ON THE GROWTH PROPERTIES OF A FUNCTION OF TWO COMPLEX VARIABLES GIVEN BY ITS POWER SERIES EXPANSION

BY

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1. **Introduction.** One of the most fundamental formulas in the theory of functions of one complex variable is the Cauchy integral formula. It is of particular value in the Weierstrass-Hadamard approach, i.e., in obtaining properties of a function from the coefficients of its power series expansion. A similar formula cannot be obtained for functions of two complex variables for an arbitrary four-dimensional domain, as is obtained, for instance, for the bicylinder, where the integration is taken over a two-dimensional surface on the boundary. Bergman(1) has shown, however, that for certain domains far more general than those previously considered, i.e., domains bounded by a finite number of analytic hypersurfaces, an analogous formula does exist, the double integral being taken essentially over the two-dimensional surface common to two or more of the analytic bounding hypersurfaces(2).

In this paper we shall obtain growth properties in terms of the coefficients of the power series expansion of a function $f(z_1, z_2)$ of two complex variables analytic in special domains of the type mentioned above; first, with the aid of Bergman's integral formula, along the two-dimensional surfaces common to the bounding hypersurfaces, and then, along a class of two-dimensional surfaces lying in only one of the bounding hypersurfaces and having a line of contact with another bounding hypersurface. We also obtain a mapping theorem which determines from the coefficients a convex region in the f_1f_2-plane, $f(z_1, z_2) = f_1 + if_2$, which must be contained in the smallest convex region of the mapping on the f_1f_2-plane of the surfaces considered.

2. **Properties of f associated with $G^2(r)$.** Let us consider a finite four-dimensional domain $\mathfrak{M}^4$ which is bounded by the hypersurfaces

$$(2.1)\qquad \begin{aligned} \overset{3}{s}_1(r) &\equiv E[z_2 = re^{i\lambda_1},\ 0 \leqq \lambda_1 \leqq 2\pi], \\ \overset{3}{s}_2(r) &\equiv E[z_1 = re^{i\lambda_2} + p(\lambda_2)z_2 \equiv h(\lambda_2, z_2),\ 0 \leqq \lambda_2 \leqq 2\pi], \end{aligned}$$

and which depends on a positive parameter r; $p(\lambda_2)$ is assumed merely to have

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(1) Bergman [2, 3]. See the bibliography at the end of this paper.

(2) Bergman calls such surfaces "distinguished boundary surfaces."

a first derivative. Let $G^2(r)$ be the two-dimensional surface on the boundary of $\mathfrak{M}^4$ which is the common part of the bounding hypersurfaces, i.e.,

$$G^2(r) \equiv \overset{3}{s}_1 \cdot \overset{3}{s}_2. \tag{2.2}$$

THEOREM I. *Given a function* $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ *regular in the domain* $\overline{\mathfrak{M}}^4(r)$; *if* $M(r)$ *is the maximum-modulus of* $f(z_1, z_2)$ *on* $G^2(r)$, *then*

$$M(r) \geqq \max_{m,n} \frac{r^{m+n} | a_{mn} |}{G(m, n; p)B(p)}, \tag{2.3}$$

where m *and* n *range over all non-negative integral values,* $B(p)$ *is a constant depending upon* p, *and* $G(m, n; p)$ *is a function of* m, n, *and* p, *given by*

$$1 + \int_0^{n+1} \left(1 + x \frac{1 + \log m}{m}\right)^m \max | p(\lambda_2) |^x dx - \frac{m}{\log p} - \frac{m}{1 + \log m}, \quad \text{when } \max | p | < 1,\ m \geqq 1,$$

$$1 + \int_0^{n+1} \left(1 + x \frac{1 + \log m}{m}\right)^m \max | p(\lambda_2) |^x dx, \quad \text{when } \max | p | \geqq 1,\ m \geqq 1, \tag{2.4}$$

$$\frac{1 - \max | p(\lambda_2) |^{n+1}}{1 - \max | p(\lambda_2) |}, \quad \text{when } m = 0 \text{ for all } p.$$

Proof of Theorem I. Keeping z_2 constant, say equal to t_2, we obtain for a particular value of z_1, say t_1,

$$f(t_1, t_2) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f[h(\lambda_2, t_2), t_2][ire^{i\lambda_2} + p'(\lambda_2)t_2]d\lambda_2}{[(re^{i\lambda_2} + p(\lambda_2)t_2) - t_1]}. \tag{2.5}$$

Since the numerator of the integrand is an analytic function of t_2, we again apply the Cauchy integral formula and obtain

$$\begin{aligned} f(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} & \frac{f[h(\lambda_2, re^{i\lambda_1}), re^{i\lambda_1}]}{[(re^{i\lambda_2} + p(\lambda_2)t_2) - t_1][re^{i\lambda_1} - t_2]} \\ & \cdot [ire^{i\lambda_2} + p're^{i\lambda_1}]ire^{i\lambda_1}d\lambda_1 d\lambda_2. \end{aligned} \tag{2.6}$$

