CONCERNING THE UNIFORMIZATION OF CERTAIN RIEMANN SURFACES ALLIED TO THE INVERSE-COSINE AND INVERSE-GAMMA SURFACES

BY

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1. **Introduction**(1). The object of this paper is to consider the three classes of open simply-connected symmetric Riemann surfaces which result from the surfaces defined by the entire functions $w=\cos z^{1/2}$, $w=\cos z$, and $w=1/\Gamma(z)$, when the branch points are displaced in an arbitrary fashion along the real axis. For each class it is proved that all members are parabolic and a representation of the corresponding entire function is obtained, which is precise in the sense that any such entire function maps the punched plane onto a surface of the class in question.

The method employed is approximation by elliptic surfaces. This avoids restrictive assumptions on the location of the branch points such as those involved in a line-complex representation.

In the terminology of Iversen [1, pp. 38–52](2), certain members of each class of surfaces exhibit indirectly critical singularities, as well as either variety of directly critical singularity. To the best of the author's knowledge, all classes of surfaces in the literature which have been proved parabolic possess only directly critical singularities of the first kind, with the exception of the surfaces treated by F. E. Ulrich [1] which possess two directly critical singularities, one of the first, and one of the second kind.

2. **Description of surfaces.** The *symmetric semi-cosinic surface* $\mathcal{J}$ covering the w-plane is determined by the sequence of real numbers a_k $(k=1, 2, \cdots)$, with $a_1>0$, $a_{2n\pm1}>a_{2n}$. $\mathcal{J}$ is composed of the sheets $S_1, S_2, \cdots, S_k, \cdots$; S_1 is a replica of the w-plane cut along the positive real axis from $w=a_1$ to $w=\infty$; S_k $(k>1)$ is a replica of the w-plane cut along the real axis except for the interval between a_{k-1} and a_k. S_1 and S_2 are joined along their cuts from a_1 to $+\infty$, forming a first order branch point over $w=a_1$; S_k and S_{k+1} are joined along their cuts from a_k to $(-)^{k-1}\infty$.

In the particular case $a_k=(-)^{k-1}$, $\mathcal{J}$ is the Riemann surface of the function $z=(\text{arc cos } w)^2$. Any symmetric semi-cosinic surface is topologically equivalent to this prototype, and therefore open and simply-connected.

The possible singularities of $\mathcal{J}$ may be classified as follows:

(1) $|a_k|<M$ $(k=1, 2, \cdots)$ and $\lim_{k\to\infty} a_k$ does not exist. Then $\mathcal{J}$ has a

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(1) The author wishes to thank Professor F. E. Ulrich for arousing the writer's interest in the type problem and for his helpful criticisms.

(2) Numbers in brackets refer to the references listed at the end of the paper.

single logarithmic branch point over $w=\infty$, that is, one directly critical singularity of the first kind.

(2) The sequence a_k is unbounded. Then $\mathcal{J}$ has one directly critical singularity of the second kind over $w=\infty$.

(3) $\text{Lim}_{k\to\infty} a_k = a \neq \infty$. Then $\mathcal{J}$ has two singularities: one directly critical of the first kind over $w=\infty$, the other directly critical over $w=a$.

The *symmetric cosinic surface* $\mathcal{J}$, covering the w-plane, bears the same relation to that of $z=\text{arc cos } w$ as the semi-cosinic surface bears to the surface of $z=(\text{arc cos } w)^2$. $\mathcal{J}$ consists of the sheets S_k ($k=0, \pm 1, \pm 2, \cdots$); S_k is a replica of the w-plane slit along the real axis except for the segment from a_k to a_{k+1}; S_k and S_{k+1} are joined along their slits extending from a_{k+1}. $\mathcal{J}$ is determined by the sequence of real numbers a_k ($k=0, \pm 1, \pm 2, \cdots$), $a_{2k\pm 1} > a_{2k}$. We shall assume $a_1 > 0$, $a_0 < 0$, so that $w=0$ is in the unslit portion of S_0.

The *symmetric gammic surface* $\mathcal{J}$ is obtained by adding a pair of logarithmic ends to the initial sheet of the semi-cosinic surface. Let the sequence a_k and the sheets S_k be as described for the semi-cosinic surface. The sheet S_1 is opened up along the negative real axis from $-\infty$ to $-a_0$, $a_0 > 0$, and a logarithmic end spiraling over ∞ and $-a_0$ is attached to each shore of this cut. For later purposes it is convenient to break these up into half sheets. The end attached to the upper shore is built up of the half sheets $Q_1, Q_2, \cdots$; Q_{2k-1} is a replica of $\Im w < 0$, Q_{2k} of $\Im w > 0$. Q_1 is joined to the upper shore of S_1 along $(-\infty, -a_0)$, Q_{2k-1} and Q_{2k} are joined on $(-a_0, \infty)$, Q_{2k} and Q_{2k+1} on $(-\infty, -a_0)$. The other logarithmic end is similarly built up of half sheets $Q_{-1}, Q_{-2}, \cdots$; $Q_{-(2k+1)}$ being an image of $\Im w > 0$, Q_{-2k} an image of $\Im w < 0$, and so on.

This class of surfaces is based on the Riemann surface defined by the entire function $w = 1/\Gamma(-z)$ (cf. Lense [1], Ginzel [1]). In this prototype surface the logarithmic branch point lies over $w=0$ rather than $-a_0$; the algebraic branch points straddle out in a uniform fashion: $a_n \sim (-)^{n-1}\Gamma(n-h_n)/\log n$, $\lim h_n = 0$. The type of the symmetric gammic surface has been considered by F. E. Ulrich [1] who proves, using Ahlfors' metric condition, that $\mathcal{J}$ is parabolic provided $\text{sgn } a_n = (-)^{n-1}$ and $|a_n| \log n > m \exp(n/\log n)$ for some positive constant m.

The possible singularities of the cosinic and gammic surfaces may be classified in terms of the sequence a_k as was done for the semi-cosinic surfaces.

