ON THE ASSOCIATE AND CONJUGATE SPACE FOR THE DIRECT PRODUCT OF BANACH SPACES

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The direct product $E_1 \otimes_N E_2$ of two Banach spaces E_1, E_2 has been defined before [5](2) as the closure of the normed linear set $\mathfrak{A}_N(E_1, E_2)$ (that is, linear set $\mathfrak{A}(E_1, E_2)$ of expressions $\sum_{i=1}^{n} f_i \otimes \phi_i$, in which N is a norm) [5, p. 200, Definition 1.3] and [6, p. 499, b].

Let N denote a crossnorm whose associate N' is also a crossnorm [5, p. 208]. Then, the cross-space $E_1 \otimes_N E_2$ determines uniquely a "conjugate space" $(E_1 \otimes_N E_2)'$ and an "associate space" $E_1' \otimes_{N'} E_2'$. It is shown [5, p. 205] that $E_1' \otimes_{N'} E_2'$ is always included in $(E_1 \otimes_N E_2)'$. While there are many known examples of cross-spaces for which the associate space coincides with the conjugate space—for example, the cross-space generated by the self-associate crossnorm constructed for Hilbert spaces by F. J. Murray and John von Neumann [3, p. 128] and [5, pp. 212–214]—it is not without interest to construct a cross-space for which the associate space forms a proper subset of the conjugate space (§§1–2).

For reflexive Banach spaces E_1, E_2 (that is, such that $E_i'' = E_i$), and a reflexive crossnorm N [6, p. 500], the reflexivity of $E_1 \otimes_N E_2$ implies $(E_1 \otimes_N E_2)' = E_1' \otimes_{N'} E_2'$ [6, p. 505]. Thus, the finding of the exact conditions imposed upon reflexive Banach spaces and a reflexive crossnorm for which the resulting cross-space is reflexive is closely connected with the above-mentioned problem.

In §1, we show that for a "natural crossnorm" N, $L' \otimes_N L'$ is a proper subset of $(L \otimes_N L)'$. In §2 we prove that for a "natural crossnorm" N, $l' \otimes_{N'} l'$ is a proper subset of $(l \otimes_N l)'$. In §3 we show that for any $p > 1$, $l_p \otimes_N l_q$ is not reflexive, provided $1/p + 1/q = 1$ and N denotes the least crossnorm whose associate is also a crossnorm [5, p. 208]. The last one is reflexive [6, p. 501].

1. Let $L_{(1)}$ and $L_{(2)}$ denote the Banach spaces of all functions integrable in the sense of Lebesgue on the interval $0 \leqq s \leqq 1$, and on the square $0 \leqq s$, $t \leqq 1$ respectively. Similarly, let $M_{(1)}$ and $M_{(2)}$ denote the Banach spaces of all functions Lebesgue measurable and essentially bounded on the interval $0 \leqq s \leqq 1$ and the square $0 \leqq s$, $t \leqq 1$ respectively [1, pp. 10, 12]. We recall that

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(2) Numerals in square brackets refer to bibliography at the end of the paper. We shall use the notation of [6].

$$\text{for} \quad f(s) \in L_{(1)}, \quad \|f(s)\| = \int_0^1 |f(s)|\,ds;$$

$$\text{for} \quad f(s, t) \in L_{(2)}, \quad \|f(s, t)\| = \int_0^1 \int_0^1 |f(s, t)|\,dsdt;$$

$$\text{for} \quad F(s) \in M_{(1)}, \quad \|F(s)\| = \underset{0 \leqq s \leqq 1}{\text{ess. l.u.b.}} |F(s)|;$$

$$\text{for} \quad F(s, t) \in M_{(2)}, \quad \|F(s, t)\| = \underset{0 \leqq s,\, t \leqq 1}{\text{ess. l.u.b.}} |F(s, t)|$$

(ess. l.u.b. stands for "essential least upper bound").

For $f_i(s) \in L_{(1)}$, $\phi_i(t) \in L_{(1)}$ let the expression $\sum_{i=1}^n f_i(s) \otimes \phi_i(t)$ denote the function $\sum_{i=1}^n f_i(s)\phi_i(t)$. The last function naturally belongs to $L_{(2)}$. For an expression $\sum_{i=1}^n f_i(s) \otimes \phi_i(t)$ in $\mathfrak{A}(L_{(1)}, L_{(1)})$ we define

$$N\left(\sum_{i=1}^n f_i(s) \otimes \phi_i(t) \right) = \int_0^1 \int_0^1 \left| \sum_{i=1}^n f_i(s)\phi_i(t) \right| dsdt.$$

LEMMA 1.1. *N is a crossnorm in $\mathfrak{A}(L_{(1)}, L_{(1)})$* [5, p. 205].

