TWO TAUBERIAN THEOREMS IN THE THEORY OF FOURIER SERIES

BY

CHING-TSÜN LOO

1. Let $f(x)$ be a function which is integrable in the sense of Lebesgue and periodic with period 2π. We consider the Fourier series of $f(x)$ and write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.1}$$

$$A_0 = a_0/2, \qquad A_n = A_n(x) = a_n \cos nx + b_n \sin nx \qquad (n > 0),$$

$$\sigma_n^\alpha = \frac{S_n^\alpha}{C_{n+\alpha,n}} = \frac{1}{C_{n+\alpha,n}} \sum_{\nu=0}^{n} C_{n+\alpha-\nu,n-\nu} A_\nu \qquad (\alpha > -1), \tag{1.2}$$

$$\phi(t) = \phi_0(t) = \{f(x+t) + f(x-t) - 2s\}/2,$$

$$\phi_p(t) = \frac{p}{t^p} \int_0^t (t-u)^{p-1} \phi(u) du \qquad (p > 0). \tag{1.3}$$

It is a theorem of Paley(1) that if $\alpha \geqq 0$ and $\sigma_n^\alpha \to s$, then $\phi_{1+\alpha+\delta} \to 0$ for every positive δ. The result is best possible of its kind(2). That is to say, we cannot replace δ by 0 in the conclusion of the above theorem. We are interested in such a problem, whether we can replace δ by 0, whenever we emphasize the hypothesis a little. This has been done by Hardy and Littlewood(3) in the case $\alpha = 0$. They proved that if (i) $A_n = O(n^{-\delta})$ for some positive δ and (ii) $\sigma_n^0 - s = o(1/\log n)$, then $\phi_1(t) \to 0$. The object of this paper is to investigate the analogous problems for the case $\alpha > 0$. We prove that for $\alpha > 0$ a single condition corresponding to condition (ii) in Hardy and Littlewood's theorem is sufficient to deduce $\phi_{1+\alpha}(t) \to 0$. Our theorem runs as follows.

THEOREM 1. *If* $\alpha > 0$, *and*

$$\sigma_n^\alpha - s = o(1/\log n), \tag{1.4}$$

then

$$\phi_{1+\alpha}(t) \to 0. \tag{1.5}$$

Presented to the Society, November 25, 1944; received by the editors March 16, 1944.

(1) R. E. A. C. Paley, *On the Cesàro summability of Fourier series and allied series*, Proc. Cambridge Philos. Soc. vol. 26 (1930).

(2) *Ibid.*

(3) G. H. Hardy and J. E. Littlewood, *Some new convergence criteria for Fourier series*, Annali Scuola Normale Superiore, Pisa, vol. 3 (1934).

It is natural to ask whether the more stringent condition $\sigma_n^\alpha - s = O(n^{-\epsilon})$ $(1>\epsilon>0)$ combined with an order condition on the coefficients, $A_n = O(n^{-\delta})$, is sufficient to deduce $\phi_\alpha(t)\to 0$. We prove that this is true when α is any positive integer, and also require a certain restriction on δ.

THEOREM 2. *If α is any positive integer such that*

$$\sigma_n^\alpha - s = O(n^{-\epsilon}) \qquad (1 > \epsilon > 0), \tag{1.6}$$

$$A_n = O(n^{-\delta}) \qquad (\delta > 1 - \epsilon), \tag{1.7}$$

then

$$\phi_\alpha(t) \to 0. \tag{1.8}$$

2. We begin by making the usual standard simplification of data, and discuss some properties of the function

$$\gamma_p(t) = \int_0^1 (1-u)^{p-1}\cos tu\, du \qquad (p > 0). \tag{2.1}$$

We suppose throughout this paper that $a_0 = 0$, $s = 0$, then

$$\phi(t) = \{f(x+t) + f(x-t)\}/2 \sim \sum_{n=1}^\infty A_n \cos nt, \tag{2.2}$$

and if $p > 0$,

$$\begin{aligned} \frac{1}{p}\phi_p(t) &= \frac{1}{t^p}\int_0^t (t-u)^{p-1}\phi(u)du = \int_0^1 (1-u)^{p-1}\phi(tu)du \\ &= \sum_{n=1}^\infty A_n \int_0^1 (1-u)^{p-1}\cos ntu\, du = \sum_{n=1}^\infty A_n\gamma_p(nt). \end{aligned} \tag{2.3}$$

If $j \geqq 1$, we denote by $\gamma_p^j(t)$ the jth derivative of $\gamma_p(t)$, while $\gamma_p^0(t)$ is defined as $\gamma_p(t)$. Then we have the following lemma.