We now proceed to treat the class of symmetric gammic surfaces in detail. The procedure for the other two classes is closely parallel and is therefore omitted.

3. **Proof that all symmetric gammic surfaces are parabolic.** Let $\mathcal{J}$, a symmetric gammic surface, be mapped onto the disc $|\zeta| < R \leq \infty$ by the normalized function

$$\zeta = \phi(w), \qquad w = f(\zeta), \tag{1}$$

$$f(0) = 0 \in S_1, \qquad f'(0) = 1. \tag{2}$$

Let $\mathcal{J}$ be cut into two hyperbolic surfaces by slicing S_1 from $-a_0$ to a_1 along the real axis, and S_k from a_{k-1} to a_k. Let $\mathcal{J}^+$ be the part containing the upper half of S_1, $\mathcal{J}^-$ the other. There is then an appropriate semi-disc D: $|\zeta| < R_1 \leqq \infty$, $\Im\zeta > 0$, such that $\mathcal{J}^+$ is the image of D by a holomorphic function $w = f_1(\zeta)$ and such that the image of $-R_1 < \zeta < R_1$ is the system of cuts separating $\mathcal{J}^+$ and $\mathcal{J}^-$. Using the Schwarz reflection principle, $f_1(\zeta)$ is holomorphic in the whole disc $|\zeta| < R_1$, which it maps onto $\mathcal{J}$. If $f_1(\zeta)$ is normalized we obtain (1). Thus the image of the branch point of $\mathcal{J}$ over $w = a_n$ is a point $\zeta = b_n$ of the real axis:

$$f(b_n) = a_n, \qquad 0 < b_1 < b_2 < \cdots < b_k \to R. \tag{3}$$

The zeros of $f'(\zeta)$ are all simple and occur at the points $\zeta = b_n$.

The fundamental regions in the ζ-plane, images of the sheets of $\mathcal{J}$, will be as follows: S_k $(k > 1)$ is mapped onto a portion of $|\zeta| < R$ bounded by two curves C_{k-1} and C_k symmetric about the real axis and intersecting the real axis at b_{k-1} and b_k respectively. The segment (a_{k-1}, a_k) of S_k corresponds to the interval (b_{k-1}, b_k); the two shores of the cut in S_k commencing at a_{k-1} correspond to the two symmetric halves of C_{k-1}, and the two shores of the cut from a_k correspond to the two halves of C_k. S_1 is mapped into a part of $|\zeta| < R$ containing $\zeta = 0$ and bounded by C_1, Γ_1, and Γ_{-1}, Γ_1 lying in the upper half circle and to the left of C_1, Γ_{-1} being its reflection in the real axis. The interval $-R < \zeta < b_1$ corresponds to the interval $-a_0 < w < a_1$ in S_1, Γ_1 to the upper shore of the cut $(-\infty, -a_0)$ in S_1, and Γ_{-1} to the lower shore. The image of Q_n $(n = 1, 2, \cdots)$ is bounded by two curves Γ_n and Γ_{n+1}, the image of Q_{-n} by curves Γ_{-n}, Γ_{-n-1} which are the reflections of Γ_n and Γ_{n+1} in the real axis. Γ_n separates Γ_{n+1} and C_1. Each of the curves C_k and $\Gamma_{\pm n}$ is a Querschnitt of the disc $|\zeta| < R$ and no two of these have points in common. These curves together with the interval $-R < \zeta < R$ of the real axis constitute the real paths of $f(\zeta)$, the locus of points for which $f(\zeta)$ is real. Aside from the symmetry, this description is topological, and indifferent to $R < \infty$ or $R = \infty$.

We now consider the *elliptic approximating surface* $\mathcal{J}_n$ which is assembled from the $n+1$ sheets $S_1, S_2, \cdots, S_n, S'_{n+1}$ and the $2n$ half sheets $Q_{\pm 1}, Q_{\pm 2}, \cdots, Q_{\pm(n-1)}, Q'_{\pm n}$. The unprimed sheets are as in $\mathcal{J}$; S'_{n+1} is obtained by closing S_{n+1} smoothly across the cut $(a_{n+1}, (-)^n\infty)$; Q'_n and Q'_{-n} dissected are the same as Q_n and Q_{-n}, but Q'_n and Q'_{-n} are now connected along their free shores rather than passing on to Q_{n+1} and $Q_{-(n+1)}$.

Thus $\mathcal{J}_n$ is a $2n+1$ sheeted simply-connected closed surface with n first order branch points over $a_1, \cdots, a_n$, an nth order branch point over $-a_0$, and a branch point of order $2n$ over $w = \infty$. This Riemann surface is the image of the closed z-plane by a rational function which we may take to be a polynomial since there is only one point of $\mathcal{J}_n$ over $w = \infty$. We normalize this map to correspond to that of the complete surface:

$$w = P_n(z) \qquad \text{of degree } 2n+1, \tag{4}$$

$$P_n(0) = 0 \in S_1, \qquad P_n'(0) = 1. \tag{5}$$

The image of the branch point over $w=a_k$ is $z=b_{n,k}$ $(k=1, 2, \cdots, n)$, $0<b_{n,1}< \cdots <b_{n,n}$. The image of the nth order branch point is $z=-c_n$, $c_n>0$. The fundamental regions are bounded by curves $C_{n,k}$ $(k=1, \cdots, n)$ through $b_{n,k}$ and by $2n$ curved rays emanating from $-c_n$.

