TWO-DIMENSIONAL SUBSONIC FLOWS OF A COMPRESSIBLE FLUID AND THEIR SINGULARITIES(1)

BY

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1. Methods for generating stream functions of two-dimensional flows of a compressible fluid. The consideration of physical phenomena, in particular the study of electric and magnetic fields, was one of the starting points from which Riemann developed his approach to the theory of integrals of algebraic functions.

In particular the consideration of two-dimensional electric and magnetic fields without singularities or with such singularities as vortices, sinks, sources, doublets, and so on, suggests the introduction of integrals of the first, second, and third kinds.

The investigation of certain phenomena in fluid dynamics, namely the consideration of two-dimensional, irrotational, steady flow patterns of an incompressible fluid, leads to the same mathematical notions as those mentioned above since these flows are, from an abstract mathematical point of view, not essentially different from electric and magnetic fields.

Generalizing this approach, one can introduce flow patterns of a compressible fluid with corresponding singularities and investigate relations between potentials and stream functions of these flows.

The compressibility equation is, however, much more complicated than Laplace's equation, and it is very questionable whether such an immediate generalization would lead to results comparable with those in the theory of functions of a complex variable.

It seems that it is preferable in this case to use the hodograph method (see below) and to link this approach with the theory of operators which transform solutions of one partial differential equation into solutions of another one.

A two-dimensional steady irrotational flow of a perfect fluid can be described either by its potential (in the following denoted by ϕ) or by the stream function, ψ. In the case of an incompressible fluid ϕ and ψ are connected by the Cauchy-Riemann equations, so that $\phi+i\psi$ is an analytic function, f, of a complex variable. Taking the real and imaginary part of f we obtain ϕ and ψ, respectively. This process can obviously be interpreted as an operation

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transforming analytic functions of a complex variable into solutions of the equation arising in the theory of an incompressible fluid (that is, Laplace's equation).

A much more complicated situation occurs in the compressible fluid case.

A flow of a fluid is initially defined in the so-called physical plane(2), that is, in the plane where the motion occurs. At every point x, y of this plane at which the flow is defined, the velocity vector is determined. See fig. 1. The pair of functions $[u(x, y), -v(x, y)]$, where u and v are cartesian

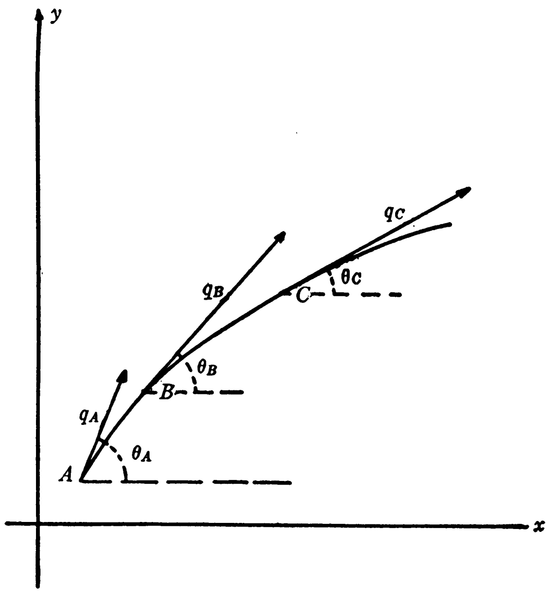

FIG. 1. A streamline in the physical plane.

components of the velocity vector, determines a mapping of the domain of the (x, y)-plane in which the motion takes place into a (not necessarily schlicht) domain of the $(u, -v)$-plane, the so-called hodograph of the flow. See figs. 1, 2, 3, 4, 6, 7, pp. 454, 463, 464, 465.

The potential, ϕ, and the stream function, ψ, can be investigated in either of these planes; that is, one can investigate either directly $\phi(x, y)$, $\psi(x, y)$ or primarily $\phi^{(1)}(u, v) = \phi[x(u, v), y(u, v)]$, $\psi^{(1)}(u, v) = \psi[x(u, v), y(u, v)]$ (the hodograph method), and from these results make conclusions concerning the flow in the physical plane.

In the case of an incompressible fluid, $\phi(x, y)$ and $\psi(x, y)$, as well as

(2) An exact description of the physical plane, as well as of the hodograph plane, will be given in §2.

