NET HOMOTOPY FOR COMPACTA[1]

BY

D. E. CHRISTIE

INTRODUCTION

The important Hurewicz theory of homotopy groups[2] is applicable only to arc-wise connected spaces. If these groups are defined for a space which is not arc-wise connected, their significance is limited to the arc-component in which the base-point is chosen.

One of the most interesting features of the homotopy group theory is its relation to homology groups. This relationship is expressed sharply in this *theorem of Hurewicz*: *the n-dimensional integral homology group and the n-dimensional homotopy group of an arc-wise connected space are isomorphic if the homotopy groups of lower dimensions vanish.* In this theorem the homology groups are, appropriately, the continuous or singular groups. More familiar homology theories for spaces are those of Čech or of Vietoris. The theorem of Hurewicz holds for these homology groups only if the space is assumed to have certain local properties such as local contractibility.

The purpose of this paper is to define homotopy groups which are significant for non-arc-wise connected spaces, and which are suitably related to the Čech homology groups for spaces which are not locally connected. These new groups are defined in terms of *nets*[3]. The theory of nets and of their homology groups is abstracted from the Čech theory. The nets which we consider here are derived from the nerves of finite coverings of compact metric spaces. By limiting the discussion to compacta we can consider simultaneously the equivalent but more intuitive theory of neighborhood homotopy. Moreover we give examples of connected compacta which have satisfactory net-homotopy groups, yet which are beyond the range of the classical theory.

The basic ingredients of any such theory are the concepts of *mapping* and of *homotopy*. In section II the standard concept of mapping is retained, but new definitions of homotopy are studied. One of these is based on nets, the other on neighborhoods. By means of machinery introduced in section I, these two homotopies are compared. In section III both basic concepts (mapping,

---

Presented to the Society, December 27, 1942; received by the editors December 21, 1942.

(1) This paper is, except for trivial changes, the author's Princeton doctoral dissertation. It was prepared under the supervision of Professor Solomon Lefschetz. The author is deeply grateful for his inspiring assistance. He is indebted also to Drs. D. Montgomery, H. Samelson, and A. D. Wallace for helpful discussion and valuable suggestions.

(2) For the Hurewicz theory, see Hurewicz [5, 6] and Eilenberg [3]. (Numbers in brackets refer to the bibliography at the end of the paper.)

(3) The theory and use of *nets* is given by Lefschetz [9, chaps. 6–7].

homotopy) are replaced by new net and neighborhood analogues. The net theory has intrinsically broader scope than the neighborhood theory; but for compacta the two are equivalent. Consequently for compacta the neighborhood results are independent of the embedding and have topological significance. In both section II and section III two distinct grades of net-homotopy, labeled weak and strong, are considered.

The new types of mappings and of homotopies can be used to define groups analogous to the Hurewicz homotopy groups. This is accomplished in section IV. As is pointed out in section VI, all the homotopy groups investigated agree for a compactum which is an absolute neighborhood retract (ANR).

In section V the range of usefulness and the relation to homology of these new groups is indicated. In the first place a property designated by C^{0*} is introduced to replace arc-wise connectedness. This property is best described in terms of nets and hence is a more complicated notion than arc-wise connectedness. It is however decidedly weaker as is shown by examples in section VI. The groups are shown to be independent of the base-point for a space which is C^{0*}. Finally an analogue to the Hurewicz theorem relating homotopy and homology is proved.

Section VI is devoted to examples. Examples are given of simple spaces which have no Hurewicz homotopy-groups, but do have informative net groups. The groups proceeding from the five types of homotopy situations studied are shown to be distinguishable.

In section VII the possibility of homotopy considerations for nets without underlying space is considered. A definition for net to net mappings is introduced. Such a mapping behaves in a reasonable way with respect to net homology groups. Moreover a space to space mapping induces a net to net mapping which in turn induces a homomorphism on weak net-homotopy groups.

Section I. Preliminaries

1. **Open coverings.** Throughout this study we shall be concerned with a *compact metric space* R. R will always be considered as a closed subset of a parallelotope P^n or P^ω. (P^n denotes the euclidean cube or parallelotope defined by the equations $0 \leqq x_i \leqq 1$, $i=1, \cdots, n$. P^ω is the "Hilbert" cube or parallelotope(4). A convenient definition is by the equations $0 \leqq x_i \leqq 1/2^{i-1}$.) It will be useful to consider finite coverings of both R and neighborhoods of R, and the corresponding *nets*(3). In this first section we shall choose a special subfamily of the possible finite coverings, and we shall point out some of the properties which justify this choice. As an introduction, we discuss some general properties of coverings of a compactum M.

The families of all finite open and of all finite closed coverings of a space M have been used extensively in deducing properties of M. Such coverings are

(4) For a discussion of these spaces, see Lefschetz [9, chap. 1, (12.8–12.10)].

ordered in a familiar way by the notion of refinement. Thus a covering $\mathfrak{U}_\mu$ is said to refine a covering $\mathfrak{U}_\lambda$ (or $\mathfrak{U}_\mu < \mathfrak{U}_\lambda$) provided that every set of $\mathfrak{U}_\mu$ is contained in some set of $\mathfrak{U}_\lambda$. When M is a compactum, this matter of refinement assumes a particularly simple form for open coverings. If $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\mu$ are finite open coverings, then the condition

$$\text{diameter of } \mathfrak{U}_\mu < \text{Lebesgue number of } \mathfrak{U}_\lambda$$

is sufficient to insure the ordering $\mathfrak{U}_\mu < \mathfrak{U}_\lambda$. This means that if we pick any sequence of coverings $\{\mathfrak{U}_i\}$ of mesh ϵ_i, where the ϵ_i's approach 0, we can choose a subsequence $\{\mathfrak{U}_{i_j}\}$ such that $\mathfrak{U}_{i_1} > \mathfrak{U}_{i_2} > \cdots > \mathfrak{U}_{i_n} > \cdots$. Moreover, this subsequence is *cofinal* in the family of all open coverings. For reference this well known fact is stated formally:

(1.1) *Any sequence $\{\mathfrak{U}_i\}$ of finite open coverings of M whose meshes ϵ_i tend to zero contains a linearly ordered subsequence cofinal in the family of all open coverings of M.*

With each finite covering we consider the *nerve*. This concept, due to Alexandroff, is well known. If $\mathfrak{U}_\lambda$ is a covering of M, its nerve will be designated by Φ_λ or Ψ_λ. Moreover, we shall use a *natural realization* of Φ_λ, defined thus:

(1.2) DEFINITION. Φ_λ is said to be *realized naturally* provided that the 0-simplex $\hat{U}_i^\lambda$ associated with the set U_i^λ of the covering is taken as a point of U_i^λ. The simplexes of higher dimension are filled in linearly. (Whenever possible, $\hat{U}_i^\lambda$ will be chosen as a point in U_i^λ but no other U_j^λ of $\mathfrak{U}_\lambda$. This is always possible for irreducible coverings(5).)

2. **Projections.** The relation of refinement $\mathfrak{U}_\mu < \mathfrak{U}_\lambda$ induces *simplicial mappings* called *projections* $\pi_\lambda^\mu: \Phi_\mu \to \Phi_\lambda$. These projections are defined as vertex transformations: $\pi_\lambda^\mu: \hat{U}_i^\mu \to \hat{U}_{j(i)}^\lambda$ for some $j(i)$ such that $U_{j(i)}^\lambda \supset U_i^\mu$. Any such vertex transformation leads to a projection since the vertices of a simplex of Φ_μ are carried onto the vertices of a simplex of Φ_λ. Unfortunately, since $j(i)$ is not necessarily unique, various choices of π_λ^μ may be eligible. In connection with later considerations, the following compensating proposition is of interest:

(1.3) *Suppose that π_λ^μ and $\bar{\pi}_\lambda^\mu$ are two projections of Φ_μ into Φ_λ. Then for every point x of Φ_μ, $\pi_\lambda^\mu(x)$ and $\bar{\pi}_\lambda^\mu(x)$ lie in the same closed simplex of Φ_λ. That is,* $\pi_\lambda^\mu \cdot - \cdot \bar{\pi}_\lambda^\mu$. (We shall use this notation frequently. Two mappings t, t' of a space S into a geometric complex K will be said to satisfy the relation $t \cdot - \cdot t'$ if for every point x of S, $t(x)$ and $t'(x)$ lie in the same closed simplex of K. This agrees with the use made of the same symbol by Alexandroff(6).)

Proof. Pick any x of Φ_μ. It lies in a simplex $\hat{A}_0 \cdots \hat{A}_j = \sigma$ of Φ_μ. Then $\pi_\lambda^\mu\sigma$ is a simplex $\hat{B}_0 \cdots \hat{B}_l$ of Φ_λ, $l \leqq j$, and $\bar{\pi}_\lambda^\mu\sigma$ is a simplex $\hat{C}_0 \cdots \hat{C}_m$ of Φ_λ, $m \leqq j$. But we know $A_0 \cap \cdots \cap A_j \neq 0$, $B_0 \cap \cdots \cap B_l \neq 0$, $C_0 \cap \cdots \cap C_m \neq 0$.

(5) See Lefschetz [9, chap. 7, pp. 247–248].

(6) Cf. Alexandroff [1, II, 8].

Furthermore, every B contains an A and every C contains an A, while every A is contained in at least one B and at least one C. Thus a point y of $A_0 \cap \cdots \cap A_j$ must be in $B_0 \cap \cdots \cap B_l \cap C_0 \cap \cdots \cap C_m$. This means that $\hat{B}_0 \cdots \hat{B}_l \hat{C}_0 \cdots \hat{C}_m$ is a simplex of Φ_λ. Thus both $\pi_\lambda^\mu(x)$ and $\bar{\pi}_\lambda^\mu(x)$ lie in faces of one simplex of Φ_λ. q.e.d.

An obvious but useful corollary to this is the fact that for any mapping α of a space S into Φ_μ, $\pi_\lambda^\mu \alpha \cdot - \cdot \bar{\pi}_\lambda^\mu \alpha$.

Since the relation $\cdot - \cdot$ implies homotopy, the non-uniqueness of projections is not significant as long as one is dealing only with homotopy classes. Certain results about projective cycles(7) are based on the employment of projections which are both *unique* and *onto*. The special coverings which we shall use have both these properties.

3. **Regular coverings.** In the following, $\mathfrak{J}$ and $\mathfrak{C}$ denote *interior* and *complement*: An upper bar denotes closure.

(1.4) Definition. *Regular closed covering.* $\mathfrak{F} = \{F_i\}$ is a regular closed covering if

$$(1) \qquad \mathfrak{J}(F_i) \cap \mathfrak{J}(F_j) = 0, \qquad i \neq j,$$

$$(2) \qquad \bar{\mathfrak{J}}(F_i) = F_i.$$

Such a covering may be described as a covering by closures of disjoint open sets.

Example. Let K be a geometric simplicial complex. Let $\{A_i\}$ be the vertices of the ith regular subdivision $K^{(i)}$. Let $\mathrm{St}A_i$ be the star of A_i as a vertex in $K^{(i+1)}$. Then the collection $\{\overline{\mathrm{St}}A_i\}$ is a regular covering of K.

In connection with these coverings, the following propositions are useful. The proofs are omitted.

(1.5) *If* $F = \bar{\mathfrak{J}}(F)$ *and* U *is open, then* $F \cap U \neq 0$ *implies that* $\mathfrak{J}(F) \cap U \neq 0$.

(1.6) $F_1 \subset F_2$ *implies* $\mathfrak{J}(F_1) \subset \mathfrak{J}(F_2)$.

An important known property of finite regular closed coverings (f.r.c.c.) is the following:

(1.7) *If* $\mathfrak{F}_\lambda$ *and* $\mathfrak{F}_\mu$ *are f.r.c.c. and* $\mathfrak{F}_\mu < \mathfrak{F}_\lambda$, *then the induced projection* π_λ^μ *is unique.*

For by (1.6), $F^\mu \subset F_i^\lambda$ implies $\mathfrak{J}(F^\mu) \subset \mathfrak{J}(F_i^\lambda)$; and by definition $\mathfrak{J}(F_i^\lambda) \cap \mathfrak{J}(F_j^\lambda) = 0$ if $i \neq j$.

We are now ready to introduce the useful theorem:

(1.8) Theorem(8). *If* $\mathfrak{F}_\lambda$ *and* $\mathfrak{F}_\mu$ *are f.r.c.c. and* $\mathfrak{F}_\mu < \mathfrak{F}_\lambda$ *then the projection* π_λ^μ *maps* Φ_μ onto Φ_λ.

This is easily deduced from

(1.9) *Under the conditions of* (1.8), *each set* F^λ *is the exact union of those* F^μ*'s which lie in* F^λ.

(7) Cf. Lefschetz [9, chap. 6, (19.1), p. 231].
(8) Cf. Steenrod [12, pp. 679–680].

The proof of (1.9), depending on (1.5), is omitted.

4. **Fringed closed coverings.** The property (1.1) of open coverings and also properties (1.7) and (1.8) of regular closed coverings are distinctly useful. For this reason, we introduce a class of open coverings enjoying properties (1.7) and (1.8) under proper circumstances. Suppose we consider any f.r.c.c. $\mathfrak{F}=\{F_i\}$. Denote by $\eta(\mathfrak{F})$ the minimum positive distance $\rho(F_i, F_j)$. Now for any $\delta<\eta/2$ we consider the δ-fringed sets defined by $U_i^{(\delta)}=U_\delta(F_i)$. ($U_\delta(F_i)$ is the set of points whose distances from F_i are less than δ.) For every such δ, $\mathfrak{U}^\delta=\{U_i^{(\delta)}\}$ is an open covering of R whose nerve is identical with that of $\mathfrak{F}$.

(1.10) Definition. We shall call $\mathfrak{U}^\delta$ the *δ-fringing of* $\mathfrak{F}$.

Suppose now that we have a sequence $\{\mathfrak{F}_i\}$ linearly ordered by refinement and with mesh ϵ_i approaching zero. For each i we consider the δ_i-fringing of $\mathfrak{F}_i$ subject to the restriction that $\delta_{i+1}<\delta_i/2$.

(1.10.1) The sequence $\{\mathfrak{U}_i^{\delta_i}\}$ will be called a *tapered fringing* of the sequence $\{\mathfrak{F}_i\}$.

