ON THE DEGREE OF POLYNOMIAL APPROXIMATION TO ANALYTIC FUNCTIONS: PROBLEM β

BY

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1. **Introduction.** Given a closed bounded point set C of the z-plane whose complement K is connected and possesses a Green's function $G(x, y)$ with pole at infinity; denote generically by C_ρ the locus $G(x, y) = \log \rho$, $1 < \rho$, in K. By Problem β we understand the following problem: If a function $f(z)$ is assumed analytic interior to a particular C_ρ, and possesses given continuity properties on or in the neighborhood of C_ρ, to study the degree of approximation by polynomials to $f(z)$ on C in the sense of Tchebycheff.

This problem has reached a fairly satisfactory solution in case $f(z)$ has generalized derivatives of various orders on C_ρ(1), and in case $f(z)$ is continuous on and within C_ρ, and its pth derivative satisfies a Lipschitz condition of order α on C_ρ(2); in the latter case, if C is bounded by a finite number of smooth mutually exterior Jordan curves, it follows (loc. cit.) that polynomials $p_n(z)$ of respective degrees n exist such that

$$|f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha}, \qquad z \text{ on } C, \tag{1}$$

where M is a constant depending on C and ρ but independent of n and z.

However, if $f(z)$ is not assumed continuous on C_ρ but merely to become infinite (if at all) sufficiently slowly, a result closely analogous to (1) exists:

$$|f(z) - p_n(z)| \leqq M n^{p+\alpha}/\rho^n, \qquad z \text{ on } C, \tag{2}$$

where $p+\alpha$ is again positive and is a measure of the rapidity with which $f(z)$ becomes infinite. Such a result has already been considered by S. Bernstein [1926] and de la Vallée Poussin [1919] for the case that C is a segment of the axis of reals, and *provided* $f(z)$ *has only isolated singularities on* C_ρ. The primary object of the present paper is to establish (2) for more general point sets C (especially when C is the closed interior of an analytic Jordan curve) and for functions $f(z)$ not required to have only isolated singularities on C_ρ.

To be more explicit, we define (§2) a hierarchy of functions, thanks to certain theorems due to Hardy and Littlewood [1932], which includes both functions whose derivatives satisfy Lipschitz conditions of various orders and functions satisfying asymptotic inequalities in the neighborhood of C_ρ. This classification of functions is highly appropriate for our present discussion, for

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(1) Sewell [1937]; numbers in brackets refer to the bibliography at the end of this paper.

(2) Walsh and Sewell [1940].

it is primarily based on (i) Lipschitz or asymptotic conditions for the functions, but is so constituted that (ii) integrals and derivatives of functions of a class belong automatically in specified new classes, likewise defined in terms of Lipschitz or asymptotic conditions; also (iii) each class implies a specific degree of approximation (Problem β), and conversely (iv) certain definite degrees of approximation imply that the function belongs to a uniquely determined class; in each case under (i) to (iv) the results are in a sense the best possible.

We study these questions of approximation (§§3, 4, 5) for the unit circle, (§8) for the line segment, annulus, and real axis, and (§7) for point sets which are the closed interiors of analytic Jordan curves. In §6 we consider approximation to functions with isolated singularities. We indicate (§9) the method of extending the above results on Tchebycheff approximation to approximation measured by a line integral. In §10 we consider the relation between integrated Lipschitz conditions and integral asymptotic conditions on the one hand and degree of approximation on the other hand. Finally in §11 we present more immediate but less thoroughgoing methods for obtaining portions of our results.

The methods and results here set forth have application to the study of approximation of harmonic functions by harmonic polynomials, an application which the writers plan to make on another occasion.

Henceforth in the present paper the degree of a polynomial is indicated consistently by its subscript; moreover the letter M with or without subscripts when used in an inequality of type (1) or (2) shall always represent a constant which may vary from inequality to inequality and depends on C and ρ but which is always independent of n and z.

2. **A classification of functions.** In the present section, the unit circle $|z|=1$ is denoted by γ. If the function $f(z)$ is analytic interior to γ, continuous in the corresponding closed region, and if $f^{(p)}(z)$, where $p \geqq 0$ is an integer, satisfies a Lipschitz condition on γ of order α, $0<\alpha\leqq 1$, we say that $f(z)$ is *of class* $L(p, \alpha)$ *on* γ. It is immaterial here whether we require that $f^{(p)}(z)$ and the Lipschitz condition should be one-dimensional or two-dimensional; compare Hardy and Littlewood [1932], Walsh and Sewell [1940]. It obviously follows that if $f(z)$ is of class $L(p, \alpha)$ on γ then the indefinite integral of $f(z)$ is of class $L(p+1, \alpha)$ on γ and (provided $p>0$) the derivative $f'(z)$ is of class $L(p-1, \alpha)$ on γ. In this connection it is appropriate to consider the following theorem due to Hardy and Littlewood [1932]:

THEOREM 2.1. *A necessary and sufficient condition that* $f(z)$, *analytic for* $|z|<1$, *should belong to class* $L(0, \alpha)$, $0<\alpha\leqq 1$, *is that*

$$|f'(re^{i\theta})| \leqq M(1-r)^{\alpha-1}, \qquad r<1, \tag{2.1}$$

where $z=re^{i\theta}$ *and where* M *is independent of* r *and* θ.

This theorem suggests a new definition: If the function $f(z)$ is analytic for $|z| < 1$ and if we have

$$|f(re^{i\theta})| \leqq M(1 - r)^{\alpha+p}, \qquad r < 1, 0 < \alpha \leqq 1, \tag{2.2}$$

where $p < 0$ is an integer, where $z = re^{i\theta}$, and where M is independent of r and θ, then $f(z)$ is said to be *of class* $L(p, \alpha)$ *on* γ. With this terminology we prove

THEOREM 2.2. *If the function $f(z)$ is of class $L(p, \alpha)$ on γ, $0 < \alpha \leqq 1$, then the indefinite integral of $f(z)$ is of class $L(p+1, \alpha)$ on γ unless $\alpha + p = -1$, and the derivative $f'(z)$ is of class $L(p-1, \alpha)$ on γ.*

We set

$$F(z) = \int_0^z f(z)dz; \tag{2.3}$$

our conclusion concerning $F(z)$ for $p \geqq 0$ has already been mentioned, and for $p = -1$ follows from Theorem 2.1. For $p < -1$ we take the path of integration in (2.3) a radius, which involves no loss of generality:

$$F(re^{i\theta}) = \int_0^r f(re^{i\theta})dr,$$

where θ is fixed. We have by (2.2)

$$|F(re^{i\theta})| \leqq M \int_0^r (1 - r)^{\alpha+p}dr \leqq M'[(1 - r)^{\alpha+(p+1)} - 1], \tag{2.4}$$

from which our conclusion on $F(z)$ (and on any indefinite integral of $f(z)$) follows.

In the case $p > 0$ the conclusion of Theorem 2.2 concerning $f'(z)$ has already been mentioned, and this conclusion for $p = 0$ follows from Theorem 2.1. Suppose now $p < 0$, so that $p + \alpha \leqq 0$. Let z be fixed interior to γ. We choose $\rho = \frac{1}{2}(1 - |z|)$ and study the integral

$$f'(z) = \frac{1}{2\pi i} \int_{|t-z|=\rho} \frac{f(t)}{(t - z)^2} dt. \tag{2.5}$$

On the path of integration we have (2.2) satisfied, whence

$$|f'(z)| \leqq \frac{2M[1 - |z| - \rho]^{\alpha+p}}{1 - |z|} \leqq M_1(1 - |z|)^{\alpha+p-1}, \tag{2.6}$$

as we were to prove. Theorem 2.2 is established.

It will be noticed that the proof of (2.4) fails in the case $\alpha + p = -1$, that is to say, in the case $p = -2$, $\alpha = 1$. In this connection it is useful to introduce a new definition, namely that $f(z)$ *shall be of class* $L'(p, 1)$, $p \geqq -1$, *provided $f(z)$ is analytic interior to γ, and provided $f^{(p+2)}(z)$ is of class $L(-2, 1)$.*

We make the following observation:

THEOREM 2.3. *If $f(z)$ is analytic and uniformly bounded interior to γ, then $f(z)$ is of class $L'(-1, 1)$.*

As above we use equation (2.5), where $\rho=(1-|z|)/2$; then we have

$$|f'(z)| \leqq \frac{M}{2\pi}\frac{2\pi\rho}{\rho^2} = 2M(1-|z|)^{-1};$$

thus $f'(z)$ is of class $L(-2, 1)$, so the theorem follows.

We shall now establish

THEOREM 2.4. *If $f(z)$ is of class $L'(p, 1)$, $p>-1$, then $f'(z)$ is of class $L'(p-1, 1)$; moreover $f^{(p+2+k)}(z)$, where k is a positive integer, is of class $L(-1-k, 1)$.*

Also, if $f(z)$ is of class $L'(p, 1)$ then for r near unity we have

$$|f^{(p+1)}(re^{i\theta})| \leqq M|\log(1-r)|, \tag{2.7}$$

and on the radii $f^{(p)}(z)$ satisfies the pseudo-Lipschitz condition for r near unity

$$|f^{(p)}(e^{i\theta}) - f^{(p)}(re^{i\theta})| \leqq M'(1-r)|\log(1-r)|, \tag{2.8}$$

where M' is independent of θ; under these conditions the qth integral of $f(z)$ is of class $L'(p+q, 1)$, $q>0$.

If $f(z)$ is of class $L'(p, 1)$, with $p>-1$, we have by definition $|f^{(p+2)}(z)| \leqq M(1-r)^{-1}$; but $f^{(p+2)}(z)$ is the derivative of order $p+1$ of $f'(z)$ and hence also by the definition of class $L'(p, 1)$, the function $f'(z)$ is of class $L'(p-1, 1)$. Furthermore it is clear from the proof of (2.6) that $|f^{(p+2+k)}(z)| \leqq M(1-r)^{-1-k}$, k a positive integer, and hence $f^{(p+2+k)}(z)$ is of class $L(-2-k, 1)$ by definition.