For the mth derivative of $f(t_1, t_2)$ with respect to t_1, we obtain

$$\begin{aligned} \frac{\partial^m f(t_1, t_2)}{\partial t_1^m} = \frac{m!}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} & \frac{f[h(\lambda_2, re^{i\lambda_1}), re^{i\lambda_1}]}{[(re^{i\lambda_2} + p(\lambda_2)t_2) - t_1]^{m+1}[re^{i\lambda_1} - t_2]} \\ & \cdot [ire^{i\lambda_2} + p'(\lambda_2)re^{i\lambda_1}]ire^{i\lambda_2}d\lambda_1 d\lambda_2. \end{aligned} \tag{2.7}$$

Let

$$H_1 \equiv (re^{i\lambda_2} + p(\lambda_2)t_2) - t_1, \qquad H_2 \equiv re^{i\lambda_1} - t_2.$$

For the nth derivative of $1/H_1^{m+1}H_2$ with respect to t_2, we obtain by Leibnitz' rule

$$\left[1+\sum_{\nu=1}^{n}\frac{(m+\nu)!}{m!\nu!}\left(\frac{H_2}{H_1}p(\lambda_2)\right)^{\nu}\right]\frac{n!}{H^{m+1}H_2^{n+1}}\cdot \tag{2.8}$$

Hence we obtain for $\partial^{m+n}f(t_1, t_2)/\partial t^m\partial t^n$ the expression

$$\begin{aligned}&\frac{m!n!}{(2\pi i)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{f[h, \lambda_1][ire^{i\lambda_2}+p'(\lambda_2)re^{i\lambda_1}]ire^{i\lambda_1}}{[(re^{i\lambda_2}+p(\lambda_2)t_2)-t_1]^{m+1}[re^{i\lambda_1}-t_2]^{n+1}}\\ &\qquad\cdot\left[1+\sum_{\nu=1}^{n}\frac{(m+\nu)!}{m!\nu!}\left(\frac{H_2}{H_1}p\right)^{\nu}\right]d\lambda_1 d\lambda_2.\end{aligned} \tag{2.9}$$

Now

$$a_{mn}=\frac{\partial^{m+n}f(0, 0)}{m!n!\partial t_1^m\partial t_2^n}\cdot$$

Hence

$$\begin{aligned}a_{mn}&=\frac{1}{(2\pi i)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{f[h, \lambda_1][ire^{i\lambda_2}+p're^{i\lambda_1}]ire^{i\lambda_1}}{r^{m+n+2}\exp\{i(m\lambda_2+n\lambda_1+\lambda_2+\lambda_1)\}}\\ &\quad\cdot\left[1+\sum_{\nu=1}^{n}\frac{(m+\nu)!}{m!\nu!}(e^{i(\lambda_1-\lambda_2)}p)^{\nu}\right]d\lambda_1 d\lambda_2.\end{aligned} \tag{2.10}$$

Taking the absolute value of a_{mn} we get

$$\begin{aligned}|a_{mn}|&\leqq\frac{1}{4\pi^2}\frac{M(r)\max_{0\leqq\lambda_2\leqq 2\pi}[1+|p'(\lambda_2)|]}{r^{m+n}}\\ &\quad\cdot\left[1+\sum_{\nu=1}^{n}\frac{(m+\nu)!}{m!\nu!}(\max|p|^{\nu})\right]4\pi^2.\end{aligned} \tag{2.11}$$

Now for $m\geqq 1$, it can be shown that

$$\begin{aligned}&1+\sum_{\nu=1}^{n}\left[\frac{(m+\nu)!}{m!\nu!}\max|p(\lambda_2)|^{\nu}\right]\\ &\qquad\leqq 1+\sum_{\nu=1}^{n}\left(1+\nu\frac{1+\log m}{m}\right)^{m}|p|^{\nu}.\end{aligned} \tag{2.12}$$

When $|p|<1$ we have

$$\begin{aligned}&1+\sum_{\nu=1}^{n}\frac{(m+\nu)!}{m!\nu!}\max|p(\lambda_2)|^{\nu}\\ &\leqq 1+\int_1^n\left[1+x\frac{1+\log m}{m}\right]^{m}\max|p|^{x}dx-\frac{m}{\log p}-\frac{m}{1+\log m},\end{aligned} \tag{2.13}$$

and when $|p|\geqq 1$,

$$(2.14)\quad 1+\sum_{\nu=1}^{n}\frac{(m+\nu)!}{m!\nu!}\max|p|^{\nu}\leqq\int_{0}^{n+1}\left[1+x\frac{1+\log m}{m}\right]^{m}\max|p|^{x}dx.$$

When $m=0$, $(1-\max|p|^{n+1})/(1-\max|p|)$ is the exact value of the left-hand side of (2.14) for all p.