LEMMA 1. $\gamma_p^j(t) = O(1)$ $(j = 0, 1, 2, \cdots)$ *for all t, and if $t \geqq 1$, then*

$$\gamma_p^j(t) = O(1/t^{j+1+\gamma}) \qquad (j = 0, 1, 2, \cdots) \tag{2.4}$$

where $\gamma = \min\,(1, p-(j+1))$.

This is well known([4]).

LEMMA 2([5]). *If $p > 0$, $q > p > 0$, then*

([4]) E. W. Hobson, *The theory of functions of a real variable*, vol. 2, 2nd ed.

([5]) S. Verblunsky, *Note on the sum of an oscillating series*, Proc. Cambridge Philos. Soc. vol. 26 (1930).

$$
(2.5)\qquad \begin{aligned} \Delta^q\gamma_p(\nu t) &= \sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q,n-\nu}\gamma_p(nt) \\ &= \int_0^1 (1-u)^{p-1}(2\sin(tu/2))^q \cos\left[\left(\nu t+\frac{q}{2}t\right)u-\frac{q}{2}\pi\right]du. \end{aligned}
$$

We have

$$
(2.6)\qquad \begin{aligned} &\sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q,n-\nu}\gamma_p(nt) \\ &\qquad = \int_0^1 (1-u)^{p-1}\left(\sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q,n-\nu}\cos ntu\right)du, \end{aligned}
$$

while

$$
\begin{aligned} &\sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q,n-\nu}\cos ntu \\ &\qquad = \cos\nu tu - C_{q,1}\cos(\nu+1)tu + C_{q,2}\cos(\nu+2)tu - \cdots + \cdots \\ &\qquad = \left[e^{i\nu tu} - C_{q,1}e^{i(\nu+1)tu} + \cdots - \cdots\right]/2 \\ &\qquad\quad + \left[e^{-i\nu tu} - C_{q,1}e^{-i(\nu+1)tu} + \cdots + \cdots\right]/2 \\ &\qquad = e^{i\nu tu}(1-e^{itu})^q/2 + e^{-i\nu tu}(1-e^{-itu})^q/2 \\ &\qquad = e^{i\nu tu}(2\sin(tu/2))^q e^{iq(tu-\pi)/2}/2 \\ &\qquad\quad + e^{-i\nu tu}(2\sin(tu/2))^q e^{-iq(tu-\pi)/2}/2 \\ &\qquad = (2\sin(tu/2))^q\cos\left[(\nu t+(q/2)t)u-(q/2)\pi\right]. \end{aligned}
$$

Substituting into the right-hand side of (2.6), we get the required result (2.5).

LEMMA 3. *If $p>0$, $q>s>0$, then*

$$
(2.7)\qquad \sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q-s,n-\nu}\Delta^s\gamma_p(nt) = \Delta^q\gamma_p(\nu t).
$$

We have by Lemma 2,

$$
(2.8)\qquad \begin{aligned} &\sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q-s,n-\nu}\Delta^s\gamma_p(nt) \\ &\qquad = \int_0^1 (1-u)^{p-1}\left(2\sin\frac{tu}{2}\right)^s \sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q-s,n-\nu} \\ &\qquad\qquad \cdot\cos\left[\left(nt+\frac{s}{2}t\right)u-\frac{s}{2}\pi\right]du, \end{aligned}
$$

and using the same method as in the proof of Lemma 2 we can deduce that

$$\sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q-s,n-\nu}\cos\left[\left(nt+\frac{s}{2}t\right)u-\frac{s}{2}\pi\right]$$

$$=\left(2\sin\frac{tu}{2}\right)^{q-s}\cos\left[\left(\nu t+\frac{s}{2}t+\frac{q-s}{2}t\right)u-\frac{s}{2}\pi-\frac{q-s}{2}\pi\right]$$

$$=\left(2\sin\frac{tu}{2}\right)^{q-s}\cos\left[\left(\nu t+\frac{q}{2}t\right)u-\frac{q}{2}u\right].$$

