TWO-TO-ONE TRANSFORMATIONS ON 2-MANIFOLDS

BY

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Introduction. An *exactly* 2-*to*-1 transformation is one for which every inverse image set consists of exactly 2 points. This notion was introduced by O. G. Harrold[1], who showed that no such continuous transformation could be defined over an arc. This result has been extended[2] to the case of the closed 2-cell. Further results concerning these transformations have been obtained by Harrold and by P. W. Gilbert[3]. The present paper is concerned with continuous 2-to-1 transformations defined on a compact 2-manifold, with or without bounding curves. The problem of the existence of such a transformation is solved, and the collection of all image spaces is determined. A precise statement of the main results is given below.

Throughout this paper the letter M will be used to denote a compact 2-manifold (absolute), or else a compact 2-manifold with boundary, the boundary consisting of a finite number of mutually exclusive simple closed curves. The set M will be considered as the whole space. T will denote some exactly 2-to-1 continuous transformation defined over M. The set of inverse images under T is[4] an upper semi-continuous collection G filling M, and every element of G is a pair of points. For each $x \, \varepsilon \, M$ let $s(x)$ be the point such that the pair $x, s(x)$ is an element of the collection G. Let $f(x) = \rho(x, s(x))$, where ρ is the metric on M. Let K denote the set of all points $x \, \varepsilon \, M$ at which f is continuous, and let L denote the subset of K consisting of those points x such that f is continuous both at x and at $s(x)$. If x is a point, then x' will denote $s(x)$; and if C is any point set, then C' will denote $s(C)$. A point set C will be called *integral* if $C' = C$.

The term *n-cell* ($n = 0, 1, 2$) will denote a closed n-cell except where the context indicates the contrary. If β denotes a closed n-cell, then β^0 will denote the open n-cell whose closure is β. If A denotes a complex, then A^* denotes

the point set covered by the complex A. If A is a complex, then $\chi(A)$ will denote $-\alpha_0+\alpha_1-\alpha_2$, where α_i is the number of i-cells in A. This is the negative of the Euler characteristic as given by Alexandroff-Hopf[5]. We assume a metric ρ on M having the following property: If a and b are points of M, then for any $\epsilon>0$ there is an arc ab whose diameter is less than $\rho(a,\ b)+\epsilon$. If σ is any metric on M, then a metric ρ with the above property can be obtained by taking $\rho(a,\ b)$ equal to the g.l.b. of the diameters of arcs joining a and b in M.

The principal results of the paper are as follows:

A necessary and sufficient condition that there be a T defined over M is that $\chi(M)$ be even. If $T(M)=B$, then $\chi(M)=2\chi(B)$. Let B_k denote a space which can be obtained from a compact manifold with k bounding curves $(k=0,\ 1,\ 2,\ \cdots)$, by the identification by pairs of a finite number of interior points of the manifold. A compact manifold M with n bounding curves $(n=0,\ 1,\ 2,\ \cdots)$, and a space B_k are said to be *properly related* if (1) $\chi(M)=2\chi(B_k)$ and (2) $\frac{1}{2}n\leqq k\leqq n$. *Given a manifold M and a space B_k, a necessary and sufficient condition that there be a T defined over M such that $T(M)=B_k$ is that M and B_k be properly related.*

These results are obtained in Part II. In order to obtain these results, it is essential to determine the nature of the discontinuities of f and the topological character of the set $M-K$. This is done in Part I.

PART I

Lemma 1. *If the point p is not on the boundary of M (i.e., if p has a 2-cell neighborhood), then p is in L if it is in K.*

Proof. If p is in K, then a sufficiently small open 2-cell containing p is mapped topologically by s into an open 2-cell containing $s(p)$. Since an open 2-cell in M is necessarily open in M, and since s has period 2, it follows that $s(p)$ is also in K, whence p is in L.

Theorem 1. *The set L is dense and open in M.*

Proof. Since K is dense and open (see *Transformations*), it follows from Lemma 1 that L is dense. If p is in L, then p and $s(p)$ are in the open set K. In view of the upper semi-continuity of the collection G it follows that if x is sufficiently close to p or to $s(p)$, then x and $s(x)$ are in K, and therefore in L. Thus L is open.

Lemma 2. *If a simple closed curve J bounds an open 2-cell U, and R is any region which contains all of J except possibly one point, then $R\cdot U$ is connected.*

Proof. Let x and y be any two points of $R\cdot U$ and let xy be an arc joining x and y in R. If $xy\cdot J=0$, then the arc xy lies in $R\cdot U$. If $xy\cdot J\neq0$, then let z

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[5] *Topologie* I, Berlin, 1935, p. 214.

and w be points on J and on the arc xy in the order $xzwy$ (possibly $z=w$) such that no point of J precedes z or follows w on the arc $xzwy$. In view of the hypothesis on J there is an arc zvw belonging to $J \cdot R$. Since finally R is open and contains zvw, there are points z_1 and w_1 on the arc xy in the order xz_1zww_1y and an arc z_1w_1 lying in $R \cdot U$. Then the arc $xz_1+z_1w_1+w_1y$ joins x and y and lies in $R \cdot U$. Since each pair x, y of $R \cdot U$ lies on an arc in $R \cdot U$, this set is connected.

LEMMA 3. *If H is a closed 2-cell in M, then H does not contain 5 arcs ad_ia' ($i=1, \cdots, 5$) such that (1) each 2 of these arcs have only their end-points a and a' in common, (2) $\sum ad_ia'$ is in L except for the points d_3, d_4, and d_5, which are not in K, (3) $d_2=d_1'$ and $s(ad_1a')=a'd_2a$, and (4) for $i=3$, 4, and 5, $s(ad_i-d_i)=a'd_i-d_i$.*

Proof. Suppose the lemma is false and there exist five arcs ad_ia' in a closed 2-cell H and properties (1), (2), (3) and (4) of the lemma hold. Now for $i=3$, 4, or 5, as $x \to d_i$ on the arc ad_i the point $s(x) \to d_i$ on the arc $a'd_i$. Let J denote the simple closed curve $ad_1a'd_2a$, and let U denote its interior with respect to H. Let R be the component of L containing J. We consider three cases.

Case 1. At least two of the points d_3, d_4, d_5 are in U. Suppose that d_3 and d_4 are in U. Then the arc ad_3a' lies in $U+a+a'$, and $U=U_1+U_2+ad_3a'-a-a'$, where U_1 and U_2 are open 2-cells bounded by $ad_1a'd_3a$ and $ad_2a'd_3a$, respectively. We suppose, without loss of generality, that d_4 is in U_2. Now let R_0, R_1, and R_2 denote respectively $R \cdot U$, $R \cdot U_1$, and $R \cdot U_2$. By Lemma 2 the sets R_0, R_1, and R_2 are connected. Furthermore, they are open subsets of the open 2-cell U. Hence $s(R_0)$, $s(R_1)$, and $s(R_2)$ are connected open sets, since s is topological over R, and an open 2-cell in M is necessarily open in M. Since $s(J)=J$, and $s(ad_ia'-d_i)=a'd_ia-d_i$ ($i=3$, 4, 5), it follows that $s(R_0)$, $s(R_1)$, and $s(R_2)$, respectively do not intersect the boundary of U, U_1 and U_2. That is, either $s(R_0) \subset U$ or else $s(R_0)$ is in $M-\overline{U}$. But the second possibility is ruled out since $s(ad_4a'-d_4)=ad_4a'-d_4$, and this set is in R_0. Therefore $s(R_0) \subset U$. Likewise $s(R_2) \subset U_2$, for d_4 is in U_2. But now let x_1, x_2, $\cdots$ be points in R_2 such that $x_n \to d_2$. Since d_2 is in L and $s(d_2)=d_1$, $s(x_n) \to d_1$. But $s(x_n) \varepsilon U_2$ and d_1 is not in $\overline{U}_2$. This is a contradiction.