Clearly the diameters in the tapered open covering still tend to zero. Hence by (1.1) the sequence is cofinal in the family of all open coverings.

We should now consider the connection between the ordering of $\{\mathfrak{F}_i\}$ and that of $\{\mathfrak{U}_i^{\delta_i}\}$. By the tapering restriction $F_j^{i+n}\subset F_j^i$ implies $U_j^{(\delta_{i+n})i+n}\subset U_j^{(\delta_i)i}$. Since the process of fringing does not alter nerves of regular coverings, this means the ordering of $\{\mathfrak{U}_i^{\delta_i}\}$ induced by that of $\{\mathfrak{F}_i\}$ is a proper ordering by refinement for $\{\mathfrak{U}_i^{\delta_i}\}$ considered just as a sequence of open coverings. These results are summarized in

(1.11) *Let $\{\mathfrak{F}_i\}$ be a linearly ordered sequence of f.r.c.c. with diameter tending to* 0 *of M, and let $\{\mathfrak{U}_i^{\delta_i}\}$ be a tapered refinement of this sequence. Then*

(a) $\{\mathfrak{U}_i^{\delta_i}\}$, *ordered by the linear ordering of $\{\mathfrak{F}_i\}$, is cofinal in the family of all finite open coverings of M.*

(b) *The nerves of $\{\mathfrak{U}_i^{\delta_i}\}$ and their projections are the same as those of $\{\mathfrak{F}_i\}$; hence projections are unique and onto.*

5. **Induced coverings.** At this stage it is necessary to examine the coverings of subsets of M induced by a covering of M.

(1.12) Definition. If N is a closed subset of M, and $\mathfrak{F}=\{F_i\}$ a finite covering of M by closed sets, then $\mathfrak{F}'=\{F_i'\}$, $F_i'=F_i\cap N$, will be called *the covering of N induced by* $\mathfrak{F}$.

We denote the nerve of $\mathfrak{F}'$ by Φ'. Φ' is isomorphic to a closed subcomplex of Φ, the nerve of $\mathfrak{F}$.

Now consider two f.r.c.c.'s of M, $\mathfrak{F}_\mu<\mathfrak{F}_\lambda$, and the induced $\mathfrak{F}_\mu'<\mathfrak{F}_\lambda'$. In this case π_λ^μ is unique and onto. What can be said for $\pi_\lambda^{\mu\prime}$? Certainly $\pi_\lambda^{\mu\prime}$ can be considered as uniquely defined since we can appropriate for it the behavior of π_λ^μ in case of ambiguity. Thus $F^\mu\subset F^\lambda$ implies $F^{\mu\prime}\subset F^{\lambda\prime}$. We can accept *this* projection and ignore any new possibilities. That $\pi_\lambda^{\mu\prime}$ is *onto* follows easily. By (1.9) each F^λ is an exact union of F^μ's. Consequently, $F^\lambda\cap N$ is an exact union of $F^\mu\cap N$'s. The rest of the argument is identical with the proof of (1.8). We have then

(1.13) *If* $\mathfrak{F}_\nu$ *is a f.r.c.c. of* M *and* $\mathfrak{F}_\nu'$ *the induced covering of* $N \subset M$, *then* $\mathfrak{F}_\mu < \mathfrak{F}_\lambda$ *implies that* $\mathfrak{F}_\mu' < \mathfrak{F}_\lambda'$ *and further that the projection* $\pi_\lambda^{\mu\prime}$: $\Phi_\mu' \mapsto \Phi_\lambda'$ *is onto.*

The corresponding steps may be repeated for *fringings* of the $\mathfrak{F}_\lambda'$.

(1.14) DEFINITION. The δ-fringing of $\mathfrak{F}_\lambda'$ on N is called the *open covering induced by the* δ-*fringing of* $\mathfrak{F}_\lambda$ *on* M. We denote it by $\mathfrak{U}_\lambda^{\delta\prime}$.

Since we can take sequences with diameters tending to zero as cofinal in the family of coverings on N, we get, corresponding to (1.11),

(1.15) *Under the hypothesis of* (1.11) *we have*:

(a) $\{\mathfrak{U}_i^{\delta_i'}\}$ *ordered by the linear ordering of* $\{\mathfrak{F}_i\}$ *is cofinal in the family of all open coverings of* N.

(b) *The corresponding nerve projections are uniquely defined and onto.*

6. **Nerves of coverings of neighborhoods.** Let N be a compactum embedded in a parallelotope P. Let $\{\mathfrak{F}_i\}$ be a sequence of f.r.c.c.'s of P with mesh ϵ_i tending to zero. For each i we consider the covering of N induced by $\mathfrak{F}_i$. We write its nerve as $\Phi_i(N)$. Likewise we consider for any $\eta > 0$ the η-neighborhood $U_\eta(N)$. Let $\Phi_i(\overline{U}_\eta(N))$ be the nerve of the induced covering of its closure.

The following lemma will be useful.

(1.16) *For each* ϵ_i *there is an* η_i *such that if* $U_{\eta_i}(N)$ *is written* $U_i(N)$, *then* $\Phi_i(\overline{U}_i(N)) = \Phi_i(N)$.

This is similar to a lemma of A. D. Wallace for a more general situation(9).

The proof for this case is simple. The nerves of $\overline{U}_i(N)$ and N differ, if at all, either because of sets of the covering meeting the neighborhood but not N, or because of intersections outside of N of sets which do meet N. Hence if η_i is picked less than the minimum distance from N to any such closed set or intersection, the nerves will be the same.

By the definition (1.14), it follows that

(1.16.1). (1.16) *holds for any tapered fringing of the sequence* $\{\mathfrak{F}_i\}$.

The following obvious modification will be used.

(1.17) *In* (1.16) *we can pick* $\{\eta_i\}$ *so that* $\eta_{i+1} < \eta_i$.

7. **Kuratowski mappings.** We shall frequently make use of a familiar tool, the *Kuratowski mapping*(10). The Kuratowski mapping is one of a class of mappings called *barycentric* by Hurewicz-Wallman(11). Let $\mathfrak{U} = \{U_i\}$ be a finite open covering of M. We shall denote by κ the particular barycentric mapping known as the Kuratowski mapping and defined by

(1.18) DEFINITION OF κ. $\kappa(x) = (x_0, \cdots, x_i, \cdots, x_n)$ where $\{x_i\}$, the barycentric coordinates of $\kappa(x)$ in the simplex of Φ whose vertices stem from all the U_i containing x, are given by $x_i = \rho(x, \mathcal{C}(U_i))$.

(9) See Lefschetz [9, chap. 7, (14.4), p. 263].

(10) See Alexandroff-Hopf [2, chap. 9, §3, N.4.]

(11) Cf. Hurewicz-Wallman [7].

It will be convenient to make available certain simple propositions about κ. When several coverings of M are involved κ will be given the index of the covering generating it. One obvious property is the following.

(1.19) *If Φ is embedded naturally with respect to $M \subset P$, then $\rho(x, \kappa(x)) < 3\epsilon$ for every point x in M, where ϵ is the mesh of the covering.* (That is, κ is a 3ϵ-deformation in a neighborhood of M.)

Now suppose $\mathfrak{U}_\mu < \mathfrak{U}_\lambda$. We prove

(1.20) $\kappa_\lambda \cdot - \cdot \pi_\lambda^\mu \kappa_\mu$.

Proof. Pick any point x of M. x belongs to certain sets, say $U_0^\mu \cap \cdots \cap U_n^\mu$ of $\mathfrak{U}_\mu$ and hence $\kappa_\mu(x)$ is a point of the simplex $\hat{U}_0^\mu \cdots \hat{U}_n^\mu$. Likewise x is in $U_0^\lambda \cap \cdots \cap U_m^\lambda$ of $\mathfrak{U}_\lambda$ and $\kappa_\lambda(x)$ is a point in $\hat{U}_0^\lambda \cdots \hat{U}_m^\lambda$. But each $\hat{U}_i^\mu$ projects into one of these $\hat{U}_j^\lambda$'s; hence $\pi_\lambda^\mu(\hat{U}_0^\mu \cdots \hat{U}_n^\mu)$ is a face of $\hat{U}_0^\lambda \cdots \hat{U}_m^\lambda$. q.e.d.

Again we take M, N, $\mathfrak{F}$, $\mathfrak{F}'$, Φ, Φ', $\mathfrak{U}$, $\mathfrak{U}'$, as in (1.14). Φ' is a subcomplex of Φ, so for a point x of N, both $\kappa(x)$ and $\kappa'(x)$ are points of Φ. $\kappa | N$ will denote κ as a mapping of N, ignoring $M-N$. These are related by

(1.21) $\kappa | N \cdot - \cdot \kappa'$.

Proof. Select any x of N. x belongs to a certain number of sets of $\mathfrak{U}$, say $U_0 \cap \cdots \cap U_n$. This means that $F_0 \cap \cdots \cap F_n \neq 0$ and $\hat{F}_0 \cdots \hat{F}_n$ is a simplex of Φ. But x also is in $U_0' \cap \cdots \cap U_m'$, $m \leqq n$. $\hat{F}_0' \cdots \hat{F}_m'$ is a simplex of Φ', hence of Φ. Considered as a simplex of Φ, $\hat{F}_0' \cdots \hat{F}_m'$ is a face of $\hat{F}_0 \cdots \hat{F}_n$. q.e.d.

8. **Special coverings.** When the nerve Φ of a covering $\mathfrak{U}$ of a parallelotope P is embedded naturally, the barycentric mapping κ gives a mapping of Φ into itself. We shall limit ourselves to coverings for which κ is homotopic to the identity *on* Φ. To do this we slice P^n successively into smaller cubes. Such cubes form a covering of P^n by regular closed sets. For instance $\mathfrak{F}_m(P^n)$ is obtained by hyper-planes $x_j = k/2^m$ for $j = 1, \cdots, n$ and for $k = 0, +1, +2, \cdots, +2^m - 1$. $\mathfrak{F}_m(P^\omega)$ is obtained by the equations on $x_1, \cdots, x_m$, used in defining the mth covering of P^m. The remaining coordinates are free except that they range only in P^ω: $0 \leqq x_i \leqq 1/2^{i-1}$. This sequence of coverings is replaced by a *tapered fringing* $\{\mathfrak{U}_i\}$. Clearly mesh $\mathfrak{U}_i \to 0$.

(1.22) Definition. Such a sequence of open coverings we shall call *special.*

Note that $\Phi_m(P^\omega) = \Phi_m(P^m)$. Hence we demonstrate the desired property of κ for $\Phi_m(P^n)$. By the construction it is apparent for any m that every set of $\mathfrak{F}_m(P^n)$ meets $2^n - 1$ other such sets at a "corner." Thus every vertex of $\Phi_m(P^n)$ is incident on a $2^n - 1$ dimensional simplex. Moreover $\Phi_m(P^n)$ is a $2^n - 1$ dimensional complex. Every such cluster of 2^n cubes forms a larger n-dimensional cube. By taking the vertices of the nerve as the midpoints of the corresponding cubes, we get the $(2^n - 1)$-simplex of the cluster well inside the larger cube. This means that every point of Φ lies in one of the 2^n sets defining the $(2^n - 1)$-simplex in which it lies. Consequently for these special coverings of P, we get

(1.23) *For special coverings of P*

$$\kappa(x) \cdot - \cdot x \qquad \textit{for every } x \textit{ in } \Phi.$$

Now let us consider a subset $M \subset P$. The special covering of P induces a covering of M. By (1.21) we have at once

(1.24) *For a covering of M induced by a special covering of P*

$$\kappa(x) \cdot - \cdot x \qquad \textit{for every } x \textit{ in } \Phi \cap M.$$

For later reference, we point out that the properties of (1.16) hold for the coverings induced by a sequence of special coverings.

(1.25) *The sequence of coverings of R induced by the special coverings of P:*

(a) *is cofinal in the family of all open coverings of R;*

(b) *has unique nerve-projections which are onto.*

Section II. Neighborhood and net homotopies

In this section we define and compare two new kinds of homotopy. (For a discussion of ordinary homotopy see §10.) The first of these, *neighborhood homotopy*, is the more intuitive geometrically. The other, *net homotopy*, uses the technique of nets[3]. The comparison of the two concepts will be carried out by means of the methods of section I. These notions can be described for more general situations. But for the present we deal with a compactum R embedded in a parallelotope P.

9. **Neighborhood homotopy.** Let S be a topological space and t, t' two mappings of S into R.

(2.1) Definition. t, t' will be called *neighborhood homotopic*, written $t \approx_U t'$, if t and t' are homotopic ($t \approx t'$) in *every* open set containing R.

Clearly $t \approx t'$ implies $t \approx_U t'$.

(2.2) **Example.** We take R as the subset of P^2 defined by: (a) $x=0$, $|y| \leqq 1$; (b) $y = \sin(1/x)$, $0 < x \leqq 2/\pi$; and (c) $y=1$, $2/\pi \leqq x \leqq (2+\pi)/\pi$. S is to be taken as the unit interval I. t, t' are the homeomorphic mappings $I \rightarrow (a)$, $I \rightarrow (c)$ respectively. Here we have t is not homotopic to t' yet $t \approx_U t'$.

(2.3) **Example.** With the preceding space, we can take t as the identity and t' as a constant mapping sending R into the point 0. The relation $t \approx_U t'$ here means that R is *neighborhood-contractible*; yet it clearly is not contractible.

(2.4) *When R is an ANR, $t \approx_U t'$ is equivalent to $t \approx t'$.*

Under our assumptions, the following is obvious.

(2.5) *$t \approx_U t'$ if $t \approx t'$ in $U_{\epsilon_i}(R)$ for a sequence of ϵ_i's approaching zero.*

10. **Digression on ordinary homotopies.** We shall be dealing at length with ordinary homotopies; so it will be worth while to establish certain conventions for their treatment.

Let us consider mappings $t_i: M \rightarrow N$.

(2.6) DEFINITION OF HOMOTOPY. To say that $t_0 \approx t_1$ means that there exists a mapping $h_1^0: M \times I \to N$ such that

$$\overset{0}{h_1}(m, 0) = t_0(m),$$
$$\overset{0}{h_1}(m, 1) = t_1(m).$$

Clearly $t_0 \approx t_1$ implies $t_1 \approx t_0$ by the homotopy $h_0^1 = h_1^0\ (m, 1-t)$.