If $f(z)$ is of class $L'(p, 1)$, an inequality on $f^{(p+1)}(re^{i\theta})$ follows directly from the inequality on $f^{(p+2)}(re^{i\theta})$:

$$\begin{aligned}|f^{(p+1)}(re^{i\theta}) - f^{(p+1)}(0)| &= \left|\int_0^r f^{(p+2)}(re^{i\theta})dr\right| \\ &\leqq M\left|\int_0^r \frac{dr}{1-r}\right| = M|\log(1-r)|,\end{aligned}$$

and since $f^{(p+1)}(0)$ is a constant we have the inequality of the theorem. The function $f^{(p)}(z)$ can be defined on the boundary as the integral of its derivative and the pseudo-Lipschitz condition (2.8) is an immediate consequence of the integration of (2.7) from r to 1 along an arbitrary radius. The remark about the qth integral follows from the definition and the fact that the derivative of an indefinite integral is the function itself under the above conditions.

The uniform pseudo-Lipschitz condition (2.8) on the radii, for functions of class $L'(0, 1)$, implies a similar condition on the circumference:

COROLLARY. *If $f(z)$ is of class $L'(0, 1)$, then $f(z)$ is continuous in the two-dimensional sense on $|z|=1$, and satisfies on that circumference a uniform pseudo-Lipschitz condition of the form*

$$|f(e^{i\theta}) - f(e^{i\theta'})| \leqq M_1 |\log |\theta - \theta'|| \cdot |\theta - \theta'|, \tag{2.9}$$

where $|\theta-\theta'|$ is sufficiently small.

Let θ and θ' be given, $|\theta-\theta'|<1$. By (2.8) we have

$$|f(e^{i\theta}) - f(re^{i\theta})| \leqq M'(1-r)|\log (1-r)|,$$

$$|f(e^{i\theta'}) - f(re^{i\theta'})| \leqq M'(1-r)|\log (1-r)|.$$

Also by (2.7) we have $|f'(re^{i\theta})| \leqq M|\log (1-r)|$,

$$|f(re^{i\theta}) - f(re^{i\theta'})| \leqq M|\log (1-r)| \cdot |\theta - \theta'|.$$

The choice $1-r=|\theta-\theta'|$ now yields (2.9).

3. **Degree of approximation, unit circle.** We now present a proof of the following theorem, which connects the class $L(p, \alpha)$ whether p is positive, negative, or zero with degree of approximation:

THEOREM 3.1. *If $f(z)$ belongs to class $L(p, \alpha)$, $0<\alpha\leqq 1$, on γ: $|z|=1$, then there exist polynomials $p_n(z)$ such that we have on the circle $|z|=1/\rho<1$*

$$|f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha}. \tag{3.1}$$

For the case $p\geqq 0$, Theorem 3.1 has already been established [Walsh and Sewell, 1940]; a new proof is given below, Theorem 10.5, second proof. For the case $p<0$ we proceed as follows. The formula

$$f(z) - \sum_{m=0}^{n} a_m z^m = \frac{1}{2\pi i}\int_{|t|=r<1} \frac{z^{n+1}f(t)}{t^{n+1}(t-z)}\,dt, \qquad |z| < r,$$

where $f(z)=\sum_{m=0}^{\infty} a_m z^m$, $|z|<1$, is well known. Thus we obtain

$$\left| f(z) - \sum_{m=0}^{n} a_m z^m \right| \leqq \frac{M}{2\pi}\left(\frac{1}{\rho}\right)^{n+1} \frac{(1-r)^{p+\alpha} 2\pi r}{r^{n+1}(r-1/\rho)}, \qquad |z| \leqq 1/\rho < r.$$

If we let $r_n=1-1/n$, we have for n sufficiently large

$$\left| f(z) - \sum_{m=0}^{n} a_m z^m \right| \leqq \frac{M_1(1/n)^{p+\alpha}}{\rho^{n+1}(1-1/n)^{n+1}} < \frac{M_2}{\rho^n n^{p+\alpha}}, \qquad |z| \leqq 1/\rho,$$

since $(1-1/n)^n$ approaches $1/e$ as n becomes infinite. For a suitably chosen constant M_2 this inequality is valid for all n, $n=1, 2, \cdots$, and the proof of the theorem is complete.

By way of complement to Theorem 3.1 we state the following theorem, whose proof is postponed until §4:

THEOREM 3.2. *If $f(z)$ belongs to class $L'(p, 1)$, $p \geqq -1$, then there exist polynomials $p_n(z)$ such that we have on the circle $|z| = 1/\rho < 1$*

$$|f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+1}. \tag{3.2}$$

Of course the hypothesis of Theorem 3.2 is less restrictive than that of Theorem 3.1 in the case $\alpha = 1$, $p \geqq -1$.

4. **Operations with approximating sequences.** It is our object in the present section to show how certain assumptions on a function imply immediate results on degree of approximation by polynomials to the various derivatives and integrals of that function.

THEOREM 4.1. *Let $f'(z)$ be of class $L(p, \alpha)$, $p \leqq -1$, $0 < \alpha \leqq 1$. Let $p_n'(z)$ denote the sum of the first $n+1$ terms of the Taylor development of $f'(z)$. Then we have for $|z| = 1/\rho < 1$*

$$\left| \int_0^z [f'(z) - p_n'(z)]dz \right| \leqq M/\rho^n \cdot n^{p+\alpha+1}.$$

We have the usual formula

$$f'(z) - p_n'(z) = \frac{1}{2\pi i} \int_{|t|=r} \frac{z^{n+1}f'(t)}{t^{n+1}(t-z)} dt, \qquad |z| \leqq 1/\rho < r < 1,$$

and hence

$$\int_0^z [f'(z) - p_n'(z)]dz = \frac{1}{2\pi i} \int_{|t|=r} \frac{f'(t)}{t^{n+1}} dt \int_0^z \frac{z^{n+1}}{t-z} dz,$$

where for simplicity the path of integration is chosen along a radius. But for $|t| > |z|$ we have

$$\int_0^z \frac{z^{n+1}}{t-z} dz = \frac{1}{t}\left[\frac{z^{n+2}}{n+2} + \frac{z^{n+3}}{(n+3)t} + \cdots\right];$$

for $|t| = r$ and $|z| \leqq 1/\rho$ the modulus of this function is dominated by

$$\frac{1}{n\rho^n}\left[\frac{1}{1 - 1/r\rho}\right].$$

Thus by the method employed in the proof of Theorem 3.1 with $r_n = 1 - 1/n$, we obtain the inequality

$$\left| \int_0^z [f'(z) - p_n'(z)]dz \right| \leqq M_1/\rho^n n^{p+\alpha+1}, \qquad |z| \leqq 1/\rho,$$

and the proof of the theorem is complete.

Theorem 4.1 is stated merely for the first integral of a function of class

$L(p, \alpha)$, but obviously extends to the iterated indefinite integrals of every order. Theorem 4.1 thus yields a new proof of Theorem 3.1 for the case $p \geqq 0$, and furnishes a proof of Theorem 3.2, which was not proved previously.

Another theorem relating to integration of approximating sequences (and which extends to iterated integrals of arbitrary order) is

THEOREM 4.2. *Let $f(z)$ be analytic interior to γ and continuous on γ. Let there exist polynomials $P_n(z)$ such that we have*

$$|f(z) - P_n(z)| \leqq \epsilon_n, \qquad z \text{ on } \gamma.$$

Let $p_n(z)$ denote the sum of the first $n+1$ terms of the Taylor development of $f(z)$. Then for $|z| = 1/\rho < 1$ we have

$$\left| \int_0^z [f(z) - p_n(z)] dz \right| \leqq M\epsilon_n/n \cdot \rho^n.$$

Theorem 4.2 admits of a relatively simple proof, but is to be reconsidered later (§7), and hence is not established in detail here. It may be noted that Theorem 4.2 with its extension to higher integrals yields by a transformation $z' = \sigma z$ a new proof of Theorem 3.1 for the case $p > 0$ by virtue of Theorem 3.1 itself for the case $p = 0$.

In connection with the differentiation of approximating sequences we also have two results analogous to Theorems 4.1 and 4.2:

THEOREM 4.3. *Let $f(z)$ be of class $L(p, \alpha)$, $p \leqq -1$, $0 < \alpha \leqq 1$. Let $p_n(z)$ denote the sum of the first $n+1$ terms of the Taylor development of $f(z)$. Then we have for $|z| = 1/\rho < 1$*

$$|f'(z) - p_n'(z)| \leqq M/\rho^n \cdot n^{p+\alpha-1}.$$

Theorem 4.3 can be proved by the method used for Theorem 4.1, and is in a sense to be generalized later as well (Theorem 7.9).

THEOREM 4.4. *Let $f(z)$ be analytic interior to γ: $|z| < 1$, and continuous on γ. Let there exist polynomials $P_n(z)$ such that we have*

$$|f(z) - P_n(z)| \leqq \epsilon_n, \qquad z \text{ on } \gamma.$$

Let $p_n(z)$ denote the sum of the first $n+1$ terms of the Taylor development of $f(z)$. Then for $|z| = 1/\rho < 1$ we have

$$|f'(z) - p_n'(z)| \leqq Mn\epsilon_n/\rho^n.$$

The proof of Theorem 4.4 is likewise postponed (compare Theorem 7.10 below). Both Theorem 4.3 and Theorem 4.4 extend at once to higher derivatives.

5. **Inverse problem. Examples.** In the direction of a converse to Theorem 3.1 we establish

THEOREM 5.1. *Let there exist polynomials $p_n(z)$ such that we have*

$$|f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha+1}, \qquad |z| = 1/\rho < 1, \tag{5.1}$$

where p is integral and $0<\alpha\leqq 1$. Then $f(z)$, if properly extended analytically from the circle $|z| = 1/\rho$, belongs to class $L(p, \alpha)$ on $|z| = 1$ if $p+\alpha+1$ is not a positive integer, and to class $L'(p, \alpha)$ if $p+\alpha+1$ is a positive integer.