Therefore for all differentiable $p(\lambda_2)$ and non-negative integral values of m and n we have

$$(2.15)\qquad |a_{mn}|\leqq\frac{M(r)B(p)}{r^{m+n}}G(m,n;p),$$

where $B(p)=\max_{0\leqq\lambda_2\leqq 2\pi}(1+|p'|)$, or

$$(2.16)\qquad M(r)\geqq\frac{r^{m+n}|a_{mn}|}{B(p)G(m,n;p)}\cdot$$

To find those values of m and n, say $\mu(r)$ and $\nu(r)$, for which the right-hand expression in (2.16) is maximum for a given r, we take the logarithm of the expression, letting $-\log|a_{mn}|=g_{mn}$ and employ a generalized Newton polygon method. Then

$$(2.17)\quad \begin{aligned} &g_{mn}-(m+n)\log r+\log B+\log G(m,n)\\ &\qquad\geqq g_{\mu\nu}-(\mu+\nu)\log r+\log B+\log G(\mu,\nu)=C.\end{aligned}$$

We choose m, n, and g_{mn} as the x-, y-, and z-axes, respectively, and plot the points (m, n, g_{mn}). Then the m and n of the first point which lies in the surface $z=x\log r+y\log r-\log G(x,y)-\log B+k$ as this surface is translated along the z-axis from $-\infty$ by varying k, i.e., until $k=C$, are the μ and ν which give the right-hand side of (2.16) a maximum. If there is more than one point lying on the surface, the one with the smaller m is chosen; if the m's are the same, the one with the smaller n is chosen. μ and ν are obviously functions of r.

We then have

$$(2.18)\qquad M(r)\geqq\frac{r^{\mu+\nu}|a_{\mu\nu}|}{BG(\mu,\nu)}\cdot$$

This gives a lower bound for the growth of $f(z_1, z_2)$ along the hypersurface $g^3\equiv S_{r=r_0}^{r_1}G^2(r)$, where r varies continuously.

3. **The mapping of the surface** $G^2(r)$. Let us introduce the function

$$(3.1)\qquad F(f,\alpha)=e^{e^{-i\alpha f(z_1,z_2)}}=\sum_{r,s=0}^{\infty}A_{rs}z_1^{r}z_2^{s},$$

where $0\leqq\alpha\leqq 2\pi$ and f is defined as in the previous sections. The coefficients $\{A_{rs}\}$ are functions of α and a combination of the a_{mn}'s such that $m\leqq r$ and $n\leqq s$. We define the region $R^2(r)$ as the product of the half-planes

$$(3.2)\qquad f_1 \cos\alpha + f_2 \sin\alpha \leqq Q(\alpha, r), \qquad 0 \leqq \alpha < 2\pi,$$

where f_1 and f_2 are cartesian coordinates in the f_1f_2-plane, and

$$(3.3)\qquad Q(\alpha, r) \equiv \log \left| A_{\mu\nu}(a, \alpha) \right| + (\mu + \nu) \log r - \log G(\mu, \nu) - \log B.$$

THEOREM II. *Let $f(z_1, z_2) = f_1 + if_2$. Then the smallest convex domain enclosing the mapping of $G^2(r)$ on the f_1f_2-plane contains the closed convex region $R^2(r)$ which depends only on the coefficients of the expansion of $f(z_1, z_2)$ and the surface $G^2(r)$.*

This gives a lower bound, so to speak, of the mapping of $G^2(r)$ on the f_1f_2-plane.

Proof of Theorem II. Let

$$(3.4)\qquad P(r) = \max \left| e^{e^{-i\alpha} f(z_1, z_2)} \right|$$

on the surface $G^2(r)$; then from (3.4) and (2.18)

$$\begin{aligned} \log P(r) &= \log \left| \exp \left\{ e^{-i\alpha} f^*(z_1, z_2) \right\} \right| \\ (3.5)\qquad &= \log \left| \exp \left\{ (f_1^* \cos\alpha + f_2^* \sin\alpha) \right\} \right| \\ &\quad \cdot \left| \exp \left\{ -i(f_1^* \sin\alpha - f_2^* \cos\alpha) \right\} \right| \\ (3.6)\qquad &= f_1^* \cos\alpha + f_2^* \sin\alpha \\ &\geqq \log \left| A_{\mu\nu}(a, \alpha) \right| + (\mu + \nu) \log r - \log G(\mu, \nu) - \log B = Q(\alpha, r), \end{aligned}$$

where the * indicates that value of f which gives $|P|$ its maximum, for a given α. Now, for each α, $Q(\alpha, r)$ has a fixed value (depending on r). It is clear from (3.6) that at least one point of the mapping, namely, (f_1^*, f_2^*), will lie in the half-plane

$$(3.7)\qquad f_1 \cos\alpha + f_2 \sin\alpha \geqq Q(\alpha, r).$$

The region $R^2(r)$ will therefore be contained in the smallest convex domain containing the mapping of $G^2(r)$ on the f_1f_2-plane. Theorem II is then proved.

It is clear that a similar theorem will hold for any surface for which we have a lower bound for the maximum of the function $f(z_1, z_2)$ on the surface. For example, we can state similar theorems for the surfaces considered in §§4 and 5.