Suppose we have given three spaces M, N, R and mappings t, $t': M \to N$ and s, $s': N \to R$. The following simple facts will be useful. (The symbol $\twoheadrightarrow$ denotes "implies.")

(2.6.1) $t \approx t' \twoheadrightarrow st \approx st'$,

(2.6.2) $s \approx s' \twoheadrightarrow st \approx s't$,

(2.6.3) $t \approx t',\ s \approx s' \twoheadrightarrow st \approx s't'$.

To prove (2.6.3) we need merely exhibit a suitable mapping of $M \times I \to R$. By the hypothesis we have mappings of $M \times I \to N$ and $N \times I$ into R. Let these be $f(m, u): M \times I \to N$, $g(n, u'): N \times I \to R$ where $f(m, 0) = t(m)$, $f(m, 1) = t'(m)$, $g(n, 0) = s(n)$, $g(n, 1) = s'(n)$. We define $h(m, w): M \times I \to R$ as follows:

$$h(m, w) = \begin{matrix} g(t(m), 2w), & 0 \leqq w \leqq 1/2, \\ s'(f(m, 2w - 1)), & 1/2 \leqq w \leqq 1. \end{matrix}$$

Obviously (2.6.1) and (2.6.2) are corollaries of (2.6.3).

We point out also a parallel statement involving the stricter relation $\cdot - \cdot$ for the case where R is a geometric complex

(2.6.2)′ $s \cdot - \cdot s' \twoheadrightarrow st \cdot - \cdot s't$.

Now suppose $t_0 \approx t_1$, $t_1 \approx t_2$ by homotopies h_1^0 and h_2^1 respectively. Then $t_0 \approx t_2$ by a homotopy $h_2^0 = (h_1^0, h_2^1)$ defined by

$$\overset{0}{h_1}(m, 2u) = \overset{0}{h_2}(m, u), \qquad \text{for } 0 \leqq u \leqq 1/2,$$

and

$$\overset{1}{h_2}(m, 2u - 1) = \overset{0}{h_2}(m, u), \qquad \text{for } 1/2 \leqq u \leqq 1.$$

(2.7) Such a homotopy (h_1^0, h_2^1) will be called a *combined homotopy*. The extension to a combination of n homotopies $(h_1, h_2, \cdots, h_n)$ is obvious.

Suppose we have four mappings t_0, t_1, t_0', t_1' of $S \to R$. We assume that the pairs t_0, t_0' and t_1, t_1' are related by homotopies f_0 and f_1. Now let h_1^0 be a homotopy $t_0 \approx t_1$. Clearly (f_0, h_1^0, f_1) give a corresponding homotopy for t_0' and t_1'. We write $(f_0, h_1^0, f_1) = h_1^{0\prime}$. Now h_1^0 and $h_1^{0\prime}$ are both mappings of $S \times I$ into R. It is important to point out that these two mappings are homotopic in a special way. We consider then $S \times I \times I$. $S \times I \times 0$ is mapped by h_1^0. $S \times I \times 1$ is mapped by $h_1^{0\prime}$. $S \times 0 \times I$ is mapped by f_0, while $S \times 1 \times I$ is mapped by f_1. We wish to extend these four mappings to a mapping on the whole of $S \times I \times I$. The equations of the extended homotopy $H(x, u, u')$ may be given by

$$(2.8.1)\quad H(x, u, u') = \begin{cases} f_0(x, u'-3u) \text{ for } u' \geqq 3u,\ u \leqq 1/3, \\ h_1^0(x, u-u'/3) \text{ for } u' \leqq \min (3u, 3(1-u)),\ 0 \leqq u \leqq 1, \\ f_1(x, u'-3u-3) \text{ for } u' \geqq 3(1-u). \end{cases}$$

Now suppose we have given four mappings t_1, t_2, t_3, t_4 of $S \rightarrow R$ and homotopies f_3^4, f_2^3, f_1^2. Clearly the combined homotopies (f_2^3, f_3^4) and (f_1^2, f_2^3) relate the pairs t_4, t_2 and t_3, t_1 respectively. It is now pointed out that

(2.8.2) $(f_2^3, f_3^4) \approx (f_1^2, f_2^3)$ *in such a way that this homotopy agrees with* f_1^2 *and* f_3^4.

Equations for such a homotopy are given by:

$H'(x, u, u') = f_3^4(x, u'-2u)$ for $u' \geqq 2u$,

$H'(x, u, u') = f_2^3(x, u-u')$ for $2u \geqq u' \geqq 2u-1$,

$H'(x, u, u') = f_1^2(x, u'-2u-2)$ for $u' \leqq 2u-1$.

11. **Net homotopy.** The second new kind of homotopy depends on the notion of nets · · · as mentioned above. The particular net of R which we consider is that derived from the family of all open coverings of R and the associated projections. We denote this net by $\Sigma_0(R) = \{\Phi_\lambda; \pi_\lambda^\mu\}$.

(2.9) Definition. *Mapping of a space into a net.* A collection of mappings $\{t_\lambda\}$, $t_\lambda: S \rightarrow \Phi_\lambda$, is called a mapping of S into Σ_0 provided that

$$t_\lambda \approx \pi_\lambda^\mu t_\mu.$$

In practice we shall wish to replace Σ_0 by the Σ defined by coverings induced by the special coverings of (1.22). By (1.15) and the discussion of special coverings, we know that

(2.10) Σ, *the net of special coverings of R, is a sequential spectrum, cofinal in Σ_0 with projections onto.*

Clearly a mapping of S into Σ_0 gives a mapping of S into Σ. On the other hand, if we have a mapping of S into Σ it can be filled out to be a mapping of S into Σ_0. For every Φ_λ not in Σ we pick the smallest λ_i such that $\lambda_i > \lambda$ and write $t_\lambda = \pi_\lambda^{\lambda_i} t_{\lambda_i}$. That the resulting system is a mapping of S into Σ_0 follows readily from (1.3) and (2.6.2)$'$.

(2.11) Definition. *Homotopy of space-net mappings.* If $\{t_\lambda\}$ and $\{t'_\lambda\}$ are space-net mappings, they are called *homotopic* if

$$t_\lambda \approx t'_\lambda \text{ on } \Phi_\lambda, \text{ every } \lambda.$$

This equivalence is written $\{t_\lambda\} \approx \{t'_\lambda\}$.

Obviously $\{t_\lambda\} \approx \{t'_\lambda\}$ on Σ_0 implies the same on the subnet Σ. Moreover, $\{t_\lambda\} \approx \{t'_\lambda\}$ on Σ is sufficient to ensure the equivalence on Σ_0. This is proved directly by use of the cofinality of Σ. For each λ pick the smallest λ_i of Σ which exceeds λ. By assumption there is a homotopy $t_{\lambda_i} \approx t'_{\lambda_i}$. This homotopy projects into a homotopy connecting $\pi_\lambda^{\lambda_i} t_{\lambda_i}$ with $\pi_\lambda^{\lambda_i} t'_{\lambda_i}$. But these are respectively homotopic to t_λ and t'_λ.

This result justifies

(2.12) *The homotopy classes of mappings of a space into a net are in* 1-1 *correspondence with those of any cofinal sequential subnet.*

By applying (2.12) twice it is seen that the word "sequential" may be omitted.

(2.13) DEFINITION. *Induced space-net mappings.* Given S, R, Σ_0, t as before, we consider the space-net mapping $\{t_\lambda\}$, $t_\lambda = \kappa_\lambda t$. ($\kappa_\lambda$ is the Kuratowski mapping of R onto Φ_λ.) This will be called the space-net mapping induced by t.

That $\{\kappa_\lambda t\}$ is a space-net mapping follows from (1.20). First we have directly

(2.14) $\{\kappa_\lambda\}$ *is a space-net mapping of* R *into* Σ_0.

In addition we point out

(2.15) *If* $\{t_\lambda\}$ *is a space-net mapping* $S \to \Sigma_0$ *and* t *is a mapping of* M *into* S, *then* $\{t_\lambda t\}$ *is a space-net mapping of* M *into* Σ_0.

The proof depends only on (2.6.1). By this

$$t_\lambda \approx \overset{\mu}{\pi}_\lambda t_\mu \rightarrow t_\lambda t \approx \overset{\mu}{\pi}_\lambda t_\mu t.$$

(2.16) DEFINITION. Two mappings t, t' of S into R are called *net-homotopic*, written $t \approx_N t'$, if their induced space-net mappings are homotopic.

(2.17) DEFINITION. *Special nets.* By (2.12) we see that it is only necessary to use a cofinal sequential subnet. By (1.25) we see that we need consider only coverings induced by special coverings of P. The corresponding net will be called a *special net* of R.

12. **Comparison.** We now give a comparison of neighborhood-homotopy and net-homotopy.

(2.18) THEOREM. *Net- and neighborhood-homotopy are equivalent.*

Proof that $t \approx_U t' \rightarrow t \approx_N t'$. By assumption $t \approx t'$ in every neighborhood of R. By (1.16) for every nerve of R there is a neighborhood whose corresponding induced covering has precisely the same nerve. For each such neighborhood $U_i(R)$ and nerve $\Phi_i(\overline{U}_i(R)) = \Phi_i(R)$ we have a mapping $\kappa(i)_i$ which carries the homotopy onto $\Phi_i(R)$, giving $\kappa(i)_i t \approx \kappa(i)_i t'$ on $\Phi_i(R)$. But by (1.21), $\kappa(i)_i | R \cdot - \cdot \kappa_i$. By an application of (2.6.2) this gives at once $\kappa(i)_i t \approx \kappa_i t$ on $\Phi_i(R)$. Thus our conclusion is $\kappa_i t \approx \kappa_i t'$ on $\Phi_i(R)$, every i. q.e.d.

(2.19) **Proof that** $t \approx_N t' \rightarrow t \approx_U t'$. For every x of S, $\rho(t(x), \kappa_i t(x)) < 3\epsilon_i$, by (1.19). But $t(x)$ is a point of R and can be joined by a segment to $\kappa_i(t(x))$, the segment lying entirely within $U_{4\epsilon_i}(R)$. Thus our assumption that $\kappa_i t \approx \kappa_i t'$ on $\Phi_i(R)$ leads at once to $t \approx t'$ in $U_{4\epsilon_i}(R)$. (Clearly $\Phi_i(R)$ is contained in this neighborhood of R.) Since $\epsilon_i \to 0$, this gives the required $t \approx_U t'$. q.e.d.

13. **Strong net homotopy.** The equivalent notions of net- and neighborhood-homotopies discussed thus far sometimes do not give as much information as might be desired. Let us look again at the example of (2.2). We now take S as a single point and map it by t onto $(0, 0)$ and by t' onto $(2/\pi, 1)$.

Here we certainly have $t \approx_U t'$. The homotopy for each ϵ-neighborhood of R is the mapping of a segment into a tube of radius ϵ around R. For successive neighborhoods, the homotopies may be taken as arcs with successively more wiggles. But here it is clear that any two of these arcs are homotopic in the larger neighborhood, the homotopy leaving the end points fixed.

As a contrasting example we look at the dyadic solenoid (to be discussed more extensively later)[12]. This too is not arc-wise connected. We let t and t' map a point S into each of two non-arc-wise connected points of the solenoid. Every ϵ-neighborhood is approximately torus-shaped; so again we get $t \approx_U t'$. But in this case small neighborhoods are coiled several times inside the larger ones. Here we find that in general the homotopy arcs are not themselves homotopic because of this multiplicity.

We wish now to strengthen our criteria for neighborhood (and net) equivalence of mappings so as to distinguish between such situations as these. The strong homotopy we need imposes a slight restriction on the families of mappings which shall be recognized as space-net mappings, so this concept will be redefined.

(2.20) DEFINITION. *Strong space-net mappings.* A collection of mappings $\{t_\lambda\}$, $t_\lambda: S \to \Phi_\lambda$, is a strong space-net mapping of $S \to \Sigma_0$ if besides

(1) $t_\lambda \approx \pi_\lambda^\mu t_\mu$ by a homotopy f_λ^μ, we have

(2) $(\pi_\lambda^\nu f_\nu^\mu, f_\lambda^\nu) \approx f_\lambda^\mu$ (cf. (2.7)). (This homotopy is fixed on t_λ and agrees with the homotopy between $\pi_\lambda^\nu \pi_\nu^\mu t_\mu$ and $\pi_\lambda^\mu t_\mu$ implied by (1.3) and (2.6.2).) Restriction (2) merely guarantees that different methods of getting a permanence relation (1) do not differ as far as homotopy is concerned.

(2.20.1) Note that when Σ is a sequential net with unique projections, (2) may be taken as an equality.

(2.21) DEFINITION. *Strong homotopy of strong space-net mappings.* Two strong space-net mappings $\{t_\lambda, f_\lambda^\mu\}$ and $\{t_\lambda', f_\lambda^{\mu\prime}\}$ are strongly homotopic if for every λ there is a homotopy $h_\lambda: t_\lambda \approx t_\lambda'$ on Φ_λ, with the set of all such homotopies $\{h_\lambda\}$ forming a mapping of $S \times I$ into Σ_0: $\{h_\lambda, \phi_\lambda^\mu\}$. It is moreover assumed that the homotopies ϕ_λ^μ agree on $S \times 0$ and $S \times 1$ with f_λ^μ and $f_\lambda^{\mu\prime}$ respectively.

The equivalence just defined will be written $\{t_\lambda\} \approx_{N^*} \{t_\lambda'\}$.