Hence

$$\sum_{n=\nu}^{\infty}(-1)^{(n-\nu)}C_{q-s,n-\nu}\Delta^{s}\gamma_{p}(nt)$$

$$=\int_0^1(1-u)^{p-1}\left(2\sin\frac{tu}{2}\right)^{q}\cos\left[\left(\nu t+\frac{q}{2}t\right)u-\frac{q}{2}\pi\right]du.$$

Our result follows from Lemma 2.

3. To prove Theorem 1, we begin by choosing r such that

$$r>(1+\alpha)/\alpha \quad \text{if} \quad \alpha\leqq 1,$$
$$r>2 \qquad\qquad \text{if} \quad \alpha>1,$$

and then write

$$l=[t^{-1}],\qquad k=[t^{-r}]. \tag{3.1}$$

We have by (2.3) that

$$\frac{1}{1+\alpha}\phi_{1+\alpha}(t)=\sum_{r=1}^{\infty}A_n\gamma_{1+\alpha}(nt)=\sum_{n=1}^{k}+\sum_{n=k+1}^{\infty}=\chi_1(t)+\chi_2(t), \tag{3.2}$$

say. Since the Fourier coefficient of any integrable function tends to zero, we notice that

$$\gamma_{1+\alpha}(nt)=\begin{cases}O((nt)^{-1-\alpha}), & 0<\alpha\leqq 1,\\ O((nt)^{-2}), & \alpha>1,\end{cases} \tag{3.3}$$

in virtue of Lemma 1. Hence we have

$$\chi_2(t)=O\left(\frac{1}{t^{1+\alpha}}\sum_{k+1}^{\infty}n^{-1-\alpha}\right)=O\left(\frac{1}{t^{1+\alpha}k^{\alpha}}\right)=O(t^{r\alpha-1-\alpha})=o(1), \tag{3.4}$$

when $\alpha\leqq 1$, and

$$\chi_2(t)=O\left(\frac{1}{t^2}\sum_{k+1}^{\infty}n^{-2}\right)=O\left(\frac{1}{t^2k}\right)=O(t^{r-2})=o(1), \tag{3.5}$$

when $\alpha>1$.

So it is enough to prove that $\chi_1(t)=o(1)$. Since

$$A_n = \sum_{\nu=0}^{n} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu} S_\nu^\alpha$$

holds for any $\alpha > -1$, hence

$$\begin{aligned} \chi_1(t) &= \sum_{n=1}^{k} A_n \gamma_{1+\alpha}(nt) = \sum_{n=1}^{k} \gamma_{1+\alpha}(nt) \sum_{\nu=0}^{n} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu} S_\nu^\alpha \\ &= \sum_{\nu=1}^{k} S_\nu^\alpha \sum_{n=\nu}^{k} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu} \gamma_{1+\alpha}(nt). \end{aligned} \tag{3.6}$$

In order to establish $\chi_1(t) = o(1)$, we require some further lemmas.

LEMMA 4.

$$\sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu} \gamma_{1+\alpha}(nt) = t^{1+\alpha}(1+\zeta_t) \int_0^1 (1-u)^\alpha u^{1+\alpha} \cos\left[\left(\nu t + \frac{\alpha+1}{2} t\right) u - \frac{\alpha+1}{2} \pi\right] du$$

where $\zeta_t \to 0$ as $t \to 0$, uniformly in n.

This is an immediate corollary of Lemma 2. From this lemma we easily see the following result.