Theorem 5.1 has already been established for the case $p\geqq 0$, $\alpha<1$ (Walsh and Sewell [1940]). From (5.1) we may now write

$$|f(z) - p_{n+1}(z)| \leqq M/\rho^{n+1} \cdot (n+1)^{p+\alpha+1}, \qquad |z| = 1/\rho,$$

whence also by (5.1) we have, whether $p+\alpha+1$ is positive or nonpositive,

$$|p_{n+1}(z) - p_n(z)| \leqq 2M_0/\rho^n \cdot n^{p+\alpha+1}, \qquad |z| = 1/\rho. \tag{5.2}$$

The extended Bernstein Lemma (e.g. Walsh [1935, p. 77]) then yields

$$|p_{n+1}(z) - p_n(z)| \leqq 2M_0\rho r^{n+1}/n^{p+\alpha+1}, \qquad |z| = r > 1/\rho. \tag{5.3}$$

We define $f(z)$ in the region $1/\rho < |z| < 1$ by means of the convergent sequence $p_n(z)$, so from (5.1) we see that $f(z)$ is analytic throughout the region $|z| < 1$. On the circle $|z| = r < 1$, $r > 1/\rho$, we can write

$$f(z) = p_1(z) + [p_2(z) - p_1(z)] + [p_3(z) - p_2(z)] + \cdots,$$

$$|f(z)| \leqq M_1 \sum_{n=1}^{\infty} r^n/n^{p+\alpha+1}. \tag{5.4}$$

If $p+\alpha<0$ we write $q = p+\alpha+1$,

$$\sum_{n=2}^{\infty} r^n/n^q \leqq \int_0^\infty r^x x^{-q} dx = \int_0^\infty e^{x \log r} x^{-q} dx$$
$$= \Gamma(1-q)(-\log r)^{q-1} \leqq M_2(1-r)^{q-1};$$

thus we have for $|z| = r < 1$

$$|f(z)| \leqq M(1-r)^{p+\alpha},$$

so the conclusion follows unless $p+\alpha+1$ is a positive integer.

If now $p+\alpha+1$ is a positive integer, we write from (5.1) by the least-square property of the Taylor development of $f(z) \equiv \sum_{k=0}^{\infty} a_k z^k$,

$$\frac{M^2}{\rho^{2n} n^{2(p+\alpha+1)}} \geqq \frac{\rho}{2\pi} \int_{|z|=1/\rho} |f(z) - p_n(z)|^2 |dz|$$
$$\geqq \frac{\rho}{2\pi} \int_{|z|=1/\rho} \left| f(z) - \sum_{k=0}^{n} a_k z^k \right|^2 |dz| = \sum_{n+1}^{\infty} |a_k|^2/\rho^{2k};$$

it follows that we have for every n

$$|a_n| \leqq M_1/n^{p+\alpha+1}, \qquad |n^{p+\alpha+1}a_n| \leqq M_1,$$

$$|(n+p+\alpha)(n+p+\alpha-1)\cdots(n+1)na_n| \leqq M_2.$$

As in the use of (5.4) for $p+\alpha<0$ it follows now that $f^{(p+\alpha+1)}(z)$ is of class $L(-2, 1)$, hence that $f(z)$ is of class $L'(p, \alpha)$; Theorem 5.1 is established.

It will be noticed that there is a discrepancy of unity in the exponents of n in Theorems 3.1 and 5.1, in such a way that those theorems are not exact converses of each other. *This discrepancy is inherent in the nature of the problem*, as we shall show by examples. Such examples have already been provided [Walsh and Sewell, 1940] for the case $p \geqq 0$; we consider now the case $p<0$.

Let $p<0$ be given, and also α, $0<\alpha<1$. If for every function of class $L(p, \alpha)$ we could establish the existence of polynomials $p_n(z)$ with

$$|f(z) - p_n(z)| \leqq \epsilon_n, \qquad |z| = 1/\rho < 1, \tag{5.5}$$

where

$$\lim_{n\to\infty} \rho^n \cdot n^{p+\alpha}\epsilon_n = 0, \tag{5.6}$$

we should have by virtue of the least-square property of the Taylor development

$$\frac{2\pi\epsilon_n^2}{\rho} \geqq \int_{|z|=1/\rho} |f(z) - p_n(z)|^2 \cdot |dz| \geqq \int_{|z|=1/\rho} |f(z) - s_n(z)|^2 \cdot |dz|,$$

where $s_n(z)$ is the sum of the first $n+1$ terms of the Taylor development $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$. Let $F^{(p)}(z)$ denote the $(-p)$th indefinite integral of $f(z)$, where the constants of integration at the origin are chosen to vanish:

$$F^{(p)}(z) = \sum_{n=0}^{\infty} \frac{a_n z^{n-p}}{(n+1)(n+2)\cdots(n-p)}.$$

Thus we have

$$\begin{aligned}
&\int_{|z|=1/\rho} \left| F^{(p)}(z) - \sum_{k=0}^{n} \frac{a_k z^{k-p}}{(k+1)(k+2)\cdots(k-p)} \right|^2 |dz| \\
&\qquad = \frac{2\pi}{\rho} \sum_{k=n+1}^{\infty} \frac{|a_k|^2}{\rho^{2k-2p}(k+1)^2(k+2)^2\cdots(k-p)^2} \\
&\qquad \leqq \frac{2\pi M}{\rho n^{-2p}} \sum_{k=n+1}^{\infty} \frac{|a_k|^2}{\rho^{2k}} \\
&\qquad = \frac{M}{n^{-2p}} \int_{|z|=1/\rho} |f(z) - s_n(z)|^2 \cdot |dz| \leqq \frac{2\pi M \epsilon_n^2}{\rho n^{-2p}}.
\end{aligned} \tag{5.7}$$

The extreme members of (5.7) form an inequality valid for an *arbitrary* function $F^{(p)}(z)$ of class $L(0, \alpha)$, an inequality which taken together with (5.6) has already been shown (loc. cit.) to be impossible.

The reasoning just given does not apply to the case $\alpha=1$, but for this case we can establish a less precise result. We shall show that it is not possible to prove for every function of class $L(p, 1)$ the existence of polynomials $p_n(z)$ such that we have (5.5) valid with

$$\epsilon_n \leqq M/\rho^n \cdot n^{p+1+\delta}, \qquad \delta > 0. \tag{5.8}$$

If (5.8) could be proved for every function $f(z)$ of class $L(p, 1)$, inequality (5.8) could be proved for every function of class $L(p+1, \delta_1)$, $0<\delta_1<\delta$, which is necessarily also of class $L(p, 1)$; but we have just proved that (5.5) and (5.8) cannot be established for all functions of the class $L(p+1, \delta_1)$; this remark completes our proof that Theorem 3.1 cannot be essentially improved, in the sense that for arbitrary p and α the exponent of n in the second member of (3.1) can be replaced by no larger number.

We show now that Theorem 5.1 cannot be improved, in the sense that in (5.1) the exponent of n in the second member can be replaced by no smaller number. Let p and α be given, $0<\alpha<1$. We choose the function

$$f(z) \equiv (1-z)^{p+\alpha} \equiv \sum a_n z^n,$$

from which there follows (e.g., de la Vallée Poussin [1914, §399])

$$|a_n| \leqq M/n^{p+\alpha+1}. \tag{5.9}$$

Thus we have

$$\left| f(z) - \sum_{k=0}^{n} a_k z^k \right| \leqq \sum_{k=n+1}^{\infty} M/\rho^k \cdot k^{p+\alpha+1}, \qquad |z| = 1/\rho.$$

Since $\rho^{k/2} \cdot k^{p+\alpha+1}$ increases with k, for k sufficiently large, we find for the last sum the bound

$$M\rho^{-n/2} \cdot n^{-p-\alpha-1} \sum_{k=n+1}^{\infty} \rho^{-k/2} = M_1/\rho^n \cdot n^{p+\alpha+1}.$$

That is to say, we have exhibited a function $f(z)$ of class $L(p, \alpha)$ *and of no higher class* for which (5.1) holds; thus for arbitrary p and $0<\alpha<1$ the exponent of n in (5.1) cannot be decreased without altering the conclusion of Theorem 5.1; this conclusion applies, by supplementary reasoning similar to that used in connection with (5.8), also for arbitrary p with $\alpha=1$.

6. **Degree of approximation—isolated singularities.** In the preceding section (§5) it was shown that if $f(z)=(1-z)^{p+\alpha}=\sum_{m=0}^{\infty} a_m z^m$ then

$$\left| f(z) - \sum_{m=0}^{n} a_m z^m \right| \leqq M/\rho^n \cdot n^{p+\alpha+1}, \qquad |z| \leqq 1/\rho, \tag{6.1}$$

a higher degree of approximation than is asserted in Theorem 3.1; for $f(z)$ is of class $L(p, \alpha)$, unless $p+\alpha$ is a non-negative integer, and is of no higher class. Functions with isolated singularities are thus of particular interest in the study of the degree of approximation; this section is devoted to an investigation of such functions.

We state a generalization of the above conclusion:

THEOREM 6.1. *Let*

$$\begin{aligned} f(z) &= F_1(z) + F_2(z) + \cdots + F_\sigma(z) \\ &\quad + k_1(z - z_1)^{h_1} + \cdots + k_\mu(z - z_\mu)^{h_\mu}, \qquad |z_j| = 1, j = 1, \cdots, \mu, \end{aligned}$$

where $F_i(z)$, $i=1, \cdots, \sigma$, is of class $L(p_i, \alpha_i)$ or $L'(p_i, \alpha_i)$. Let $H_i = p_i+\alpha_i$ and $h=\min(h_j-1, H_i)$. Then there exist polynomials $p_n(z)$ such that

$$|f(z) - p_n(z)| \leqq M/\rho^n \cdot n^h, \qquad |z| = 1/\rho.$$

The proof simply consists in applying Theorem 3.1 and the conclusion (6.1), and is left to the reader. If here $f(z) = \sum a_n z^n$, we have $|a_n| \leqq M_1/n^h$.