Proof. The proof is elementary. In particular, the invariance of the norm under equivalence can be readily verified, since the equivalence of two expressions $\sum_{i=1}^n f_i(s) \otimes \phi_i(t)$, $\sum_{j=1}^m g_j(s) \otimes \psi_j(t)$ implies $\sum_{i=1}^n f_i(s)\phi_i(t) = \sum_{j=1}^m g_j(s)\psi_j(t)$ for almost every s and almost every t.

LEMMA 1.2. $L_{(1)} \otimes_N L_{(1)} = L_{(2)}$.

Proof. Obviously, $\mathfrak{A}_N(L_{(1)}, L_{(1)}) \subset L_{(2)}$. Since $L_{(2)}$ is complete, the closure of $\mathfrak{A}_N(L_{(1)}, L_{(1)})$, that is, $L_{(1)} \otimes_N L_{(1)} \subset L_{(2)}$. On the other hand, it is well known that functions in $L_{(2)}$ can always be approximated in norm by a sequence of expressions $\{\sum_{k=1}^{p_n} f_k^{(n)}(s)\phi_k^{(n)}(t)\}$, where $f_k^{(n)}(s) \in L_{(1)}$, $\phi_k^{(n)}(t) \in L_{(1)}$. This completes the proof.

LEMMA 1.3. $L_{(i)}' = M_{(i)}$ for $i = 1, 2$.

Proof. The proof may be found in [1, p. 65].

LEMMA 1.4. $(L_{(1)} \otimes_N L_{(1)})' = M_{(2)}$.

Proof. This is a consequence of Lemmas 1.2 and 1.3.

THEOREM 1. *$L_{(1)}' \otimes_{N'} L_{(1)}'$ is a proper subset of $(L_{(1)} \otimes_N L_{(1)})'$.*

Proof. Clearly, $L_{(1)}' \otimes_{N'} L_{(1)}' \subset (L_{(1)} \otimes_N L_{(1)})'$ [5, p. 205]. Due to Lemmas 1.3 and 1.4, the last statement may be expressed as $M_{(1)} \otimes_{N'} M_{(1)} \subset M_{(2)}$. We shall prove our theorem by showing that not every function in $M_{(2)}$ can be approximated in norm by a sequence of functions $\{\sum_i^{p_n} F_i^{(n)}(s)\Phi_i^{(n)}(t)\}$ where $F_i^{(n)}(s)$, $\Phi_i^{(n)}(t)$ belong to $M_{(1)}$. We shall show in particular that the function $K(s, t)$ defined for $0 \leqq s, t \leqq 1$ as follows: $K(s, t)=1$ if $s \leqq t$, otherwise $K(s, t)=0$, cannot be approximated in norm by such a sequence of expressions. Suppose to the contrary, that

$$\lim_{n\to\infty} \operatorname*{ess.\,l.u.b.}_{0\leqq s, t\leqq 1} \left| \sum_{i=1}^{p_n} F_i^{(n)}(s)\Phi_i^{(n)}(t) - K(s, t) \right| = 0.$$

Put

$$K_n(s, t) = \sum_{i=1}^{p_n} F_i^{(n)}(s)\Phi_i^{(n)}(t) - K(s, t).$$

Thus, there exists a set E_0 of points (s, t) in the square $0 \leqq s, t \leqq 1$, and a sequence $\{\epsilon_n\}$ of positive numbers such that:

(a) $mE_0=0$,

(b) $\epsilon_n \to 0$,

(c) $|K_n(s, t)| \leqq \epsilon_n$ for $(s, t) \notin E_0$.

Let $H(s, t)$ denote the characteristic function of E_0. Its Lebesgue integral over the square $0 \leqq s, t \leqq 1$ is 0. Fubini's theorem [7, p. 77] gives

$$\int_0^1 \left(\int_0^1 H(s, t)dt \right) ds = 0.$$

Therefore, there exists a linear set S of measure 1 in the interval $0 \leqq s \leqq 1$ such that, for every $s_0 \in S$,

$$\int_0^1 H(s_0, t)dt = 0.$$

The last statement implies for each $s \in S$ the existence of a linear set T_s of measure 1 in the interval $0 \leqq t \leqq 1$ such that $s \in S$ and $t \in T_s$ implies $H(s, t)=0$, consequently $(s, t) \notin E_0$, and therefore $|K_n(s, t)| \leqq \epsilon_n$ for $n=1, 2, \cdots$. This proves the existence of a linear set S in $0 \leqq s \leqq 1$, of measure 1, such that, for every $s_0 \in S$,

$$(1) \qquad \lim_{n\to\infty} \operatorname*{ess.\,l.u.b.}_{0\leqq t\leqq 1} \left| \sum_{i=1}^{p_n} F_i^{(n)}(s_0)\Phi_i^{(n)}(t) - K(s_0, t) \right| = 0.$$

