to ∞. Therefore, ξ is not a prime-end, and so cannot correspond to a *single* point. This is a contradiction and proves the theorem.

References

1. L. Ahlfors, *Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen*, Acta Academiae Scientiarum Fennicae, vol. 1 (1930), no. 7.

2. C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

3. M L. Cartwright, *Some inequalities in the theory of functions*, Mathematische Annalen, vol. 111 (1935), pp. 98–118.

4. E. Lindelöf, *Sur un principe général de l'analyse et ses applications à la théorie de la representation conforme*, Acta Academiae Scientiarum Fennicae, vol. 46 (1915).

5. P. Montel, *Leçons sur les Familles Normales*, Paris, 1927.

6. J. E. Littlewood, *On inequalities in the theory of functions*, Proceedings of the London Mathematical Society, (2), vol. 23 (1925), pp. 481–519.

(26) See Carathéodory [2]. Carathéodory develops his theory only for schlicht functions; here we require its extension to finitely valent functions. The extension is, however, trivial.

Massachusetts Institute of Technology,
Cambridge, Mass.