LEMMA 5. *For all positive t,*

$$\sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu} \gamma_{1+\alpha}(nt) = O(t^{1+\alpha}). \tag{3.7}$$

LEMMA 6. *If $\nu t > 1$, then*

$$\sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu} \gamma_{1+\alpha}(nt) = O\left(\frac{1}{\nu^{1+\alpha}}\right). \tag{3.8}$$

Putting, in Lemma 4, $\nu t = X$, $(\alpha+1)(ut-\pi)/2 = \beta$, we are going to prove

$$\int_0^1 (1-u)^\alpha u^{1+\alpha} \cos(Xu+\beta) du = O\left(\frac{1}{X^{1+\alpha}}\right). \tag{3.9}$$

If α is an integer, we have by successive integration by parts

$$\left| \int_0^1 (1-u)^\alpha u^{1+\alpha} \cos(Xu+\beta) du \right| = \left| \frac{1}{X^\alpha} \int_0^1 \cos\left[Xu + \beta + (\alpha+1)\frac{\pi}{2}\right] \sum_{j=0}^{\alpha} A_j (1-u)^{\alpha-j} u^{1+j} du \right|,$$

where the A's are numbers which depend on α. Each term can be integrated by parts again, whence our result follows.

If α is not an integer, let $[\alpha]$ denote the greatest integer which does not exceed α. Integrating by parts gives

$$\left|\int_0^1 (1-u)^\alpha u^{1+\alpha} \cos (Xu+\beta)du\right|$$
$$= \left| \frac{1}{X^{[\alpha]+1}} \int_0^1 \cos\left[Xu+\beta+([\alpha]+1)\frac{\pi}{2}\right] \sum_{j=0}^{[\alpha]+1} A_j (1-u)^{\alpha-j} u^{\alpha-[\alpha]+j} du\right|.$$

Each of the first $[\alpha]+1$ terms can be integrated by parts again and is thus seen to be numerically less than

$$K/X^{[\alpha]+2},$$

which is the form required, since $X>1$, and $[\alpha]+2>\alpha+1$. The numerical value of the last term, on taking $x(1-u)$ as a new variable under the integrand, is seen to be

$$\left|\frac{A_{[\alpha]+1}}{X^{1+\alpha}} \int_0^X \cos\left[u-X-\beta-([\alpha]+1)\frac{\pi}{2}\right]\left(1-\frac{u}{x}\right)^{1+\alpha} u^{\alpha-[\alpha]-1} du\right|,$$

which is again of order $O(1/X^{1+\alpha})$, since the last integral converges.

4. We are in a position to prove $\chi_1(t)=o(1)$. First, we shall confine our proof to the case $0<\alpha\leqq 1$. We have by (3.6)

$$\begin{aligned} \chi_1(t) &= \sum_{\nu=1}^{k} S_\nu^\alpha \sum_{n=\nu}^{k} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu}\gamma_{1+\alpha}(nt) \\ &= \sum_{\nu=1}^{k} S_\nu^\alpha \sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu}\gamma_{1+\alpha}(nt) - \sum_{\nu=1}^{k} S_\nu^\alpha \sum_{n=k+1}^{\infty}. \end{aligned} \tag{4.1}$$

We put

$$\sum_{\nu=1}^{k} S_\nu^\alpha \sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu}\gamma_{1+\alpha}(nt) = \sum_{\nu=1}^{l} + \sum_{l+1}^{k} = \chi_3(t)+\chi_4(t) \tag{4.2}$$

say. In $\chi_3(t)$, $nt\leqq 1$, we have by Lemma 5,

$$\chi_3(t) = t^{1+\alpha}\sum_{\nu=1}^{l} o(\nu^\alpha) = o(t^{1+\alpha}l^{1+\alpha}) = o(1). \tag{4.3}$$

On the other hand, in $\chi_4(t)$, $nt\geqq 1$, we have by Lemma 6,

$$\begin{aligned} \chi_4(t) &= o\left(\sum_{l+1}^{k} \frac{\nu^\alpha}{\nu^{\alpha+1}\log n}\right) = o\left(\sum_{l+1}^{k} \frac{1}{n\log n}\right) \\ &= o(\log\log t^{-\nu} - \log\log t^{-1}) = o(1). \end{aligned} \tag{4.4}$$