$\phi^{(1)}(u, v)$ and $\psi^{(1)}(u, v)$, satisfy Laplace's equation and, except in some special problems, are investigated directly in the physical plane. In the case of a compressible fluid, $\phi(x, y)$ and $\psi(x, y)$ satisfy a system of complicated nonlinear equations, while $\phi^{(1)}(u, v)$ and $\psi^{(1)}(u, v)$ satisfy a system of *linear* equations. See §2 for details. Therefore, in order that the operators which transform functions f of one variable into potentials or stream functions of a compressible fluid should be linear, we have to consider ϕ and ψ in the hodograph (or an allied) plane.

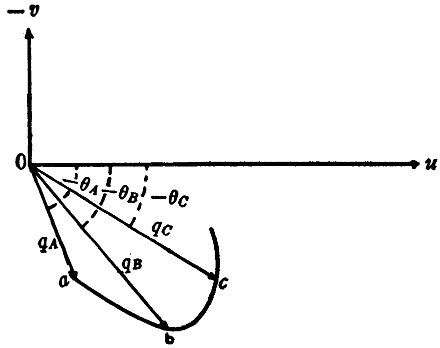

FIG. 2. The image in the hodograph plane of the streamline indicated in Fig. 1.

Several procedures for generating $\phi^{(1)}$ and $\psi^{(1)}$ from functions of one variable are known.

1. Chaplygin(3) [13], who introduced the hodograph method into the theory of compressible fluids, applied the method of separation of variables in order to obtain solutions of this equation. If, in the power series development for the stream function of an incompressible fluid, the powers of the speed, q, are replaced by certain hypergeometric functions of q, then, as Chaplygin has shown, the series obtained in this manner is a solution of the compressibility equation(4). (We designate this procedure as the Chaplygin operator, P_1.)

2. In [3, pp. 23–24] and [5, §2] the author of the present paper introduced a new operator which generates solutions $\phi^{(1)}$ and $\psi^{(1)}$ of the compressibility equations. The main idea of this method is as follows.

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(3) The numbers in brackets refer to the bibliography at the end of the paper. Acquaintance with the contents of these publications is not assumed in the present paper.

(4) Recently several authors, for example, Ringleb [19], Kraft and Dibble [15], and so on, using Chaplygin's method, have given a number of highly interesting flow patterns.

Here and hereafter we understand by the somewhat vague expression "compressibility equation" either the equation for the potential function, that is, (2.11a), or that for the stream function, that is, (2.11b), or the system (2.8) connecting these two functions.

Let θ and H denote cartesian coordinates. Functions ϕ and ψ defined by

$$\phi + i\psi = (\theta + i\mathrm{H})^{[n]}$$

$$= \Big\{\Big[\theta^n - 2!C_{n,2}\theta^{n-2}\int_0^{\mathrm{H}} l(\mathrm{H}_2)d\mathrm{H}_2\int_0^{\mathrm{H}_2} d\mathrm{H}_1 + \cdots\Big]$$

$$\text{(1.1a)} \qquad + i\Big[1!C_{n,1}\theta^{n-1}\int_0^{\mathrm{H}} d\mathrm{H}_1$$

$$- 3!C_{n,3}\theta^{n-3}\int_0^{\mathrm{H}} d\mathrm{H}_3\int_0^{\mathrm{H}_3} l(\mathrm{H}_2)d\mathrm{H}_2\int_0^{\mathrm{H}_2} d\mathrm{H}_1 + \cdots\Big]\Big\},$$

$$\phi + i\psi = i \odot (\theta + i\mathrm{H})^{\{n\}}$$

$$= -\Big\{\Big[1!C_{n,1}\theta^{n-1}\int_0^{\mathrm{H}} l(\mathrm{H}_1)d\mathrm{H}_1$$

$$\text{(1.1b)} \qquad - 3!C_{n,3}\theta^{n-3}\int_0^{\mathrm{H}} l(\mathrm{H}_3)d\mathrm{H}_3\int_0^{\mathrm{H}_3} d\mathrm{H}_2\int_0^{\mathrm{H}_2} l(\mathrm{H}_1)d\mathrm{H}_1 + \cdots\Big]$$

$$- i\Big[\theta^n - 2!C_{n,2}\theta^{n-2}\int_0^{\mathrm{H}} d\,\mathrm{H}_2\int_0^{\mathrm{H}_2} l(\mathrm{H}_1)d\,\mathrm{H}_1 + \cdots\Big]\Big\}$$