Case 2. Exactly one of the points d_3, d_4, d_5 is in U. Suppose that $d_5 \varepsilon U$ and d_3 and d_4 are not in U. The sum of ad_3a' and one of the two arcs ad_1a', ad_2a' is a simple closed curve bounding a 2-cell U_1 which lies in H and contains the other of these arcs, except for end-points. We suppose that the boundary of U_1 is $ad_3a'd_2a$, and let U_2 be the 2-cell in H bounded by $ad_3a'd_1a$. To summarize, $H \supset \overline{U}_1$, and $U_1=U+U_2+ad_1a'-a-a'$. Let R_0, R_1, and R_2 denote respectively $R \cdot U$, $R \cdot U_1$, and $R \cdot U_2$. Then it follows that $s(R_0) \subset U$ and $s(R_1) \subset U_1$, since both U and U_1 contain the set $ad_5a'-a-a'$, and $s(ad_5a'-d_5)=ad_5a'-d_5$. Now let x_1, x_2, $\cdots$ be points in R_2 converging to d_1.

Then $s(x_1)$, $s(x_2)$, $\cdots$ are points in U_1 (for $R_2 \subset R_1$) converging to d_2. But for n sufficiently large $s(x_n)$ is in U, hence in R_0. But $s(R_0) \subset U$, and $s(s(x_n)) = x_n$. Thus x_n is in both U and U_2 and we have a contradiction.

Case 3. None of the points d_3, d_4, d_5 is in U. Then there is a 2-cell in H bounded by the sum of one arc from the set $\{ad_ia'\}$ ($i = 1, 2$) and one arc from the set $\{ad_ia'\}$ ($i = 3, 4, 5$), and containing one of d_1, d_2, and one of d_3, d_4, d_5. For definiteness we may suppose there are 2-cells U, U_1, and U_2 bounded respectively by curves $ad_1a'd_2a$, $ad_3a'd_2a$, and $ad_3a'd_1a$, such that d_4 is in U_2 and $U_1 = U + U_2 + ad_1a' - a - a'$. Let $R_2 = R \cdot U_2$. Since d_4 is in U_2, it follows by earlier arguments that $s(R_2) \subset U_2$. Let x_1, x_2, $\cdots$ be points in R_2 converging to d_1. Then $s(x_n) \to d_2$. But d_2 is not in $\overline{U}_2$ while $s(x_n) \subset U_2$. This contradiction completes the proof of the lemma

LEMMA 4. *Suppose q is a point, H is a closed 2-cell, V is an open set, and ϵ is a positive number, and the following properties hold:*

(1) *H contains a neighborhood of q;*

(2) *$V \supset q$;*

(3) *q is a limit point of $M - K$, and*

(4) *if $p \, \epsilon \, K \cdot V$, $f(p) < \epsilon$, and pt is any arc in V and in $K + t$, where t is not in K, then $f(x) \to 0$ as $x \to t$ on the arc pt.*

Then it follows that there exists an $\epsilon_1 > 0$ such that no arc cc' lies in $K \cdot S(q, \epsilon_1)$.

Proof. Suppose the lemma is false. Then for each positive integer n there is an arc c_nc_n' which is a subset of each of the sets H, K, V, and $S(q, 1/n)$, and for n sufficiently large the arc $s(c_nc_n')$ is in $H \cdot K \cdot V$. The set $c_nc_n' + s(c_nc_n')$ contains[6] a simple closed curve J, the sum of two arcs utu' and $u't'u$ such that $s(utu') = u't'u$. Then J bounds an open 2-cell H_1 which is a subset of H. Let R be the component of K which contains J. Then by Lemma 2 $R \cdot H_1$ is connected. We want to get a simple closed curve $ad_1a'd_2a$ lying in $R \cdot H_1$ and such that $s(ad_1a') = a'd_2a$. This will follow readily if we prove that $s(R \cdot H_1) = R \cdot H_1$, for then any point p in this set can be joined to p' by an arc pp' in this set, and some subset of $pp' + s(pp')$ will be the desired curve. If there is no point of $M - K$ in H_1, then s is topological over $\overline{H}_1$, and $s(\overline{H}_1)$ is a closed 2-cell with J for boundary. Either $s(H_1) \subset H_1$ or $s(H_1) \subset M - \overline{H}_1$. But the first possibility is ruled out, because under it s is a topological mapping of the closed 2-cell $\overline{H}_1$ into itself which has no fixed point. Under the second possibility $\overline{H}_1 + s(\overline{H}_1)$ is a sphere, hence is M. But $\overline{H}_1 + s(\overline{H}_1)$ is in K, contrary to the fact that there are points in $M - K$ (e.g., the point q). Thus there is some point of $M - K$ in H_1. Join a point p of $R \cdot H_1$ to a point t of $M - K$ in H_1 by an arc pt lying in H_1. Then $f(x) \to 0$ as $x \to t$ on the arc pt, and therefore for x near enough to t, $s(x)$ is also in H_1. Hence $s(R \cdot H_1) \subset H_1$.

Now by Lemma 1, $s(R \cdot H_1) \subset K$. It follows that $s(R \cdot H_1) = R \cdot H_1$, and the desired simple closed curve exists. That is, there is a simple closed curve

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[6] See *Transformations*, §8, for a proof.

$ad_1a'd_2a$ lying in H_1 and in K, and such that $s(ad_1a') = a'd_2a$. By Lemma 1, this curve lies in L. There is an arc joining a to some point of $M-K$, such that this arc is in $H \cdot V$, has only the point a on the simple closed curve $ad_1a'd_2a$, and has no point, except possibly an end-point, on the boundary of M. On this arc let d_3 be the first point of $M-L$. Then d_3 is in $M-K$. For if d_3 is the other end-point it is by definition in $M-K$. In the other case d_3 is not on a bounding curve of M and hence by Lemma 1 is in $M-K$ if it is in $M-L$. Then $f(x) \to 0$ as $x \to d_3$ on the arc ad_3. It follows from Theorem 3 of *Transformations*[7] that $ad_3 + s(ad_3 - d_3)$ contains two arcs, atd_3 and $a't'd_3$ which have only d_3 in common and such that $s(atd_3 - d_3) = a't'd_3 - d_3$. Denote the sum $atd_3 + d_3t'a'$ by ad_3a'.