Let D_n be the z-plane cut along the real axis except for the interval $(-c_n, b_{n,n})$. Let Δ_n be that region of the ζ-plane containing the origin and bounded by $|\zeta| =R$, Γ_{n+1}, $\Gamma_{-(n+1)}$, C_{n+1}, and the interval (b_n, b_{n+1}). As is readily seen D_n and Δ_n correspond to the same portion of $\mathcal{J}$ by the maps $w=P_n(z)$ and $w=f(\zeta)$ respectively. The composite function

$$\zeta = \psi_n(z) = \phi(P_n(z)) \tag{6}$$

maps D_n schlichtly onto Δ_n, and by (2) and (5)

$$\psi_n(0) = 0, \qquad \psi_n'(0) = 1. \tag{7}$$

The domain D_n contains the disc $|z| <\min(c_n, b_{n,n})$. Applying the Koebe 1/4-theorem to the map of this disc by $\psi_n(z)$ we obtain

$$R > 4^{-1} \min (c_n, b_{n,n}). \tag{8}$$

Now $P_n'(z)$ has simple zeros at $z=b_{n,k}$ and an nth order zero at $z=-c_n$. Consulting (5), we have

$$P_n'(z) = (1 + z/c_n)^n \prod_{k=1}^{n} (1 - z/b_{n,k}), \tag{9}$$

$$P_n(z) = \int_0^z P_n'(z)dz. \tag{10}$$

In particular,

$$- a_0 = \int_0^{-c_n} P_n'(z)dz, \qquad a_1 = \int_0^{b_{n,1}} P_n'(z)dz. \tag{11}$$

Let b_n^* be defined by

$$n/b_n^* = \sum_{k=1}^{n} 1/b_{n,k}. \tag{12}$$

Then $b_{n,1}<b_n^*<b_{n,n}$. For $-c_n<z<b_{n,1}$ all the factors in the product (9) are positive, and by the comparison between the geometric and arithmetic means

$$P_n'(z) < (1 + z/c_n)^n \left\{\frac{1}{n} \sum_{k=1}^{n} (1 - z/b_{n,k})\right\}^n = (1 + z/c_n)^n(1 - z/b_n^*)^n.$$

We consider two possibilities:

(1) $b_n^* \geqq c_n$. Then for $-c_n < z < 0$, $P_n'(z) < (1+z/c_n)^n(1-z/c_n)^n = (1-z^2/c_n^2)^n$ and by (11)

$$a_0 = \int_{-c_n}^{0} P_n'(z)dz < \int_{-c_n}^{0} (1 - z^2/c_n^2)^n dz = c_n \int_0^{\pi/2} \cos^{2n+1} t dt$$
$$= c_n \pi^{1/2}\Gamma(n + 1)/2\Gamma(n + 3/2) \sim \pi^{1/2}c_n/2n^{1/2}.$$

Thus $b_n^* \geqq c_n > An^{1/2}$, $A > 0$.

(2) $c_n > b_n^*$. Then for $0 < z < b_{n,1}$, $P_n'(z) < (1+z/b_n^*)^n(1-z/b_n^*)^n = (1-z^2/b_n^{*2})^n$ and by (11)

$$a_1 = \int_0^{b_{n,1}} P_n'(z)dz < \int_0^{b_{n,1}} (1 - z^2/b_n^{*2})^n dz < \int_0^{b_n^*} (1 - z^2/b_n^{*2})^n dz$$
$$= b_n^* \int_0^{\pi/2} \cos^{2n+1} tdt \sim \pi^{1/2}b_n^*/2n^{1/2}.$$

Thus $c_n \geqq b_n^* > An^{1/2}$, $A > 0$.

In either case, by (12)

$$(13) \qquad c_n > An^{1/2}, \qquad b_{n,n} > b_n^* > An^{1/2}, \qquad A > 0.$$

By (8) and (13) $R = \infty$, that is, $\mathcal{J}$ is parabolic.

4. **The structure of the entire function** $w = f(z)$. The following is a standard theorem (cf. Bieberbach [1, pp. 13–15]) on families of schlicht mappings: let A_n be a sequence of schlicht domains in the z-plane, all containing the origin. Let B_n be a sequence of schlicht domains in the ζ-plane, all containing the origin. Let $\zeta = F_n(z)$ map A_n schlichtly onto B_n, $F_n(0) = 0$, $F_n'(0) = 1$. If the sequence A_n converges to its kernel A then a necessary and sufficient condition that $F_n(z)$ converge uniformly in any closed subset of A is that the B_n's converge to their kernel B. The limit function $F(z)$ maps A schlichtly onto B.

We have just shown that the sequence of domains D_n converges to the punched z-plane, and Δ_n converges to the punched ζ-plane. Hence $\psi_n(z) \to F(z)$, $F(0) = 0$, $F'(0) = 1$, and $\zeta = F(z)$ maps the finite z-plane onto the finite ζ-plane. Therefore $F(z) = z$. Thus by (1), (6)

$$(14) \qquad P_n(z) \to f(z), \qquad P_n'(z) \to f'(z)$$

uniformly for $|z|$ bounded. By Hurwitz' theorem the zeros of $P_n'(z)$ tend to those of $f'(z)$,

$$(15) \qquad \lim_{n\to\infty} b_{n,k} = b_k, \qquad k = 1, 2, \cdots,$$

and the multiple zero of $P_n'(z)$ at $z = -c_n$ passes out of the picture by (13). Note that (15) is of course not uniform for all k.

For $|z| < b_1/2$ and $n > n_0$, $P_n'(z)$ and $f'(z)$ do not vanish and log $P_n'(z)$

$\to \log f'(z)$, where we take that determination of the logarithm for which $\log P_n'(0) = \log f'(0) = 0$. From (9)

$$(16) \quad \log P_n'(z) = \left(n/c_n - \sum_{k=1}^{n} 1/b_{n,k}\right) z - \frac{1}{2}\left(n/c_n^2 + \sum_{k=1}^{n} 1/b_{n,k}^2\right) z^2 + \cdots,$$

and therefore the following limits exist:

$$(17) \quad \lim_{n\to\infty} \left(n/c_n - \sum_{k=1}^{n} 1/b_{n,k}\right) = \sigma_1,$$

$$(18) \quad \lim_{n\to\infty} \frac{1}{2}\left(n/c_n^2 + \sum_{k=1}^{n} 1/b_{n,k}^2\right) = \sigma_2 > 0.$$