are, in the special case where $l(\mathrm{H})=1$, connected by the Cauchy-Riemann equations, and can be interpreted as the potential and the stream function of an incompressible fluid flow. In the case of an arbitrary $l(\mathrm{H})$, ϕ and ψ are connected by the equations

$$\text{1.2)} \qquad \phi_\theta = \psi_{\mathrm{H}}, \ \phi_{\mathrm{H}} = -l(\mathrm{H})\psi_\theta, \ \phi_\theta = \partial\phi/\partial\theta, \cdots$$

and if we choose $\mathrm{H}=\int q^{-1}\rho dq$, $l(\mathrm{H})=(1-M^2)/\rho^2$ where $\rho=\rho(q)$ is the density and M the local Mach number, then ϕ and ψ can be interpreted as the potential and the stream function of a *compressible* fluid. For details see [5, §2].

Let us denote as indicated above by $(\theta+i\mathrm{H})^{[n]}$ and $i\odot(\theta+i\mathrm{H})^{\{n\}}$ the expressions (1.1a) and (1.1b), respectively, obtained from $(\theta+i\mathrm{H})^n$ by the above procedure. We defined in general the operation P_2 by the relation

$$\text{(1.3)} \quad \mathrm{P}_2\Big[\sum(\alpha_n + i\beta_n)(\theta + i\mathrm{H})^n\Big] = \sum\Big[\alpha_n(\theta + i\mathrm{H})^{[n]} + i \odot \beta_n(\theta + i\mathrm{H})^{\{n\}}\Big]$$

where α_n, β_n are real constants. (P_2 can obviously be applied to finite sums and in some cases to infinite ones, producing potentials and stream functions of a compressible fluid flow.)

In a joint investigation, Bers and Gelbart [11], independently of the author of the present paper, found the same operator, which they then investigated in a subsequent publication [12]. They term the functions obtained Σ-monogenic. Recently Díaz [14] generalized this procedure to the case of equations of a higher order.

3. The operators P_1 and P_2 can be applied only to a power series development of an analytic function of a complex variable; they transform these series into potentials and stream functions of *both* subsonic and supersonic flows(5). Operators (to be denoted henceforth as $P_{3,\kappa}$) introduced recently in the general theory of partial differential equations (see for example [9]), when specialized to the case of functions ϕ and ψ which satisfy the compressibility equations, have the advantage that they can be applied directly to functions (and not only to their power series developments) and they generate solutions which are defined in an arbitrary simply-connected domain (and not necessarily in a circle with the center at the origin). They transform analytic functions of a complex variable into stream functions (or potential functions) of subsonic flows and differentiable functions of one real variable into stream functions of supersonic flows, in both cases preserving many properties of the functions to which the operator has been applied, thus serving as a useful tool in the investigation of flow patterns of a compressible fluid(6).

In the case of the compressibility equation, operators $P_{3,\kappa}$ can be represented in the following form. Let $Z=\theta+i\lambda(q)$, $\overline{Z}=\theta-i\lambda(q)$, where $\lambda(q)$ is a certain (fixed) function of the speed $q=(u^2+v^2)^{1/2}$ (see formula (2.7)). Let further $F(Z, \overline{Z})$ (see (2.14), (2.16b)) and $R_\kappa(Z, \overline{Z})$ be some (fixed) functions of Z, $\overline{Z}$ and let $Q_\kappa^{(n)}$ be a set of functions connected by the recurrence relations

$$i\overset{(1)}{Q}_{\kappa,s} + 2F = 0,\ i(2n+1)\overset{(n+1)}{Q}_{\kappa.\overline{Z}} + 2(\overset{(n)}{Q}_{\kappa,Z\overline{Z}} + F\overset{(n)}{Q}) = 0,\quad \overset{(n)}{Q}_{\kappa,\overline{Z}} \equiv \partial\overset{(n)}{Q}_\kappa/\partial\overline{Z}.$$

Then the expression

$$\psi = \operatorname{Im}\,[P_{3,\kappa}(g)],$$

$$\text{(1.4a)}\qquad P_{3,\kappa}(g) = R_\kappa\Big[g(Z) + \sum_{n=1}^{\infty}\frac{\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}Q_\kappa^{(n)} \cdot\int_{Z_0}^{Z}\cdots\int_{Z_0}^{Z_{n-1}} g(Z_n)dZ_n\cdots dZ_1\Big]$$

where g is an arbitrary analytic function of a complex variable, which function is regular at Z_0, represents a possible stream function of a subsonic compressible fluid flow. In particular, two operators, that of the first kind and

(5) The potential and stream function of a supersonic flow each satisfy an equation of hyperbolic type. Thus P_1 and P_2 transform solutions of an equation of elliptic type (Laplace's equation) into solutions of a hyperbolic one. It is well known that the solutions of both equations have a quite different character, and therefore these operators *can not* preserve various properties of the functions upon which they act.