In a similar way we obtain successively arcs ad_4a' and ad_5a' such that for $i = 4, 5$, d_i is in $M-K$ but $ad_ia' - d_i$ is in L, ad_ia' has only a and a' in common with the sum of the other four arcs ad_ja', and $s(ad_i - d_i) = a'd_i - d_i$. But then the five arcs ad_ia' ($i = 1, \cdots, 5$) have the properties stated in Lemma 3, and we have reached a contradiction.

LEMMA 5. *Suppose H is a closed 2-cell which contains a neighborhood V of a point q, pq is an arc in V and in $L+q$ such that $f(x) \to 0$ as $x \to q$ on the arc pq. Let R be the component of $L \cdot V$ which contains p, and suppose $R \cdot s(R) \cdot V = 0$. Let ϵ be any positive number. Then there exists in V an open set W with boundary J such that*

(1) *$W \supset q$ and $W+J$ is of diameter less than ϵ, and $\overline{W}$ is a closed 2-cell;*

(2) *if q is on a bounding curve of M, then J is an arc xax', where x and x' are on a bounding curve of M, a is in $M-K$, $xa-a$ is in R, and $s(xa-a) = x'a-a$;*

(3) *if q is not on a bounding curve of M, then J is a simple closed curve $axbx'a$, where a and b are in $M-K$, $axb-a-b$ is in R, and $s(axb-a-b) = ax'b-a-b$.*

Proof. The proof given for Theorem 5 of *Transformations* requires only a trivial change in order to apply here.

LEMMA 6. *Suppose q_1 is a point, H is a closed 2-cell, V is an open set and cq_1 is an arc, and the following properties hold:*

(1) *$H \supset V \supset cq_1$;*

(2) *if $x \, \epsilon \, H$, then either $f(x) < \epsilon$ or $f(x) > 3\epsilon$, where $4\epsilon = f(q_1)$;*

(3) *$c \, \epsilon \, L$ and $f(c) < \epsilon$;*

(4) *there is no arc connecting any point d to d' and lying in $V \cdot K$; and*

(5) *if ef is any arc lying in $V \cdot L+f$, where $f(e) < \epsilon$ and f is not in K, then $f(x) \to 0$ as $x \to f$ on the arc ef.*

Then there exists an arc from c to q_1 and lying in $V \cdot L+q_1$.

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(7) Theorem 3 of *Transformations* is false as stated. The proof given is based on the assumption that K is an integral set, i.e., that $s(K) = K$. Now L is an integral set, and the argument given suffices to prove the theorem as stated if K is replaced by L. In our application the the arc ad_3 lies in $L+d_3$, hence the modified theorem applies.

Proof. Let R_1 be the component of $L \cdot V$ that contains c. Then $R_1 \cdot s(R_1) = 0$. Let E denote the set of all x in $R_1 + s(R_1)$ for which $f(x) < \epsilon$. Let t be the last point of $\overline{E}$ on the arc cq_1.

Suppose that t is not accessible from R_1 by an arc ct such that $f(x) \rightarrow 0$ as $x \rightarrow t$ on ct. Then t is not accessible by any arc ut lying in $R_1 + s(R_1) + t$ and containing a point of E. Then we will show that there is an infinite sequence $c_1, c_2, c_3, \cdots$ such that (1) $c_n \, \epsilon \, R_1$ and $f(c_n) < \epsilon$, (2) $c_n \rightarrow t$ as $n \rightarrow \infty$, and (3) there is a fixed positive δ such that every arc joining c_i and c_j $(i \neq j)$ in R_1 has diameter greater than δ. To prove this assertion consider the following hypothesis:

Given any positive number β and any component R_β of $R_1 \cdot S(t, \beta)$ having t on its boundary, it is true that for every k there is some component of $R_\beta \cdot S(t, 1/k)$ which has t on its boundary. If this is true, then it follows that t is accessible from R_1 by an arc ct such that $f(x) \rightarrow 0$ as $x \rightarrow t$ along ct. But this contradicts a supposition made above. Hence the above hypothesis is false. This means that there is a positive number β such that there is an infinite set $R_1^1, R_1^2, R_1^3, \cdots$ of components of $R_1 \cdot S(t, \beta)$ such that, for every i, t is not a limit point of R_1^i, but t is a limit point of $\sum_{i=1}^{\infty} R_1^i$. And this implies that the sequence $c_1, c_2, c_3, \cdots$ exists.

There exist three open sets W_1, W_2, and W_3 containing t and bounded respectively by P_1, P_2, and P_3, these being simple closed curves or arcs[8], and there is an integer N, such that

(1) $V \supset \overline{W}_1$ and $W_i \supset \overline{W}_{i+1}$ $(i = 1, 2)$;

(2) if $n > N$, there is an arc $c_n d_n e_n$ in R_1 and in $\overline{W}_1$, where e_n and d_n are on the boundaries of W_1 and W_2, respectively, and c_n is in W_3; and

(3) if $n > N$, $m > N$, and $n \neq m$, then no component of $R_1 \cdot \overline{W}_1$ contains both c_n and c_m.

Suppose $n > N$. Let x_n and y_n be the first points of the boundary of R_1 on the circle P_2 starting from d_n in the two senses, and let $x_n d_n y_n$ denote the indicated arc of the circle P_2. Then $s(x_n d_n y_n - x_n - y_n) + x_n + y_n$ is an arc $x_n d_n' y_n$ in $s(R_1) + x_n + y_n$. Since $x_n d_n' y_n$ does not intersect $c_n d_n e_n$ (because $R_1 \cdot s(R_1) = 0$), there is a positive γ independent of n such that $d(x_n d_n' y_n) > \gamma$. But $d(x_n d_n y_n) \rightarrow 0$ as $n \rightarrow \infty$. If we drop to a subsequence, we may suppose $\limsup_{n \rightarrow \infty} x_n d_n y_n$ is a point r. Then $\limsup_{n \rightarrow \infty} x_n d_n' y_n$ is contained in $r + s(r)$. But this is clearly impossible, and we have thus proved that t is accessible by an arc ct lying in $R_1 + t$, and therefore in $V \cdot L + t$. It follows that $f(x) \rightarrow 0$ as $x \rightarrow t$ on the arc ct.

Suppose now that $t \neq q_1$. Choose $\epsilon_1 < \rho(t, q_1)$. Let W and J be sets given by Lemma 5, where t and the arc ct replace q and the arc pq, and ϵ_1 replaces ϵ. Then the arc tq must contain a point r $(r \neq t)$ on J. But J lies in $\overline{E}$, and t is the last point of $\overline{E}$ on the arc cq_1. This contradiction proves Lemma 6.

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(8) If t is on a bounding curve of M, then P_i is an arc with end-points on this bounding curve.

THEOREM 2. *If q is a point in $M-K$ and pq is an arc in $K+q$, then $f(x)\to 0$ as $x\to q$ on the arc pq.*

Proof. We suppose the theorem is false. Then there is an arc pq lying in $K+q$, q not in K, such that $f(x)\to f(q)$ as $x\to q$ on the arc pq. Then by Theorem 4 of *Transformations* there is an open set U, an arc p_1q_1, and a positive ϵ, such that

(1) $U\supset q_1$;

(2) $p_1q_1-q_1$ is in K, but q_1 is not in K;

(3) $f(x)\to f(q_1)=4\epsilon$ as $x\to q_1$ on the arc p_1q_1;

(4) if $x\,\varepsilon\,U$, then either $f(x)<\epsilon$ or $f(x)>3\epsilon$;

(5) if p_2q_2 is any arc in $U\cdot(K+q_2)$, and q_2 is not in K, and if $f(p_2)<\epsilon$, then $f(x)\to 0$ as $x\to q_2$ on the arc p_2q_2.