In a similar way we obtain for the special function $f(z) = \log(1-z)$, which is of class $L'(-1, 1)$, a stronger result than that of Theorem 3.2:

THEOREM 6.2. *Let $f(z) = \log(1-z) = \sum_{m=1}^\infty z^m/m$. Then we have*

$$\left| f(z) - \sum_{m=1}^{n} z^m/m \right| \leqq M/\rho^n \cdot n, \qquad |z| = 1/\rho < 1.$$

These theorems can be extended to somewhat more general functions by means of certain inequalities concerning multiplication of series. [See, e.g., Hardy and Littlewood, 1935.] If $f(z) = \sum_{m=0}^\infty a_m z^m$, $g(z) = \sum_{m=0}^\infty b_m z^m$, $f(z)g(z) = \sum_{m=0}^\infty c_m z^m$, and if $|a_m| \leqq M_1/r^m \cdot m^h$, $|b_m| \leqq M_2/\rho^m$, $1 \leqq r < \rho$, then $|c_m| \leqq M/r^m m^h$. Thus we have

THEOREM 6.3. *Under the hypothesis of Theorem* 6.1 *or* 6.2, *the conclusion is valid if the function $f(z)$ is replaced by the product $f(z)g(z)$, where $g(z)$ is analytic in $|z| \leqq 1$.*

We also have in the above notation as a consequence of the inequalities $|a_m| \leqq M_1/m^h$, $|b_m| \leqq M_2/m^l$, the following inequalities:

$$\begin{aligned} |c_m| &\leqq M_3/m^{h+l-1}, && 1 > h \geqq l, \\ |c_m| &\leqq M_3/m^h, && l \geqq h > 1,\ l > h \geqq 1, \\ |c_m| &\leqq M_3 \log m/m^{h+l-1}, && 1 = h \geqq l. \end{aligned}$$

Thus we have

THEOREM 6.4. *Let $f(z) = (z-z_1)^{h_1-1}(z-z_2)^{h_2-1} = \sum_{m=0}^\infty a_m z^m$, $|z_1| = |z_2| = 1$, $z_1 \neq z_2$. Then we have*

$$\left| f(z) - \sum_{m=0}^{n} a_m z^m \right| \leqq \epsilon_n, \qquad |z| = 1/\rho,$$

where

$$\epsilon_n = M/\rho^n \cdot n^{h_1+h_2-1}, \qquad 1 > h_1 \geqq h_2;$$
$$\epsilon_n = M/\rho^n \cdot n^{h_1}, \qquad h_2 \geqq h_1 > 1, \text{ or } h_2 > h_1 \geqq 1;$$
$$\epsilon_n = M \log n/\rho^n \cdot n^{h_1+h_2-1}, \qquad 1 = h_1 \geqq h_2.$$

The extension of Theorem 6.4 to functions of the type $\prod_{k=1}^{n}(z-z_k)^{h_k-1}$ is immediate; details can be supplied by the reader. Also, Theorems 6.1–6.4 extend with identical conclusions to approximation on an arbitrary analytic Jordan curve, by replacing the Taylor development by the Faber [1920] development of the function.

7. **Extensions to more general regions.** It is obvious that much of the preceding discussion can be applied to the study of approximation on point sets more general than circles; we proceed to discuss some of the details of this extension. Broadly considered, the extension (for instance Theorems 7.5 7.9) applies to Jordan curves which are required to be smooth but not necessarily analytic; however, some of the following methods of proof (Theorems 7.7 and 7.8) apply only to analytic Jordan curves, so for simplicity we restrict ourselves to that case.

The reader may notice that some of the following treatment (e.g., Theorems 7.5, 7.9, 7.10, 7.11) applies also to approximation on point sets which are not connected but are bounded by disjoint analytic Jordan curves, provided C_ρ has no multiple points.

DEFINITION. *Let* Γ *be an analytic Jordan curve in the z-plane. Let the interior of* Γ *be mapped conformally onto the interior of* γ: $|w| = 1$, *by the transformation* $w = \Phi(z)$, $z = \Psi(w)$. *The function* $f(z)$ *analytic interior to* Γ *is said to be of class* $L(p, \alpha)$ *on* Γ *if the function* $f[\Psi(w)]$ *(suitably defined on* γ *if necessary) is of class* $L(p, \alpha)$ *on* γ, *where* $0 < \alpha \leqq 1$ *and* p *is an integer, positive, negative, or zero.*

Thanks to the analyticity of the Jordan curves considered, and of the consequent continuity of the derivatives of the mapping functions in the closed regions, the following theorems are immediate consequences of the discussion of §2:

THEOREM 7.1. *If the function* $f(z)$ *is continuous on and within the analytic Jordan curve* Γ, *then a necessary and sufficient condition that* $f(z)$ *be of class* $L(p, \alpha)$ *on* Γ *with* $p \geqq 0$ *is that* $f^{(p)}(z)$ *satisfy on* Γ *a Lipschitz condition of order* α.

THEOREM 7.2. *Let* Γ *be an analytic Jordan curve, and let* $\Gamma(\rho)$ *be a sequence of analytic Jordan curves interior to* Γ *defined for all values of* ρ *in an interval* $\rho_0 \leqq \rho < \rho_1$ *by an equation of the form*

$$(7.1) \qquad \Gamma(\rho): \quad |F(z)| = \rho,$$

where $F(z)$ is analytic on Γ, with $F'(z)$ different from zero on Γ, and with the property $|F(z)| = \rho_1$ on Γ. Then a necessary and sufficient condition that a function $f(z)$ be of class $L(p, \alpha)$ on Γ with $p < 0$ is

$$(7.2) \qquad |f(z)| \leqq N(\rho_1 - \rho)^{p+\alpha}, \qquad z \text{ on } \Gamma(\rho),$$

where N is independent of z and ρ.

The property expressed by (7.1) and (7.2) is independent of the particular analytic function $F(z)$ considered.

THEOREM 7.3. *If the function $f(z)$ is of class $L(p, \alpha)$ on the analytic Jordan curve Γ, $0 < \alpha < 1$, then the indefinite integral of $f(z)$ is of class $L(p+1, \alpha)$ on Γ, and the derivative $f'(z)$ is of class $L(p-1, \alpha)$ on Γ.*

The proof is easy and is left to the reader.

The class $L'(p, 1)$, $p \geqq -1$ on Γ is defined as the transform of the class $L'(p, 1)$ on γ. Analogous to Theorem 7.3 we have

THEOREM 7.4. *If the function $f(z)$ is analytic and bounded interior to Γ, it is of class $L'(-1, 1)$ on Γ.*

If $f(z)$ is of class $L'(p, 1)$, $p > -1$, on Γ, then $f'(z)$ is of class $L'(p-1, 1)$ on Γ; moreover $f^{(p+2+k)}(z)$, $k > 0$, is of class $L(-2-k, 1)$ on Γ; the qth integral of $f(z)$ is of class $L'(p+q, 1)$.

Theorems 7.3 and 7.4 are the respective extensions of Theorems 2.2 and 2.3 together with 2.4. Likewise the study of degree of approximation for Γ can be treated precisely like the study for the unit circle (§3). We leave to the reader the proof of the extension of Theorem 3.1, already established [Walsh and Sewell, 1940] for $p \geqq 0$, and to which the method of Theorem 3.1 applies for $p < 0$ with the interpolation formula for equidistributed points:

THEOREM 7.5. *Let C be an analytic Jordan curve, and let the function $f(z)$ analytic interior to C_ρ be of class $L(p, \alpha)$ on C_ρ. Then there exist polynomials $p_n(z)$ such that we have on C*

$$(7.3) \qquad |f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha}.$$

We shall indicate the proof of the extension of Theorem 3.2:

THEOREM 7.6. *Let C be an analytic Jordan curve, and let the function $f(z)$ analytic interior to C_ρ be of class $L'(p, 1)$ on C_ρ, $p \geqq -1$. Then there exist polynomials $p_n(z)$ such that we have (7.3) valid on C.*

Theorem 3.2 follows from Theorem 4.1 precisely as Theorem 7.6 follows from a general result of which Theorem 4.1 is a limiting case:

THEOREM 7.7[3]. *Let C be an analytic Jordan curve and let $f(z)$ be analytic in the interior of C_ρ. Let $f'(z)$ be of class $L(p, \alpha)$, $p \leqq -1$, $0<\alpha\leqq 1$ on C_ρ, and let $p_n'(z)$ denote the polynomial of degree n defined by interpolation to $f'(z)$ in points uniformly distributed on a suitable level curve $C_{1-\delta}$ interior to C belonging to the analytic family of curves C_r. Then we have for z on C*

$$\left| \int_a^z [f'(z) - p_n'(z)]dz \right| \leqq M/\rho^n n^{p+\alpha+1},$$

where a is an arbitrary point interior to $C_{1-\delta}$, and where the path of integration contains no point exterior to C.

Theorem 7.7 is stated merely for the first integral, but extends at once to an arbitrary integral.

We use the well known Lagrange-Hermite interpolation formula for z interior to C_r

$$f'(z) - p_n'(z) = \frac{1}{2\pi i}\int_{C_r} \frac{\omega_n(z)f'(t)}{\omega_n(t)(t-z)}dt, \qquad 1 < r < \rho,$$

where $\omega_n(z) \equiv (z-z_1)(z-z_2) \cdots (z-z_{n+1})$, the points z_j lying interior to the curve C_r. Then we have for z interior to C_r

$$\int_a^z [f'(z) - p_n'(z)]dz = \frac{1}{2\pi i}\int_{C_r} \frac{f'(t)dt}{\omega_n(t)}\int_a^z \frac{\omega_n(z)dz}{t-z}.$$

Let $w=\phi(z)$ map the exterior of C onto $|w|>1$ with $\phi(\infty)=\infty$, and let $\delta>0$ be chosen so small that the locus $C_{1-\delta}$: $|\phi(z)|=1-\delta$ is an analytic Jordan curve interior to C, with $\phi(z)$ analytic (except at infinity) and univalent throughout the closed exterior of $C_{1-\delta}$. We set $e^g=(1-\delta)/|\phi'(\infty)|=(1-\delta)\Delta$.

We make use of the inequality [Curtiss, 1935 or Walsh and Sewell, 1940]

$$e^{-M} \leqq \left| \frac{\omega_n(z)}{e^{(n+1)g}(w^{n+1}-1)} \right| \leqq e^M, \quad z \text{ on or exterior to } C_{1-\delta},$$

where the points z_j are chosen as equally distributed points on $C_{1-\delta}$, and where w now and henceforth represents the function $w=\phi(z)/(1-\delta)$ which maps the exterior of $C_{1-\delta}$ onto $|w|>1$. Thus we have for z on $C_{1-\delta}$

$$|\omega_n(z)| \leqq M_1\Delta^{n+1}(1-\delta)^{n+1}.$$

The function $\omega_n(z)/w^{n+1}$ is analytic for z in the closed exterior of $C_{1-\delta}$ even at infinity, and w has the modulus unity for z on $C_{1-\delta}$; so we may write for z on and exterior to $C_{1-\delta}$

$$|\omega_n(z)| \leqq M_2\Delta^{n+1}(1-\delta)^{n+1}|w|^{n+1}. \tag{7.4}$$

[3] Some of the details of the present proof are due to the referee, replacing incorrect details of our original draft.