Again we must try to replace Σ_0 by Σ. It is still clear that a mapping into Σ_0 induces a mapping into Σ. The procedure used before for extending a mapping into Σ to one into Σ_0 is still valid since projection preserves homotopy (2.6.1). The real problem lies in this question: does $\{t_\lambda\} \approx_{N^*} \{t_\lambda'\}$ with respect to Σ imply the same for Σ_0? To prove this affirmatively, we must find a scheme for assigning homotopies to all nerves not in Σ. As before to each such λ we associate the smallest index from Σ which exceeds λ, say λ_i. For λ_i there is by assumption a suitable homotopy h_{λ_i}. Its projection into Φ_λ gives a homotopy between $\pi_\lambda^{\lambda_i} t_{\lambda_i}$ and $\pi_\lambda^{\lambda_i} t_{\lambda_i}'$. These are homotopic to t_λ and

[12] See (6.6).

t_λ' by $f_\lambda^{\lambda_i}$ and $f_\lambda^{\lambda_i\prime}$ respectively. We denote by h_λ this combined homotopy $(f_\lambda^{\lambda_i}, \pi_\lambda^{\lambda_i} h_{\lambda_i}, f_\lambda^{\lambda_i\prime})$. Thus far we have followed the method used for ordinary space-net mappings. It remains to ascertain whether the whole collection $\{h_\lambda\}$ satisfies the restrictions of (2.21). This means first that we must assign permanence homotopies ϕ_λ^μ agreeing with f_λ^μ and $f_\lambda^{\mu\prime}$. (f_λ^μ, $f_\lambda^{\mu\prime}$ are given since the whole strong mapping is taken as given.) The construction of $\phi_\lambda^{\lambda_i}$ is given in (2.8). It remains to fill the gaps between pairs of Φ's both of which are outside of Σ. Suppose we have $\mu > \lambda$. Then $h_\mu = (f_\mu^{\mu_i}, \pi_\mu^{\mu_i} h_{\mu_i}, f_\mu^{\mu_i\prime})$ and $h_\lambda = (f_\lambda^{\lambda_i}, \pi_\lambda^{\lambda_i} h_{\lambda_i}, f_\lambda^{\lambda_i\prime})$. A first obvious step toward such a ϕ_λ^μ is the combined homotopy $\phi_\lambda^{\mu\prime} = (\pi_\lambda^\mu \phi_\mu^{\mu_i}, \eta_\lambda^{\mu_i}(\mu, \lambda_i), \pi_\lambda^{\lambda_i}\phi_{\lambda_i}^{\mu_i}, \phi_\lambda^{\lambda_i})$, where $\eta_\lambda^{\mu_i}(\mu, \lambda_i)$ denotes the homotopy between $\pi_\lambda^{\lambda_i}\pi_{\lambda_i}^{\mu_i} h_{\mu_i}$ and $\pi_\lambda^\mu \pi_\mu^{\mu_i} h_{\mu_i}$ given by (1.3) combined with (2.6.2). This is indeed a mapping of $S \times I \times I$ onto Φ_λ mapping $(S\times I\times 0)$ by $\pi_\lambda^\mu h_\mu$ and $(S\times I\times 1)$ by h_λ. But in the form given, the final requirement of (2.21) is not expressly satisfied. This fault is easily surmountable by use of restriction (2) of (2.20) as applied to the f_λ^μ's. Our mapping of $S\times I\times I$ is a double homotopy, giving a homotopy also between the induced mappings of $S\times 0\times I$ and $S\times 1\times I$. But each of these mappings is in turn homotopic to f_λ^μ and $f_\lambda^{\mu\prime}$ by application of this requirement (2) to the mappings from μ_i to λ. By these homotopies the homotopy ϕ_λ^μ is altered so that (2.21) is fully satisfied. The result is the desired ϕ_λ^μ.

The conclusion of this is the following analogue of (2.12):

(2.22) *The strong homotopy classes of strong mappings of a space into a net are in* 1-1 *correspondence with those of any cofinal sequential subnet.*

(2.23) $\{\kappa_\lambda\}$ *is a strong space-net mapping of R into Σ_0.*

This follows at once from the extremely strict conclusions of (1.20) and (1.3).

An immediate corollary is

(2.24) *If t maps S into R, then $\{\kappa_\lambda t\}$ is a strong space-net mapping of S into Σ_0.*

Analogous to (2.16) we make the

(2.25) DEFINITION. Two mappings t, t' of S into R are *strongly net-homotopic* (written $t \approx_{N^*} t'$) if their induced space-net mappings are strongly homotopic.

14. Strong neighborhood homotopy. The net and neighborhood techniques are still parallel. This is shown in the following concept.

(2.26) DEFINITION. t and t' are strongly neighborhood homotopic (written $t \approx_{U^*} t'$) if $t \approx t'$ in every neighborhood U_λ of R, and if moreover we add the restriction that these homotopies $\{h_\lambda\}$ are themselves homotopic; that is, $h_\lambda \approx h_\mu$ (keeping t and t' fixed) in U_λ when $U_\mu \subset U_\lambda$.

In other words, $t \approx_{U^*} t'$ if there is a family of mappings $\{h_\lambda\}$ such that h_λ

maps $S\times I$ into U_λ, and in addition if there is a family of mappings $\{\phi_\lambda^\mu\}$ sending $S\times I\times I$ into U_λ satisfying

$$\phi_\lambda^\mu(S \times I \times 0) = h_\mu(S \times I), \tag{1}$$

$$\phi_\lambda^\mu(S \times I \times 1) = h_\lambda(S \times I), \tag{2}$$

$$\phi_\lambda^\mu(S \times 0 \times I) = t(S) \tag{3}$$

for every value of the second parameter,

$$\phi_\lambda^\mu(S \times 1 \times I) = t'(S) \tag{4}$$

for every value of the second parameter.

If we have a sequence of neighborhoods getting arbitrarily close to R, it is clear that every neighborhood of R has one out of the sequence inside it. Hence strong neighborhood homotopy with respect to such a cofinal sequence implies strong neighborhood homotopy as defined above. Thus we have a new version of (2.5).

(2.27) $t\approx_{U^*}t'$ *if there is a sequence* $\{h_i\}$ *of homotopies satisfying the conditions imposed in* (2.26) *for the* $U_{\epsilon_i}(R)$ *generated by a sequence of* ϵ_i *approaching zero.*

The results (2.22) and (2.27) enable us to consider net- and neighborhood-homotopies in the strong sense with reference only to sequences of nets or neighborhoods.

The following theorem will be a corollary of a more general result of a later section.

(2.28) THEOREM. $t\approx_{U^*}t'\leftrightarrows t\approx_{N^*}t'$. (See (3.10).)

(2.29) THEOREM ON SIMPLICIAL APPROXIMATION. *If K is a geometric complex, $\{t_\lambda\}$ a mapping of K into Σ_0, and Σ a cofinal sequential subnet of Σ_0, then there is a space-net mapping $\{t_\lambda^*\}$ strongly homotopic to $\{t_\lambda\}$ so that the mappings $\{t_\lambda^*\}$ into Σ are simplicial.*

By (2.22) it suffices to prove

(2.30) *If $\{t_i\}$ is a strong space-net mapping of K into a special sequential net Σ, then there is a strongly homotopic simplicial space-net mapping $\{t_i^*\}$.*

Proof. We define t_i^* inductively on sufficiently fine regular subdivisions of K. Assume that t_i^*, $t_i^*\approx t_i$, has been defined simplicial on $K^{(j(i))}$. Now define $t_{i+1}^*=t_{i+1}$ on a subdivision of K sufficiently fine and so that $j(i+1)\geqq j(i)$.

For each i, we know that $t_i^*\cdot - \cdot t_i$.

$\{t_i^*\}$ is a mapping but not yet a strong space-net mapping. To get this we must assign permanence homotopies $\{f_i^{*i+k}\}$. By (2.20.1) we consider only permanences of the form f_i^{*i+1}, defined as $(h_i, f_i^{i+1}, \pi_i^{i+1}h_{i+1})$. A direct application of the method of (2.8) shows that (2.21) is satisfied if we take as h_i the linear homotopy indicated by $t_i^* \cdot - \cdot t_i$.

As an example of the manner in which the strong homotopy conditions may be fitted together, we give the following lemma. It is of course implied by the stronger (2.22).

(2.31) *Let* $\{t_i\}$ *be a strong space-net mapping, and then consider* $\{\pi_i^{\eta(i)} t_{\eta(i)}\}$ *where* $\eta(i)$ *is a monotone increasing function changing indices. Then* $\{t_i\}$ *and* $\{\pi_i^{\eta(i)} t_{\eta(i)}\}$ *are strongly homotopic.*

Proof. We show merely that $f_i^{\eta(i)}$ given by the following combined homotopy $(\pi_i^{\eta(i)-1} f_{\eta(i)-1}^{\eta(i)}, \cdots, \pi_i^{i+1} f_{i+1}^{i+2}, f_i^{i+1})$ will serve as in h_i. This follows directly by the construction given in (2.9). The four mappings in this lemma are replaced for every i by $\pi_i^{\eta(i+1)} t_{\eta(i+1)}$, $\bar{\pi}_i^{\eta(i)} t_{\eta(i)}$, $\pi_i^{i+1} t_{i+1}$, t_i. The lemma yields $\pi_i^{i+1} h_{i+1} \approx h_i$ with the proper restrictions specified in (2.21).

Thus far in this section we have introduced four principal kinds of homotopy. These homotopies have been shown to fall into pairs which turn out to be equivalent. The two sets of pairs are distinct, as was shown by the example involving the Vietoris or dyadic solenoid.

Section III. A generalization of the notion of mapping

15. **Mappings towards a space.** In dealing with nets we began with space-net mappings induced by space-space mappings (2.13). Yet most of the discussion of such space-net mappings has been independent of any space-space mapping[13]. The neighborhood point of view has until now been focused on particular mappings of a space S into R. It is to be remarked that the collection of homotopies $\{h_\lambda\}$ of (2.26) is of a different character if we think of it as a function relating $S \times I$ and R. Neighborhood concepts parallel to space-net and strong space-net mappings will now be defined.

(3.1) Definition. Let $\{U_\lambda\}$ be the set of all neighborhoods of R and $\{t_\lambda\}$ a set of mappings such that for each λ, $t_\lambda(S) \subset U_\lambda$. $\{t_\lambda\}$ is a *mapping of S towards R* if $\mu > \lambda$ implies that $t_\mu \approx t_\lambda$ in U_λ; that is, provided that there is a collection $\{f_\lambda^\mu\}$, $\mu > \lambda$, of homotopies satisfying

(1) $$f_\lambda^\mu : S \times I \to U_\lambda,$$

(2) $$f_\lambda^\mu(S \times 0) = t_\mu(S),$$

(3) $$f_\lambda^\mu(S \times 1) = t_\lambda(S).$$

(3.2) Definition. If $\{t_\lambda\}$ and $\{t'_\lambda\}$ are two mappings of S towards R, then $\{t_\lambda\} \approx_U \{t'_\lambda\}$ means that there exists a set $\{h_\lambda\}$ where h_λ is a homotopy in U_λ of t_λ with t'_λ.

(3.3) *For any set of ϵ_i approaching zero it is sufficient to consider only the sequence of neighborhoods* $(U_{\epsilon_i}(R))$.

(13) Note for instance (2.21) and (2.29).

This statement involves the following facts, similar to those discussed before:

(i) the sequence of neighborhoods mentioned is cofinal, and hence

(ii) there is a 1-1 correspondence between homotopy classes as defined in (3.2) and homotopy classes defined with respect to the subsequence specified.

The proof of (i) is obvious. To demonstrate (ii) it is necessary only to show that $\{t_\lambda\} \approx_U \{t'_\lambda\}$ with respect to the subsequence implies the result for the whole family of neighborhoods. This is clearly a result of the transitivity of the homotopy relation.

As an analogue of strong space-net mappings we now state

(3.4) DEFINITION. *Strong mapping towards a space.* A mapping towards R is said to be strong if its homotopies f_λ^μ satisfy

$$(f_\nu^\mu, f_\lambda^\nu) \approx f_\lambda^\mu,$$

this homotopy leaving t_μ, t_λ fixed. (The notation (f, f') for combined homotopy was explained in (2.7).)

To relate these stricter mappings we use a slight extension of (2.26) namely

(3.5) DEFINITION. *Strong homotopy of strong mappings towards a space.* We write $\{t_\lambda\} \approx_{U^*} \{t'_\lambda\}$ if there are homotopies $\{h_\lambda\}$ as in (3.2) which form themselves a mapping of $S \times I$ towards R with permanence relations ϕ_λ^μ agreeing with f_λ^μ and $f_\lambda^{\mu'}$. This last stipulation replaces conditions (3) and (4) of (2.26) by

$$(3)' \qquad \phi_\lambda^\mu(S \times 0 \times I) = f_\lambda^\mu(S \times I),$$

$$(4)' \qquad \phi_\lambda^\mu(S \times 1 \times I) = f_\lambda^{\mu'}(S \times I).$$

The proof that we can limit our study to strong mappings defined with respect to a cofinal sequence of neighborhoods follows precisely the lines of the proof of (2.22). It is slightly simplified by the absence of projections and of the function η. We state the equivalence in

(3.6) *The strong homotopy classes of strong mappings towards a space R are in* 1-1 *correspondence with those defined with respect to any cofinal sequence of neighborhoods.*

(3.6.1) Any strong mapping towards a space considered with respect to a cofinal sequence is strongly equivalent to a similar mapping where $f_j^{j+2} = (f_{j+1}^{j+2}, f_j^{j+1})$, and so on.

This is obvious if we take h_j as the identity.

(3.7) **Examples.** The following items will illustrate the scope and use of the concepts just introduced.

(3.7.1) Any mapping of S onto R is a strong mapping towards R.

(3.7.2) The mappings t and t' of the two examples mentioned on pages 285 and 286 are mappings towards which are homotopic. But for the second of these examples the mappings are not strongly homotopic.