Let $\mathfrak{M}$ denote the closed linear manifold determined by all $\Phi_i^{(n)}(t)$; $n=1, 2, 3, \cdots; i=1, 2, \cdots, p_n$. Clearly, $\mathfrak{M}$ is separable, and a subset of $M_{(1)}$. For a fixed point $s_0 \in S$, $\sum_{i=1}^{p_n} F_i^{(n)}(s_0)\Phi_i^{(n)}(t)$ is a function of one variable t, $0 \leqq t \leqq 1$, and obviously belongs to $\mathfrak{M}$. Since $\mathfrak{M}$ is closed, $K(s_0, t) \in \mathfrak{M}$ by virtue of (1). Furthermore, for $s_0 \in S$, $s_1 \in S$, and $s_0 \neq s_1$,

$$\operatorname*{ess.\,l.u.b.}_{0\leqq t\leqq 1} | K(s_0, t) - K(s_1, t) | = 1.$$

Thus, $\mathfrak{M}$ contains a "continuum number" of elements $K(s_0, t)$ whose "distance" from each other is 1. The last implication contradicts the separability of $\mathfrak{M}$ [2, p. 126]. This completes the proof.

2. Let l denote the space of all sequences of real numbers $\{x_i\}$ for which $\sum_{i=1}^{\infty}|x_i| < \infty$, and m the space of all bounded sequences of real numbers [1, pp. 11–12]. Let $\mathfrak{a}$ denote the Banach space of all infinite matrices $(a_{i,j})$ for which $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{i,j}| < \infty$, and $\mathfrak{b}$ the Banach space of all bounded matrices $(b_{i,j})$. We recall that

$$\text{for} \quad (x_1, x_2, \cdots) \in l, \quad \|(x_1, x_2, \cdots)\| = \sum_{i=1}^{\infty} |x_i|;$$

$$\text{for} \quad (\alpha_1, \alpha_2, \cdots) \in m, \quad \|(\alpha_1, \alpha_2, \cdots)\| = \sup_{1 \leq i < \infty} |\alpha_i|;$$

$$\text{for} \quad (a_{i,j}) \in \mathfrak{a}, \quad \|(a_{i,j})\| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{i,j}|;$$

$$\text{for} \quad (b_{i,j}) \in \mathfrak{b}, \quad \|(b_{i,j})\| = \sup_{1 \leq i,j < \infty} |b_{i,j}|.$$

Obviously l is equivalent to $\mathfrak{a}$ [1, p. 180]. Similarly, $\mathfrak{b}$ is equivalent to m. For $(x_1^{(k)}, x_2^{(k)}, \cdots) \in l$, $(y_1^{(k)}, y_2^{(k)}, \cdots) \in l$, the expression $\sum_{k=1}^{n}(x_1^{(k)}, x_2^{(k)}, \cdots)\otimes(y_1^{(k)}, y_2^{(k)}, \cdots)$ will mean the infinite matrix $(a_{i,j})$ of rank not greater than n, where $a_{i,j}=\sum_{k=1}^{n}x_i^{(k)}y_j^{(k)}$. Clearly, two expressions $\sum_{k=1}^{n}(x_1^{(k)}, x_2^{(k)}, \cdots)\otimes(y_1^{(k)}, y_2^{(k)}, \cdots)$, $\sum_{k=1}^{m}(\bar{x}_1^{(k)}, \bar{x}_2^{(k)}, \cdots)\otimes(\bar{y}_1^{(k)}, \bar{y}_2^{(k)}, \cdots)$ are equivalent [5, p. 196] if and only if $\sum_{k=1}^{n}x_i^{(k)}y_j^{(k)} =\sum_{k=1}^{m}\bar{x}_i^{(k)}\bar{y}_j^{(k)}$, for $i, j=1, 2, \cdots$ [5, p. 202, Theorem 2.1]. Let

$$N\left(\sum_{k=1}^{n}(x_1^{(k)}, x_2^{(k)}, \cdots)\otimes(y_1^{(k)}, y_2^{(k)}, \cdots)\right) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\sum_{k=1}^{n} x_i^{(k)} y_j^{(k)}\right| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{i,j}|.$$

LEMMA 2.1. *N is a crossnorm in* $\mathfrak{A}(l, l)$ [5, p. 205].