4. **Further properties of the function on other surfaces of the type** $G^2(r)$. Let us consider the finite four-dimensional region $\mathfrak{M}^4(r)$ bounded by the three infinite hypersurfaces:

$$\begin{aligned} \overset{3}{s}_1(r) &\equiv E\left[z_2 = re^{i\lambda_1},\ 0 \leqq \lambda_1 \leqq 2\pi\right], \\ (4.1)\qquad \overset{3}{s}_2(r) &\equiv E\left[z_1 = re^{i\lambda_2} + C_2 z_2,\ 0 \leqq \lambda_2 \leqq 2\pi\right], \\ \overset{3}{s}_3(r) &\equiv E\left[z_1 = re^{i\lambda_3} - C_3 z_2,\ 0 \leqq \lambda_3 \leqq 2\pi\right], \end{aligned}$$

where, as above, r is a parameter and C_2 and C_3 are positive constants less than unity. This restriction on C_2 and C_3 is necessary in order that the hypersurfaces of (3.1) form the boundary of a finite closed domain. Let $G^2_{ks}(r)$ be that part of $s^3_k(r)\cdot s^3_s(r)$ which belongs to the boundary of $\mathfrak{M}^4$. Now let

$$G^{2}(r) \equiv G^{2}_{12}(r) + G^{2}_{13}(r) + G^{2}_{23}(r). \tag{4.2}$$

Let also $g^3 \equiv \mathfrak{S}^{r_1}_{r=r_0} G^2(r)$, and $g^3_{ks} \equiv \mathfrak{S}^{r_1}_{r=r_0} G^2_{ks}(r)$, where r varies continuously and $r_1 < \infty$.

Let $f(z_1, z_2)$, as before, be an analytic function regular in $\overline{\mathfrak{M}}^4$. We now apply Bergman's integral formula(3) for functions of two complex variables which states that at a point (t_1, t_2) in $\mathfrak{M}^4$,

$$\begin{aligned} f(t_1, t_2) &= \frac{1}{2} \sum_{k,s}{}' M_{ks}(t_1, t_2) \\ &= \frac{1}{2(2\pi i)^2} \sum_{k,s}{}' \int\int_{B^2_{ks}} \frac{f(\overset{(1)}{\phi}_{ks}, \overset{(2)}{\phi}_{ks}) B_{ks}(t_1, t_2, \lambda_k, \lambda_s) d\lambda_k d\lambda_s}{\Phi_k(t_1, t_2, \lambda_k)\Phi_s(t_1, t_2, \lambda_s)}, \end{aligned}$$

$$B_{ks}(t_1, t_2, \lambda_k, \lambda_s) = \frac{Z_{ks}(t_1, t_2, \lambda_k, \lambda_s)}{(\phi^{(1)}_{ks} - t_1)(\phi^{(2)}_{ks} - t_2)}, \qquad k \neq s,$$

$$\begin{aligned} Z_{ks}(t_1, t_2, \lambda_k, \lambda_s) = \frac{D(\overset{(1)}{\phi}_{ks}, \overset{(2)}{\phi}_{ks})}{D(\lambda_k, \lambda_s)} &[\Phi_s(t_1, t_2, \lambda_s)\Phi_k(t_1, \phi^{(1)}, \lambda_k) \\ &- \Phi_k(t_1, t_2, \lambda_k)\Phi_s(t_1, \phi^{(2)}, \lambda_s)], \end{aligned}$$

where B^2_{ks} is the surface range of integration. We have in our case

$$\begin{aligned} \Phi_1 &= z_2 - re^{i\lambda_1}, \\ \Phi_2 &= z_1 - re^{i\lambda_2} - C_2 z_2, \\ \Phi_3 &= z_1 - re^{i\lambda_3} + C_3 z_2; \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\begin{cases} z_1 = \overset{(1)}{\phi}_{12} \equiv re^{i\lambda_2} + C_2 re^{i\lambda_1}, \\ z_2 = \overset{(2)}{\phi}_{12} \equiv re^{i\lambda_1}, \end{cases} \\ &\begin{cases} z_1 = \overset{(1)}{\phi}_{13} \equiv re^{i\lambda_3} - C_3 re^{i\lambda_1}, \\ z_2 = \overset{(2)}{\phi}_{13} \equiv re^{i\lambda_1}, \end{cases} \\ &\begin{cases} z_1 = \overset{(1)}{\phi}_{23} \equiv \dfrac{1}{C_2+C_3}[C_2 re^{i\lambda_3} + C_3 re^{i\lambda_2}], \\ z_2 = \overset{(2)}{\phi}_{23} \equiv \dfrac{1}{C_2+C_3}[re^{i\lambda_3} - re^{i\lambda_2}]; \end{cases} \end{aligned} \tag{4.4}$$

(3) Bergman [2, p. 97] and [3, p. 861].