Let H be a closed 2-cell such that $U\supset H\supset q_1$, and H contains a neighborhood of q_1. By Lemma 4 there exists an $\epsilon_1>0$ such that if c is a point and $\rho(c,q_1)<\epsilon_1$, then there does not exist an arc of diameter less than ϵ_1 lying in K and joining c to $s(c)$. Let V denote an open set containing q_1 and lying in $H\cdot S(q_1,\epsilon_1)$. Since f is not continuous at q_1, there is a point c in $L\cdot V$ such that $f(c)<\epsilon$. There is an arc cq_1 in V. Then all the hypotheses of Lemma 6 are satisfied. Hence there is an arc cq_1 lying in $V\cdot L+q_1$. Then on this arc $f(x)\to 0$ as $x\to q_1$ (since $f(c)<\epsilon$).

Let ϵ_2 be the smaller of $\rho(p_1,q_1)$ and $\rho(c,q_1)$. Let H, V, q_1, cq_1, and ϵ_2, respectively, play the roles of H, V, p_1, cp_1, and ϵ in Lemma 5, and let J be the corresponding arc or simple closed curve having properties (2) and (3) of Lemma 5. Then J separates p_1 and q_1 and also c and q_1. Then for $x\,\varepsilon\,J$, $x\,\varepsilon\,K$ we have $f(x)<\epsilon$. But for x on p_1q_1 we have $f(x)>3\epsilon$. This is a contradiction, and the theorem is proved.

THEOREM 3. *If q is a limit point of $M-K$, then there is a positive number ϵ_1 such that there does not exist, for any point c, an arc joining c to $s(c)$ lying in K and in $S(q,\epsilon_1)$.*

Proof. Let H denote a closed 2-cell in M which contains a neighborhood of q. Let V be any open set containing q and let ϵ be any positive number. Then with the help of Theorem 2 it follows immediately that q, H, V, and ϵ have the properties stated in the hypothesis of Lemma 4. The number ϵ_1 given in the conclusion of Lemma 4 has the required property.

THEOREM 4. *If R is a component of K and q is on the boundary of R, then q is arc-wise accessible from R.*

Proof. If q is not a limit point of $M-K$, the result is obvious. If q is a limit point of $M-K$, then it is possible, with the help of Theorems 2 and 3, to define H, V, cq, having properties as stated in the hypothesis of Lemma 6, with the additional hypothesis that $c\,\varepsilon\,R$. Then the arc cq given by Lemma 6 will lie in $R+q$.

THEOREM 5. *The set L is identical with K.*

Proof. It is sufficient to show that if p is in K, then $s(p)$ is in K. This result was shown in Lemma 1 except for the case where p is on a bounding curve of M. Suppose then that p is in K and on a bounding curve of M. There is a simple closed curve J bounding an open 2-cell H, such that (1) J is the sum of an arc apb on that bounding curve of M which contains p, and an arc aqb having only a and b on the boundary of M, and (2) $\overline{H} \cdot s(\overline{H}) = 0$. Then the transformation s is topological over $\overline{H}$, whence $s(\overline{H})$ is a closed 2-cell having $s(q)$ on its boundary. Let cp be an arc in $H+p$. Then $s(cp)$ is an arc $c'p'$ in $s(H)+p'$ and by Lemma 1, $s(H)$ is in L, hence in K. Suppose that p' is not in K. Then, by Theorem 2, $f(x) \to 0$ as $x \to p'$ on the arc $c'p'$. But $f(x) = f(s(x))$, and as $x \to p'$ on the arc $c'p'$ the point $s(x) \to p$ on the arc cp, and $f(s(x)) \to f(p) \neq 0$. This contradiction shows that p' is in K.

LEMMA 7. *If an integral subset of K is the sum of two mutually separated connected sets R_1 and R_2, then either $s(R_1) = R_2$ or else $s(R_1) = R_1$.*

This lemma follows immediately from the facts that the continuous image of a connected set is connected, and that s is of period 2; i.e., that $s(s(A)) = A$.

THEOREM 6. *The set K has at most two components. If it has two components R_1 and R_2, then $s(R_1) = R_2$, and R_1 and R_2 have the same boundary.*

Proof. Let R_1 be a component of K. Then $s(R_1)$ is also a component of K. If q is on the boundary of R_1, then there is an arc pq in R_1+q. On this arc $f(x) \to 0$ as $x \to q$ (Theorem 2). Hence $s(pq-q)+q$ is an arc $p'q$ in $s(R_1)+q$, and therefore q is on the boundary of $s(R_1)$. Similarly, every boundary point of $s(R_1)$ is on the boundary of R_1. The proof will be completed by showing that $\overline{R}_1+s(R_1) = M$.

Let $N = \overline{R}_1+s(R_1)$, and suppose that there exists a point t in $M-N$. There is an arc tq having only q in the closed point set N. Clearly q is on the boundary of R_1. Furthermore, q is a limit point of $M-K$, for if it were an isolated point in this set then some point of R_1 could be joined to t by an arc not hitting the boundary of R_1. Then we can apply Lemmas 4 and 5 and get a closed point set J which separates t from q, and such that J is a subset of $R_1+s(R_1)$, except possibly for one or two points on the boundary of R_1. Then $J \cdot tq = 0$, since q is the only point of N on tq. This contradicts the fact that J separates t and q.

LEMMA 8. *If p is any point of $M-K$, then for every $\epsilon > 0$ there is an open set $W \supset p$ such that*
   (1) *$\overline{W}$ is a closed 2-cell of diameter less than ϵ;*
   (2) *J, the boundary of W with respect to M, is a simple closed curve or an arc with both end-points on a single bounding curve of M; and*
   (3) *$J \cdot (M-K)$ consists of 0, 1, or 2 points.*

If p is not a limit point of $M-K$, then the result is obvious. If p is a limit point of $M-K$, then the proof results from an application of Lemmas 4 and 5. The sets W and J given by Lemma 5 have the desired properties.

THEOREM 7 *Let N be a component of $M-K$. Then N is a point, an arc, or a simple closed curve, and no point of N is a limit point of $M-K-N$.*

Proof. It follows from Lemma 8 that every point of the closed and compact set $M-K$ is of Menger order 0, 1, or 2 with respect to this set. Hence each component of $M-K$ is a point, an arc, or a simple closed curve. Let q be a point of a component N of $M-K$, and suppose that q is a limit point of $M-K-N$. We can apply Lemma 5 and get an open set W, with boundary J, having properties (1), (2), and (3) of Lemma 5, and the property that no arc cc' exists in $K \cdot \overline{W}$, for any point c. We suppose that the set $\overline{W} \cdot (M-K)$ $= T_1 + T_2$, mutually separated sets. Then there exists a point u such that u and u' are in K and in $\overline{W}^0$ (the open 2-cell whose closure is $\overline{W}$). Then there exist two arcs aub and $au'b$ such that (1) a and b are in T_1 and T_2, respectively, (2) $s(aub-a-b)=au'b-a-b$, and (3) $(aub+au'b)-a-b$ is in $K \cdot \overline{W}^0$. Then $aub+au'b$ is a simple closed curve J_1 bounding an open 2-cell W_1. Neither of the mutually exclusive closed sets $T_1 \cdot \overline{W}_1$ and $T_2 \cdot \overline{W}_1$ separates u from u' in $\overline{W}_1$, but their sum does. But this is impossible. Hence $(M-K) \cdot \overline{W}$ is connected. Since it contains q it is in N. Then q is not a limit point of $M-K-N$.