(6) In contrast to operators of type $P_{3,\kappa}$ (which mainly have been introduced in order to "translate" properties of solutions of simpler equations into properties of more complicated ones), P_1 and P_2 can be used only in a few cases as a tool for investigation of properties of the functions which they generate.

that of the second kind, have been investigated in detail. In the case of the operator of the first kind ($\kappa=1$), R_1 and $Q_1^{(n)}$ are determined by the initial conditions, $R_1(Z-Z_0,0)=1$, $Q_1^{(n)}(Z-\overline{Z}_0, 0)=0$. In the case of the operator of the second kind ($\kappa=2$), the corresponding conditions are $R_2(Z, Z-2Z_0)=1$, $Q_2^{(n)}(Z, Z-2\overline{Z}_0)=0$, where iZ_0 is now a *real* number; $R_2 \equiv H(2\lambda)$ and $Q_2^{(n)}(2\lambda)$ are functions of λ alone.

We remark that the operator $\mathbf{P}_{3,\kappa}$ can be represented also in the form

$$(1.4b) \qquad \mathbf{P}_{3,\kappa}(g) = \int_{t=-1}^{1} \mathbf{E}_\kappa(Z, \overline{Z}, t) f[Z(1-t^2)/2]dt/(1-t^2)^{1/2}$$

where $\mathbf{E}_\kappa = R_\kappa[1+\sum_{n=1}^{\infty} Z^n t^{2n} Q^{(n)}]$, and f is connected with g by the relation(7) $g(\overline{Z})=\int_{t=-1}^{1} f[\overline{Z}(1-t^2)/2]dt/(1-t^2)^{1/2}$.

We note further that operator $\mathbf{P}_{3,1}$ (integral operator of the first kind) can also be written in the form

$$(1.4c) \qquad \begin{aligned} \mathbf{P}_{3,1}(g) = R_1\Big[g &- \int_{z_0}^{Z}\int_{\overline{Z}_0}^{\overline{Z}} FgdZ_1 d\overline{Z}_1 \\ &+ \int_{Z_0}^{Z}\int_{\overline{Z}_0}^{\overline{Z}}\left(F\int_{Z_0}^{Z_1}\int_{\overline{Z}_0}^{\overline{Z}_1} FgdZ_2 d\overline{Z}_2\right) dZ_1 d\overline{Z}_1 + \cdots \Big]. \end{aligned}$$

See [9] pp. 317–318.

In the case of the integral operator of the second kind the representation (1.4b) holds for $|Z|<2|\lambda|$, as does formula (1.4). For $2|\lambda|<|Z|$ the generating function of the integral operator of the second kind can be represented in the form

$$(1.5) \qquad \mathbf{E}_2 = H(2\lambda)\sum_{\mu=1}^{2}\left[\frac{Z(1-t^2)}{2}\right]^{2(\mu-1)/3} A_\mu \sum_{n=0}^{\infty}\frac{q^{(n,\mu)}}{(-t^2 iZ)^{n-1/2+2\mu/3}}$$

where A_μ are constants, and

$$(1.6) \qquad q^{(n,\mu)} = \sum_{\nu=0}^{\infty} C_\nu^{(n,\mu)}(-\lambda)^{n-1/2+2(\mu+\nu)/3}$$

are connected by equations

$$(1.7) \qquad \begin{gathered} q^{(0,\mu)} + 4Fq^{(0,\mu)} = 0, \\ 2(n+2\mu/3)q_\lambda^{(n,\mu)} + q_{\lambda\lambda}^{(n+1,\mu)} + 4Fq^{(n+1,\mu)} = 0, \\ n = 1, 2, 3, \cdots; \mu = 1, 2. \end{gathered}$$

(7) In the case of the operator of the first kind, $\mathbf{P}_{3,1}$, the connection between $\psi(Z, \overline{Z})$ and function $g(Z)$ can be defined also as follows. Let us continue ψ to complex values of the arguments λ and θ, that is, let us consider that Z, $\overline{Z}$ are two *independent* variables which are not necessarily conjugate to each other; $\psi(Z-Z_0, 0)$ is then the value which this function assumes in the "characteristic" plane $Z=0$. (See [9 p. 317].) We have then $\psi(Z-Z_0, 0)=g(Z)+\overline{R}_1(0, Z)\cdot$const. (See [9, p. 303].)