and consequently,

$$
\begin{aligned}
(4.5)\qquad f(t_1, t_2) &= M_{12}(t_1, t_2) + M_{13}(t_1, t_2) + M_{23}(t_1, t_2) \\
&= \frac{1}{(2\pi i)^2}\iint_{B_{12}^2} \frac{f(\overset{(1)}{\phi_{12}}, \overset{(2)}{\phi_{12}})(r^2 e^{i(\lambda_1+\lambda_2)})d\lambda_1 d\lambda_2}{(re^{i\lambda_1} - t_2)(re^{i\lambda_2} + C_2 t_2 - t_1)} \\
(4.6)\qquad &\quad + \frac{1}{(2\pi i)^2}\iint_{B_{13}^2} \frac{f(\overset{(1)}{\phi_{13}}, \overset{(2)}{\phi_{13}})(r^2 e^{i(\lambda_1+\lambda_3)})d\lambda_1 d\lambda_3}{(re^{i\lambda_1} - t_2)(re^{i\lambda_3} - C_3 t_2 - t_1)} \\
&\quad + \frac{1}{(2\pi i)^2}\iint_{B_{23}^2} \frac{f(\overset{(1)}{\phi_{23}}, \overset{(2)}{\phi_{23}})(r^2 e^{i(\lambda_2+\lambda_3)})d\lambda_2 d\lambda_3}{(re^{i\lambda_2} + C_2 t_2 - t_1)(re^{i\lambda_3} - C_3 t_2 - t_1)}.
\end{aligned}
$$

As in §2 we have that

$$
(4.7)\qquad a_{mn} = \frac{\partial^{m+n} f(0, 0)}{m!n!\partial t_1^m \partial t_2^n} = \left.\frac{\partial^{m+n}[M_{12}(t_1, t_2) + M_{13}(t_1, t_2) + M_{23}(t_1, t_2)]}{m!n!\partial t_1^m \partial t_2^n}\right|_{t_1, t_2=0},
$$

$$
\begin{aligned}
&\frac{\partial^{m+n} M_{12}(t_1, t_2)}{\partial t_1^m \partial t_2^n} \\
(4.8)\qquad &= \frac{m!}{(2\pi i)^2}\iint_{B_{12}^2} \frac{\partial^n}{\partial t_2^n}\left[\frac{f(\overset{(1)}{\phi_{12}}, \overset{(2)}{\phi_{12}})(r^2 e^{i(\lambda_1+\lambda_2)})}{(re^{i\lambda_1} - t_2)(re^{i\lambda_2} + c_2 t_2 - t_1)^{m+1}}\right] d\lambda_1 d\lambda_2 \\
&= \frac{m!n!}{(2\pi i)^2}\iint_{B_{12}^2} \frac{f(\overset{(1)}{\phi_{12}}, \overset{(2)}{\phi_{12}})(r^2 e^{i(\lambda_1+\lambda_2)})d\lambda_1 d\lambda_2}{[-\Phi_1(t_2)]^{n+1}[-\Phi_2(t_1, t_2)]^{m+1}},
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\partial^{m+n} M_{12}(0, 0)}{\partial t_1^m \partial t_2^n} &= \frac{m!n!}{(2\pi i)^2}\iint_{B_{12}^2} \frac{f(\overset{(1)}{\phi_{12}}, \overset{(2)}{\phi_{12}})(r^2 e^{i(\lambda_1+\lambda_2)})}{r^{m+n+2}e^{i(m\lambda_2+n\lambda_1+\lambda_2+\lambda_1)}} \\
(4.9)\qquad &\quad \cdot \sum_{\nu=0}^{n} \frac{(m+\nu)!}{m!\nu!}(c_2 e^{i(\lambda_1-\lambda_2)})^\nu d\lambda_1 d\lambda_2.
\end{aligned}
$$

This yields, by a process analogous to that used in §2,

$$
(4.10)\qquad \frac{1}{m!n!}\left|\frac{\partial^{m+n} M_{12}(0, 0)}{\partial t_1^m \partial t_2^n}\right| \leqq \frac{B_{12}(g_{12}^3)M(r)G_{12}(m, n)}{r^{m+n}},
$$

where $M(r)$ is the maximum-modulus of f on $G^2(r)$, $B(g_{12}^3)$ is a constant depending on the hypersurface $g_{12}^3 = S_{r-r_0}^{r_1} G_{12}^2(r)$ and $G_{12}(m, n)$ is a function of m and n, also depending on g_{12}^3 and is defined in a way similar to $G(m, n)$ of §2.