All terms in the parenthesis of (18) are positive, hence bounded for all n:

$$(19) \quad n/c_n^2 < M, \qquad \sum_{k=1}^{n} 1/b_{n,k}^2 < M.$$

For any fixed N, by (15), $2^{-1}\sum_{k=1}^{N} 1/b_k^2 = \lim_{n\to\infty} 2^{-1}\sum_{k=1}^{N} 1/b_{n,k}^2 \leqq \liminf_{n\to\infty} 2^{-1} \cdot \sum_{k=1}^{n} 1/b_{n,k}^2 \leqq \sigma_2$, and allowing N to become infinite

$$(20) \quad \sigma_3 = \frac{1}{2} \sum_{k=1}^{\infty} 1/b_k^2 \leqq \sigma_2.$$

Now from (19), (20), and the monotone character of the sequences b_k, $b_{n,k}$

$$(21) \quad c_n > An^{1/2}, \qquad b_{n,k} > Ak^{1/2}, \qquad b_k > Ak^{1/2}$$

for some positive constant A. For $p \geqq 3$ and any N

$$\limsup_{n\to\infty} \left| \sum_{k=1}^{\infty} b_k^{-p} - \sum_{k=1}^{n} b_{n,k}^{-p} \right| \leqq \limsup_{n\to\infty} \sum_{k=1}^{N} \left| b_k^{-p} - b_{n,k}^{-p} \right|$$
$$+ \sum_{k=N+1}^{\infty} b_k^{-p} + \limsup_{n\to\infty} \sum_{k=N+1}^{n} b_{n,k}^{-p}.$$

If we use (15) and (21) the first term on the right vanishes and the last two are $O(N^{1-p/2})$; if we allow N to become infinite

$$(22) \quad \lim_{n\to\infty} \sum_{k=1}^{n} 1/b_{n,k}^p = \sum_{k=1}^{\infty} 1/b_k^p, \qquad p \geqq 3.$$

If we combine (16), (17), (18), (21), (22)

$$(23) \quad \log f'(z) = \sigma_1 z - \sigma_2 z^2 - \sum_{p=3}^{\infty} (z^p/p) \sum_{k=1}^{\infty} b_k^{-p}.$$

By (20) the canonical product

$$\Pi(z) = \prod_{k=1}^{\infty} \{(1 - z/b_k)e^{z/b_k}\} \tag{24}$$

converges, and for $|z|$ small

$$\log \Pi(z) = -(z^2/2)\sum_{k=1}^{\infty} b_k^{-2} - \sum_{p=3}^{\infty}(z^p/p)\sum_{k=1}^{\infty} b_k^{-p}.$$

Comparing this with (23) and (20): $\log f'(z) = \sigma_1 z - (\sigma_2 - \sigma_3)z^2 + \log \Pi(z)$ with $\sigma_2 - \sigma_3 \geqq 0$. Writing $\sigma_2 - \sigma_3 = \alpha$, $\sigma_1 = \beta$, we have

$$f'(z) = e^{-\alpha z^2 + \beta z}\Pi(z) \qquad \alpha \geqq 0, \beta \text{ real.} \tag{25}$$

Now $f(z)$ must have a finite asymptotic value along the negative end of the real axis corresponding to the logarithmic branch point of $\mathcal{J}$ over $w = -a_0$. Hence the integral $\int_0^{-\infty} \exp(-\alpha x^2 + \beta x)\Pi(x)dx$ must converge. So we must have either $\alpha > 0$ or $\beta + \sum_{k=1}^{\infty} 1/b_k > 0$. If $\sum 1/b_k$ diverges this condition is met, but as we shall see later (Theorem Ia) this sum may converge.

Summarizing these results we have the following theorem.

THEOREM I. *The symmetric gammic surface $\mathcal{J}$ determined by any sequence of real numbers a_k $(k = 1, 2, \cdots)$, $a_1 > 0$, $a_{2n\pm1} > a_{2n}$, and any $a_0 > 0$ is always parabolic. Furthermore, $\mathcal{J}$ is the* (1-1) *image of the z-plane by an entire function*

$$w = f(z) = \int_0^z f'(t)dt, \tag{26a}$$

$$f'(z) = e^{-\alpha z^2 + \beta z}\prod_{k=1}^{\infty}\{(1 - z/b_k)e^{z/b_k}\}, \tag{26b}$$

$$0 < b_1 < b_2 < \cdots, \sum b_k^{-2} < \infty, \tag{26c}$$

$$\alpha \geqq 0, \quad \beta \text{ real}, \quad \max\left\{\alpha, \beta + \sum_{k=1}^{\infty} 1/b_k\right\} > 0, \tag{26d}$$

the branch point over $w = a_k$ corresponding to $z = b_k$.

Remark. If the normalization $w(0) = 0 \in S_1$, $w'(0) = 1$, is dropped we obtain: any entire function $w = g(z)$ which maps the z-plane (1-1) onto $\mathcal{J}$ is of the form $g(z) = f(pz + q)$, p, q constants, $p \neq 0$.

In a very similar fashion we obtain the following results for the other two classes of surfaces described in §2.

THEOREM II. *The symmetric semi-cosinic surface $\mathcal{J}$ is always parabolic; it is the* (1-1) *image of the z-plane by an entire function*

$$w = f(z) = \int_0^z f'(t)dt, \tag{27a}$$

(27b) $$f'(z) = e^{-\delta z}\prod_{k=1}^{\infty}(1 - z/b_k),$$

(27c) $$0 < b_1 < b_2 < \cdots, \quad \sum 1/b_k < \infty,$$

(27d) $$\delta \geqq 0,$$

the branch point over $w = a_k$ *corresponding to* $z = b_k$, *and* $f(0) = 0 \in S_1$.

THEOREM III. *The symmetric cosinic surface* $\mathcal{J}$ *is always parabolic; it is the* (1-1) *image of the z-plane by an entire function*

(28a) $$w = f(z) = \int_0^z f'(t)dt,$$

(28b) $$f'(z) = e^{-\alpha z^2+\beta z}\prod_{k=-\infty}\left\{(1 - z/b_k)e^{z/b_k}\right\},$$

(28c) $$\cdots < b_{-1} < b_0 < 0 < b_1 < b_2 < \cdots, \sum b_k^{-2} < \infty,$$

(28d) $$\alpha \geqq 0, \qquad \beta \text{ real},$$

the branch point over $w = a_k$ *corresponding to* $z = b_k$, *and* $f(0) = 0 \in S_0$.