THEOREM 8. *If a component N of $M-K$ is an arc, then each of its end-points, but no other point, is on a boundary curve of M; if N is a point or a simple closed curve, then no point of N is on any boundary curve of M.*

Proof. If p is an end-point of an arc N which is a component of $M-K$, then by Lemma 4 $M-K$ locally separates M at p, and hence by Theorem 7 N locally separates M at p. But this is impossible if p has an open 2-cell neighborhood in M. Hence p is on a bounding curve of M.

Let p be a point of order 2 on some component N of $M-K$, and let apb be an arc which is a subset of N. There is an arc aqb having only a and b in $M-K$ and such that the simple closed curve $aqb+apb$ bounds an open 2-cell U which is in K. Then it follows that $U+s(U)+apb$ contains an open 2-cell containing p.

In a similar way it can be shown that if the point p is a component of $M-K$, then it lies in an open 2-cell.

<div align="center">PART II</div>

LEMMA 1. *If* $\lim \sup A_n = A$, *then* $\lim \sup s(A_n) \subset A + s(A)$.

Proof. Let p be a point of $\lim \sup s(A_n)$. Then there exists a sequence of points $\{p_n\}$, $p_n \ \varepsilon \ s(A_n)$ $(n=1, 2, \cdots)$ such that p is a sequential limit point of $\{p_n\}$. Since G is upper semi-continuous, the sequence $\{s(p_n)\}$, or some sub-

quence, has as limit point either p or $s(p)$. But clearly $s(p_n) \varepsilon A_n$, since $p_n \varepsilon s(A_n)$. Hence if p is a limit point of $\{s(p_n)\}$, then $p \varepsilon \lim \sup A_n$. If $s(p)$ is a limit point of $\{s(p_n)\}$, $s(p) \varepsilon \lim \sup A_n = A$ and $p \varepsilon s(A)$. In either case the lemma is proved.

Lemma 2. *Given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* A *is a point set of diameter less than* δ, *then* $s(A)$ *is the sum of two sets, each of diameter less than* ϵ.

Proof. Suppose the lemma is false. Then there exists an $\epsilon > 0$ and a sequence of sets, $\{A_n\}$, such that $d(A_n) < 1/n$, but for no n can $s(A_n)$ be expressed as the sum of two sets, each of diameter less than ϵ. Let $\{A_{n_i}\}$ be a subsequence of $\{A_n\}$ such that $\lim \sup A_{n_i} = \lim \inf A_{n_i} = p$, a point. Then by Lemma 1, $\lim \sup s(A_{n_i}) \subset p + s(p)$. But then clearly, for n_i large enough, $s(A_{n_i}) \subset s(p, \epsilon/2) + s(s(p), \epsilon/2)$, and thus $s(A_{n_i})$ is the sum of two sets, each of diameter less than ϵ. This contradiction proves the lemma.

We now have as a corollary of Lemma 2 the following:

Lemma 3. *Given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $d(A) < \delta$ *and* $s(A)$ *is connected, then* $d(s(A)) < \epsilon$.

Lemma 4. *If* p *is a point of* $M - K$ *and if* $q_1 p$, $q_2 p$, $q_3 p$ *and* $q_4 p$ *are arcs in* $K + p$ *with only the point* p *in common and such that* $s(q_1 p - p) = q_2 p - p$ *and* $s(q_3 p - p) = q_4 p - p$, *then these arcs have the cyclic order* $q_1 p$, $q_3 p$, $q_2 p$, $q_4 p$ *about* p.

Proof. Suppose the lemma is false. Choose $\epsilon > 0$ so that the sphere $S(p, 4\epsilon)$ is an open 2-cell which contains no q_i ($i = 1, 2, 3, 4$) and intersects no component of $M - K$ other than the one to which p belongs. Then for every n we can find a point b_n of $q_1 p$ and an arc $b_n b_n'$ in K such that $\rho(b_n, p) < \min(1/n, \epsilon)$ and $d(b_n b_n') < 1/n$ and $b_n b_n' \cdot \sum_{i=1}^{4} q_i p = b_n + b_n'$.

Now $b_n b_n' + s(b_n b_n')$ is a connected integral set, and hence by Lemma 3, for $n > n_0$, $d(b_n b_n' + s(b_n b_n')) < \epsilon$. Now $b_n b_n' + s(b_n b_n')$ contains a simple closed curve J_n, which is an integral subset of K[9]. But since, for $n > n_0$, $J_n \subset S(p, 4\epsilon)$ and $J_n \cdot \sum_{i=1}^{4} q_i p \subset b_n + b_n'$, it follows that J_n contains neither p nor any other point of $M - K$ in its interior. Now let R be the component of K which contains J_n, and let I_n be the interior of J_n. Since $s(R) \cdot R \neq 0$, it follows from Theorem 6 of Part I that $s(R) = R$. For n sufficiently large, it is easy to see that $R - (J_n + I_n)$ is connected. Now we cannot have $s(J_n + I_n) = J_n + I_n$, for that would contradict the principal theorem of *Transformations*. Hence by Lemma 7 of Part I, $s(I_n) = R - (J_n + I_n)$. But there is a positive number ϵ_1 such that, for $n > n_1$, $d(R - (J_n + I_n)) > \epsilon_1$, and hence, by Lemma 3, a corresponding positive number δ exists such that $d(I_n) > \delta$, for every $n > n_1$. But $\lim d(J_n) = 0$, and since M is compact it follows that $\lim d(I_n) = 0$. This contradiction proves the lemma.

We are now in a position to prove

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[9] See the argument early in §8 of *Transformations* proving the existence of a simple closed curve.

THEOREM 1. *If T is a 2-to-1 transformation defined over M, then M can be so triangulated that the image under s of every n-cell α of the triangulation is an n-cell of the triangulation different from α ($n = 0, 1, 2$). Hence $\chi(M)$ is even, and $\chi(T(M)) = \chi(M)/2$.*

Proof. It follows from Theorem 5 of Part I that $M - K$ is an integral set. Also, $M - K$ is a compact metric space, and consists of a finite number of simple closed curves $J_1, J_2, \cdots, J_{n_1}$, of isolated points $p_1, p_2, \cdots, p_{n_2}$, and of arcs $v_1, v_2, \cdots, v_{n_3}$ whose end-points lie on the bounding curves of M; and each bounding curve of M contains either two or no points of $M - K$. (This follows easily from Theorems 7 and 8 of Part I.) Moreover, from Theorem 5 of Part I, if p is an isolated point of $M - K$, then $s(p)$ is also an isolated point of $M - K$. Hence T is a 2-to-1 transformation defined over $M - K$, and the point set K_1, over which s is continuous relative to $M - K$, is open and dense in $M - K$. In view of this it can be shown[10] that $K_1 \supset \sum_{i=1}^{n_2} p_i + \sum_{i=1}^{n_3} v_i$, and that J_i contains either no point or exactly two points, u_{1i} and u_{2i}, of $M - K - K_1$.