We may restate (2.26) in

(3.8) *Two neighborhood-homotopic mappings are strongly neighborhood-homotopic if their homotopies are a mapping towards.*

16. **Net mappings and mappings towards.** We wish now to prove the equivalence of the concepts of net mappings and mappings towards in both the weak and strong senses. A fact which we have used before, that the nerves of our special sequential subnet get arbitrarily close to R, serves to point out that every nerve lies in a neighborhood and every neighborhood contains a nerve. We limit ourselves to the sequential case since its equivalence to the general situation has been proved. The mere fact that every neighborhood contains a nerve tells us at once that any space-net mapping is a mapping towards when the nerves are imbedded naturally. For by connecting every point of Φ_{i+1} with its image under π_i^{i+1} in Φ_i, we get segments in the neighborhood containing Φ_i along which a homotopy may be defined. Another way of saying this is that the projections π_i^{i+1} are the result of deformations on the "fundamental complex"[14] of R, imbedded naturally. We now need a correspondence going in the other direction. Let $\{t_i\}$ be a mapping towards R defined with reference to our cofinal sequence of neighborhoods $U_{\epsilon_i}(R)$. Referring to (1.16) and (1.17), we now consider the collection $\{\kappa(i)_i t_i\}$. As before (in proof of (2.18)) $\kappa(i)_i$ is the Kuratowski mapping of $\overline{U}_i(R)$ onto $\Phi_i(U_i(R))$ which by (1.16) is the same as $\Phi_i(R)$. Thus for every i, $\kappa(i)_i t_i$ is a mapping of S into $\Phi_i(R)$. It remains to show that condition (1) of (2.20) is satisfied, since by (2.20.1) restriction (2) is trivial for the sequential case. Since $\{t_i\}$ is a (strong) mapping towards R, we have the homotopy $t_{i+1} \approx t_i$ on $U_i(R)$. As a consequence of (2.6.1), this yields

$$(\alpha) \qquad \kappa(i)_i t_{i+1} \approx \kappa(i)_i t_i$$

on $\Phi_i(R)$. The general result of (1.20) may be applied thus:

$$(\beta') \qquad \pi(i+1)_i^{i+1} \kappa(i+1)_{i+1} \cdot - \cdot \kappa(i+1)_i$$

which by (2.6.2)′ becomes

$$(\beta) \qquad \pi(i+1)_i^{i+1} \kappa(i+1)_{i+1} t_{i+1} \cdot - \cdot \kappa(i+1)_i t_{i+1}.$$

Now $\Phi_i(\overline{U}_{i+1}(R)) = \Phi_i(\overline{U}_i(R)) = \Phi_i(R)$; so we may apply (1.21) which here gives

$$(\gamma') \qquad \kappa(i+1)_i \cdot - \cdot \kappa(i)_i \mid \overline{U}_{i+1}(R),$$

or by (2.6.2)′

$$(\gamma) \qquad \kappa(i+1)_i t_{i+1} \cdot - \cdot \kappa(i)_i t_{i+1},$$

(14) Cf. Lefschetz [10, p. 325].

since t_{i+1} maps S into the smaller neighborhood. Combining α, β, γ we have

$$(\delta') \qquad \pi(i+1)_i^{i+1}\kappa(i+1)_{i+1}t_{i+1} \approx \kappa(i)_i t_i.$$

Now since $\Phi_{i+1}(\overline{U}_{i+1}(R)) = \Phi_{i+1}(R)$ and $\Phi_i(\overline{U}_{i+1}(R)) = \Phi_i(R)$, we know that $\pi(i+1)_i^{i+1} = \pi_i^{i+1}$. Thus we have finally the desired permanence homotopy

$$(\delta) \qquad \pi_i^{i+1}\kappa(i+1)_{i+1}t_{i+1} \approx \kappa(i)_i t_i.$$

This completes the proof of:

(3.9) $\{\kappa(i)_i t_i\}$ *is a strong space-net mapping.*

We wish now to prove the following:

(3.10) (*Strong*) *homotopy classes of* (*strong*) *space-net mappings are in one-one correspondence with* (*strong*) *classes of* (*strong*) *mappings towards.*

We have set up a method of going from one family to another. It is clear that the step of passing from nets to neighborhoods carries homotopies, since the transition is merely in point of view. It should be noted that any class of mappings towards can be derived from a space-net mapping. Let $\{t_i\}$ be a mapping towards. From it we get $\{\kappa(i)_i t_i\}$, a space-net mapping. Let us now consider t_i as a mapping into $U_{4\epsilon_i}(R) = U_i'$. The set of neighborhoods U_i' is cofinal, and $\{t_i\}$ considered in this way is strongly homotopic to the original. Moreover U_i' is large enough so that $\kappa(i)_i$ is a $3\epsilon_i$-deformation along segments in U_i'. It follows that $\{\kappa(i)_i t_i\}$ considered as a mapping towards is strongly homotopic to $\{t_i\}$.

In the other direction $\kappa(i)_i h_i$ gives a homotopy on Φ_i between $\kappa(i)_i t_i$ and $\kappa(i)_i t_i'$. In fact, if we use (3.9) with h_i in place of t_i, we get the desired homotopies directly for the strong case. We have then:

(3.11) If $\{t_i\} \approx_{U^*} \{t_i'\}$ then it follows that $\{\kappa(i)_i t_i\} \approx_{N^*} \{\kappa(i)_i t_i'\}$. Likewise $\{t_i\} \approx_U \{t_i'\}$ implies $\{\kappa(i)_i t_i\} \approx_N \{\kappa(i)_i t_i'\}$.

To prove (3.10) we need show now only that every class of mappings in the net sense can be derived from a neighborhood class by the procedure of (3.9). We suppose now that $\{t_i\}$ is a strong space-net mapping. Following the procedure used before, we consider for each Φ_i a neighborhood (U_{j_i}) of R containing it $\cdots$ this choice being made monotone and cofinal in the sequence of neighborhoods of R given by (1.16)–(1.17). Since our sequence of nerves is assumed special, we can apply (1.24) which yields $\kappa(j_i)_i t_i \cdot - \cdot t_i$ on $\Phi_i(R) = \Phi_i(\overline{U}_i(R)) \subset \Phi_i(\overline{U}_{j_i}(R))$ and hence $\pi((j_i))_{j_i}^i \kappa(j_i)_i t_i \cdot - \cdot \pi(j_i)_{j_i}^i t_i$ on $\Phi_{j_i}(U_{j_i}(R)) = \Phi_{j_i}(R)$. But by (1.20)

$$\pi(j_i)_{j_i}^i \kappa(j_i)_i t_i \cdot - \cdot \kappa(j_i)_{j_i} t_i.$$

Therefore

$$\kappa(j_i)_{j_i} t_i \cdot - \cdot \pi(j_i)_{j_i}^i t_i = \pi_{j_i}^i t_i.$$

By (2.31), we know that $\{\pi_{j_i}^i t_i\} \approx_{N^*} \{t_i\}$. It remains to point out that the permanence relations f_i^{i+1} are affected in exactly the same way: that is,

$$\kappa(j_i)_{j_i}\overset{i+1}{f_i} \cdot - \cdot \overset{i}{\pi}_{j_i}\overset{i+1}{f_i}.$$

Since this $\cdot - \cdot$ relation holds for both the mappings and their permanences, we get

$$\{\kappa(j_i)_{j_i}t_i\} \approx_{N^*} \{\overset{i}{\pi}_{j_i}t_i\} \approx_{N^*} \{t_{j_i}\} \approx_{N^*} \{t_i\}.$$

This completes the proof of (3.10).

SECTION IV. HOMOTOPY GROUPS

In this section we shall introduce groups associated with the mappings and homotopies presented in sections II and III. These groups will be patterned after the homotopy groups of Hurewicz. The kth homotopy group (in the sense of Hurewicz) of a space R at a point p has as elements *restricted* homotopy classes of *admissible* mappings([15]). A very brief review of admissible mappings and restricted equivalence follows. An admissible mapping is either a mapping of an oriented k-sphere (on which a base-point q has been selected) into R in such a way that q goes into p, or a mapping of a closed oriented k-cell into R in such a way that the boundary of the closed cell is mapped into p. There is a one-one correspondence between admissible mappings of the k-sphere with base-point q and admissible mappings of the closed cell. This is a consequence of the fact that a punctured k-sphere is homeomorphic to a k-cell without its boundary. Since this correspondence exists, it is convenient to think of elements of the kth homotopy group as classes of admissible sphere-mappings (that is, mappings of spheres) or admissible cell-mappings, quite interchangeably.

Two admissible mappings of a k-sphere into R are *restrictedly* homotopic if they are homotopic in such a way that q goes into p throughout the homotopy. A similar rule applies for admissible cell-mappings. Two admissible mappings belong to the same restricted homotopy class if they are related by a finite chain of equivalences such as these just given: between two cell-mappings, two sphere-mappings, or between a cell-mapping and a sphere-mapping.

The restricted homotopy classes of admissible mappings are elements of the kth homotopy group $\pi_k(R, p)$. Later in this section, a brief summary of the procedure for combining these elements is given. In the following section certain well known properties are reviewed.

We proceed now to a description of the analogous concepts for net- and neighborhood-homotopies.

17. Admissible space-net mappings and admissible mappings towards. If T is a closed subset of S, and M a closed subset of R, we can consider the net of M (a subnet of that of R) and space-net mappings of T into this net. Then it makes sense to deal with those space-net mappings of S into the net

(15) The term "admissible" is taken from Eilenberg who uses it in this sense in some unpublished notes on homotopy.

of R which are extensions of the T to M mappings. Furthermore we can define a restricted homotopy among the S to R mappings by insisting that T be sent into the subnet of M throughout all homotopies. Similarly we can limit our study to mappings of S towards R which send T towards M. This possibility will not be developed in detail since in this paper we shall not use the relative homotopy groups which require such machinery. Throughout our discussions we shall need only two types of restricted mappings. In particular the subset M will always be a single base-point p. Hence we can replace the phrase "T towards M" by "T towards p." We shall moreover be considering only two main choices for S and T. When S is a sphere S^n, T will be a fixed base-point q of S. When S is a cell E^n, T will be the boundary of E^n, $\mathcal{B}(E^n)=S^{n-1}$. For $S=S^n\times I$, T will be $q\times I$, and so on.

(4.1.1) Definition. We apply the term *basic simplex* to the σ_λ in each Φ_λ whose sets in the corresponding covering contain p, the base-point of R. Clearly $\pi^\lambda_\mu\sigma_\lambda\subset\sigma_\mu$.

(4.1) Definition. $\{t_\lambda\}$ is an *admissible strong space-net mapping* if $t_\lambda(T)=p_\lambda\subset\sigma_\lambda$ and $f^\mu_\lambda(T\times I)\subset\sigma_\lambda$. $t_\mu(T)\subset\sigma_\mu$ suffices for a *weak admissible mapping*.

(4.2) Definition. Two admissible mappings $\{t_\lambda\}$, $\{t'_\lambda\}$ are *restrictedly homotopic* if all the homotopies needed for unrestricted homotopy leave the image of T in the basic simplex at every level; that is, for every λ.

For example, two strongly homotopic admissible strong space-net mappings $\{t_\lambda\}$ and $\{t'_\lambda\}$ are strongly *restrictedly* homotopic if $h_\lambda(T\times I)\subset\sigma_\lambda$, and moreover $\phi^\lambda_\mu(T\times I\times I)\subset\sigma_\lambda$.

Since projections of basic simplexes lie in basic simplexes, and since the function η used for (2.22) is a homotopy carrying no point out of its closed simplex, the proof of (2.22) goes through step by step for restricted homotopies. We have then

(4.3) *The strong (weak) restricted homotopy classes of space-net mappings are in one-one correspondence with those defined with respect to any cofinal sequential subnet.*

Similarly (3.6) becomes at once

(4.4) *The strong (weak) restricted homotopy classes of mappings towards are in one-one correspondence with those defined for cofinal sequences of neighborhoods.*

This means that we pick a suitable sequence of $\epsilon_i\to 0$ and consider homotopies which send $T\times I$ into $U_{\epsilon_i}(p)$, and so on.

We need now to show that the theorem of equivalence of the net and neighborhood (3.10) notions is valid when admissible mappings and restricted homotopies are substituted. In the first place, it was pointed out that a strong (weak) space-net mapping may be considered as a strong (weak) mapping towards.

By the cofinality assumptions we have nerves and basic simplexes getting arbitrarily close to R and p respectively. Moreover every set $U_{\epsilon_i}(R)$ must contain a nerve whose basic simplex lies in $U_{\epsilon_i}(p)$. This situation maintains the desired easy passage from net to neighborhood. In the other direction, we point out that $\kappa(i)_i$ maps a neighborhood of p (in fact the intersection of the open sets of $\mathfrak{U}_i$ which include p) into the ith basic simplex. Thus we have the transfer from *admitted mappings towards* to *admitted space-net mappings*. The rest of the argument for equivalence is unchanged by the new assumptions; so we conclude that

(4.5) *Strong* (*weak*) *restricted classes of admissible space-net mappings are in one-one correspondence with strong* (*weak*) *restricted classes of admissible mappings towards.*

We might point out that when S is a polyhedron, T a subpolyhedron, and M is an ANR the restriction "T towards M" may be replaced by "T into M." For, for a cofinal sequence of neighborhoods, every mapping into a neighborhood is homotopic to a mapping into M, and since S is a polyhedron the homotopy on T may be extended to S. That this modification is not possible in general is shown by the following example. We take the space given in (2.2) plus its reflection in the y-axis. Let R be the whole space thus defined between $-(2+\pi)/\pi$ and $(2+\pi)/\pi$. M is the closed subset given by $-2/\pi \leqq x \leqq 2/\pi$. S is again a unit interval AB while the end point B is taken as T. We consider two mappings of AB, one, t, into the segment $2/\pi \leqq x \leqq (2+\pi)/\pi$ and t', the reflected mapping. Then $t(B) = (2/\pi, 1)$ and $t'(B) = (-2/\pi, 1)$. Since these are both mappings into R, they may be considered as mappings towards. We have the restricted homotopies in the sense defined, but there is no homotopy between them sending B actually into M at all stages.

18. Strong and weak homotopy groups. The study of restricted homotopies makes it possible to introduce the corresponding homotopy groups. As in the Hurewicz homotopy group theory, we can pass freely from admissible mappings of a sphere to admissible mappings of a cell. Henceforth this will be done without mention.

In order to make our classes of admissible mappings into a group, we must introduce an operation and an inverse. These steps again follow well known procedures in the theory of the Hurewicz homotopy groups[16].

Let t and t' be two admissible mappings of the closed cell $\overline{E}^k$ into R. We define a third such mapping as follows. Let $\overline{E}_1^k$ and $\overline{E}_2^k$ be two disjoint closed k-cells in the interior of $\overline{E}^k$. Now let t'' be the mapping of $\overline{E}_1^k$ as if by t and $\overline{E}_2^k$ as if by t', the rest of $\overline{E}^k$ being mapped into p, the base-point in R. We denote the class of t'' by $\tau+\tau'$, where τ is the class of t and τ' that of t'. We may even write $t+t'=t'' \cdots$ but of course it must be remembered that such

---

([16]) The particular presentation which we review follows some lectures given by Ralph H. Fox at Princeton in 1940. A similar method for the case $k=2$ is given by H. Robbins, Trans. Amer. Math. Soc. vol. 49 (1941) p. 310.

a sum is unique only within homotopy. It is easily shown that this method of combination is a real class operation, giving a unique sum for the classes τ and τ'. The inverse of an element $\tau=\{t\}$ is merely the class of admissible mappings $\{t'\}$ where t' is the mapping t defined on $\overline{E}^k$ after its orientation has been reversed. These definitions of sum and inverse lead readily to the group $\pi_k(R, p)$. The construction of $\tau+\tau'$ (written $\tau\tau'$ for $k=1$) shows that $\pi_k(R, p)$ is abelian for $k>1$. In general, as is well known, the group $\pi_1(R, p)$ is not abelian.