Remarks. In Theorems I and III, $\sum 1/b_k$ may or may not converge (§5 will show that both possibilities may occur). If $\sum 1/b_k$ converges then the representation (26b) or (28b) may be simplified. Note that then (26d) and (27d) prevent overlapping between (26b) and (27b).

5. **The converse problem.** The converses of the preceeding theorems are also true, that is:

THEOREM Ia. *Any entire function of the form* (26abcd) *maps the z-plane* (1-1) *onto the symmetric gammic surface with* $a_k = f(b_k)$, $-a_0 = \lim_{x\to-\infty} f(x)$, *x real.*

THEOREM IIa. *Any entire function of the form* (27abcd) *maps the z-plane* (1-1) *onto the symmetric semi-cosinic surface with* $a_k = f(b_k)$.

THEOREM IIIa. *Any entire function of the form* (28abcd) *maps the z-plane* (1-1) *onto the symmetric cosinic surface with* $a_k = f(b_k)$.

The proofs of these three theorems are similar, and we consider only the first. The essence of the proof is to construct the fundamental regions for $w = f(z)$ by finding the real paths of $f(z)$. Now it is obvious that any $f(z)$ of the form (26) has, among others, the following real paths: the real axis, and one curve C_k through each point $z = b_k$ ($k = 1, 2, \cdots$) symmetric about the real axis. Furthermore, no two real paths intersect except at the critical points b_k. But it is not obvious that $f(z)$ also has the appropriate real paths $\Gamma_{\pm k}$ described in §3, nor is it immediate that there are no more real paths between the C_k's. The easiest procedure is to use a sequence of polynomials which approximate $f(z)$ and which correspond to easily constructed surfaces.

Let

$$\beta_n = \beta + \sum_{k=1}^{n} 1/b_k. \tag{29}$$

For any sequence of positive integers λ_n which increase rapidly enough the polynomials

$$Q_n(z) = \int_0^z Q_n'(t)dt, \tag{30}$$

$$Q_n'(z) = (1 - \alpha z^2/\lambda_n)^{\lambda_n}(1 + \beta_n z/\lambda_n)^{\lambda_n}\prod_{k=1}^{n}(1 - z/b_k) \tag{31}$$

approximate $f(z)$:

$$\lim_{n\to\infty} Q_n'(z) = f'(z), \qquad \lim_{n\to\infty} Q_n(z) = f(z) \tag{32}$$

uniformly for $|z|$ bounded. This depends on

$$\lim_{N\to\infty}(1 + z/N)^N = e^z, \qquad \text{uniformly for } |z| \leqq R. \tag{33}$$

Thus

$$\lim_{n\to\infty}(1 - \alpha z^2/\lambda_n)^{\lambda_n} = e^{-\alpha z^2}. \tag{34}$$

Also

$$\begin{aligned} &\left|(1 + \beta_n z/\lambda_n)^{\lambda_n}\prod_{k=1}^{n}(1 - z/b_k) - e^{\beta z}\prod_{k=1}^{\infty}\{(1 - z/b_k)e^{z/b_k}\}\right| \\ &\leqq \left|(1 + \beta_n z/\lambda_n)^{\lambda_n} - e^{\beta_n z}\right| \cdot \left|\prod_{k=1}^{n}(1 - z/b_k)\right| \\ &+ \left|e^{\beta_n z}\prod_{k=1}^{n}(1 - z/b_k) - e^{\beta z}\prod_{k=1}^{\infty}\{(1 - z/b_k)e^{z/b_k}\}\right|. \end{aligned} \tag{35}$$

The last term on the right of (35) tends to zero, uniformly for $|z| \leqq R$, by the nature of an infinite product. If we choose λ_n so that, by virtue of (33) $|(1+\beta_n z/\lambda_n)^{\lambda_n} - e^{\beta_n z}| < [n\prod_{k=1}^{n}(1+n/b_k)]^{-1}$ for $|z| \leqq n$, then the left-hand member of (35) tends to zero uniformly for $|z|$ bounded. Combining this with (34) we have the first part of (32); the second follows immediately.

We shall now construct the Riemann surface of the function $z = \phi(w)$, inverse to $w = f(z)$, for the case $\alpha > 0$. The general idea is that the real paths of $Q_n(z)$ tend to those of $f(z)$. Increasing the terms in the sequence λ_n does not affect (32), so we shall assume that $\lambda_n/|\beta_n| > (\lambda_n/\alpha)^{1/2} > b_n$. By (31) it is readily seen that the real paths of $Q_n(z)$ are as follows: (1) the real axis; (2) curves

$C_{n,k}$ $(k=1, \cdots, n)$ through b_k; (3) $2\lambda_n$ curved rays $\Gamma_{n,\pm k}$ $(k=1, \cdots, \lambda_n)$ emanating from $z=-(\lambda_n/\alpha)^{1/2}$, those in the upper half-plane being $\Gamma_{n,1}$, $\Gamma_{n,2}$, $\cdots$, Γ_{n,λ_n} in counterclockwise order about their origin, and $\Gamma_{n,-k}$ being the reflection of $\Gamma_{n,k}$ in the real axis; (4) $2\lambda_n$ curved rays Γ' emanating from $z=(\lambda_n/\alpha)^{1/2}$; and (5) $2\lambda_n$ curved rays Γ'' emanating from $z=-\lambda_n/\beta_n$. This whole scheme is symmetric about the real axis.

The rays Γ' and Γ'' disappear in the limit, for if not, some circle $|z| \leqq R$ would contain sections of the limiting curves of all the $C_{n,k}$ and $\Gamma_{n,k}$ (since $\lambda_n/|\beta_n| > (\lambda_n/\alpha)^{1/2} > b_n$), and $f(z)$ would possess sections of infinitely many different real paths in $|z| \leqq R$, which is impossible for an entire function.