It remains to prove

$$\sum_{\nu=1}^{k} S_\nu^\alpha \sum_{n=k+1}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu} \gamma_{1+\alpha}(nt) = o(1). \tag{4.5}$$

We have $\gamma_{1+\alpha}(nt) = O(1/(nt)^{1+\alpha})$ for $0<\alpha\leqq 1$, hence the left-hand side of (4.5) is

$$O\left(\frac{1}{t^{1+\alpha}} \sum_{\nu=1}^{k} \nu^\alpha \sum_{n=k+1}^{\infty} \frac{1}{n^{1+\alpha}(n-\nu)^{\alpha+2}}\right) = O\left(\frac{1}{t^{1+\alpha}k} \sum_{\nu=1}^{k} \sum_{n=k+1}^{\infty} \frac{1}{(n-\nu)^{\alpha+2}}\right)$$

$$= O\left(\frac{1}{t^{1+\alpha}k} \sum_{\nu=1}^{k} \frac{1}{(k+1-\nu)^{1+\alpha}}\right) = O\left(\frac{1}{t^{1+\alpha}k}\right) = O(t^{r-\alpha-1}) = o(1),$$

for $r>(\alpha+1)/\alpha \geqq \alpha+1$, when $0<\alpha\leqq 1$.

Collecting our results from (4.1), (4.2), (4.3), (4.4), and (4.5), we get $\chi_1(t)=o(1)$.

We notice that (4.3) and (4.4) are established for all $\alpha>0$, while (4.5) holds in general only for $0<\alpha\leqq 1$; this is certainly the sole reason for us to treat the case $0<\alpha\leqq 1$ first. As for $\alpha>1$, by repeated use of Abel's transformation $[\alpha]$ times, we have

$$\begin{aligned} \chi_1(t) &= \sum_{n=1}^{k} A_n \gamma_{1+\alpha}(nt) = \sum_{n=1}^{k-[\alpha]} S_n^{[\alpha]-1} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) \\ &\quad + \sum_{j=0}^{[\alpha]-1} S_{k-j}^{j} \Delta^{j} \gamma_{1+\alpha}((k-j)t). \end{aligned} \tag{4.6}$$

Also(6),

$$\begin{aligned} \Delta^1\gamma_{1+\alpha}(mt) &= \gamma_{1+\alpha}(mt) - \gamma_{1+\alpha}((m+1)t) \\ &= -t\gamma'_{1+\alpha}((m+\theta_1)t) \qquad (0<\theta_1<1), \\ \Delta^2\gamma_{1+\alpha}(mt) &= \Delta^1\gamma_{1+\alpha}(mt) - \Delta^1\gamma_{1+\alpha}((m+1)t) \\ &= t^2\gamma''_{1+\alpha}((m+\theta_1+\theta_2)t) \qquad (0<\theta_2<1), \end{aligned}$$

in general,

$$\begin{aligned} \Delta^{j}\gamma_{1+\alpha}(mt) &= (-1)^{j} t^{j} \gamma^{j}_{1+\alpha}((m+\theta_1+\theta_2+\cdots+\theta_j)t) \\ &= (-1)^{j} t^{j} \gamma^{j}_{1+\alpha}((m+\Theta)t) \end{aligned} \tag{4.7}$$

where $0<\theta_i<1$ $(i=1, 2, \cdots, j)$.