Proof. The proof is elementary. In particular, the previous remark shows that the norm is invariant under equivalence.

LEMMA 2.2. $l\otimes_N l=\mathfrak{a}$.

Proof. Clearly, every matrix with a finite number of rows in $\mathfrak{a}$, and every infinite matrix $(a_{i,j})\in\mathfrak{a}$ is the limit of the sequence of finite rowed square matrices $(a_{i,j})_{j=1,2,\cdots,n}^{i=1,2,\cdots,n}$; $n=1, 2, \cdots$. Thus, $\mathfrak{a}\subset l\otimes_N l$. On the other hand, every element of $\mathfrak{A}(l, l)$ belongs to $\mathfrak{a}$. Since $\mathfrak{a}$ is complete, $l\otimes_N l\subset\mathfrak{a}$. This completes the proof.

LEMMA 2.3. $(l\otimes_N l)'=\mathfrak{a}'=\mathfrak{b}$.

Proof. The proof is analogous to the case $l'=m$ [1, p. 97].

For $(\alpha_1^{(k)}, \alpha_2^{(k)}, \cdots) \in m$, $(\beta_1^{(k)}, \beta_2^{(k)}, \cdots) \in m$, let the expression $\sum_{k=1}^{n} (\alpha_1^{(k)}, \alpha_2^{(k)}, \cdots) \otimes (\beta_1^{(k)}, \beta_2^{(k)}, \cdots)$ in $\mathfrak{A}(m, m)$ denote the infinite bounded matrix $(b_{i,j})$ of rank not greater than n where $b_{i,j} = \sum_{k=1}^{n} \alpha_i^{(k)} \beta_j^{(k)}$.

LEMMA 2.4. $(l \otimes_N l)' \supset l' \otimes_{N'} l' = m \otimes_{N'} m$.

Proof. This is a consequence of [1, p. 97] and [5, p. 205].

THEOREM 2. *$l' \otimes_{N'} l'$ is a proper subset of $(l \otimes_N l)'$.*

Proof. It is sufficient to show that $m \otimes_{N'} m$ is a proper subset of $\mathfrak{b}$ (Lemmas 2.3, 2.4), or that not every bounded matrix can be approximated by a sequence of matrices of finite rank. We shall show that the bounded infinite matrix $(\delta_{i,j})$, where $(\delta_{i,j}) = 1$ if, and only if, $i = j$, otherwise $\delta_{i,j} = 0$, cannot be approximated by a sequence of bounded matrices of finite rank, that is, does not belong to $m \otimes_{N'} m$. To see that, we notice first that every bounded matrix $(b_{i,j})$ represents a linear transformation T from l into m. Let $(l_1, l_2, \cdots) \in l$. Put $T(l_1, l_2, \cdots) = (m_1, m_2, \cdots)$ where $m_i = \sum_{j=1}^{\infty} b_{i,j} l_j$. We prove

$$|||T||| = \sup_{i,j} |b_{i,j}| \qquad (|||T||| \text{ denotes the bound of } T).$$

By definition

$$|||T||| = \sup_{\|(l_1, l_2, \cdots)\| = 1} \|T(l_1, l_2, \cdots)\|.$$

The last number may be written as

$$\sup_{\|(l_1, l_2, \cdots)\| = 1} \sup_i \left| \sum_{j=1}^{\infty} b_{i,j} l_j \right| = \sup_i \sup_{\|(l_1, l_2, \cdots)\| = 1} \left| \sum_{j=1}^{\infty} b_{i,j} l_j \right|.$$

For a fixed i, $\sum_{j=1}^{\infty} b_{i,j} l_j$ denotes a linear functional on l [1, p. 97]; the bound of this linear functional is

$$\sup_{\|(l_1, l_2, \ldots)\| = 1} \left| \sum_{j=1}^{\infty} b_{i,j} l_j \right| = \sup_j |b_{i,j}|.$$

Substituting the last number in the previous equation, we get

$$|||T||| = \sup_i \sup_j |b_{i,j}| = \sup_{i,j} |b_{i,j}|,$$

or the norm of the bounded matrix $(b_{i,j})$ is equal to the bound of the linear transformation T it represents.

It is easy to see that if the matrix is of finite rank, the corresponding linear transformation T is finite-dimensional. The linear transformation T_δ corresponding to the matrix $(\delta_{i,j})$ is obviously not completely continuous, therefore it can not be considered a limit of linear transformations whose ranges are finite-dimensional [1, p. 96]. Therefore, $(\delta_{i,j})$ is not a limit of bounded matrices of finite rank. This completes the proof.