Let $\{t_\lambda, f_\lambda^\mu\}$ and $\{t_\lambda', f_\lambda^{\mu\prime}\}$ be two strong admissible mappings of $\overline{E}^n$ towards R. The sum of two such mappings will be defined by means of the sum just described for each λ. $t_\lambda+t_\lambda'$ is to be a (not unique) mapping which agrees with t_λ on one interior n-cell $\overline{E}_1^n$ of $\overline{E}^n$, with t_λ' on another $\overline{E}_2^n$ and sends the rest of $\overline{E}^n$ into $U_\lambda(p)$. The existence of such a mapping of $\overline{E}^n$ follows from the fact that all ordinary homotopy groups of $U_{\epsilon_\lambda}(p)$ vanish with the result that any mapping defined on a subpolyhedron $(\overline{E}_1^n \cup \overline{E}_2^n)$ can be extended to the whole polyhedron $(\overline{E}^n)$[17]. We must now consider what sort of permanence relations may be used to make $\{t_\lambda+t_\lambda'\}$ a mapping towards. Consider the two mappings $t_\lambda+t_\lambda'$ and $\pi_\lambda^\mu(t_\mu+t_\mu')$ as mapping $\overline{E}^n-((\mathfrak{I}\overline{E}_1^n)\cup(\mathfrak{I}\overline{E}_2^n))$ into $U_{\epsilon_\lambda}(p)$. By the fact that $\pi_k(\overline{U}_{\epsilon_\lambda}(p))=0$, all k, homotopies f_λ^μ and $f_\lambda^{\mu\prime}$ already defined for $\mathfrak{B}(\overline{E}_1^n)$ and $\mathfrak{B}(\overline{E}_2^n)$ respectively can be extended to the rest of $\overline{E}^n-((\mathfrak{I}\overline{E}_1^n)\cup(\mathfrak{I}\overline{E}_2^n))$. We denote this extension by $f_\lambda^{(+)\mu}$. So we write

$$\{t_\lambda; f_\lambda^\mu\} + \{t_\lambda'; f_\lambda^{\mu\prime}\} = \{t_\lambda+t_\lambda', f_\lambda^{(+)\mu}\}.$$

We should next show that any sums of two strongly homotopic pairs of mappings yield strongly homotopic sums.

Assume then that $\{t_\lambda^{(1)}; f_\lambda^{(1)\mu}\} \approx_{U^*} \{t_\lambda^{(2)}; f_\lambda^{(2)\mu}\}$ and $\{t_\lambda^{(3)}; f_\lambda^{(3)\mu}\} \approx_{U^*} \{t_\lambda^{(4)}; f_\lambda^{(4)\mu}\}$ by suitable homotopies $\{h_\lambda; \phi_\lambda^\mu\}$ and $\{h_\lambda'; \phi_\lambda^{\mu\prime}\}$ respectively. First we define a new homotopy $h_\lambda^{(+)}$ simply by extending the homotopies h_λ and h_λ' in the manner just indicated for $f_\lambda^{(+)}$. Finally the extension of ϕ_λ^μ on $\overline{E}_1^n\times I\times I$, $\phi_\lambda^{\mu\prime}$ on $\overline{E}_2^n\times I\times I$, $h_\lambda^{(+)}$, $h_\mu^{(+)}$ on $\overline{E}^n\times I\times 0$ and $\overline{E}^n\times I\times 1$ respectively, and $f_\lambda^{(+)\mu}$, $f_\lambda^{(+)\mu\prime}$ on $\overline{E}^n\times 0\times I$ and $\overline{E}^n\times 1\times I$ respectively to $\phi_\lambda^{(+)\mu}$ on the whole of $\overline{E}^n\times I\times I$ is again just a matter of extending the induced mapping to $(\overline{E}^n-((\mathfrak{I}\overline{E}_1^n)\cup(\mathfrak{I}\overline{E}_2^n))\times I\times I$.

The discussion of $-\{t_\lambda\}=\{-t_\lambda\}$ follows the same pattern, and likewise for associativity.

For weak homotopies the machinery is simplified but follows similar lines. For net homotopies the discussion is identical except for notation. Without further ado, then, we consider the groups formed by these various homotopy classes. The theorems on equivalence of net- and neighborhood-restricted homotopies guarantee that we have essentially only two kinds of groups—weak and strong. For any dimension k we denote them by $\Pi_k(R)$ and $\Pi_k^*(R)$.

---

[17] For this theorem see Eilenberg [3, (6.2)].

(4.6) DEFINITION. $\Pi_k(R, p)$, the weak homotopy group of R at p, is the group of weak restricted classes of admissible mappings of $\bar{E}^k$ or S^k towards R (or, equivalently, the group of corresponding weak classes of space-net mappings into the net of R).

(4.7) DEFINITION. $\Pi_k^*(R, p)$, the strong homotopy group of R at p, will denote the group of strong restricted classes of admissible mappings of S^k or $\bar{E}^k$ towards R (or, equivalently, the group of corresponding space-net classes).

SECTION V. A HUREWICZ THEOREM FOR $\Pi_n(R, p)$

19. Standard results for geometric complexes. In the theory of homotopy groups as developed by Hurewicz, one finds the following basic theorems:

(5.1) *If R is arc-wise connected, then $\pi_k(R, p)$ is independent of p.*

(5.2) *If R is arc-wise connected, and if $\pi_k(R)=0$, $k<n$, then $\pi_n(R)\cong\mathcal{H}^n(R)$.*

(In this conclusion, π_n is taken modulo its commutator subgroup. This is significant only for $n=1$.)

Theorem (5.2) holds when certain hypotheses are made about R. These hypotheses depend on the type of homology groups being used. That is, whether Vietoris-Čech or continuous cycles are under consideration. The assumption about vanishing homotopy groups can be replaced by an equivalent one about the nullhomotopy of any continuous $(n-1)$-dimensional complex on R. This equivalent property is actually the one ordinarily used in the proof of the theorem. We shall need this notion in a generalized form; so we give it a name "$(n-1)$-contractibility," denoted by c^{n-1}.

(5.3) DEFINITION. R is said to be *k-contractible*, or c^k, if every finite continuous k-dimensional complex on R is homotopic to a point. The following remark is obvious.

(5.4) *0-contractibility is equivalent to arc-wise connectedness.*

The Hurewicz Theorem may now be restated:

(5.5) *If R is c^{n-1}, then $\pi_n(R)\cong\mathcal{H}^n(R)$ (π_n mod commutator for $n=1$).*

Since our generalized situation will deal with a net of geometric complexes, we point out the following.

(5.6) *If K is a geometric complex, then these statements are equivalent:*

(5.6.1) *K^n can be deformed to a point.*

(5.6.2) *K is c^n.*

(K^n is the n-section of K.)

Proof. (1) (5.6.1)$\rightarrow$(5.6.2). Let (L^n, t) be any continuous complex on K. We take a simplicial approximation (L'^n, t'). Since t' is simplicial, $t'(L^n)\subset K^n$. When K^n is deformed to a point, the continuous complex (L'^n, t') is deformed into a new one (L'^n, t'') where $t''(L'^n)$ is a point. Combining this deformation with the one connecting (L^n, t) and (L'^n, t'), the result is attained.

(2) (5.6.2)$\rightarrow$(5.6.1). For K^n itself is a continuous complex, namely $(K^n, 1)$ on K.

20. **Net contractibility.** We now state the corresponding notions in terms of nets.

(5.7) DEFINITION. R is *weakly k-contractible* or C^k if every finite open covering has a refinement whose nerve is c^k.

By (5.6) this definition can be weakened to a statement about the contractibility of Φ^k.

The analogous *strong* notion can most easily be defined in terms of regular closed coverings, since they lead to projections *onto* for the k-sections of nerves. That is, $\pi_\lambda^\mu \Phi_\mu^k = \Phi_\lambda^k$ when the Φ's are nerves of finite closed regular coverings.

(5.8) DEFINITION. R is *strongly k-contractible* or C^{k*} if in the family of finite regular closed coverings of R there is a cofinal subsequence $\{\mathfrak{F}_i\}$ such that

(i) each nerve Φ_i is c^k,

(ii) for any contraction η_i deforming Φ_i^k to a point p_i there is a contraction η_{i+1} deforming Φ_{i+1}^k to a point p_{i+1} (where $\pi_i^{i+1}p_{i+1}=p_i$) in such a way that projection and contraction commute within homotopy, this homotopy leaving Φ_{i+1}^k and p_{i+1} fixed. Symbolically, this relation may be written

$$\pi_i^{i+1}\eta_{i+1} \approx \eta_i\pi_i^{i+1}.$$

By (1.11) this implies a similar situation for a cofinal sequence in the family of all open coverings.

As an analogue of (5.1) we have

(5.9) $\Pi_k^*(R, p) \cong \Pi_k^*(R, p')$ *when R is C^{0*}.*

(5.10) $\Pi_k(R, p) \cong \Pi_k(R, p')$ *when R is C^{0*}.*

It is convenient here to select a projective sequence of vertices from the projective sequence of basic simplexes corresponding to p. That is, we pick vertices $\bar{p}_i$ from σ_i such that $\pi_i^{i+1}\bar{p}_{i+1}=\bar{p}_i$. Likewise we pick $\bar{p}_i'$ from σ_i', the ith basic simplex corresponding to the base-point p'. By the C^{0*} assumption, there is an arc $\bar{p}_i\bar{p}_i'$ in each Φ_i such that $\bar{p}_i\bar{p}_i'$ is homotopic to $\pi_i^{i+1}(\bar{p}_{i+1}\bar{p}_{i+1}')$, the homotopy leaving the end points fixed.

Let now $\{t_i\}$ be a mapping admissible with respect to p. Then $t_i(q)=p_i \in \sigma_i$. For each i we take the segment $p_i\bar{p}_i$ and the arc $\bar{p}_i\bar{p}_i'$ as the means of getting a mapping admissible with respect to p'. The procedure of setting up the correspondence between elements of $\Pi_k^*(R, p)$ and of $\Pi_k^*(R, p')$ or between those of $\Pi_k(R, p)$ and $\Pi_k(R, p')$ is identical with the standard proof of (5.1).

Since strong contractibility is unwieldy, the following result is stated:

(5.11) *In* (5.9), (5.10) *C^{0*} may be replaced by C^1.*

This is trivial, since $C^1 \rightarrow C^0$ and $C^1 \rightarrow \pi_1(\Phi_i) = 0$. By C^0 we can pick arcs $p_i p_i'$ as above. The fact that $\pi_1(\Phi_i) = 0$ guarantees that the two arcs $p_i p_i'$ and $\pi_i^{i+1}(p_{i+1} p_{i+1}')$ are properly homotopic.

The following remark is a consequence of (5.4) or of the preceding arguments.

(5.11.1) If R is C^k, then for any point p of R we can assume that the k-deformations are onto points p_i in $\sigma_i(p)$, the basic simplexes.

(5.11.1) enables us to use any fixed point of R as a reference point for C^k properties as well as for homotopy groups.

21. **Weak contractibility, weak homotopy, and homology.** Now we compare net homotopy and net homology for a space subject to assumptions parallel to those of (5.5). We assume that R is C^{n-1}. Moreover we assume $n > 1$. The modifications for $n = 1$ are the same as in the classical theorem. This means that we can pick from the family of finite open coverings of R a cofinal subsequence with c^{n-1} nerves. Then for this subsequence we can choose (by (2.29)) a simplicial representative $\{t_i\}$ out of any element τ in $\Pi_n(R)$. For every i, $t_i(S^n)$ induces an n-cycle of Φ_i (image of fundamental cycle of the subdivision of S^n on which t_i is defined) which we denote by $\gamma_i^n(t_i)$. The collection $\{\gamma_i^n(t_i)\}$ is a cycle since $\pi_i^{i+1}\gamma_{i+1}^n(t_{i+1}) \sim \gamma_i^n(t_i)$. This follows from the stronger homotopy relation, $\pi_i^{i+1} t_{i+1} \approx t_i$, which gives a mapping f_i^{i+1} of $S^n \times I$ into Φ_i. $f_i^{i+1}(S^n \times 0)$ is continuously homologous to $\gamma_i^n(t_i)$—where $S^n \times 0$ is thought of as the fundamental cycle of $S^n \times 0$—and likewise $f_i^{i+1}(S^n \times 1)$ is continuously homologous to $\pi_i^{i+1}\gamma_{i+1}^n(t_{i+1})$. Hence $\gamma_i^n(t_i)$ and $\pi_i^{i+1}\gamma_{i+1}^n(t_{i+1})$ are continuously homologous, which assures ordinary homology(18).

Suppose that two simplicial representatives of the same class were selected. For every i we should then have two mappings t_i, t_i' of subdivisions of S^n into Φ_i. These are related by a homotopy h_i. The $\{h_i\}$ may be taken as simplicial for proper subdivisions. This transfer to a simplicial h_i replaces t_i and t_i' by slightly different mappings. But different simplicial approximations of the same continuous cycle are homologous; so the desired homology obtains.

Clearly this correspondence carries sums of homotopy elements into sums of cycles, and so on. Thus we have at first a homomorphism

$$\Pi_n(R) \rightarrow \mathcal{H}^n(R).$$

The homomorphism which we have described is based upon the one which ordinarily would be used in proving the corresponding theorem (5.5), for each separate complex Φ_i. So we review briefly the course which such a proof would follow(19). We have assumed that Φ_i is c^{n-1}. By (5.5) we then have $\pi_n(\Phi_i) \cong \mathcal{H}^n(\Phi_i)$. This isomorphism is reached by means of the homomorphism as-

(18) See for instance Alexandroff-Hopf [2, p. 338, Satz Ia].