The curves $C_{n,k}$ will tend to limiting curves C_k. They cannot pass out of the picture since $C_{n,k}$ contains the fixed point $z=b_k$ for all n, but it is conceivable that C_k might consist of several pieces, all "ends" being at $z=\infty$. The strip $D_{n,k}$, bounded by $C_{n,k}$ and $C_{n,k+1}$, is mapped by $Q_n(z)$ onto a w-plane, $\Delta_{n,k}$, slit along the real axis except for the segment $(Q_n(b_k), Q_n(b_{k+1}))$. The sequence $\Delta_{n,k}$ converges to its kernel Δ_k, the w-plane slit along the real axis except for the segment $(a_k, a_{k+1})=(f(b_k), f(b_{k+1}))$. Applying the theorem stated at the beginning of §4 to the sequence of functions $Q_n^{-1}(w)$ (the condition $F_n(0)=0$, $F_n'(0)=1$, is replaced by $Q_n^{-1}((a_k+a_{k+1})/2)\rightarrow A$, $Q_n^{-1}((a_k+a_{k+1})/2) \rightarrow B \neq 0$) we see: $w=f(z)$ maps D_k, the kernel of the sequence $D_{n,k}$, onto Δ_k. Thus D_k is simply-connected. Furthermore C_k is all in one piece, for if we apply similar considerations to the map of the part of $D_{n,k-1}+D_{n,k}$ in the upper half-plane, $f(z)$ is analytic at every point of C_k except one at infinity, that is, C_k is one connected curve and the regions D_1, D_2, $\cdots$, fill out the section of the z-plane bounded by C_1 and not containing $z=0$. Comparing this with §3, this part of the z-plane is mapped onto the sheets S_2, S_3, $\cdots$, of the "semicosinic end" of a gammic surface.

The logarithmic ends are obtained in a similar fashion: the region of the upper half-plane bounded by $C_{n,1}$, $\Gamma_{n,2}$, the real axis, and containing $\Gamma_{n,1}$ is mapped by $Q_n(z)$ onto a plane slit along the real axis from $+\infty$ to $Q_n(-(\lambda_n/\alpha)^{1/2})$, $\Gamma_{n,1}$ mapping into the remainder of the real axis. Applying the same theorem to this sequence of maps, there are two possibilities: (1) $\Gamma_{n,1}$ (and hence all $\Gamma_{n,k}$) disappear in the limit and the (appropriate) region bounded by the real axis and C_1 is mapped by $w=f(z)$ onto an upper half-plane. This is impossible, for this would imply that $\lim_{x\to-\infty} f(x)=-\infty$ (x real), whereas equations (26) imply a finite asymptotic value $=-a_0$ along the negative real axis. Hence we have (2), the curves Γ_1 and Γ_2, limits of $\Gamma_{n,1}$ and $\Gamma_{n,2}$, actually exist; the region bounded by Γ_2, the real axis, and C_1 is mapped by $w=f(z)$ onto a plane slit along $(-a_0, +\infty)$. The curve Γ_1 is mapped onto $(-\infty, -a_0)$ and hence Γ_1 is all in one piece. Repeating this argument for the various regions between $\Gamma_{n,k}$ and $\Gamma_{n,k+2}$, we see that $f(z)$ has an infinite sequence of real paths Γ_k as described in §3, and the associated strips map into the half-sheets of a logarithmic end with singularities over $w=-a_0$ and

$w=\infty$. Since $f(z)$ is symmetric in the real axis we obtain the other logarithmic end, which completes the proof of Theorem Ia, for $\alpha>0$. For $\alpha=0$ the procedure is virtually the same except that the paths Γ_k arise from the multiple zero of $Q_n'(z)$ at $z=-\lambda_n/\beta_n$ (less than 0 for n sufficiently large by (26d)).

6. **Some properties of the fundamental regions.** The sequence of polynomials $Q_n(z)$ may be used to derive properties of the real paths of $w=f(z)$ other than the topological and symmetric properties mentioned in §3.

First it may be shown that the line $y=y_0>0$ $(z=x+iy)$ intersects all the real paths of $Q_n(z)$ which lie in the upper half-plane exactly once. These intersections correspond to the roots of the equation

$$\mathfrak{I}Q_n(x+iy_0)=0 \tag{36}$$

which, for $\alpha>0$, is of degree $3\lambda_n+n$ in x. Considering the fact that the real paths of a polynomial divide the angle at $z=\infty$ equally, the line $y=y_0$ must intersect each of the following at least once: $C_{n,1},\cdots,C_{n,n};\Gamma_{n,1},\cdots,\Gamma_{n,\lambda_n}$; λ_n of the Γ', and λ_n of the Γ''. But this makes up the precise degree of (36), so there is exactly one simple intersection of the line with each curve mentioned. Since the roots of (36) are real and simple, the roots of

$$\partial\,\mathfrak{I}Q_n(x+iy_0)/\partial x=0 \tag{37}$$

are real, simple, and alternate with those of (36). That these facts still obtain in the limit may be proved in various ways, either elementary or by use of Hurwitz' theorem. *The roots of*

$$\mathfrak{I}f(x+iy_0)=0, \qquad y_0\neq 0, \tag{38}$$

are simple and alternate with the (simple) roots of

$$\partial\,\mathfrak{I}f(x+iy_0)/\partial x=0. \tag{39}$$

For $\alpha=0$ this is proved in a similar manner. This result does not state that the line $y=y_0$ actually intersects C_k and Γ_k for all values of y_0 and k; some of these intersections might disappear in the limit, for example Γ_1 might have a horizontal asymptote $y=y_1>0$. We consider this question now. Equation (39) is equivalent to $\mathfrak{I}f'(x+iy_0)=0$, or

$$\Theta=\arg f'(x+iy_0)=-k\pi, \qquad k=0,\pm 1,\pm 2,\cdots. \tag{40}$$

Consulting (26b), we have

$$\Theta=-2\alpha xy+\beta y+\sum_{k=1}^{\infty}\left\{\frac{y}{b_k}-\tan^{-1}\frac{y}{b_k-x}\right\} \tag{41}$$

where we choose $0<\tan^{-1}(y/(b_k-x))<\pi$. With this determination the sum in (41) is convergent, and Θ is a continuous function of both x and y for all x and for $y>0$. Differentiating (41), we have

$$\frac{\partial\Theta}{\partial x} = -2\alpha y - \sum_{k=1}^{\infty} y/[y^2 + (b_k - x)^2]. \tag{42}$$