We have by Lemma 1 that

$$\begin{aligned} \gamma^{j}_{1+\alpha}((m+\Theta)t) &= O(1/(mt)^{j+2}) \qquad (j=0, 1, 2, \cdots, [\alpha]-1), \\ \gamma^{[\alpha]}_{1+\alpha}(mt) &= O(1/(mt)^{1+\alpha}), \end{aligned} \tag{4.8}$$

and by (4.7),

(6) Hereafter $\gamma^{j}(mt)$ denotes differentiation with respect to the argument.

$$\Delta^j\gamma_{1+\alpha}(mt) = O(t^j/(mt)^{j+2}) = O(1/m^{j+2}t^2) \tag{4.9}$$
$$(j = 0, 1, 2, \cdots, [\alpha] - 1),$$

$$\Delta^{[\alpha]}\gamma_{1+\alpha}(mt) = O(t^{[\alpha]}/(mt)^{1+\alpha}) = O(1/m^{1+\alpha}t^{\alpha-[\alpha]+1}). \tag{4.10}$$

We notice that $A_n = o(1)$, hence $S_n^j = o(n^{j+1})$. Thus

$$\begin{aligned} S_{k-j}^{j}\Delta^{j}\gamma_{1+\alpha}((k-j)t) &= O((k-j)^{j+1}/(k-j)^{j+2}t^{2}) \\ &= O(1/kt^{2}) = O(t^{r-2}) = o(1) \end{aligned} \tag{4.11}$$
$$(j = 0, 1, \cdots, [\alpha] - 1)$$

since $r > 2$. That is to say, each term in the second sum of the right-hand side of (4.6) is $o(1)$. We thus obtain

$$\chi_1(t) = \sum_{n=1}^{k-[\alpha]} S_n^{[\alpha]-1}\Delta^{[\alpha]}\gamma_{1+\alpha}(nt) + o(1). \tag{4.12}$$

We notice that if $p > q > -1$, $S_n^q = \sum_{\nu=0}^{n}(-1)^{(n-\nu)}C_{p-q,n-\nu}S_\nu^p$. Hence

$$\begin{aligned} &\sum_{n=1}^{k-[\alpha]} S_n^{[\alpha]-1}\Delta^{[\alpha]}\gamma_{1+\alpha}(nt) \\ &\qquad = \sum_{\nu=1}^{k-[\alpha]} \Delta^{[\alpha]}\gamma_{1+\alpha}(nt) \sum_{\nu=1}^{n} (-1)^{(n-\nu)}C_{\alpha-[\alpha]+1,n-\nu}S_\nu^{\alpha} \\ &\qquad = \sum_{\nu=1}^{k-[\alpha]} S_\nu^{\alpha} \sum_{n=\nu}^{k-[\alpha]} (-1)^{(n-\nu)}C_{\alpha-[\alpha]+1,n-\nu}\Delta^{[\alpha]}\gamma_{1+\alpha}(nt) \\ &\qquad = \sum_{\nu=1}^{k-[\alpha]} S_\nu^{\alpha} \sum_{n=\nu}^{\infty} (-1)^{(n-\nu)}C_{\alpha-[\alpha]+1,n-\nu}\Delta^{[\alpha]}\gamma_{1+\alpha}(nt) + o(1). \end{aligned} \tag{4.13}$$

The last formula is justified by

$$\sum_{\nu=1}^{k-[\alpha]} S_\nu^{\alpha} \sum_{n=k-[\alpha]+1}^{\infty} (-1)^{(n-\nu)}C_{\alpha-[\alpha]+1,n-\nu}\Delta^{[\alpha]}\gamma_{1+\alpha}(nt) = o(1), \tag{4.14}$$

which can be easily deduced by noticing that $S_\nu^\alpha = o(\nu^\alpha)$ and (4.10). Hence the left-hand side of (4.14) is

$$\begin{aligned} &O\left(\frac{1}{t^{\alpha-[\alpha]+1}} \sum_{\nu=1}^{k-[\alpha]} \nu^{\alpha} \sum_{n=k-[\alpha]+1}^{\infty} \frac{1}{(n-\nu)^{\alpha-[\alpha]+2}n^{\alpha+1}}\right) \\ &\qquad = O\left(\frac{1}{t^{\alpha-[\alpha]+1}k} \sum_{\nu=1}^{k-[\alpha]} \sum_{n=k-[\alpha]+1}^{\infty} \frac{1}{(n-\nu)^{\alpha-[\alpha]+2}}\right) \\ &\qquad = O\left(\frac{1}{t^{\alpha-[\alpha]+1}k} \sum_{\nu=1}^{k-[\alpha]} \frac{1}{(n-\nu)^{\alpha-[\alpha]+1}}\right) \\ &\qquad = O(1/t^{\alpha-[\alpha]+1}k) = O(t^{r-(\alpha-[\alpha]+1)}) = o(1), \end{aligned}$$