(19) For more detail see Hurewicz [6, §7].

signing the homology class of $\gamma_i^n(t_i)$ to the mapping-class of t_i. That this homomorphism is univalent is demonstrated by showing that $\gamma_i^n(t_i)$ can bound only when t_i is homotopic to zero. In proving that the isomorphism into $\mathcal{H}^n(\Phi_i)$ is actually *onto* $\mathcal{H}^n(\Phi_i)$, the assumption c^{n-1} is employed. By it each cycle γ_i^n can be deformed into a continuous "spherical cycle" $\bar{\gamma}_i^n$ which determines an element of $\pi_n(\Phi_i)$, $\tau_i(\gamma_i^n)$.

The method of passing from "spherical cycle" to admissible mapping-class is as follows. Let γ_i^n be the continuous cycle (K, γ^n, s_i). Then the spherical cycle $\bar{\gamma}_i^n$ is the continuous cycle (K, γ^n, s_i') where $s_i'(K^{n-1}) = p_i$. s_i' is homotopic to s_i, so $\bar{\gamma}_i^n \sim \gamma_i^n$. Let γ^n in K be given by $a^j\sigma_j^n$. The mapping $s_i' \mid \bar{\sigma}_j^n$ is an admissible mapping of class τ_j. We assign to $\bar{\gamma}_i^n$ the class $\tau_i(\bar{\gamma}_i^n) = a^j\tau_j$. Actually we might assign a specific mapping $t_i(\bar{\gamma}_i^n) = a^j s_i \mid \bar{\sigma}_j^n$. But since this sum of mappings is not unique, the other assignment is preferable. Now it is shown that homology relations for "spherical cycles" are transmitted into homotopy relations for the corresponding elements of $\pi_n(\Phi_i)$. Moreover, the reciprocal nature of the two correspondences (from homotopy to homology and vice versa) is proved by showing that $\gamma_i^n(\bar{t}_i)$ for any $\bar{t}_i \in \tau_i(\gamma_i^n)$ satisfies $\gamma_i^n(\bar{t}_i) \sim \gamma_i^n$. Since no significant ambiguity is involved in this context we use the same symbol $\sim$ for both singular homology and ordinary homology.

These relationships and notations will now be applied to the net case. We have established a homomorphism of $\Pi_n(R)$ into $\mathcal{H}^n(R)$. First we show that it is *univalent*. For every i we have $\gamma_i^n(t_i) \sim 0 \rightarrow t_i \approx 0$. Hence if the space-net mapping $\{t_i\}$ induces a cycle $\{\gamma_i^n(t_i)\}$ which is a bounding cycle, we have $\{t_i\} \approx 0$.

Now take any cycle $\{\gamma_i^n\}$ and consider the collection $\{\bar{\gamma}_i^n\}$ of spherical cycles formed with respect to points p_i in basic simplexes as indicated in (5.11.1), and any corresponding collection of mappings $\bar{t}_i$ where $\bar{t}_i \in \tau_i(\bar{\gamma}_i^n)$. We need show merely that this collection is a space-net mapping: that is $\pi_i^{i+1}\bar{t}_{i+1} \approx \bar{t}_i$. By the very nature of the method of assigning homotopy elements to cycles we have the relationship $\pi_i^{i+1}\tau_{i+1}(\bar{\gamma}_{i+1}^n) = \tau_i(\pi_i^{i+1}\bar{\gamma}_{i+1}^n)$. $\pi_i^{n+1}\bar{\gamma}_{i+1}^n$ is homologous (continuously) to $\bar{\gamma}_i^n$. Hence both determine the same homotopy element, that is, $\tau_i(\pi_i^{i+1}\bar{\gamma}_{i+1}^n) = \tau_i(\bar{\gamma}_i^n)$. But the relationship $\pi_i^{i+1}\tau_{i+1}(\bar{\gamma}_{i+1}^n) = \tau_i(\bar{\gamma}_i^n)$ implies $\pi_i^{i+1}\bar{t}_{i+1} \approx \bar{t}_i$. This proves that the isomorphism is *onto*, so the following theorem has been established:

(5.12) THEOREM. *If R is C^{n-1}, $n>1$, then the weak net homotopy group $\Pi_n(R)$ and the integral net homology group $\mathcal{H}^n(R)$ are isomorphic.*

As in the classical theory, the following special theorem holds.

(5.13) THEOREM. *If R is C^0, then $\Pi_1(R)$ modulo its commutator group is isomorphic with the integral net homology group $\mathcal{H}^1(R)$.*

It seems not unlikely that an assumption that R is C^{k*} might lead to an analogous relationship between $\Pi_n^*(R)$ and a suitably defined homology group.

SECTION VI. EXAMPLES

In the foregoing sections, five types of homotopy have been discussed. The first of these is ordinary homotopy or *absolute* homotopy. This is the basic concept which is generalized in the others. In section II neighborhood and net homotopies of actual mappings into a compactum were introduced. These notions were shown to be equivalent. Two distinct types of net-neighborhood homotopies were discussed—the weak and the strong homotopies. Thus for ordinary mappings we now have weak, strong, and absolute homotopies. Besides these equivalences for mappings into a space, strong and weak homotopy relations were defined for mappings towards a space and for space-net mappings. For compacta these generalized mappings were shown to lead to equivalent theories.

22. **Five homotopy groups.** A suitable basis for comparison of the homotopy concepts is found in the homotopy groups to which they lead. Absolute homotopy leads to the classical Hurewicz homotopy groups. Weak and strong equivalence classes of space-net mappings and mappings towards a space have been made into groups in section IV. The groups of weak and strong classes of mappings *into* a space have not been explicitly defined; but their formulation lies exactly between the absolute and net groups already discussed at length; so no detail will be given. It is clear that we can arrive at such groups by considering the subgroup of Π_k or of Π_k^* stemming from mappings towards a compactum R which actually are into R, or correspondingly considering space-net mappings which are induced by mappings into.

We shall denote the five groups as follows: for

(i) mappings into, homotopies absolute . . . . $\pi_k(R)$,
(ii) mappings into, homotopies strong . . . . $\pi_k[N^*](R)$,
(iii) mappings into, homotopies weak . . . . $\pi_k[N](R)$,
(iv) mappings towards, homotopies strong . . . . $\Pi_k^*(R)$,
(v) mappings towards, homotopies weak . . . . $\Pi_k(R)$.

23. **Specific spaces.** We consider first the circle.

(6.1) *The one-sphere* S^1. $\pi_1=\pi_1[N]=\pi_1[N^*]=\Pi_1^*=\Pi_1=$ *the infinite cyclic group.*

It is well known that the fundamental group of a circle is the infinite cyclic group. Since S^1 is an ANR it follows trivially that no new classes or equivalences are introduced by the neighborhood procedure.

It should be emphasized that *these groups are identical for any compactum which is an ANR.*

(6.2) Consider the subspace of the Euclidean plane consisting of the four sets (a) $x=0$, $-3\leqq y\leqq 1$; (b) $y=\sin(1/x)$, $0<x\leqq 2/\pi$; (c) $x=2/\pi$, $-3\leqq y\leqq 1$; (d) $y=-3$, $0\leqq x\leqq 2/\pi$. For it $\pi_1=\pi_1[N]=\pi_1[N^*]=0$. These groups vanish trivially since although the space is arc-wise connected there can be only uninteresting mappings of S^1 *into* it. This is a consequence of the break in local connectivity.

On the other hand, every ϵ-neighborhood for small ϵ is annulus-like, and every such annulus will receive essential mappings of the one-sphere. (By *annulus* we mean a homeomorph of the open plane set between two concentric circles.) The homotopy properties of an annulus are those of a circle, since the latter may be considered as a deformation-retract of the annulus. Moreover for successively smaller ϵ each new and smaller ϵ-neighborhood is a deformation retract of the larger ones. Thus every mapping into one such neighborhood can be deformed into the smaller ones. In this way mappings towards the space are generated, and their classes are in one-one correspondence with the classes of mappings for any one of the neighborhoods. We have then $\Pi_1 = \Pi_1^* =$ *infinite cyclic group.*

In this example real information is given by the Π-groups while the π-groups reveal nothing. Their failure is due to their basic dependence on local connectedness.

(6.3) Next we take the space defined by (a) $x=0$, $|y| \leqq 1$; (b) $y=\sin(1/|x|)$, $0<|x| \leqq 2/\pi$; (c) $|x| = 2/\pi$, $-3 \leqq y \leqq 1$; (d) $y=-3$, $|x| \leqq 2/\pi$. This again is a one-dimensional space. It is not arc-wise connected, so the standard homotopy group theory does not apply significantly. In this case, however, each of the two arc-components has the first three groups, π_1, $\pi_1[N]$, and $\pi_1[N^*]$, all zero. Hence the groups do have accidental significance for the space as a whole. As in the preceding example, $\Pi_1 = \Pi_1^* =$ infinite cyclic group.

This space is *not* c^0, that is, *not arc-wise connected*, but *it is* C^{0*}. This again is a consequence of the possibility of a cofinal sequence of neighborhoods which are constructed by successive deformation-retractions.

Since the property C^{0*} has a role in the net theory parallel to that of arc-wise connectedness in the Hurewicz theory, this is a simple example of the type of space to which the new theory applies but the old does not.

This important distinction is brought out more sharply in the following related examples.

(6.4) Consider the one-dimensional subset of Euclidean two-space defined by (a) $x=0$, $2 \leqq |y| \leqq 4$; (b) $|y| = 3+\sin(1/|x|)$, $0<|x| \leqq 2/\pi$; (c) $|x| = 2/\pi$, $|y| \leqq 4$. The space is a connected compactum.

It is an example of a space which is C^{0*} but not arc-wise connected. Thus here the net theory applies perfectly while the standard theory is useless.

(6.5) We now rotate the space of (6.4) about the x-axis. The classical homotopy groups clearly do not apply for this space. They can be defined for each of the arc-components but they are not attributes of the space as a whole. This space is 2-sphere-like, and its only irregularity is the equatorial band of non-local-connectedness. The central arc-component is a closed annulus, and hence has an infinite cyclic first homotopy group. The other two arc-components are two-cells, and hence have vanishing groups. All four of the remaining groups: $\pi_1[N]$, $\pi_1[N^*]$, Π_1, Π_1^* vanish.

When we consider two-dimensional groups, we see again that the Π-groups are the ones which are consistently informative. All the π-groups vanish by default because of the break in local connectedness. But this space, like the preceding ones, has a cofinal sequence of neighborhoods each of which is a deformation retract of the preceding one in the sequence. It follows that the net-groups of the space are identical with π_2 for each such neighborhood. We have then $\Pi_2=\Pi_2^*=$ the infinite cyclic group; while $\pi_2[N]=\pi_2[N^*]=0$.

(6.6) *The dyadic solenoid.*

We have previously cited the dyadic solenoid as a space which distinguishes between the concepts of weak and strong net-homotopy. Since the solenoid is one-dimensional, it shows this difference for mappings of the 0-sphere. Thus the solenoid stands out as an example of a space which is C^0 *but neither* c^0 *nor* C^{0*}. That it is not c^0 is well known. One can easily designate points which are not joined by an arc in the solenoid. To do this we replace the torus construction[20] by a sequential inverse-mapping system[21]. In this mapping system the coordinate spaces are all circles, S_i^1, and the projections π_i^{i+1} are mappings of degree two. We may define these mappings by representing each of the circles by the real numbers mod 1 and by taking each projection as ordinary multiplication by two. We designate a particular point by a sequence of coordinates $(x_1, x_2, \cdots)$, each x_i being a positive real number less than 1. Then, for example, the points given by $(0, 0, 0, \cdots)$ and $(1/2, 3/4, 3/8, 11/16, 11/32, 43/64, 43/128, \cdots)$ are not arc-wise connected.

On the other hand, any two points of the solenoid can be joined by an arc in any neighborhood. For consider merely the cofinal sequence of neighborhoods given by the solid torus construction—each torus being regarded as a neighborhood of the space. This neighborhood property is reflected in the C^0 net property. For suitably chosen subdivisions we can regard the circles of the inverse mapping system as simplicial complexes, with the projections as simplicial mappings. Each covering of one of these circles induces a covering of the limit space. We use the coverings of the circles by the stars of vertices. The nerve of such a covering of a complex is isomorphic to the complex itself. So it turns out that each subdivision of an S_i^1 may be taken as the nerve of a covering of the solenoid, and each projection as a true projection in the net. This sequence of coverings is cofinal (for consider the parallel case for the torus construction), and each nerve is c^0. Hence the space is C^0.

Now let us suppose that the solenoid is C^{0*}. We consider in each nerve of the sequential net the coordinate-points of the two non-arc-wise-connected points just specified. If the space is C^{0*} it must be possible to join the corresponding pair of points on the ith nerve by a homotopy h_i such that the sequence $\{h_i\}$ is a space-net mapping of $\overline{E}^0\times I$ into the net. We assign a degree $\eta(h_i)$ to each h_i. This is defined to be the number of times the arc h_i wraps

(20) Referred to in section II.

(21) See Lefschetz [9, chap. 1, pp. 31 ff.].

around the ith circle. It may be defined more specifically as the degree of a circle-to-circle mapping induced by the arc-mapping. We shall be satisfied here, however, with a rapid description which avoids unnecessary complications. Since π_i^{i+n} is of degree 2^n, we have the relation

$$\eta(\pi_i^{i+n} h_{i+n}) \geqq 2^n \eta(h_{i+n}).$$

If $\{h_i\}$ is a suitable space-net mapping, we must have $\pi_i^{i+n} h_{i+n} \approx h_i$ with the end points fixed during the homotopy. This implies that we must have

$$\eta(\pi_i^{i+n} h_{i+n}) = \eta(h_i).$$

Consequently for every n, $\eta(h_{i+n}) \leqq \eta(h_i)/2^n$. This is possible only for all $\eta(h_i) = 0$. But consider the possible arcs joining 0 to 3/8 in S_3^1. π_1^3 sends them into arcs of degree greater than 0. So the choice $\eta = 0$ throughout is impossible.

The argument sketched here serves to show incidentally that there is no *essential* (that is, degree nonzero) mapping of S^1 into the net of the solenoid. The degree η above may be regarded without essential alteration as the degree of a circle-mapping. Thus we have $\Pi_1 = 0$, $\Pi_1^* = 0$.