In the same way we obtain

$$
(4.11)\qquad \frac{1}{m!n!}\left|\frac{\partial^{m+n} M_{13}(0, 0)}{\partial t_1^m \partial t_2^n}\right| \leqq \frac{B_{13}(G_{13}^2)M(r)G_{13}(m, n)}{r^{m+n}}.
$$

From (4.3), we have

$$\Phi_2(t_1, t_2) = (t_1 - re^{i\lambda_2} - C_2t_2), \qquad \Phi_3(t_1, t_2) = (t_1 - re^{i\lambda_3} + C_3t_2). \tag{4.12}$$

Hence

$$\frac{\partial^m}{\partial t_1^m}\left[\frac{1}{\Phi_2\Phi_3}\right] = (-1)^m m! \sum_{\nu=0}^{m} \frac{1}{\Phi_2^{\nu+1}\Phi_3^{m-\nu+1}}, \tag{4.13}$$

and

$$\frac{\partial^m}{\partial t_1^m}\left[\frac{1}{\Phi_2\Phi_3}\right] = (-1)^{m+n} m!n! \sum_{\nu=0}^{m}\sum_{\mu=0}^{n} \frac{(m+n-\nu-\mu)!C_2^{\mu}C_3^{n-\mu}}{(m-\nu)!(n-\mu)!\Phi_2^{\nu+\mu+1}\Phi_3^{m+n-\nu-\mu+1}} \tag{4.14}$$

$$\begin{aligned} &= \frac{(-1)^{m+n}m!n!}{\Phi_3^{m+n+2}} \sum_{\mu=0}^{n} \frac{1}{(n-\mu)!}\left(\frac{C_2}{C_3}\frac{\Phi_3}{\Phi_2}\right)^{\mu} \\ &\quad \cdot \sum_{\nu=0}^{m} \frac{(m+n-\nu-\mu)!}{(m-\nu)!}\left(\frac{\Phi_3}{\Phi_2}\right)^{\nu}. \end{aligned} \tag{4.15}$$

Then

$$\begin{aligned} \frac{1}{m!n!}\left|\frac{\partial^{m+n} M_{23}(0, 0)}{\partial t_1^m \partial t_2^n}\right| &\leqq \frac{1}{4\pi^2}\int\int_{B_{23}^2} \frac{\left|f(\overset{(1)}{\phi_{23}}, \overset{(2)}{\phi_{23}})\right|}{r^{m+n}} \sum_{\mu=0}^{n} \frac{1}{(n-\mu)!}\left(\frac{C_2}{C_3}\right)^{\mu} \\ &\quad \cdot \sum_{\nu=0}^{m} \frac{(m+n-\nu-\mu)!}{(m-\nu)!} d\lambda_2 d\lambda_3 \\ &\leqq \frac{B_{23}(\overset{2}{G_{23}})M(r)}{r^{m+n}} \sum_{\mu=0}^{n} \frac{1}{(n-\mu)!}\left(\frac{C_2}{C_3}\right)^{\mu} \\ &\quad \cdot \sum_{\nu=0}^{m} \frac{(m+u-\nu-\mu)!}{(m-\nu)!}. \end{aligned} \tag{4.16}$$

The constant $B_{23}(\overset{3}{g_{23}})$ is given by $(1/4\pi^2)\int\int_{B_{23}^2} d\lambda_2 d\lambda_3$, where the precise limits of integration are obtained by a tedious process and can be omitted here since they are not necessary for our purpose; it may be noted, however, that $0 < B_{23} < 1$. We shall denote by $G_{23}(m, n)$ the expression

$$\sum_{\mu=0}^{n} \frac{1}{(n-\mu)!}\left(\frac{C_2}{C_3}\right)^{\mu} \sum_{\nu=0}^{m} \frac{(m+n-\nu-\mu)!}{(m-\nu)!}. \tag{4.17}$$

This gives

$$\begin{aligned} |a_{mn}| &\leqq \frac{1}{m!n!}\left(\left|\frac{\partial^{m+n}M_{12}(0, 0)}{\partial t_1^m \partial t_2^n}\right| + \left|\frac{\partial^{m+n}M_{13}(0, 0)}{\partial t_1^m \partial t_2^n}\right| + \left|\frac{\partial^{m+n}M_{23}(0, 0)}{\partial t_1^m \partial t_2^n}\right|\right) \\ &\leqq \frac{M(r)}{r^{m+n}} \frac{1}{2} \sum_{k,s=1}^{3}{}' B_{ks}(\overset{3}{g_{ks}})G_{ks}(m, n). \end{aligned} \tag{4.18}$$

Therefore

$$M(r) \geqq \frac{r^{m+n}\,|\,a_{mn}\,|}{\frac{1}{2}\sum_{k,s=1}^{\prime 3} B(g_{ks})G_{ks}(m,\,n)}. \tag{4.19}$$

Those values of μ and ν which make the right-hand side of (4.19) a maximum can be obtained by a process similar to that employed in §2. Hence

$$M(r) \geqq \frac{r^{\nu+\mu}\,|\,a_{\mu\nu}\,|}{\frac{1}{2}\sum_{k,s=1}^{\prime 3} B_{ks}(g_{ks}^3)G_{ks}(\mu,\,\nu)}. \tag{4.20}$$