3. THEOREM 3. *Let N denote the least crossnorm whose associate is also a crossnorm* [5, p. 208]. *If $p>1$, $1/p+1/q=1$ and l_p denotes the Banach space of all sequences of real numbers $\{x_i\}$ for which $\sum_{i=1}^{\infty}|x_i|^p<\infty$* [1, p. 12], *then $l_p\otimes_N l_q$ is not reflexive.*

Proof. Let $\Phi_1, \Phi_2, \Phi_3, \cdots$ and $\phi_1, \phi_2, \phi_3, \cdots$ denote the sequence of elements $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$, $(0, 0, 1, \cdots)$ in l_p and l_q respectively. Clearly, $\Phi_i(\phi_j)=0$ if $i\neq j$, and $\Phi_i(\phi_i)=1$; $i, j=1, 2, \cdots$. With a fixed sequence of real numbers $\{\lambda_i\}$ converging towards 0, consider the sequence of expressions

$$\lambda_1\Phi_1\otimes\phi_1, \qquad \sum_{i=1}^{2}\lambda_i\Phi_i\otimes\phi_i, \qquad \sum_{i=1}^{3}\lambda_i\Phi_i\otimes\phi_i, \cdots .$$

First we prove, if $n>m$, $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)=\max_{m\leqq i\leqq n}|\lambda_i|$. By definition [5, p. 208], $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)=\sup|\sum_{i=m}^{n}\lambda_i\Phi_i(\phi)\Phi(\phi_i)|$ where sup, that is, the least upper bound, is taken for all $\Phi\in l_p$, $\phi\in l_q$, such that $\|\Phi\|=\|\phi\|=1$. Substituting in the last equation Φ_i for Φ and ϕ_i for ϕ, we obtain $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)\geqq|\lambda_i|$. Thus,

$$N\left(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i\right)\geqq\max_{m\leqq i\leqq n}|\lambda_i|.$$

On the other hand, if $\Phi\in l_p$ and $\Phi=x_1\Phi_1+x_2\Phi_2+\cdots$, then $\|\Phi\|=1$ if, and only if, $\sum_{i=1}^{\infty}|x_i|^p=1$. Similarly, if $\phi\in l_q$ and $\phi=y_1\phi_1+y_2\phi_2+\cdots$, then $\|\phi\|=1$ if, and only if, $\sum_{i=1}^{\infty}|y_i|^q=1$. Furthermore, $N(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i)=\sup|\sum_{i=m}^{n}\lambda_i x_i y_i|$ where sup is taken over the set of all sequences of real numbers $\{x_i\}$, $\{y_i\}$, for which $\sum_{i=1}^{\infty}|x_i|^p=1$ and $\sum_{i=1}^{\infty}|y_i|^q=1$. Hölder's inequality gives:

$$N\left(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i\right)\leqq\sup\left\{\left(\max_{m\leqq i\leqq n}|\lambda_i|\right)\left(\sum_{i=m}^{n}|x_i|^p\right)^{1/p}\left(\sum_{i=m}^{n}|y_i|^q\right)^{1/q}\right\}$$

where sup is as stated in the previous equation. Since

$$\sum_{i=m}^{n}|x_i|^p\leqq\sum_{i=1}^{\infty}|x_i|^p=1, \qquad \sum_{i=m}^{n}|y_i|^q\leqq\sum_{i=1}^{\infty}|y_i|^q=1,$$

$$N\left(\sum_{i=m}^{n}\lambda_i\Phi_i\otimes\phi_i\right)\leqq\max_{m\leqq i\leqq n}|\lambda_i|.$$

Since $\lambda_i\to 0$, the sequence of expressions $\lambda_1\Phi_1\otimes\phi_1, \sum_{i=1}^{2}\lambda_i\Phi_i\otimes\phi_i, \sum_{i=1}^{3}\lambda_i\Phi_i\otimes\phi_i, \cdots$ is fundamental. Therefore, it may be considered as an element of $l_p\otimes_N l_q$. Its norm is obviously [5, p. 205]

$$\sup_{1 \leq i < \infty} |\lambda_i|.$$

Thus, the well known non-reflexive space c_0 [1, p. 181] ($c_0^{\prime\prime} = l' = m$ [1, pp. 66–67]) of all converging towards 0 sequences of real numbers may be considered a subspace of $l_p \otimes_N l_q$. Since a subspace of a reflexive space is also reflexive [4, p. 423], $l_p \otimes_N l_q$ is not reflexive. This completes the proof.

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