Thus Θ decreases as we move to the right along $y = y_0$. Since Θ is continuous, the value of k in (40) associated with the intersection between two given C or Γ curves is a constant. The particular value of this constant may be determined by setting $x = (b_k + b_{k+1})/2$ in (41) and letting $y \to 0$:

$$\lim_{y\to 0} \Theta((b_k + b_{k+1})/2, y) = -k\pi.$$

Let the root of $\Theta(x, y) = -k\pi$ be $x = \xi_k(y)$. Let the equation of the curve C_k be $x = \chi_k(y)$ and the equation of Γ_k be $x = \chi_{-k}(y)$. Then we have

$$\begin{aligned} &\xi_{k-1}(y) < \chi_k(y) < \xi_k(y), && k = 1, 2, \cdots, \\ &\xi_{-k}(y) < \chi_{-k}(y) < \xi_{-k+1}(y), && k = 1, 2, \cdots. \end{aligned} \tag{43}$$

There is one immediate fact to be drawn from (42) if $\alpha > 0$, for then

$$|\Theta(x_2, y) - \Theta(x_1, y)| = \int_{x_1}^{x_2} \left|\frac{\partial\Theta}{\partial x}\right| dx > 2\alpha y |x_2 - x_1|.$$

Therefore $0 < \xi_{k-1}(y) - \xi_k(y) < \pi/2\alpha y$, and by (43),

$$\begin{aligned} &0 < \chi_{k+1}(y) - \chi_k(y) < \pi/\alpha y, && k = 1, 2, \cdots, \\ &0 < \chi_1(y) - \chi_{-1}(y) < \pi/\alpha y, && \\ &0 < \chi_{-k}(y) - \chi_{-k-1}(y) < \pi/\alpha y, && k = 1, 2, \cdots. \end{aligned} \tag{44}$$

In this case, $\alpha > 0$, we see that the real paths have no horizontal asymptotes, and the horizontal width of all strips decreases uniformly as $1/y$.

Now for $x < 0$, $\tan^{-1}(y/(b_k - x)) < y/(b_k - x) < y/b_k$, and since the sum in (41) is a monotone function of x we have for any N

$$\lim_{x\to-\infty} \sum_{k=1}^{\infty} \left\{\frac{y}{b_k} - \tan^{-1}\frac{y}{b_k - x}\right\} \geqq \lim_{x\to-\infty} \sum_{k=1}^{N} \left\{\frac{y}{b_k} - \tan^{-1}\frac{y}{b_k - x}\right\} = \sum_{k=1}^{N} \frac{y}{b_k}.$$

Also

$$\lim_{x\to-\infty} \sum_{k=1}^{\infty} \left\{\frac{y}{b_k} - \tan^{-1}\frac{y}{b_k - x}\right\} \leqq \sum_{k=1}^{\infty} \frac{y}{b_k}.$$

Thus this limit is $y\sum_{k=1}^{\infty} 1/b_k$. If $\alpha = 0$, then

$$\lim_{x\to-\infty} \Theta(x, y) = y\left(\beta + \sum_{k=1}^{\infty} 1/b_k\right). \tag{45}$$

Thus if $\sum 1/b_k$ diverges all curves Γ are intersected by any horizontal line, but if $\sum_{k=1}^{\infty} 1/b_k = t < \infty$ then $y = y_0$ intersects Γ_k for $y_0 > k\pi/(\beta + t)$ but does not intersect Γ_k for $y_0<(k-1)\pi/(\beta+t)$.

On the other side of the picture, Θ is a monotone function of x and so $\lim_{x\to+\infty}\Theta(x, y_0)$ exists and the value determines roughly which curves C_k are intersected by $y=y_0$. Here we consider only $\alpha=0$, since we know the answer by (44) when $\alpha>0$.

For $b_n<x<b_{n+1}$, $b_n>y$, we have from (42)

$$\frac{\partial\Theta}{\partial x} < -\sum_{k=1}^{n}(y/(y^2+x^2)-\sum_{k=n+1}^{\infty}y/(y^2+b_k^2)$$

$$< -\sum_{k=1}^{n}y/(x^2+x^2) = -ny/2x^2,$$

$$\Theta(b_n, y)-\Theta(b_{n+1}, y)=\int_{b_n}^{b_n+1}-\frac{\partial\Theta}{\partial x}dx>\frac{ny}{2}\left(\frac{1}{b_n}-\frac{1}{b_{n+1}}\right).$$

Adding these inequalities for all $n\geqq m$, where $b_m>y$:

$$\Theta(b_m, y)-\lim_{x\to\infty}\Theta(x, y)>\frac{y}{2}\sum_{n=m}^{\infty}\left(\frac{n}{b_n}-\frac{n}{b_{n+1}}\right)$$

$$=\frac{y}{2}\left(m/b_m+\sum_{n=m+1}^{\infty}1/b_n\right).$$

Therefore if $\sum 1/b_n$ diverges, $\lim_{x\to\infty}\Theta(x, y)=-\infty$ and C_n is intersected by every horizontal line.

If $\sum 1/b_n$ converges we proceed as follows. From (41), $\alpha=0$,

$$\Theta=\beta y+y\sum_{n=1}^{\infty}1/b_n-\sum_{n=1}^{\infty}\tan^{-1}\frac{y}{b_n-x}. \tag{46}$$

Then since $\tan^{-1}(y(b_n-x))>0$ we have for $x>b_m$, $\Theta<\beta y+y\sum_{n=1}^{\infty}1/b_n-\sum_{n=1}^{m}\pi/2=Ay-m\pi/2$ and $\lim_{x\to\infty}\Theta(x, y)=-\infty$, and again, *$C_k$ is intersected by every horizontal line.*

Sharper results on the course of the real paths may obviously be obtained using the relation between (39) and (40), especially when a reasonably regular behaviour is assumed for the sequence b_n.