Now let m be a positive number such that the distance between any two components of $M - K$ is greater than m, and the distance between any two points of $M - K - K_1$ is greater than m. We now choose four positive numbers $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$ with the following properties:

   (1) $4\epsilon_1 < m$;

   (2) if B and $s(B)$ are connected sets and if $d(B) < \epsilon_2$, then $d(s(B)) < \epsilon_1$ (see Lemma 3);

   (3) any simple closed curve of M of diameter less than ϵ_3 bounds a 2-cell of M of diameter less than ϵ_2[11]; and

   (4) if A and $s(A)$ are connected sets, and if $d(A) < \epsilon_4$, then $d(s(A)) < \epsilon_3$ (see Lemma 3).

Now consider an isolated point p_j of $M - K$. Let q_1 be a point of K such that $\rho(q_1, p_j) < \epsilon_4$ and let γ_1 be an arc of diameter less than ϵ_4 from q_1 to p_j. Let γ_2 be the arc $p_j + s(\gamma_1 - p_j)$. Let q_3 be a point of K such that $\rho(p_j, q_3) < \epsilon_4$, and $q_3 \notin \gamma_1 + \gamma_2$. Let γ_3 be an arc of diameter less than ϵ_4 from q_3 to p_j such that $\gamma_3 \cdot (\gamma_1 + \gamma_2) = p_j$. Let γ_4 be the arc $p_j + s(\gamma_3 - p_j)$. Now by Lemma 4 these arcs have the cyclic order, $\gamma_1, \gamma_3, \gamma_2, \gamma_4$, about p. We now find two points, $r_1 \in \gamma_1$ and $r_3 \in \gamma_3$, and two arcs, β_1 from r_1 to r_3, and β_2 from r_3 to r_1' with the following properties:

   (1) $(\sum_{i=1}^4 \gamma_i)(\beta_1 + \beta_2 + \beta_1' + \beta_2') = r_1 + r_3 + r_1' + r_3'$; and

   (2) if $r_1' = r_2$ and $r_3' = r_4$, and if δ_i denotes the subarc of γ_i from p_j to r_i ($i = 1, 2, 3, 4$), then $d(\beta_1 + \delta_1 + \delta_3) < \epsilon_4$ and $d(\beta_3 + \delta_3 + \delta_2) < \epsilon_4$. It follows from the definition of ϵ_3 and the fact that $\epsilon_4 < \epsilon_3$, that $\beta_1 + \delta_1 + \delta_3$ and $\beta_3 + \delta_3 + \delta_2$ are simple closed curves which are the boundaries of closed 2-cells, λ_1 and λ_3,

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[10] K_1 will be used to denote this set throughout the rest of the paper. The proofs of the statements of this sentence, while not trivial, are sufficiently straightforward to be omitted.

[11] It is easy to see that, since M is compact, ϵ_3 can be chosen to satisfy this property.

respectively. It is easy to see that $s(\beta_1)$ is an arc β_2 from r_2 to r_4, and $s(\beta_3)$ is an arc β_4 from r_4 to r_1; and further, that $s(\lambda_1^0)$ is an open 2-cell λ_2^0 bounded by $\beta_2 + \delta_2 + \delta_4$, and $s(\lambda_3^0)$ is an open 2-cell λ_4^0 bounded by $\beta_4 + \delta_4 + \delta_1$. Thus the neighborhood of p_j has been triangulated in accordance with the theorem.

We suppose that this has been done for every p_j ($j = 1, 2, \cdots, n_1$). In a similar manner we triangulate the neighborhood of each of the simple closed curves $J_1, J_2, \cdots, J_{n_1}$ and each of the arcs $v_1, v_2, \cdots, v_{n_3}$ in such a way that these simple closed curves and arcs appear in the triangulation as sums of 1-cells and vertices of the triangulation which map under s into 1-cells and vertices of the triangulation. We take care, also, to make every point of $M - K - K_1$ a vertex of the triangulation.

We now let A_1 denote the complex which is composed of all these neighborhoods of the components of $M - K$ so triangulated, and let $H = M - A_1^*$. Then $\overline{H}$ is a closed and compact point set over which f is continuous and positive. Hence there is a positive number ψ such that $f(x) > 2\psi$ if $x \, \varepsilon \, \overline{H}$. Now we choose three positive numbers $\psi_1, \psi_2,$ and ψ_3 as follows: (1) if γ is a simple closed curve of diameter less than $4\psi_1$, then γ bounds a 2-cell of diameter less than ψ (this implies $4\psi_1 < \psi$); and (2) if B and $s(B)$ are connected sets, then if $d(B) < \psi_2$, it follows that $d(s(B)) < \psi_1$; and if $d(B) < \psi_3$, it follows that $d(s(B)) < \psi_2$. (See Lemma 3. These conditions imply $\psi_1 > \psi_2 > \psi_3$.) If A_1 contains a 1-cell α such that $d(\alpha) \geq \psi_3$, let A_2 be a subdivision of A_1 containing no such 1-cell but still having the property that the image under s of an n-cell of A_2 is an n-cell of A_2 ($n = 0, 1, 2$). If A_1 contains no such 1-cell, $A_2 = A_1$.

Now it may be possible to find either one, two, or three arcs whose end-points are vertices of A_2 but which otherwise lie in $M - A_2^*$ and which, together with two, one, or no arcs, respectively, of A_2^*, bound a 2-cell ϕ of diameter less than ψ whose interior lies in $M - A_2^*$, but not in $S(A_2^*, \psi_3)$. If such a possibility exists, we add to A_2 this 2-cell ϕ and also $s(\phi)$. Since $d(\phi) < \psi$, $\phi \cdot s(\phi) = 0$. After extending A_2 in this manner as many times as possible, successively, we call the extended complex A_3.

If $M - A_3^*$ contains an open 2-cell α^0 such that the boundary of α^0 consists of three vertices and three 1-cells of A_3 and $s(\alpha^0) \cdot \alpha^0 = 0$, then we add α^0 and $s(\alpha^0)$ to A_3. After adding all such 2-cells to A_3, we call the new complex A_4. We then obtain A_5 from A_4 by subdivision, in the same way that we obtained A_2 from A_1.