We integrate from an arbitrary z_0 on $C_{1-\delta}$ to z on C, choosing as path the image ("Radiusbild") in the z-plane of a radius of the unit circle in the plane of $w=\phi(z)/(1-\delta)$; for all t on C_r, where r is sufficiently near ρ, we have

$$\left|\int_{z_0}^{z}\frac{\omega_n(z)dz}{t-z}\right| \leqq M_3\int_{z_0}^{z}\Delta^{n+1}|\phi(z)|^{n+1}|dz| \leqq M_4\int_{z_0}^{z}\Delta^{n+1}|\phi(z)|^{n+1}|\phi'(z)dz|$$
$$= M_4\Delta^{n+1}[|\phi(z)|^{n+2}-|\phi(z_0)|^{n+2}]/(n+2) \leqq M_5\Delta^{n+2}/(n+2).$$

Here M_5 is independent of n, t, z_0, and z. If a is a fixed point interior to $C_{1-\delta}$, further use of (7.4) yields for z on C

$$\left|\int_{a}^{z}\frac{\omega_n(z)dz}{t-z}\right| \leqq \left|\int_{a}^{z_0}\frac{\omega_n(z)dz}{t-z}\right| + \left|\int_{z_0}^{z}\frac{\omega_n(z)dz}{t-z}\right| \leqq M_6\Delta^{n+2}/(n+2).$$

Also for t on or exterior to C we have $|\omega_n(t)| \geqq M_7\Delta^{n+1}|\phi(t)|^{n+1}>0$; hence for z on C we have

$$\left|\int_{a}^{z}[f'(z)-p_n'(z)]dz\right| \leqq M_8\cdot\max\,[|f'(t)|,\ t \text{ on } C_r]/r^n\cdot n.$$

From Theorem 7.2 it follows that $|f'(t)|\leqq N(\rho-r)^{p+\alpha}$ *on* C_r; if we put $r=\rho(1-1/n)$ we obtain the conclusion of the theorem.

This reasoning cannot be carried through if the points of interpolation are chosen equidistributed on C itself. Let C be the unit circle $|z|=1$, whence $\omega_n(z)\equiv z^{n+1}-1$; then for z on C we have

$$\int_0^z (z^{n+1}-1)dz = \frac{z^{n+2}}{n+2}-z,$$

so no additional factor n appears in the denominator due to the integration. For the particular function $f'(z)\equiv(t-z)^{-1}$, $|t|=\rho>1$, of class $L(-2,1)$ on C_ρ, with C the circle $|z|=1$ and $\omega_n(z)=z^{n+1}-1$, the conclusion of Theorem 7.7 is false.

An extension of Theorem 4.2, which likewise extends to higher integrals, is:

THEOREM 7.8. *Let C be an analytic Jordan curve. Let $f(z)$ be analytic interior to C_ρ, continuous in the corresponding closed region, and let polynomials $P_n(z)$ exist such that we have on C_ρ*

$$|f(z)-P_n(z)| \leqq \epsilon_n.$$

Let $p_n(z)$ denote the polynomial which interpolates to $f(z)$ in $n+1$ points equally distributed on $C_{1-\delta}$, $\delta>0$, where $1-\delta$ is sufficiently small. Then we have

$$\left|\int_a^z [f(z)-p_n(z)]dz\right| \leqq M\epsilon_n/n\rho^n, \qquad z \text{ on } C,$$

where a is an arbitrary point interior to $C_{1-\delta}$, and the integral is taken along an arbitrary path containing no point exterior to C.

The proof here is similar to that of the preceding theorem. Instead of the Lagrange-Hermite interpolation formula we use the following form employed by the authors [1940]

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\omega_n(z)[f(t) - P_n(t)]}{\omega_n(t)(t-z)}\,dt, \qquad z \text{ on } C, \tag{7.5}$$

where $P_n(z)$ is the polynomial mentioned above and $p_n(z)$ is the polynomial of degree n which interpolates to $f(z)$ in the roots of $\omega_n(z)$, namely, the points z_j on $C_{1-\delta}$ used in the proof of Theorem 7.7. The procedure goes through with only obvious modifications(4).

An analogue of Theorem 4.3 is

THEOREM 7.9. *Let C be an analytic Jordan curve; let $f(z)$ be of class $L(p, \alpha)$, $p \leqq -1$, $0<\alpha\leqq 1$, on C_ρ. Let $p_n(z)$ denote the polynomial of degree n which interpolates to $f(z)$ in $n+1$ points equally distributed on C. Then we have*

$$|f'(z) - p_n'(z)| \leqq M/\rho^n\cdot n^{p+\alpha-1}, \qquad z \text{ on } C.$$

In the formula

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_r} \frac{\omega_n(z)f(t)}{\omega_n(t)(t-z)}\,dt, \qquad z \text{ on or within } C,\ 1 < r < \rho,$$

let us differentiate with respect to z:

$$f'(z) - p_n'(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(t)}{\omega_n(t)}\left[\frac{(t-z)\omega_n'(z) + \omega_n(z)}{(t-z)^2}\right] dt.$$

But $|\omega_n(z)| \leqq M\Delta^{n+1}$, z on C, and hence by an extension of Bernstein's theo-

(4) We mention here the following theorem, analogous to Theorem 7.8:

Let C be an analytic Jordan curve, let $f(z)$ be analytic interior to C and continuous in the corresponding closed region, and let there exist polynomials $P_n(z)$ such that we have for z on C_ρ

$$|f(z) - P_n(z)| \leqq \epsilon_n.$$

Then we have

$$|f(z) - a_0p_0(z) - a_1p_1(z) - \cdots - a_np_n(z)| \leqq M\epsilon_n/\rho^n, \qquad z \text{ on } C,$$

where $\sum a_kp_k(z)$ is the expansion of $f(z)$ in Faber polynomials belonging to C.

This theorem follows from the formulas (in the notation of Faber [1920])

$$a_k = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(t)}{w^{k+1}}\,dt = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(t) - P_n(t)}{w^{k+1}}\,dt, \qquad k > n;$$

for the expansion of $P_n(z)$ in Faber polynomials is unique, whence

$$\int_{C_\rho} \frac{P_n(t)}{w^{k+1}}\,dt = 0, \qquad k > n.$$

rem [Sewell, 1937] it follows that $|\omega_n'(z)| \leqq M\Delta^{n+1}(n+1)$, z on C. Thus we have for z on C

$$|f'(z) - p_n'(z)| \leqq M_1 n \cdot \max [|f(t)|, \; t \text{ on } C_r]/r^{n+1};$$

by using Theorem 7.2 and putting $r=\rho(1-1/n)$ we obtain the inequality of the theorem; the proof is complete.

A direct analogue of Theorem 4.4 is

THEOREM 7.10. *Let $f(z)$ be analytic interior to the analytic Jordan curve C, continuous in the corresponding closed region. Let there exist polynomials $P_n(z)$ such that we have on C_ρ*

$$|f(z) - P_n(z)| \leqq \epsilon_n.$$

Let $p_n(z)$ denote the polynomial of degree n which interpolates to $f(z)$ in $n+1$ points equally distributed on C. Then we have

$$|f'(z) - p_n'(z)| \leqq Mn\epsilon_n/\rho^n, \qquad z \text{ on } C.$$

The proof here is similar to that of the preceding theorem except that we use the formula (7.5). The details are left to the reader.

Theorem 7.10 includes the conclusion of Theorem 7.9 with the restriction $p \geqq 0$, $0<\alpha\leqq 1$; in the boundary case $p+\alpha=0$ we set $\epsilon_n = M_0$.

The theorems just established are the analogues of those of §§3 and 4; the latter are limiting cases but not properly special cases of the former. In the converse direction we have the following analogue of Theorem 5.1:

THEOREM 7.11. *Let C be an analytic Jordan curve and let $f(z)$ be defined on C. For each n, $n=1, 2, \cdots$, let a polynomial $p_n(z)$ exist such that*

$$|f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha+1}, \qquad z \text{ on } C, \; \rho > 1.$$

Then $f(z)$, when suitably defined, is analytic interior to C_ρ and is of class $L(p, \alpha)$ on C_ρ if $p+\alpha+1$ is not a positive integer, and of class $L'(p, \alpha)$ if $p+\alpha+1$ is a positive integer.

The extension of Bernstein's lemma [Walsh, 1935, p. 77] applies here and there are no essential changes necessary in the proof of Theorem 5.1 as given, except that in the case where $p+\alpha+1$ is a positive integer we now use the polynomials $q_n(z)$ normal and orthogonal on C. The function $f(z)$ is analytic throughout the interior of C_ρ [Walsh, 1935, p. 78], and we have $f(z) \equiv \sum_{k=0}^{\infty} a_k q_k(z)$ throughout the interior of C_ρ, uniformly on any closed set interior to C_ρ. By virtue of the given $p_n(z)$ and the least-square property of the $q_n(z)$ we have

$$|a_n| \leqq M_1/\rho^n \cdot n^{p+\alpha+1}.$$

The polynomials $q_n(z)$ are uniformly bounded on C [Szegö, 1939, p. 365], so by Bernstein's inequality in an extended form we have

$$\left| q_n^{(p+\alpha+1)}(z) \right| \leqq M_2 n^{p+\alpha+1}, \qquad z \text{ on } C,$$

$$\left| f^{(p+\alpha+1)}(z) \right| \leqq \sum_{k=0}^{\infty} M_3 \cdot r^k/\rho^k, \qquad z \text{ on } C_r;$$

the reasoning used in connection with (5.4) now applies.

A consequence of such theorems as 7.5–7.10 is inequalities on degree of approximation of polynomials *of best approximation in the sense of Tchebycheff.* For such polynomials and others (e.g., as in Theorem 7.11), Problem γ, namely, the study of degree of convergence on C_σ, $1<\sigma<\rho$, can be treated by the methods that we have already developed.

It is to be observed that the methods we use in §7 apply to much more general measures of degree of convergence and asymptotic conditions than those exhibited by functions of class $L(p, \alpha)$. In fact we have the following theorems:

THEOREM 7.12. *Let C be an analytic Jordan curve and suppose on each C_r for which r lies in an interval $r_0<r<\rho$ we have $|f(z)| \leqq \phi(\rho-r)$, where the function $\phi(x)$ is defined in some interval $0<x<x_0$. Then there exist polynomials $p_n(z)$ such that we have*

$$\left| f(z) - p_n(z) \right| \leqq \frac{M \cdot \phi(\rho/n)}{\rho^n}, \qquad z \text{ on } C.$$

The usual results hold also for approximation to integrals and derivatives of $f(z)$; additional factors n appear on the right in denominator or numerator.