since $r>2>\alpha-[\alpha]+1$. Combined with (4.12) and (4.13) it follows that

$$\chi_1(t) = \sum_{n=1}^{k-[\alpha]} S_\nu^\alpha \sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha-[\alpha]+1,n-\nu} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) + o(1). \tag{4.15}$$

Moreover, by applying Lemma 3 to the above sum we obtain

$$\chi_1(t) = \sum_{n=1}^{k-[\alpha]} S_\nu^\alpha \sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1,n-\nu}\gamma_{1+\alpha}(nt) + o(1) = o(1), \tag{4.16}$$

in virtue of (4.2), (4.3) and (4.4). Theorem 1 is thus completely proved.

5. In the proof of Theorem 2, we may suppose $0<\delta<1$. We write

$$\frac{1}{\alpha}\phi_\alpha(t) = \sum_{n=1}^{\infty} A_n\gamma_\alpha(nt) = \sum_{n=1}^{k} + \sum_{n=k+1}^{\infty} = \chi_1(t) + \chi_2(t). \tag{5.1}$$

Notice that $A_n = O(n^{-\delta})$, and that

$$\gamma_\alpha(nt) = \begin{cases} O\left(\dfrac{1}{(nt)^\alpha}\right), & 0 < \alpha \leqq 2, \\[2ex] O\left(\dfrac{1}{(nt)^2}\right), & \alpha > 2, \end{cases}$$

and write

$$l = [t^{-1}], \qquad k = [t^{-r}], \tag{5.2}$$

where r is so chosen that

$$1/(1-\epsilon) > r > 1/\delta. \tag{5.3}$$

Thus we have for $\alpha = 1$,

$$\begin{aligned} \chi_2(t) = \sum_{n=k+1}^{\infty} A_n\gamma_\alpha(nt) &= O\left(\frac{1}{t}\sum_{k+1}^{\infty} n^{-1-\delta}\right) = O\left(\frac{1}{k^\delta t}\right) \\ &= O(t^{r\delta-1}) = o(1), \end{aligned} \tag{5.4}$$

and for $\alpha = 2, 3, \cdots$ we have

$$\begin{aligned} \chi_2(t) &= O\left(\frac{1}{t^2}\sum_{k+1}^{\infty} n^{-2-\delta}\right) = O\left(\frac{1}{k^{1+\delta}t^2}\right) \\ &= O(t^{(1+\delta)r-2}) = o(1), \end{aligned} \tag{5.5}$$

since $r>1/\delta>2/(1+\delta)$.

It is enough to prove $\chi_1(t) = o(1)$. By repeated use of Abel's transformation, we obtain

$$\chi_1(t) = \sum_{n=1}^{k} A_n\gamma_\alpha(nt) = \sum_{n=1}^{k-\alpha-1} S_n^\alpha \Delta^{\alpha+1}\gamma_\alpha(nt) + \sum_{j=0}^{\alpha} S_{k-j}^{j}\Delta^{j}\gamma_\alpha((k-j)t). \tag{5.6}$$

Making use of Lemma 1, we have

$$(5.7)\qquad \overset{j}{\gamma}_\alpha(mt) = O(1)$$

for all t, and

$$(5.8)\qquad \overset{j}{\gamma}_\alpha(mt) = \begin{cases} O(1/(mt)^{j+2}) & (j = 0, 1, \cdots, \alpha - 2), \\ O(1/(mt)^2) & (j = \alpha - 1, \alpha, \alpha + 1), \end{cases}$$

when $mt > 1$.