The problem of the relation between the numbers α, β, b_n and the branch points, a_n, of the Riemann surface is not simple. The following however may easily be established: *if $\alpha>0$ then*

$$\sum_{k=n}^{\infty}|a_{k+1}-a_k|=O(e^{-(\alpha-\epsilon)b_n^2})=o(e^{-\gamma n}) \tag{47}$$

for any positive γ. For the canonical product in (26) is $O(e^{\epsilon r^2})$ for any $\epsilon>0$. Thus $\sum_{k=n}^{\infty}|a_{k+1}-a_k|=\int_{b_n}^{\infty}|f'(z)|dz=O(e^{-(\alpha-\epsilon)b_n^2})$, which is the first part of (47). The rest follows since $b_n^2/n\to\infty$.

Results similar to the above may be stated for the other two classes of surfaces. The *semi-cosinic fundamental regions*, associated with (27), have the following properties:

(1) The real paths of $w=f(z)$ consist of the real axis and one curve C_n through each critical point b_n, symmetric about the real axis.

(2) As the point P travels along the upper half of C_n from b_n to infinity its ordinate increases monotonically.

(3) Each C_n eventually enters the half-plane $\Re(z)<b_1$. The upper half of C_n has a finite negative slope at every point of this half-plane. C_n has precisely one simple intersection with the line $\Re(z)=x_0<b_1$ in the upper half-plane. Let this point be $(x_0, y_n(x_0))$.

(4) The values $y_n(x_0)$ alternate with the simple roots of $\arg f'(x_0-iy) =(2k-1)\pi/2$. If $\arg f'(x_0)=0$ and $\arg f'(x_0-i\eta_k)=(2k-1)\pi/2$, then $0<\eta_1 <y_1<\eta_2<y_2< \cdots$.

(5) Each C_n is simply visible from any point $z<b_1$ of the real axis.

(6) C_k lies entirely in the half-plane $\Re(z)<b_{2k-1}$.

(7) If $\delta>0$, then $\lim_{x\to\infty}f(x)=a\neq\infty$, x real, and

$$\sum_{k=n}^{\infty}\left| a_{k+1}-a_k\right| < A(\epsilon)e^{-(\delta-\epsilon)b_n}.$$

If $\delta=0$ and $\lim_{x\to\infty}f(x)=a\neq\infty$, then there exists an infinite subsequence of indices n for which $\left| a_n - a\right| > K(\epsilon)e^{-\epsilon b_{n+1}}$.

The *cosinic fundamental regions* associated with (28) have the following properties:

(1) The real paths of $w=f(z)$ consist of the real axis and a curve C_n through each critical point b_n, symmetric about the real axis.

(2) The line $y=y_0$ intersects each C_n exactly once.

(3) If $\alpha>0$, the horizontal width of the strip between C_n and C_{n+1} is $O(1/y)$ uniformly for all n.

(4) If $\alpha>0$, then

$$\sum_{k=n}^{\infty}\left| a_{k+1}-a_k\right| = O(e^{-(\alpha-\epsilon)b_n^2}), \qquad \sum_{k=-n}^{-\infty}\left| a_{k+1}-a_k\right| = O(e^{-(\alpha-\epsilon)b_{-n}^2}).$$

7. **Remarks.** As is immediately seen upon examination of the proofs of both the direct and converse theorems, the restriction $a_{2n\pm1}>a_{2n}$, or its equivalent $b_{n+1}>b_n$, may be replaced by the weaker $b_{n+1}\geqq b_n$; that is, a group of p consecutive first-order branch points may be collapsed into a single branch point of order p, with no essential change in the statement of the theorem involved.

Concerning the product representations for $w=f(z)$ we may say:

THEOREM Ib. *If $\mathcal{J}$ is a symmetric gammic surface with $a_{2n+1}>0$, $a_{2n}<0$, then* $w=f(z)$ (cf. (26)) *is of the form*

$$f(z) = ze^{-\gamma z^2+\delta z}\prod_{n=1}^{\infty}\{(1 - z/c_n)e^{z/c_n}\}$$

with $\gamma \geqq 0$, δ *real*, $0 < c_1 < c_2 < \cdots$, *and* $\max(\gamma,\ \delta + \sum_{n=1}^{\infty} 1/c_n) > 0$. *And conversely, an entire function of this form maps the z-plane onto a symmetric gammic surface* $\mathcal{J}$ *with* $a_{2n+1} > 0$, $a_{2n} < 0$.

The proof of this theorem is obvious. Similar results obtain for the other two classes of surfaces.

This enables us to conclude at once that the Riemann surface corresponding to the entire function $w = z^{-n}J_n(z)$, where $J_n(z)$ is the Bessel function of order $n > -1$, is a symmetric cosinic surface. For the zeros of $w(z)$ are real and (cf. Watson [1, pp. 482 and 498]) $w\Gamma(n+1)2^n = \prod_{k=-\infty}^{\infty}\{(1 - z/c_k)e^{z/c_k}\}$. The Riemann surfaces connected with the Bessel functions have been considered by Lense [2].

When the condition sgn $a_n = (-)^{n-1}$ is not fulfilled $f(z)$ has certain complex zeros occurring in conjugate pairs, and the story is not so simple; for example, if $f(z) = z(1 + z^2/3)\prod_{n=1}^{\infty}(1 - z/b_n)$, $b_n > 0$, $f'(z)$ will have zeros in the vicinity of $\pm i$ if the sequence b_n is chosen large enough, since the derivative of $z(1 + z^2/3)$ has zeros at $\pm i$, and hence $f(z)$ will not correspond to a symmetric semi-cosinic surface.

A comparison of Theorems I and III suggests that it should be possible to replace a pair of logarithmic ends by a symmetric "semi-cosinic end" without altering the type of the Riemann surface. That this is true in fairly general situations will be shown in a later paper.

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