We can then state

THEOREM III. *Given a function*

$$f(z_1,\,z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$$

regular in the closed domain $\overline{\mathfrak{M}}^4(r)$; *then along*

$$g^3 = \mathop{\mathrm{S}}_{r=r_0}^{r_1} G^2(r) = \mathop{\mathrm{S}}_{r=r_0}^{r_1} (G_{12}^2(r) + G_{13}^2(r) + G_{23}^2(r)),$$

$$M(r) \geqq \frac{r^{\nu+\mu}\,|\,a_{\mu\nu}\,|}{\frac{1}{2}\sum_{,s=1}^{\prime 3} B_{ks}(g_{ks}^3)G_{ks}(\mu,\,\nu)}.$$

5. **Properties for the function on certain classes of surfaces lying in the boundary and different from $G^2(r)$.** We next wish to consider the growth of the function $f(z_1, z_2)$ over a special class of surfaces $H^2(r)$ belonging to the boundary of $\mathfrak{M}^4$. Let

$$H^2(r) \equiv E[z_1 = \zeta(r,\,\lambda_1,\,\sigma),\; z_2 = re^{i\lambda_1},\; \lambda_1^{(1)} \leqq \lambda_1 \leqq \lambda_1^{(2)},\; \sigma_1 \leqq \sigma \leqq \sigma_2] \tag{5.1}$$

where for all σ satisfying $\sigma_1 \leqq \sigma < \sigma_2$, and for any fixed λ_1 in the range considered,

$$\begin{aligned} &\zeta(r,\,\lambda_1,\,\sigma) \in \mathfrak{J}_1^2(r,\,\lambda_1),\\ &\mathfrak{J}_1^2(r,\,\lambda_1) \equiv A_1^2(r,\,\lambda_1)\cdot A_2^2(r,\,\lambda_1);\\ &A_1^2(r,\,\lambda_1) \equiv E[\,|\,z_1 - C_2 z_2\,| \leqq r,\; z_2 = re^{i\lambda_1}],\\ &A_2^2(r,\,\lambda_1) \equiv E[\,|\,z_1 + C_3 z_2\,| \leqq r,\; z_2 = re^{i\lambda_1}]; \end{aligned} \tag{5.2}$$

and for $\sigma = \sigma_2$, with λ_1 again fixed,

$$\zeta(r,\,\lambda_1,\,\sigma_2) \in s_1^1(r,\,\lambda_1),$$

where s_1 is the boundary of $\mathfrak{J}_1^2(r,\,\lambda_1)$. It will be assumed that the set of all points of $H^2(r)$ for which λ_1 has an arbitrary fixed value in the range considered is a continuous curve $h^1(r)$(4) with an initial point $z_1=\zeta(r, \lambda_1, \bar{\sigma}_1(\lambda_1))$, and a terminal point on $s_1^1(r, \lambda_1)$. The surface $H^2(r)$ lies completely in that part of $s_1^3(r)$ which belongs to the boundary of the $\mathfrak{M}^4(r)$ of the previous section. A portion of the boundary of $H^2(r)$ lies on $G^2(r)$ of (4.2).

Let the maximum-modulus of $f(z_1, z_2)$ on $H^2(r)$ be $\gamma(r)$. We now map (using for simplicity the same notation for the mapped region) $\mathfrak{J}_1^2$ into the unit circle so that $z_1=0$ goes into itself and the direction of the real axis, $\mathfrak{R}(z_1)=0$, at the point $z_1=0$, remains unchanged. The curve $h^1(r)$ maps into a segment of a continuous curve, its initial point determined by $\sigma=\bar{\sigma}_1(\lambda_1)$ and its terminal point lies on the unit circle. Now let $\theta=|z_1|=|\zeta(r, \lambda_1, \bar{\sigma}_1(\lambda_1))|$ for $\lambda_1^{(1)}\leqq\lambda_1\leqq\lambda_1^{(2)}$. The quantities θ and $\alpha=\lambda_1^{(2)}-\lambda_1^{(1)}$ were introduced by Bergman and are the characteristic numbers of the surface(5).

One form of the Milloux theorem is(6): Let J be a continuous finite arc lying in the unit circle $|z|\leqq 1$ joining a point z_0 within the circle to a point on the boundary. Let $W(z)$ be regular, single-valued, and $|W(z)|<1$ inside the unit circle, and let $|W(z)|\leqq\omega$ on J. Then

$$|W(0)| < \omega^{(2/\pi)\sin^{-1}(1-\theta')/(1+\theta')}, \tag{5.3}$$

where $\theta'=|z_0|$.

Using this theorem for the mapped region $\mathfrak{J}_1^2$ with

$$W(z_1) = \frac{f(z_1, z_2^*)}{M(r)}, \tag{5.4}$$

we have

$$|W(z_1)| = \frac{|f(z_1, z_2^*)|}{M(r)} \leqq \frac{\gamma(r)}{M(r)} < 1, \tag{5.5}$$

and get, letting $\Theta=(2/\pi)\sin^{-1}(1-\theta)/(1+\theta)$,

$$|f(0, z_2)| < M^{1-\Theta}\gamma^{\Theta}, \tag{5.6}$$

where $M(r)$ is the maximum-modulus of $f(z_1, z_2)$ in $\overline{\mathfrak{M}}^4$, λ_1^* is an arbitrarily chosen value of λ_1 in the range considered, and $z_2=z_2^*=re^{i\lambda_1^*}$.