From (5.7), (5.8) and

$$(5.9)\qquad \Delta^{j}\gamma_\alpha(mt) = O(t^{j} \,|\, \overset{j}{\gamma}_\alpha(mt) \,|\,),$$

we easily deduce that

$$(5.10)\qquad \Delta^j\gamma_\alpha(mt) = O(t^j)$$

for all positive t, and if $mt > 1$

$$(5.11)\qquad \begin{gathered} \Delta^j\gamma_\alpha(mt) = O(1/m^{j+2}t^2) \quad (j = 0, 1, \cdots, (\alpha - 2)), \\ \Delta^{\alpha-1}\gamma_\alpha(mt) = O(1/m^\alpha t),\ \Delta^\alpha\gamma_\alpha(mt) = O(1/m^\alpha),\ \Delta^{\alpha+1}\gamma_\alpha(mt) = O(t/m^\alpha). \end{gathered}$$

All of the following estimates depend upon (5.10) or (5.11).

Since $A_n = O(n^{-\delta})$, so that $S_n^j = O(n^{j+1-\delta})$, we have for $j = 0, 1, 2, \cdots, (\alpha - 2)$,

$$(5.12)\qquad \begin{aligned} S_{k-j}^{j}\Delta^{j}\gamma_\alpha((k - j)t) &= O((k - j)^{j+1-\delta}1/(k - j)^{j+2}t^{2}) \\ &= O(1/k^{1+\delta}t^{2}) = O(t^{(1+\delta)r-2}) = o(1) \end{aligned}$$

since $r > 1/\delta > 2/(1 + \delta)$; and for $j = \alpha - 1$ we have

$$(5.13)\qquad \begin{aligned} S_{k-(\alpha-1)}^{\alpha-1}\Delta^{(\alpha-1)}\gamma_\alpha((k - (\alpha - 1))t) &= O(k^{\alpha-\delta}/k^{\alpha}t) \\ &= O(1/k^{\delta}t) = O(t^{r\delta-1}) = o(1). \end{aligned}$$

Moreover, since $S_n^\alpha = o(n^{\alpha-\epsilon})$, because of our hypothesis we have

$$(5.14)\qquad S_{k-\alpha}^{\alpha}\Delta^{\alpha}\gamma_\alpha((k - \alpha)t) = O\left(k^{\alpha-\epsilon}\frac{1}{k^\alpha}\right) = O(k^{-\epsilon}) = o(1).$$

It follows that each term in the second sum of the right hand side of (5.6) is $o(1)$.

Lastly, we write

$$(5.15)\qquad \sum_{n=1}^{k-\alpha-1} S_n^{\alpha}\Delta^{\alpha+1}\gamma_\alpha(nt) = \sum_{n=1}^{l} + \sum_{l+1}^{k-\alpha-1} = \chi_3(t) + \chi_4(t)$$

say; we note that in $\chi_3(t)$, $nt < 1$,

$$\chi_3(t) = \sum_{n=1}^{l} S_n^{\alpha} \Delta^{\alpha+1} \gamma_\alpha(nt) = O\left(t^{\alpha+1} \sum_{n=1}^{l} n^{\alpha-\epsilon}\right) = O(l^{\alpha+1-\epsilon} t^{\alpha+1})$$
$$= O(t^{\epsilon-\alpha-1+\alpha+1}) = O(t^{\epsilon}) = o(1), \tag{5.16}$$

while in $\chi_4(t)$, $nt > 1$,

$$\chi_4(t) = O\left(t \sum_{n=1}^{k-\alpha-1} n^{\alpha-\epsilon} \frac{1}{n^{\alpha}}\right) = O\left(t \sum_{n=1}^{k-\alpha-1} n^{-\epsilon}\right) = O(k^{1-\epsilon} t)$$
$$= O(t^{\nu(\epsilon-1)+1}) = o(1), \tag{5.17}$$

since $r < 1/(1-\epsilon)$. Collecting our results from (5.6), (5.12), (5.13), (5.14), (5.15), (5.16), (5.17) we get $\chi_1(t) = o(1)$. Theorem 2 is thus proved.

NATIONAL UNIVERSITY OF CHEKIANG,
KWEICHOW, CHINA.