Now let $r_1, r_2, \cdots, r_k$ denote the vertices of A_5 which are on the boundary of $M - A_5^*$. Let r_{k+1} be a point of $M - A_5^*$ for which $\psi_3 < \rho(r_{k+1}, A_5^*) < \psi_2$. Join r_{k+1} to two points, r_{i_2} and r_{i_3}, which are end-points of the same 1-cell α_1 of A_5 by arcs α_2 and α_3 in such a way that (1) $\alpha_2 \cdot \alpha_3 = r_{k+1}$, (2) $\alpha_2 \cdot A_5^* = r_{i_2}$ and $\alpha_3 \cdot A_5^* = r_{i_3}$, and (3) $d(\alpha_i) < 2\psi_1$ ($i = 1, 2$). To see that this is possible, we draw an arc β of diameter less than ψ_2[12], from r_{k+1} to some point p in the interior

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[12] This is possible since the metric ρ which we are using has the property that if H is closed and $\rho(x, H) < \epsilon$, then there exists an arc from x to a point of H of diameter less than ϵ.

of a 1-cell α_1 on the boundary of A_5^*. Since no 1-cell of A_5 has diameter as great as ψ_3, we can draw an arc in $M - A_5^*$ from r_{k+1}, running along very close to β and then running along close to α_1 until we get near an end-point r_{i_2} of α_1. Then we run our arc into r_{i_2} and call the arc α_2. Similarly, on the other side of β, we draw α_3 from r_{k+1} to r_{i_3}. If in drawing these arcs we stay close enough to β and then to α_1, neither α_2 nor α_3 can have diameter greater than $\psi_2 + \psi_3$, which is less than $2\psi_1$. Moreover, $\alpha_1 + \alpha_2 + \alpha_3$ is a simple closed curve of diameter less than $4\psi_1$, and hence by definition of ψ_1 bounds a 2-cell η of diameter less than ψ. It follows that $s(\eta)$ is a 2-cell such that $\eta \cdot s(\eta) = 0$ and $s(\eta) \cdot A_5 = s(\alpha_1)$. We add η and $s(\eta)$ to A_5 and call the resulting complex A_6.

We now begin over again, obtaining A_7 from A_6 in the same way that we obtained A_2 from A_1, etc. It is clear that the method of extension from A_5 to A_6 can be carried out only a finite number of times. For otherwise M would contain an infinite point set with no limit point, since we require that the new vertex have a distance greater than ψ_3 from the point set covered by the complex. But when this method of extension cannot be repeated that means that the complex we have, A_n, has the property that there is no point of $M - A_n^*$ which has a distance greater than ψ_3 from A_n^*. When this is the case, it is easy to see that a finite number of applications of the methods of extension from A_2 to A_3 and from A_3 to A_4 will give a complex which covers M. Hence the theorem is proved.

LEMMA 5. *Let p be an interior point of M. If p is neither an isolated point of $M - K$, nor a point of $M - K - K_1$, then $T(p)$ has a 2-cell neighborhood in $T(M)$. Otherwise $T(p)$ has a neighborhood in $T(M)$ which is homeomorphic to a neighborhood of the vertex of the double cone $x^2 = y^2 + z^2$.*

Hereafter when we speak of a manifold with identifications we shall always mean by "identifications" a finite number of points with neighborhoods homeomorphic to a neighborhood of the vertex of the double cone.

Proof. If p belongs to K, the result follows immediately; for s is locally a homeomorphism at every point of K, and if U is an open subset of K such that $U \cdot s(U) = 0$, then $s(U)$ and $T(U)$ are homeomorphic. But this means that $T(U)$ contains an open set which is an open 2-cell containing $T(p)$.

If p belongs to $(M - K) \cdot K_1$, but is not an isolated point of $M - K$, then p belongs to an arc or a simple closed curve of $M - K$. Let c denote the arc or the simple closed curve. By triangulating the neighborhood of p and of $s(p)$ in the same way in which they were triangulated in the proof of Theorem 1, and taking care to have neither p nor $s(p)$ be a vertex in this triangulation, we obtain mutually exclusive open 2-cells V and W containing p and $s(p)$, respectively, and having the following properties:

    (1) $V - c = V_1 + V_2$, mutually separated open 2-cells such that $s(V_1) = V_2$;

    (2) $V \cdot c$ is an open 1-cell lying in K_1;

    (3) $V \cdot (M - K) = V \cdot c$;

(4) $W \cdot s(V \cdot c) = W \cdot (M - K)$;

(5) $W - s(V \cdot c) = W_1 + W_2$, mutually separated open 2-cells such that $s(W_1) = W_2$.

It follows that $s(V \cdot c)$ is an open 1-cell, and that $T(V \cdot c) = T(s(V \cdot c))$ is an open 1-cell. Likewise, each of the sets $T(V)$ and $T(W)$ is homeomorphic to the intersection of the interior of the unit sphere in the Euclidean plane with the half-plane $x \geqq 0$. Moreover, $T(V) \cdot T(W) = T(V \cdot c)$, an open 1-cell. Hence $T(V + W)$ is an open 2-cell containing $T(p)$. Furthermore, it is clear that $T(V + W)$ is an open set in $T(M)$, since $V + W$ is an integral open set in M.

If p is an isolated point of $M - K$, then, as we remarked in the proof of Theorem 1, $s(p)$ is also. We triangulate the neighborhood of p and of $s(p)$ as we did in the proof of Theorem 1. From the description of that triangulation, it is easy to verify that if $\pi(p)$ denotes the open 2-cell containing p whose closure is $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ (see the proof of Theorem 1 for the meaning of λ_i), then $T(\pi(p))$ is an open 2-cell. Similarly for $\pi(p')$. But $T(\pi(p)) \cdot T(\pi(p')) = T(p)$. Hence $T(\pi(p) + \pi(p'))$ is homeomorphic to a neighborhood of the vertex of the double cone $x^2 = y^2 + z^2$, and since $\pi(p) + \pi(p')$ is an integral open set in M, $T(\pi(p) + \pi(p'))$ is open in $T(M)$, and the lemma follows for this case.

The case in which p belongs to $M - K - K_1$ is handled in a somewhat similar manner.

LEMMA 6. *Let p be on one of the boundary curves of M. Then $T(p)$ has a neighborhood in $T(M)$ which is homeomorphic to a neighborhood of p in M.*

The proof follows in much the same way that the proof of the preceding lemma followed.

THEOREM 2. *If M is a compact manifold, then $T(M)$ is a compact manifold or can be obtained from a compact manifold by the identification by pairs of a finite number of points. If M is a compact manifold with boundary, then $T(M)$ is a compact manifold with boundary or can be obtained from a compact manifold with boundary by the identification by pairs of a finite number of interior points.*

Proof. The first statement follows as a corollary of Lemma 5. To prove the second statement, we first show that if c is a boundary curve of M, then $T(c)$ is a simple closed curve. For suppose first that $K \not\supset c$. Then by Theorem 8 of Part I, $c \cdot K \neq 0$. Let p be a point of $c \cdot K$, and let pq be a subarc of c which lies in $K + q$ but not in K. Then $p'q = s(pq - p) + q$ is a subarc of c, by virtue of Theorem 2 of Part I and the fact that the image under s of a boundary point of M is a boundary point of M. Let pr be a subarc of c such that $pr \cdot pq = p$ and $K + r \supset pr$ but $K \not\supset pr$. (It is easy to see that there must be a point $r \neq q$ such that $r \in c \cdot (M - K)$. For otherwise, as a variable point x

moved continuously along $c - pq - p'q + p$ from p, $s(x)$ would move continuously along $c - pq - p'q + p'$ and a two-to-one transformation would be defined on the arc $c - pq - p'q + p + p'$, in contradiction to the result of O. G. Harrold([1]).) Then as above, $p'r = s(pr - r) + r$ is a subarc of c. Then clearly $c = pq + p'q + pr + p'r$, and a direct argument shows that $s(q) = r$. Hence in this case $T(c)$ is a simple closed curve.