Now

$$a_{0n} = \frac{1}{2\pi i}\left\{\int_0^{\lambda_1^{(1)}} \frac{f(0, re^{i\lambda_1})}{r^n e^{in\lambda_1}}\,d\lambda_1 + \int_{\lambda_1^{(1)}}^{\lambda_1^{(2)}} \frac{f(0, re^{i\lambda_1})}{r^n e^{in\lambda_1}}\,d\lambda_1 + \int_{\lambda_1^{(2)}}^{2\pi} \frac{f(0, re^{i\lambda_1})}{r^n e^{in\lambda_1}}\,d\lambda_1\right\}, \tag{5.7}$$

(4) The restriction that $h^1(r)$ be continuous is not essential since theorems of the Milloux type hold for more general one-dimensional sets.

(5) Bergman [1, pp. 347–348, Corollary]; and [4, pp. 200–201].

(6) R. Nevanlinna [5].

$$
(5.8)\qquad \begin{aligned} |a_{0n}| \leqq \frac{1}{2\pi}\Bigg\{ \int_0^{\lambda_1^{(1)}} \frac{|f(0, re^{i\lambda_1})|}{r^n}\, d\lambda_1 + \int_{\lambda_1^{(1)}}^{\lambda_1^{(2)}} \frac{|f(0, re^{i\lambda_1})|}{r^n}\, d\lambda_1 \\ + \int_{\lambda_1^{(2)}}^{2\pi} \frac{|f(0, re^{i\lambda_1})|}{r^n}\, d\lambda_1 \Bigg\}, \end{aligned}
$$

and

$$
(5.9)\qquad |a_{0n}| \leqq \frac{1}{2\pi r^n}\left[(2\pi - \alpha)\overline{M} + \alpha M\left(\frac{\gamma}{M}\right)^{\Theta}\right],
$$

where $\overline{M} = \max |f(0, z_2)| = \max |\sum_{\nu=0}^{\infty} a_{0\nu} z_2^{\nu}|$. Then

$$
(5.10)\qquad \frac{2\pi}{\alpha}\left[\frac{|a_{0n}|\, r^n}{M} - \frac{\overline{M}}{M}\right] + \frac{\overline{M}}{M} \leqq \left(\frac{\gamma}{M}\right)^{\Theta}.
$$

Let Δ be defined by the equation

$$
(5.11)\qquad \Delta = 1 - \frac{|a_{0\mu}|\, r^{\mu}}{\overline{M}},
$$

μ being that n which maximizes $|a_{0n}| r^n$, and μ depends on r. Then Δ is positive.

If $\alpha > 2\pi\Delta$, we have that

$$
(5.12)\qquad \gamma(r) \geqq M(r)\left[\frac{\overline{M}}{M}\left(1 - \frac{2\pi}{\alpha}\Delta\right)\right]^{\Theta^{-1}(\theta)},
$$

where the right-hand side is positive.

From these results we can state

THEOREM IV. *Given the function*

$$
f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^{m} z_2^{n},
$$

regular in $\mathfrak{M}^4(r)$. *Let* $\max |f(z_1, z_2)| \leqq \gamma(r)$ *on the surface* $H^2(r)$ *of* (5.1) *having the characteristic numbers* $\theta(r)$ *and* α, $\alpha = \lambda_1^{(2)} - \lambda_1^{(1)} > 2\pi\Delta$, *where* $\Delta \equiv 1 - |a_{0\mu}| r^{\mu}/\overline{M}$; *then*

$$
(5.13)\qquad \gamma(r) \geqq M(r)\left[\frac{\overline{M}(r)}{M(r)}\left(1 - \frac{2\pi\Delta}{\alpha}\right)\right]^{\Theta^{-1}(\theta)},
$$

where $M = \max |f(z_1, z_2)|$ *and* $\overline{M} = \max |f(0, z_2)|$.

Since

$$
(5.14)\qquad \frac{|a_{0\mu}|\, \rho^{\mu+1}}{\rho - r} > \overline{M}(r), \qquad \rho > r,
$$

a lower bound for $\gamma(r)$ can be obtained in terms of the coefficients of $f(z_1, z_2)$ by replacing in (5.12) $M(r)$ by the right-hand side of (4.20), $\overline{M}(r)$ by $|a_{0\mu}|r^\mu$, and the $\overline{M}(r)$ in Δ by

$$\frac{|a_{0\mu'}|\rho_1^{\mu'+1}}{\rho_1 - r} \tag{5.15}$$

where

$$|a_{0\mu'}|\rho_1^{\mu'} = \underset{1+\epsilon<\rho<\infty}{\text{l.u.b.}} \left[\max_n |a_{0n}|\rho^n\right], \tag{5.16}$$

for an arbitrary positive ϵ. ρ_1 is a function of r and of the coefficients $\{a_{0n}\}$, and can be determined by a process similar to the Newton polygon method.

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