THEOREM 7.13. *Let $f(z)$ be defined on C and polynomials $p_n(z)$ exist such that*

$$\left| f(z) - p_n(z) \right| \leqq \frac{\phi(1/n)}{\rho^n}, \qquad z \text{ on } C,$$

where $\phi(x)$ is defined and monotonic throughout some interval $0<x<x_0 \geqq 1$. Then we have

$$\left| f(z) \right| \leqq M \sum_{n=1}^{\infty} \phi(1/n) r^n/\rho^n, \qquad z \text{ on } C_r,\ r<\rho,$$

provided this series converges.

8. **Approximation on a line segment. Trigonometric approximation.** Approximation on a finite line segment is analogous to approximation on an analytic Jordan curve, provided the approximated function is analytic on the given segment. In §7 we studied approximation on Jordan curves by interpolation in equally distributed points; these same points serve in the study of approximation on a line segment.

The roots of the polynomial $T_n(z) = 2^{-n+1} \cos (n \cos^{-1} z)$ are equally distributed on the segment C: $-1 \leqq z \leqq 1$. The inequality

$$\left| \frac{T_n(z)}{T_n(t)} \right| \leqq \frac{M}{\rho^n}, \qquad z \text{ on } C,\ t \text{ on } C_\rho,$$

is known [for instance Walsh and Sewell, 1940]. We may choose M independent of ρ, for $\rho > \rho_0 > 1$. Consequently the discussion of §7 concerning direct approximation on C is valid in the present case; we state

THEOREM 8.1. *If C is the segment $-1 \leqq z \leqq 1$ and $f(z)$ is analytic interior to C_ρ and of class $L(p, \alpha)$ or $L'(p, \alpha)$ on C_ρ, then there exist polynomials $p_n(z)$ such that we have on C*

$$| f(z) - p_n(z) | \leqq M/\rho^n \cdot n^{p+\alpha}.$$

For the case $p \geqq 0$, Theorem 8.1 has already been established [Walsh and Sewell, 1940]; for the case $p < 0$, the proof follows that of Theorem 3.1. The polynomials $p_n(z)$ are chosen as the polynomials interpolating to $f(z)$ in the zeros of $T_{n+1}(z)$. For the class $L'(p, \alpha)$, compare the remarks on integration below.

In the direction of a converse we have

THEOREM 8.2. *Let C be the segment $-1 \leqq z \leqq 1$ and let $f(z)$ be defined on C. For each n, $n = 1, 2, \cdots$, let a polynomial $p_n(z)$ exist such that*

$$| f(z) - p_n(z) | \leqq M/\rho^n \cdot n^{p+\alpha+1}, \qquad z \text{ on } C,\ \rho > 1.$$

Then $f(z)$ when suitably defined, is of class $L(p, \alpha)$ on C_ρ if $p+\alpha+1$ is not a positive integer, and of class $L'(p, \alpha)$ if $p+\alpha+1$ is a positive integer.

The proof of Theorem 8.2 is essentially the same as that of Theorem 5.1 for $p+\alpha+1$ not a positive integer; for $p+\alpha+1$ a positive integer we proceed as in Theorem 7.11, using polynomials normal and orthogonal on a particular C_σ, $1 < \sigma < \rho$, and the inequality

$$| f(z) - p_n(z) | \leqq M_1 \sigma^n / \rho^n n^{p+\alpha+1}, \qquad z \text{ on } C_\sigma.$$

The entire discussion of §7 concerning differentiation and integration of sequences remains essentially valid, except that in differentiation the additional factor n is to be replaced by n^2; on the segment $-1 \leqq z \leqq 1$, we have $| T_n'(z) | \leqq n^2/2^{n-1}$.

In the study of integration of sequences (compare Theorems 7.7 and 7.8) we use the following evaluation. From the interpolation formula

$$f(z) - p_{n-1}(z) = \frac{1}{2\pi i} \int_{C_r} \frac{T_n(z)}{T_n(t)} \frac{f(t) - P_{n-1}(t)}{t - z} dt, \qquad -1 \leqq z \leqq 1,$$

we have by integration

$$\int_0^z [f(z) - p_{n-1}(z)]dz = \frac{1}{2\pi i}\int_{C_r} \frac{[f(t) - P_{n-1}(t)]}{T_n(t)}\left[\int_0^z \frac{T_n(z)}{t-z}\,dz\right]dt.$$

Thus we have to consider merely

$$\int_0^z \frac{\cos(n\cos^{-1} z)dz}{t-z} = \frac{1}{t-z}\int \cos(n\cos^{-1} z)dz\Big]_0^z$$

$$- \int_0^z \left[\int \cos(n\cos^{-1} z)dz\right]\frac{dz}{(t-z)^2};$$

for all t on C_r and z on C we have

$$\left|\int_0^z \frac{\cos(n\cos^{-1} z)}{t-z}\,dz\right| \leqq M_1/n.$$

Consequently we obtain the same inequalities for integration of sequences in the case of a line segment C as for C an analytic Jordan curve.

This completes our study of the line segment. It is of interest to note that Theorems 8.1 and 8.2 might have been proved by mapping the complement of C conformally on the exterior of the unit circle γ: $|w| = 1$, and applying the results already obtained (Theorems 3.1, 3.2, 5.1) for the unit circle. However the above method is more direct.

We now consider the unit circle and functions analytic in the annulus γ_ρ: $\rho > |z| > 1/\rho < 1$. Suppose $f(z) = \sum_{m=-\infty}^{m=\infty} c_m z^m$ is analytic in γ_ρ; it is well known that we may write $f(z) \equiv f_1(z) + f_2(z)$, where

$$f_1(z) = \sum_{m=0}^{\infty} c_m z^m, \qquad |z| < \rho, \qquad f_2(z) = \sum_{m=-1}^{-\infty} c_m z^m, \qquad |z| > 1/\rho.$$

If $f_1(z)$ and $f_2(1/z)$ belong to the class $L(p, \alpha)$ or $L'(p, \alpha)$ on $|z| = \rho$ we say that $f(z)$ belongs to the class $L(p, \alpha)$ or $L'(p, \alpha)$ on γ_ρ. With this definition it is easy to establish theorems analogous to Theorems 3.1, 3.2, and 5.1.

THEOREM 8.3. *Let $f(z)$ belong to the class $L(p, \alpha)$ or $L'(p, \alpha)$ in the annular region $\rho > |z| > 1/\rho < 1$, and let $f(z) = \sum_{m=-\infty}^{\infty} c_m z^m$. Then with the notation $a_m = c_m + c_{-m}$, $b_m = i(c_m - c_{-m})$ we have the relation*

$$\left| f(e^{i\theta}) - \left[\frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos m\theta + b_m \sin m\theta)\right]\right| \leqq M/\rho^n \cdot n^{p+\alpha}.$$

In the converse direction we are concerned with a polynomial $p_n(z, 1/z)$ of degree n in z and $1/z$, namely a function of the form

$$p_n(z, 1/z) \equiv a_{-n}z^{-n} + \cdots + a_0 + \cdots + a_n z^n.$$

THEOREM 8.4. *Let $f(z)$ be defined on $|z| = 1$ and let polynomials $p_n(z, 1/z)$ exist such that*

$$|f(z) - p_n(z, 1/z)| \leqq M/\rho^n \cdot n^{p+\alpha+1}, \qquad |z| = 1, \rho > 1.$$

Then $f(z)$, if properly extended from the unit circle, belongs to the class $L(p, \alpha)$ in the annulus $\rho > |z| > 1/\rho$ if $p+\alpha+1$ is not a positive integer, and to the class $L'(p, \alpha)$ if $p+\alpha+1$ is a positive integer.

For $p \geqq 0$, $\alpha < 1$ Theorems 8.3 and 8.4 have already been proved [Walsh and Sewell, 1938]; for $p<0$, $0<\alpha\leqq 1$, and for $p\geqq 0$, $\alpha=1$ the methods for the unit circle may be applied to $f_1(z)$ and $f_2(z)$; in the latter case we make use of S. Bernstein's theorem concerning the derivative of a trigonometric polynomial of order n. Theorems 8.3 and 8.4 may be interpreted as results on trigonometric approximation (loc. cit.); in fact the transformation $w=e^{iz}$ suggests directly the definitions involved in the following theorems; formal definitions and proofs may be easily supplied by the reader:

THEOREM 8.5. *Let the function $f(z)$ be periodic with period 2π and of class $L(p, \alpha)$ or $L'(p, \alpha)$ in the band $|y| < \log \rho > 0$, $z=x+iy$. Then there exist trigonometric polynomials $t_n(z)$ such that we have for all real z*

$$|f(z) - t_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha}.$$

THEOREM 8.6. *Let the function $f(z)$ be defined for all real z and periodic with period 2π. Let trigonometric polynomials $t_n(z)$ exist such that for all real $z=x+iy$*

$$|f(z) - t_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha+1}.$$

Then $f(z)$ belongs to the class $L(p, \alpha)$ on $|y| < \log \rho$ if $p+\alpha+1$ is not a positive integer, and to the class $L'(p, \alpha)$ if $p+\alpha+1$ is a positive integer.

Results analogous to those of the present section have already been established by de la Vallée Poussin [1919] and S. Bernstein [1926], who study approximation by trigonometric polynomials and approximation on the segment $(-1, 1)$, for the case that the function $f(z)$ has only isolated singularities.

9. **Approximation measured by an integral.** Well known methods apply to our results of §§3–8 on approximation, and give us theorems on approximation by polynomials as measured by line integrals. For instance under the hypothesis of Theorem 7.5 or 8.1 there exist polynomials $p_n(z)$ such that we have

$$\int_C |f(z) - p_n(z)|^m \cdot |dz| \leqq M/\rho^{mn} n^{m(p+\alpha)}, \qquad m > 0. \tag{9.1}$$

Conversely an inequality of form (9.1) implies that $f(z)$ is of class $L(p-1, \alpha)$ on C_ρ if $p+\alpha$ is not a positive integer and of class $L'(p-1, \alpha)$ on C_ρ if $p+\alpha$ is a positive integer; but of course when (9.1) is given, the function $f(z)$ appears in our hypothesis merely almost everywhere, and the characterization of $f(z)$ just given contemplates a revision of the definition of $f(z)$ on a set of measure zero.