(6.7) A space of dimension two for which Π_1 and Π_1^* do not agree may be constructed by taking the join of the solenoid and two points. Each of the coordinate spaces in the inverse mapping system is a two-sphere. The arguments used above go through with dimensions raised. We get $\Pi_1 = 0$ but $\Pi_1^* \neq 0$, $\Pi_2 = 0$ and $\Pi_2^* = 0$.

Section VII. Net homotopy and net-net mappings

24. **General remarks.** In the previous sections we have dealt with nets of simplicial complexes associated with coverings of spaces. But several of the topics discussed have only incidental connection with these underlying spaces. Many directions for generalization are possible. Our discussion was limited to compacta in order to maintain an easy parallel between net and neighborhood concepts. The transfer to nets connected with more general spaces or to nets divorced from spaces is often quite simple. Another direction of generalization is from nets to inverse mapping systems in general.

In the present section a few almost purely net concepts will be discussed cursorily. To maintain the previous point of view, however, applications will be made to nets of coverings of compacta.

In dealing with nets aside from nets related to spaces and their coverings, we shall wish to assume certain properties which actually have been realized in the examples we have used. The following is a modification of the definition of Lefschetz(22).

Definition. A net Σ is a system of finite simplicial complexes $\{\Phi_\lambda\}$ indexed by a directed set $\Lambda = \{\lambda; >\}$ and with the following properties:

(22) See Lefschetz [9, chap. 6, (2.1), p. 214].

N.1. When $\lambda > \mu$, there exists one or more simplicial mappings or projections, $\pi_\mu^\lambda: \Phi_\lambda \to \Phi_\mu$.

N.2. When $\lambda > \mu > \nu$ and π_μ^λ, π_ν^μ are projections so is $\pi_\nu^\mu \pi_\mu^\lambda$.

N.3. Any two projections π_μ^λ, $\pi_\mu^{\lambda\prime}$, $\lambda > \mu$, are homotopic.

Thus our net will be a simplicial net with N.3 strengthened.

The term spectrum will apply to a net where π_μ^λ is unique.

25. Homotopy groups for spectra.

(7.1) $\Pi_k(\Sigma, p_\lambda, l_\mu^\lambda)$.

In our discussion of net groups for R, we used a point p in R as a point of reference from which our "basic simplexes" and "base points" on the various nerves were obtained. Now we wish to speak of nets quite apart from the spaces they may represent. We could choose as base points selected points in a projective set of simplexes. But for the moment it will be convenient to become more general in this regard, even though counter-restrictions will be necessary elsewhere. In the spectrum $\Sigma = \{\Phi_\lambda; \pi_\mu^\lambda\}$ we now pick base-points $p_\lambda \in \Phi_\lambda$ quite at random. We do assume that each Φ_λ is connected. Now when $\lambda > \mu$ we consider an arc l_μ^λ joining the points p_μ and $\pi_\mu^\lambda p_\lambda$. We shall always assume that our choice of arcs can be made homotopically consistent: for example when $\nu > \mu > \lambda$ we have two arcs, namely l_λ^ν and $\pi_\lambda^\mu l_\mu^\nu + l_\lambda^\mu$ joining the points p_λ and $\pi_\lambda^\nu p_\nu$. We shall assume that these arcs have been chosen so as to be homotopic with the end points fixed throughout the homotopy. A sufficient condition for making a random choice of such arcs behave in this way is that all complexes Φ_λ of the spectrum are c^1, or in other words that the spectrum is C^1. It is clear also that the use of a projective collection of basic simplexes makes it possible to elect as arcs the segments joining the points in question. Thus, as far as base points are concerned, the present discussion is a generalization of the previous one (section IV).

Now consider the classical homotopy group of Φ_λ at p_λ: $\pi_k(\Phi_\lambda, p_\lambda)$. Let p_λ' be any other point of Φ_λ. By the assumption that Φ_λ is a connected simplicial complex it follows (by (5.1)) that the groups $\pi_k(\Phi_\lambda, p_\lambda)$ and $\pi_k(\Phi_\lambda, p_\lambda')$ are isomorphic. There may however be several distinct isomorphisms. A particular isomorphism is determined by every choice of an arc from p_λ to p_λ'. But two homotopic paths determine the same isomorphism.

A projection π_μ^λ determines a homomorphism of $\pi_k(\Phi_\lambda, p_\lambda)$ into $\pi_k(\Phi_\mu, \pi_\mu^\lambda p_\lambda)$. The arc l_μ^λ determines a unique isomorphism between $\pi_k(\Phi_\mu, \pi_\mu^\lambda p_\lambda)$ and $\pi_k(\Phi_\mu, p_\mu)$. Thus for each pair of indices $\lambda > \mu$ there is a unique homomorphism ϕ_μ^λ of $\pi_k(\Phi_\lambda, p_\lambda)$ into $\pi_k(\Phi_\mu, p_\mu)$. The collection of groups $\pi_k(\Phi_\lambda, p_\lambda)$ and homomorphisms ϕ_μ^λ forms an inverse homomorphism system. The limit group may be called a kth *homotopy group* of the spectrum. It depends on the choice of base-points and on the arcs. Thus we write it as $\Pi_k(\Sigma, p_\lambda, l_\mu^\lambda)$. This is a generalization of $\Pi_k(R)$ for each admissible space-net mapping $\{t_\lambda\}$, where $t_\lambda \approx \pi_\mu^\lambda t_\lambda$, determines a set of elements $\{\tau_\lambda\}$ where $t_\lambda \in \tau_\lambda \in \pi_k(\Phi_\lambda)$ and such that $\pi_\mu^\lambda \tau_\lambda = \tau_\mu$. We have already mentioned sufficient conditions for making

$\Pi_k(\Sigma, p_\lambda, l_\mu^\lambda)$ independent of the arcs l_μ^λ. Moreover we have shown in section V sufficient conditions for making a group of this sort independent of base points.

The main purpose of this sub-section is to point out that $\Pi_k(R)$ may be considered as an inverse homomorphism system, and that immediate methods of generalization are available. It should be added that the type of machinery indicated here applies equally well for any inverse mapping system. This point of view is mentioned by Freudenthal(23) and for $k=1$ is used by Komatu(24).

26. **Net to net mappings.** Let $\Sigma = \{\Phi_\lambda; \pi_\lambda^\mu\}$ and $\Sigma' = \{\Phi'_\alpha; \pi_\alpha^{\beta\prime}\}$ be two nets. Consider a collection of simplicial mappings $\{T_\alpha^\lambda\}$ where T_α^λ maps Φ_λ into Φ'_α.

(7.2) Definition. $\{T_\alpha^\lambda\}$ is a *net-net mapping* of Σ into Σ' provided that

(i) for every pair of indices λ', α' there is in the collection $\{T_\alpha^\lambda\}$ a T_α^λ with $\lambda > \lambda'$, $\alpha > \alpha'$,

(ii) for every pair of mappings T_α^λ, T_β^μ with λ, μ and α, β ordered, permanence relations such as

$$T_\beta^\mu \pi_\mu^\lambda \approx \pi_\beta^{\alpha\prime} T_\alpha^\lambda, \qquad \text{for } \lambda > \mu,\ \alpha > \beta,$$

hold.

Essentially a net-net mapping is a cofinal collection of nerve to nerve simplicial mappings which obey permanence relations.

Simplicial net-net mappings for spectra of a more strict sort (the $\approx$ replaced by $\cdot - \cdot$) have been considered by Nakasawa(25).

(7.2.1) Definition. Such a net-net mapping will be called a *strict net-net mapping*.

(7.3) Definition. Two net-net mappings $\{T_\alpha^\lambda\}$, $\{T_\beta^{\mu\prime}\}$ are *homotopic* if all permanence relations such as

$$T_\beta^\mu \pi_\mu^\lambda \approx \pi_\beta^{\alpha\prime} T_\alpha^{\lambda\prime} \qquad \text{for } \lambda > \mu,\ \alpha > \beta$$

hold.

This may be shown directly to be a true equivalence relation.

These mappings are essentially "weak" in the sense of the weak homotopy of previous sections. The possibility of "strong" net-net mappings will be ignored here.

(7.4) *If* $\{t_\lambda\}$ *is a space-net mapping of* S *into* Σ, *then* $\{T_\alpha^\lambda t_\lambda\}$ *is a space-net mapping of* S *into* Σ' *for any net-net mapping* $\{T_\alpha^\lambda\}$.

Proof. We must show that $\alpha > \beta$ implies $\pi_\beta^{\alpha\prime} T_\alpha^\lambda t_\lambda \approx T_\beta^\mu t_\mu$. Since our nets are ordered by directed sets, we can pick a $\nu' > \lambda$ or μ. Then there is a T_γ^ν where $\nu > \nu'$ and $\gamma > \alpha$. Since $\{T_\alpha^\lambda\}$ is a net-net mapping, we have:

(23) See Freudenthal [4, pp. 227–228].
(24) See Komatu [8].
(25) See Nakasawa [11].

(i) $$\pi_\alpha^{\gamma\prime} T_\gamma^\nu \approx T_\alpha^\lambda \pi_\lambda^\nu$$

and

(ii) $$\pi_\beta^{\gamma\prime} T_\gamma^\nu \approx T_\beta^\mu \pi_\mu^\nu$$

but (i)→(iii)

(iii) $$\pi_\beta^{\alpha\prime} \pi_\alpha^{\gamma\prime} T_\gamma^\nu \approx \pi_\beta^{\alpha\prime} T_\alpha^\lambda \pi_\lambda^\nu$$

and (ii)+(iii)→(iv)

(iv) $$T_\beta^\mu \pi_\mu^\nu t_\nu \approx \pi_\beta^{\alpha\prime} T_\alpha^\lambda \pi_\lambda^\nu t_\nu.$$

But $\pi_\mu^\nu t_\nu \approx t_\mu$ and $\pi_\lambda^\nu t_\nu \approx t_\lambda$; so we get

(v) $$T_\beta^\mu t_\mu \approx \pi_\beta^{\alpha\prime} T_\alpha^\lambda t_\lambda.$$

q.e.d.

27. **Induced net-net mappings.**

(7.5) *If T maps R into R', then T induces a strict net-net mapping of Σ_0 into* Σ_0' (R, R' compacta; Σ_0, Σ_0' their nets).

Proof. For every covering $\mathfrak{U}_\alpha'$ of R' and consequently for every induced covering of $T(R)$, the inverse of T leads to a finite open covering of R, $\mathfrak{U}_\lambda$, whose sets are the inverse image of the sets $U'\cap t(R)$ for every $U'\in\mathfrak{U}_\alpha'$. As has been shown by Lefschetz[26], the nerve of $\mathfrak{U}_\lambda$ is isomorphic to the corresponding nerve for $t(R)$ and hence to a closed subcomplex of the nerve of $\mathfrak{U}_\alpha'$. The corresponding simplicial mapping we denote by T_α^λ. If $\mathfrak{U}_\mu$ is a refinement of $\mathfrak{U}_\lambda$ we have $T_\alpha^\mu = T_\alpha^\lambda \pi_\lambda^\mu$. Consider the collection $\{T_\gamma^\nu\}$ of simplicial mappings of these types. It is cofinal with respect to the two nets jointly. Moreover it has the strict permanence relations such as

$$T_\beta^\mu \pi_\mu^\lambda \cdot - \cdot \pi_\beta^{\alpha\prime} T_\alpha^\lambda.$$

This follows from (1.3).

An easy consequence of (7.5) is the following:

(7.6) *The strict net-net mapping of* (7.5) *induces a homomorphism of the weak homotopy groups of the net Σ_0 into the groups of Σ_0'.*

Since the mapping is strict, basic simplexes for p in R go into basic simplexes for $T(p)$ in R'. By (7.4) then admissible mappings are carried into admissible mappings. That homotopic mappings go into homotopic mappings is trivial.

Homology properties require less stringent definitions both of nets and of net-net mappings. It may however be pointed out incidentally that the proof of (7.4) is formally identical with the proof that $\{\overline{T}_\alpha^\lambda \gamma_\lambda^k\}$ is a k-cycle of the net Σ' if $\{\gamma_\lambda^k\}$ is a k-cycle of Σ. $\overline{T}_\alpha^\lambda$ denotes the chain mapping induced by the simplicial mapping T_α^λ. To change from the homotopy proof to the homology

([26]) See Lefschetz [9, chap. 7, proof of (5.11), p. 251].

analogue one merely replaces t_λ by γ_λ^k and $\approx$ by the weaker $\sim$. On this basis it is easy to show that a net-net mapping induces a homomorphism on net homology groups.

Bibliography

1. P. Alexandroff, *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Ann. of Math. vol. 30 (1929) pp. 101–187.

2. P. Alexandroff and H. Hopf, *Topologie*, I, Berlin, Springer, 1935.

3. S. Eilenberg, *Extension and classification of continuous mappings*, Lectures in Topology, University of Michigan Press, 1941, pp. 57–99.

4. H. Freudenthal, *Entwicklungen von Räumen und ihren Gruppen*, Compositio Math. vol. 4 (1937) pp. 145–234.

5. W. Hurewicz, *Beiträge zur Topologie der Deformationen*. I. *Hoherdimensionale Homotopiegruppen*, Proceedings of Koninklijke Akademie van Wetenschappen te Amsterdam vol. 38 (1935) pp. 112–119.

6. ———, *Beiträge zur Topologie der Deformationen*. II. *Homotopie- und Homologiegruppen*, ibid. vol. 38 (1935) pp. 521–528.

7. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1941.

8. A. Komatu, *Bemerkungen über die Fundamentalgruppe eines Kompaktums*, Proc. Imp Acad. Tokyo vol. 13 (1937) pp. 56–58.

9. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, 1942.

10. ———, *Topology*, Amer. Math. Soc. Colloquium Publications, vol. 12, 1930.

11. T. Nakasawa, *Über die Abbildungskette vom Projektionsspektrum*, Science Reports of the Tokyo Bunrika Daigaku vol. 3 (1938) No. 64.

12. N. E. Steenrod, *Universal homology groups*. Amer. J. Math. vol. 58 (1936) pp. 661–701.

Bowdoin College,
Brunswick, Me.