Now suppose that $K \supset c$. Then $s(c)$ is a simple closed curve which is a boundary curve of M, and hence either $s(c) = c$ or else $s(c) \cdot c = 0$. In either case $T(c)$ is a simple closed curve.

Now by combining the fact that $T(c)$ is a simple closed curve with Lemma 5 and Lemma 6, we see that the second statement of our theorem is proved.

Moreover, it is easy to see that we have also proved

LEMMA 7. *If M has n boundary curves, then $T(M)$ has k boundary curves, where $n/2 \leq k \leq n$.*

We now prove

THEOREM 3. *Given a space M and a space B_k, a necessary and sufficient condition that there exist a 2-to-1 transformation T such that $T(M) = B_k$ is that M and B_k be properly related. (See the introduction.)*

Proof. The necessity follows immediately from Theorems 1 and 2 and Lemmas 5, 6, and 7. The sufficiency will be proved by actually constructing the transformation T.

Let M be an orientable manifold, and let M be embedded in Euclidean 3-space in such a way that M is symmetric with respect to the xy-plane, and the common part of M and the xy-plane consists of h simple closed curves $c_1, c_2, \cdots, c_h$ if $\chi(M) = 2(h - 2)$. If p is a point of M not in the xy-plane, we define $s(p)$ as the reflection of p in the xy-plane. If B_k is an orientable manifold, then h is even and we let s map c_1 into c_2, c_3 into c_4, $\cdots$, c_{h-1} into c_h topologically. When s is defined, T is determined, and the theorem is proved for this case. If B_k is a non-orientable manifold, we define s exactly as before except that s maps c_h into itself by identifying diametrically opposite points, in case h is odd, and it maps both c_h and c_{h-1} into themselves in this manner if h is even. If B_k is an orientable manifold with b identifications, then $h - b$ is even and non-negative. For if a manifold N has $\chi(N) = v$, and if N_1 is obtained from N by identifying b pairs of points, then $\chi(N_1) = v + b$. Hence we define $s(p)$ as before for a point p not in the xy-plane, and we let x map c_i $(i = 1, 2, \cdots, b)$ into itself by a reflection in a diameter and the identification of the two points of c_i which lie on the diameter. And we let s map c_{b+1} into c_{b+2}, c_{b+3} into c_{b+4}, $\cdots$, c_{h-1} into c_h, topologically. If B_k is a non-orientable manifold with b identifications, s is defined as in the preceding case except that if $h - b$ is odd, then s maps c_h into itself by identifying diametrically

opposite points, and if $h - b$ is even then s maps both c_{h-1} and c_h into themselves in that manner. This completes the cases in which M is an orientable manifold.

If M is a non-orientable manifold, and $\chi(M)$ is even, then either M is a Klein's bottle, in which case $\chi(M) = 0$, or else M can be obtained from a Klein's bottle by inserting h handles, in which case $\chi(M) = 2h$. If M is a Klein's bottle, let M be triangulated as in the figure. The numbers in this figure refer to the vertices by which they are placed and the letters denote the
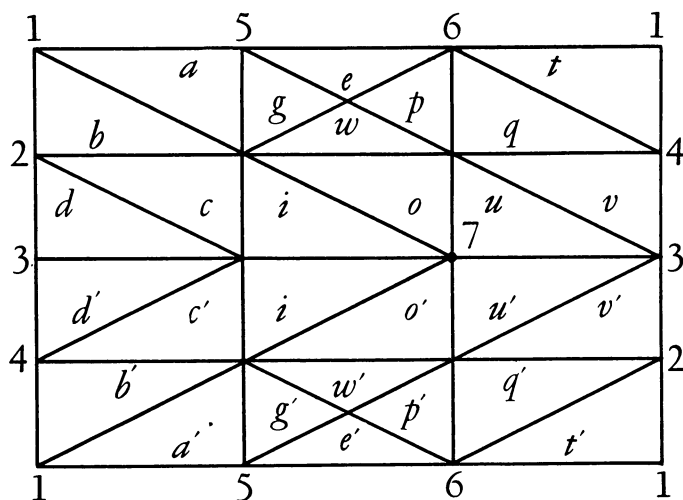
2-cells in which they are placed. The top edge of the figure is identified with the bottom edge, and the points of the left edge reading up are identified with those of the right edge reading down, as the numbers indicate. Let s map the 2-cells above the horizontal bisector 373 into those below 373 by reflection in 373, as the primes indicate. This defines s for every point except those points of the horizontal bisector 373 and of the edge 1561; we call these two simple closed curves c_1 and c_2. If B_k is a torus, we let s map c_1 into c_2 topologically; if B_k is a projective plane with one identification we let s map c_1 into itself by identifying diametrically opposite points and c_2 into itself by identifying points by reflection in a diameter and identifying the points of c_2 on the diameter; if B_k is a Klein's bottle, we let s map c_i into itself by identification of diametrically opposite points $(i = 1, 2)$; if B_k is a sphere with two identifications, we let s map c_i into itself by reflection in a diameter and identification of the points of c_i on the diameter $(i = 1, 2)$.

If M is non-orientable and $\chi(M) = 2h > 0$, then M can be constructed as follows: Let N_1 be the manifold with boundary obtained from the figure by deleting the open 2-cells w and w', and let s map $N_1 - (c_1 + c_2)$ into itself in the manner described above. Let N_2 be an orientable manifold for which

$\chi(N_2) = 2h - 4$. Let N_2 be embedded in Euclidean 3-space so that it is symmetric with respect to the xy-plane and the common part of N_2 and the xy-plane consists of h simple closed curves. Let s map the points of N_2 above the xy-plane into those below the xy-plane by reflection in that plane. Let α be an open 2-cell in N_2 lying above the xy-plane and having a simple closed curve as boundary, and let $\alpha' = s(\alpha)$. Delete α and α', and identify the boundaries of α and α' with the boundaries of w and w', respectively. Then we have defined s over M except for $h+2$ simple closed curves. We define s over these simple closed curves, using one of, or a combination of, the three methods already described for defining s over simple closed curves, depending on the character of the image space B_k.

The definition of T in the cases in which M is a manifold with bounding curves is analogous to its definition in the cases already treated.

Conclusion. It is known that in some cases a space M may be mapped into a space B in a continuous, exactly 2-to-1 fashion, in at least two essentially different ways. For example, a sphere may be mapped into a projective plane (1) by identifying diametrically opposite points, or (2) by identifying point pairs which are symmetrical with respect to the equatorial plane, and then identifying diametrically opposite points on the equator. It is easy to show that the two mappings so defined are not topologically equivalent[13]. The following problem naturally arises: *For a given M and B how many topologically different 2-to-1 continuous mappings of M into B are there?* It seems very likely that this number is finite.

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[13] For a definition of this term see G. T. Whyburn, *Interior transformations on compact sets*, Duke Mathematical Journal, vol. 3 (1937), p. 373, footnote 8.

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