The statements just made have already been established [Walsh and Sewell, 1940, 1940a] for the case $p+\alpha-1>0$, $p+\alpha$ not a positive integer, and can be established for the remaining case by standard methods [Walsh, 1935, p. 92]; compare the proof of Theorem 8.2.

These remarks on approximation as measured by an integral apply likewise if a suitably restricted norm function is introduced.

10. **Integrated Lipschitz conditions and integral asymptotic conditions.** We described in §2 a classification of functions based on results of Hardy and Littlewood, a classification which we have seen (§§3–5) to be highly appropriate in the study of both direct and indirect theorems under Problem β. Still another classification, likewise based on results of Hardy and Littlewood, is of interest and also appropriate in the study of Problem β. But this new classification is far less elementary and intuitive than the former one, and also has been far less used; for this reason we have emphasized the one rather than the other. Nevertheless the more sophisticated classification deserves some treatment, which we proceed to develop in the special case of the circle, and to apply in the study of approximation.

If the function $f(z)$ is analytic for $|z|<1$, we use the definition ($|z|=r<1$)

$$M_m = M_m(f) = M_m(r, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^m d\theta\right)^{1/m};$$

this has a meaning for every $m>0$, but is to be used below primarily for $m=2$.

We shall say that the function $f(z)$ analytic for $|z|<1$ is *of class* $L_2(p, \alpha)$, where p is a negative integer and $0<\alpha\leqq 1$, provided we have

$$M_2(f) \leqq M(1-r)^{p+\alpha}. \tag{10.1}$$

We shall say that the function $f(z)$ analytic for $|z|<1$ and with boundary values almost everywhere on $|z|=1$ is *of class* $L_2(0, \alpha)$, $0<\alpha\leqq 1$ provided there is satisfied the integrated Lipschitz condition of order α:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta+ih}) - f(e^{i\theta})|^2 d\theta \leqq Mh^{2\alpha}. \tag{10.2}$$

With these definitions, Hardy and Littlewood [1932] prove three important theorems:

THEOREM 10.1. *If $p+\alpha\leqq 0$ and if $f(z)$ is of class $L_2(p, \alpha)$, then $f^{(k)}(z)$ is of class $L_2(p-k, \alpha)$.*

THEOREM 10.2. *If $p+\alpha<0$, $p-k+\alpha<0$, and if $f(z)$ is of class $L_2(p-k, \alpha)$, then the kth integral of $f(z)$ is of class $L_2(p, \alpha)$.*

THEOREM 10.3. *A necessary and sufficient condition that $f(z)$ be of class $L_2(0, \alpha)$, $0<\alpha\leqq 1$, is that $f'(z)$ be of class $L_2(-1, \alpha)$.*

It is now natural to say that $f(z)$ is of class $L_2(p, \alpha)$, where p is a non-negative integer and $0<\alpha\leqq 1$ provided $f^{(p+1)}(z)$ is of class $L_2(-1, \alpha)$. With this understanding we have at once for every p and k

THEOREM 10.4. *If $f(z)$ is of class $L_2(p, \alpha)$, then the function $f^{(k)}(z)$ is of class $L_2(p-k, \alpha)$ and if $(p+\alpha)(p+k+\alpha+1)$ is not a negative integer the kth integral of $f(z)$ is of class $L_2(p+k, \alpha)$.*

As in §2, there is here an exception if $\alpha=1$. We define the class $L_2'(-1, 1)$ as the class of integrals of functions of class $L_2(-2, 1)$, and the class $L_2'(p, 1)$ as the class of $(p+2)$th iterated integrals of functions of the class $L_2(-2, 1)$, where $p>-2$. It follows that if $f(z)$ is of class $L_2'(p, 1)$, with $p>-2$, then $f^{(k)}(z)$, $0<k<p+2$, is of class $L_2'(p-k, 1)$; also $f^{(k)}(z)$, $k\geqq p+2$, is of class $L_2(p-k, 1)$; if $f(z)$ is of class $L_2(p, 1)$, $p<-2$, then $f^{(k)}(z)$ is of class $L_2(p-k, 1)$ and the kth iterated integral of $f(z)$ is of class $L_2(p+k, 1)$ or $L_2'(p+k, 1)$ according as $p+k\leqq -2$ or $p+k>-2$.

These preliminaries completed we are in position to study approximation:

THEOREM 10.5. *If $f(z)$ is of class $L_2(p, \alpha)$ or of class $L_2'(p, \alpha)$, there exist polynomials $p_n(z)$ such that we have for $|z|=1/\rho<1$*

$$|f(z) - p_n(z)| \leqq M/\rho^n\cdot n^{p+\alpha}. \tag{10.3}$$

In the case $p+\alpha\leqq 0$, the method previously given (§3) is applicable; we employ (10.1) and the Schwarz inequality; in the case $p+\alpha>0$, we use that same method but applied now to the function $f^{(p+1)}(z)$, and integrate $p+1$ times under the integral sign in the interpolation formula.

We present an alternative proof of Theorem 10.5 for the class $L_2(p, \alpha)$, $p\geqq 0$. If $f(z)=\sum a_n z^n$ is of class $L_2(0, \alpha)$ we may write [this method is well known]

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta})}{e^{in\theta}}\,d\theta \\ &= \frac{-1}{2\pi}\int_0^{2\pi}\frac{f(e^{i(\theta+\pi/n)})}{e^{in\theta}}\,d\theta = \frac{1}{4\pi}\int_0^{2\pi}\left[f(e^{i\theta}) - f(e^{i(\theta+\pi/n)})\right]\frac{d\theta}{e^{in\theta}}, \end{aligned}$$

whence by Schwarz's inequality and the fundamental definition of class $L_2(0, \alpha)$, we have $|a_n|\leqq M_0/n^\alpha$. If $f(z)=\sum a_n z^n$ is of class $L_2(p, \alpha)$, $p>0$, we have by p-fold differentiation and use of the preceding relation, $|a_n| \leqq M'/n^{p+\alpha}$(5). Consequently on the circle $|z|=1/\rho<1$ we have

$$\left|f(z) - \sum_{\nu=0}^{n} a_\nu z^\nu\right| \leqq \sum_{n+1}^{\infty}|a_\nu z^\nu| \leqq M'\sum_{n+1}^{\infty}\frac{1}{\rho^\nu \nu^{p+\alpha}} \leqq \frac{M'}{n^{p+\alpha}}\sum_{n+1}^{\infty}\frac{1}{\rho^\nu} \leqq \frac{M_1}{n^{p+\alpha}\rho^n},$$

which establishes (10.3) for the case $p\geqq 0$.

(5) This last inequality is readily proved for functions of class $L(p, \alpha)$, $L'(p, \alpha)$, $L_2(p, \alpha)$, and $L_2'(p, \alpha)$, for every p and $0<\alpha\leqq 1$.

The indirect approximation problem is similarly handled:

THEOREM 10.6. *Let there exist polynomials $p_n(z)$ such that*

$$|f(z) - p_n(z)| \leqq M/\rho^n \cdot n^{p+\alpha+1/2} \tag{10.4}$$

is valid for $|z| = 1/\rho < 1$; then $f(z)$ is of class $L_2(p, \alpha)$ if $p+\alpha+1$ is not a positive integer and is of class $L_2'(p, \alpha)$ if $p+\alpha+1$ is a positive integer.

Our proof of Theorem 10.6 uses not (10.4) directly, but the inequality

$$\sum_{\nu=n+1}^{\infty} |a_\nu|^2/\rho^{2\nu} \leqq M_1/\rho^{2n}n^{2p+2\alpha+1}, \qquad f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \tag{10.5}$$

which is a direct consequence of (10.4) by virtue of the least-square property of the polynomials $s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ on the circle $|z| = 1/\rho$:

$$\begin{aligned} \sum_{\nu=n+1}^{\infty} |a_\nu|^2/\rho^{2\nu} &= \frac{\rho}{2\pi}\int_{|z|=1/\rho} |f(z) - s_n(z)|^2 |dz| \\ &\leqq \frac{\rho}{2\pi}\int_{|z|=1/\rho} |f(z) - p_n(z)|^2 |dz|. \end{aligned}$$

An inequality which follows from (10.5) is

$$|a_n|^2 \leqq M_2/n^{2p+2\alpha+1}. \tag{10.6}$$

Let us now choose the non-negative integer k in such a manner that we have $2k > 2p+2\alpha+1$; we have from (10.6)

$$\begin{aligned} f^{(k)}(z) = \sum_{\nu=0}^{\infty} b_\nu z^\nu, \qquad b_\nu &= (\nu + k)(\nu + k - 1) \cdots (\nu + 1)a_{\nu+k}, \\ |b_n|^2 &\leqq M_3 n^{2k-2p-2\alpha-1}. \end{aligned} \tag{10.7}$$

A consequence of (10.7) is (see below)

$$\begin{aligned} \frac{1}{2\pi r}\int_{|z|=r} |f^{(k)}(z)|^2 |dz| = \sum_{n=0}^{\infty} |b_n|^2 r^{2n} &\leqq M_3 \sum_{n=0}^{\infty} n^{2k-2p-2\alpha-1} r^{2n} \\ &\leqq M_4(1 - r^2)^{2(p+\alpha-k)}, \end{aligned} \tag{10.8}$$

whence $f^{(k)}(z)$ is of class $L_2(p-k, \alpha)$. By Theorem 10.4 the kth integral of $f^{(k)}(z)$ is of class $L_2(p, \alpha)$ if $p+\alpha+1$ is not a positive integer; and the kth integral of $f^{(k)}(z)$ is of class $L_2'(p, \alpha)$ if $p+\alpha+1$ is a positive integer, so Theorem 10.6 is established.

It remains to justify the last inequality in (10.8); this is accomplished by the method used in the treatment of inequality (5.4).

It has been noted that our proof of Theorem 10.6 uses not (10.4) as hypothesis, but rather (10.5). It is of interest to remark that *the hypothesis may be taken as*

$$\int_{|z|=1/\rho} |f(z) - p_n(z)|^m |dz| \leqq M/\rho^{mn} n^{m(p+\alpha+1/2)}, \tag{10.9}$$

where m is an arbitrary positive number. For (10.9) implies by standard algebraic inequalities (e.g., Walsh [1935, p. 93])

$$\int_{|z|=1/\rho} | p_{n+1}(z) - p_n(z) |^m | dz | \leqq M_1/\rho^{mn} n^{m(p+\alpha+1/2)},$$

which in turn yields (Walsh [1935, p. 92])

$$| p_{n+1}(z) - p_n(z) | \leqq M_2/\rho_1^n n^{p+\alpha+1/2} \tag{10.10}$$

on the circle $|z| = 1/\rho_1$, with $1 < \rho_1 < \rho$. Inequality (10.10) implies by the method of proof of (10.5)

$$\sum_{\nu=n+1}^{\infty} | a_\nu |^2/\rho_1^{2\nu} \leqq M_3/\rho_1^{2n} n^{2p+2\alpha+1},$$

which is precisely of form (10.5) with ρ replaced by ρ_1, and which suffices to prove the conclusion of Theorem 10.6.

We mention the following beautiful result, stated without proof by Hardy and Littlewood [1928]: *The class of functions $f(\theta)$ satisfying an integrated Lipschitz condition of order α is identical with the class of functions which can be approximated in the mean by trigonometric polynomials of degree n with error not greater than M/n^α.* It is to be noted that Theorems 10.5 and 10.6 have been proved without the help of this result. Nevertheless this result can readily be used to give a new proof of Theorem 10.5 for the case $p \geqq 0$, by the methods already developed by the authors [1940].

Theorems 10.5 and 10.6 are obviously to be compared with Theorems 3.1 and 5.1. The discrepancy between the exponents of n in (10.3) and (10.4) is only $\frac{1}{2}$, whereas that between the exponents of n in (3.1) and (5.1) is unity, so in this respect Theorems 10.5 and 10.6 are an improvement over Theorems 3.1 and 5.1. It may be remarked, however, that the proof of Theorem 10.6 as given does not admit of direct extension to an arbitrary analytic Jordan curve C.

Theorems 10.5 and 10.6 are in a sense the best possible results, namely in the sense that we cannot replace $p+\alpha$ in (10.3) by any $\alpha' > p+\alpha$, and that we cannot replace $p+\alpha$ in (10.4) by any $\alpha'' < p+\alpha$; we proceed to illustrate this fact by specific examples.

For an example in connection with Theorem 10.5 we set

$$f(z) = \sum_{n=1}^{\infty} 2^{\beta n} z^{2^n}, \qquad \beta > 0;$$

we have for $|z| = r$

$$\left|\sum_{n=1}^{\infty} 2^{\beta n} z^{2^n}\right| \leqq \int_1^{\infty} (2^x)^{\beta} r^{2^x} dx \leqq M(1 - r)^{-\beta},$$

and hence a fortiori $f(z)$ is of class $L_2(p, \alpha)$ where $0<\alpha\leqq 1$, p is a negative integer, and $-\beta=p+\alpha$. Also it follows by the method used in §5 that there exist no polynomials $p_n(z)$ such that for every n we have

$$|f(z) - p_n(z)| \leqq M/\rho^n n^{p+\alpha+\delta}, \qquad \delta > 0, \ |z| = 1/\rho < 1.$$

Thus we see that in the sense mentioned, Theorem 10.5 cannot be improved for $p+\alpha\leqq 0$; by integration and Theorems 10.2 and 10.3 the conclusion extends to non-integral positive $p+\alpha$. The case of integral non-negative $p+\alpha$ can be treated as in §5.

For an example in connection with Theorem 10.6 we choose

$$f(z) = (1 - z)^{\beta-1/2} = \sum_{m=0}^{\infty} a_m z^m, \qquad \beta \leqq 0,$$

where [e.g., de la Vallée Poussin, 1914, §399]

$$\frac{M_2}{m^{\beta+1/2}} \leqq |a_m| \leqq \frac{M_1}{m^{\beta+1/2}}, \qquad M_2 > C. \tag{10.11}$$

In §5 we have seen that

$$\left|f(z) - \sum_{m=0}^{n} a_m z^m\right| \leqq M_3/n^{\beta+1/2}\rho^n, \qquad |z| = 1/\rho.$$

But we have

$$\frac{1}{2\pi r}\int_{-\pi}^{\pi} |1 - z|^{2\beta-1} d\theta = \sum_{m=0}^{\infty} |a_m|^2 r^{2m}, \qquad |z| = r,$$

and hence by inequality (10.11) we see that $f(z)$ is of class $L_2(p, \alpha)$, if $\beta=p+\alpha<0$, and of class $L_2'(p, \alpha)$ if $\beta=p+\alpha=0$; in each case $f(z)$ is of no higher class. The same method as above serves to extend the scope of the example to all values of $p+\alpha$. Consequently Theorems 10.5 and 10.6 are the best possible in the sense mentioned.

It may be observed that for $\delta>1$ the Hölder inequality

$$\int |f| \leqq M\left(\int |f|^{\delta}\right)^{1/\delta}$$

and for $\delta=1$ more elementary inequalities establish the conclusion of Theorem 10.5, where now $f(z)$ is an arbitrary function of class $L_\delta(p, \alpha)$ or $L_\delta'(p, \alpha)$;

suitable definitions of these classes are fairly obvious from our previous definitions. We remark that $L_\delta(-1, 1)$ is identical with the class of functions H_δ studied by F. Riesz [1923], namely functions $f(z)$ analytic for $|z| < 1$ such that

$$\int_{|z|=r} |f(z)|^\delta |dz|$$

is uniformly bounded for all $r < 1$.

Theorems 10.5 and 10.6 extend at once to the situations of §8. Theorem 10.5 extends also to the case that C is an arbitrary analytic Jordan curve; but the writers have not as yet extended Theorem 10.6 to this more general case. We have in the present paper insisted on ordinary Lipschitz conditions and asymptotic conditions rather than on integrated Lipschitz conditions and mean asymptotic conditions because the theory of the latter concepts is not as yet widely developed, and because the former concepts are relatively simple and more direct.

11. **Direct methods on Problems α and β.** In the above sections we have established various results on Problem β, results which are as favorable in many respects as can be obtained. On the other hand, our methods have been in part relatively high-powered, for instance in our proofs of Theorems 7.7 and 7.8. However, some results only slightly less favorable than those obtained above and elsewhere can be established by thoroughly immediate and elementary methods, with a minimum of machinery, as we now proceed to indicate for both Problem α(6) and Problem β. For the present we restrict ourselves to the case of functions analytic in the unit circle γ: $|z| = 1$.

If $f(z)$ is of class $L(p, \alpha)$, $p + \alpha \leqq 0$, our results as already established (Theorem 3.1) for approximation on $|z| = 1$ are obtained by elementary methods; these results refer to Problem β, and Problem α does not properly present itself.

If $f(z)$ is assumed to satisfy a uniform *radial* Lipschitz condition we can also proceed by elementary methods:

THEOREM 11.1. *Let $f(z)$ be analytic and bounded for $|z| < 1$ and satisfy a uniform radial Lipschitz condition, in the sense*

$$|f(e^{i\theta}) - f(re^{i\theta})| \leqq M(1 - r)^\alpha, \qquad 0 < r < 1, \tag{11.1}$$

where M is independent of r and θ. Then there exist polynomials $p_n(z)$ such that we have for all θ

$$|f(e^{i\theta}) - p_n(e^{i\theta})| \leqq M'(\log n)^\alpha / n^\alpha; \tag{11.2}$$

(6) For the set C: $|z| \leqq 1$, Problem α is the study of degree of approximation on C of a function analytic in $|z| < 1$ satisfying given conditions of continuity on or in the neighborhood of γ: $|z| = 1$.

indeed, the $p_n(z)$ may be defined as $s_n(r_n z)$, where $s_n(z)$ is the sum of the first $n+1$ terms of the Taylor development of $f(z)$ and $r_n = 1-(2\log n)/n$.

We set $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $s_n(z)=\sum_{k=0}^{\infty} a_k z^k$, and by the boundedness of $f(z)$

$$|f(re^{i\theta}) - s_n(re^{i\theta})| = \left| \frac{1}{2\pi i}\int_{|t|=1} \frac{(re^{i\theta})^{n+1}}{t^{n+1}} \frac{f(t)}{(t-z)}\,dt \right| \leqq \frac{M_1 r^{n+2}}{1-r}, \qquad r<1,$$

for $z=re^{i\theta}$, where M_1 is a constant depending only on $f(z)$. We have from (11.1) by addition

$$|f(e^{i\theta}) - s_n(re^{i\theta})| \leqq \frac{M_1 r^{n+2}}{1-r} + M(1-r)^{\alpha}.$$

Corresponding to each n we choose

$$r = r_n = 1 - 2\log n/n,$$

whence by writing

$$\left(1-\frac{2\log n}{n}\right)^n = \left[\left(1-\frac{2\log n}{n}\right)^{n/2\log n}\right]^{2\log n},$$

which is asymptotic to n^{-2}, we obtain the inequality (11.2).

Theorem 11.1 is a result on Problem α; the corresponding result on Problem β is

THEOREM 11.2. *Let $f(z)$ satisfy the hypothesis of Theorem 11.1. Then there exist polynomials $p_n(z)$ such that we have*

$$|f(z) - p_n(z)| \leqq M''(\log n)^{\alpha}/\rho^n\cdot n^{\alpha}, \qquad |z| = 1/\rho < 1.$$

We write

$$\begin{aligned} f(z) - p_n(z) &= \frac{1}{2\pi i}\int_{|t|=1} \frac{z^{n+1}f(t)}{t^{n+1}(t-z)}\,dt \\ &= \frac{1}{2\pi i}\int_{|t|=1} \frac{z^{n+1}f(rt)}{t^{n+1}(t-z)}\,dt + \frac{1}{2\pi i}\int_{|t|=1} \frac{z^{n+1}[f(t)-f(rt)]}{t^{n+1}(t-z)}\,dt, \\ &\qquad\qquad 0<r<1. \end{aligned}$$

For each n we choose $r=r_n=1-\log n/n$, whence by the method used in Theorem 11.1 we have the inequality of the theorem.

The method of Theorem 11.2 extends in an elementary way to yield results on approximation to integrals and derivatives of $f(z)$; compare §4.

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