Here F is the function introduced in (2.15b). F can be represented in the neighborhood of $\lambda=0$ in the form

$$F = \frac{1}{\lambda^2}\sum_{\nu=0}^{\infty} \alpha_\nu(-\lambda)^{2\nu/3}, \quad \alpha_0 = 5/36, \ \alpha_1 = 0, \cdots . \tag{1.8}$$

We wish to add that the representation (1.5) of E_2 can be used to obtain "mixed" flow patterns. In the supersonic region λ has to be replaced by $i\Lambda$, where Λ is a real variable, see (49) of [7].

If we choose in (1.5) the complex constants A_κ so that $\operatorname{Im}\{i^{-4/3}A_1\overline{A}_2\} \neq 0$, and if we assume that the integration in (1.4b) is carried out along a curve $\mathfrak{C}$ in the complex t-plane which connects $t=-1$ and $t=1$ and lies in $|t|>1$, then the following inversion formula holds. Let

$$\begin{aligned} \lim_{M\to 1^-} \psi(M,\theta) &= \chi_1(\theta) = \sum_{\nu=0}^{\infty} \alpha_\nu^{(1)}\theta^\nu, \\ \lim_{M\to 1^-} (\partial\psi/\partial M) &= \chi_2(\theta) = 3^{1/3}2^{2/3}(k+1)^{-2/3}\sum_{\nu=0}^{\infty} \alpha_\nu^{(2)}\theta^\nu, \end{aligned} \tag{1.9}$$

then (in the case of the integral operator (1.4b) with E_2 given by (1.5)) we have

$$\begin{aligned} f(\zeta) = 3^{1/2}[2\pi \operatorname{Im}\{D_0\overline{D}_2\}]^{-1}\Big\{&(-2i\zeta)^{7/6}\overline{D}_0\int_0^1 \chi_1'(\sigma)\tau^{-1/3}d\tau \\ &+ (-2i\zeta)^{1/6}\sum_{\kappa=1}^{2} D_\kappa\int_0^1 [\chi_\kappa(\sigma) - \alpha_0^{(\kappa)}](1-\tau)^{-1}\tau^{-2/3}d\tau \\ &- \frac{1}{3}(-2i\zeta)^{-1/6}\sum_{\kappa=1}^{2}\overline{D}_\kappa\int_0^1 \overset{*}{\chi}_\kappa(\sigma) - 3\alpha_0^{(\kappa)}(1-\tau)^{-4/3}\tau^{-2/3}d\tau \\ &+ (-2i\zeta)^{1/6}\Big[D_0\alpha_0^{(1)} - \sum_{\kappa=1}^{2}\overline{D}_\kappa\alpha_0^{(\kappa)}\Big] \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} \overset{*}{\chi}_\kappa(\sigma) &= \int_0^\sigma \omega^{-2/3}\chi_\kappa(\omega)d\omega, \qquad \sigma = -2i\zeta(1-\tau), \\ D_0 &= -2^{1/2(k-1)+4/3}3^{7/6}i^{3/2}(k+1)^{-1/2(k+1)-1/6}A_2, \\ D_1 &= 5^{-1}2^{1/3}3^{-1/3}i^{1/6}(k+1)^{-1/3}(2k+5)A_1, \\ D_2 &= -2^{1/2(k-1)+1/3}3^{-1/6}i^{1/6}(k+1)^{-1/2(k+1)-1/6}A_1. \end{aligned}$$

REMARK 1.1. In the case where the pressure density relation has the form $p=\sigma/\rho+A$, σ and A being constants, that is, in the case of Chaplygin-von Kármán-Tsien, $R=1$, $Q^{(n)}=0$ for $n\geqq 1$. The same holds in the incompressible fluid case; in the latter case $\lambda(q)=\bar{\lambda}(q)=\lg q$.

REMARK 1.2. We wish to indicate another possible simplification of the theory which consists in replacing the aforementioned function F (see (2.16b)) by $\tilde{F}=C/\lambda^2$, where C is a constant. It is possible in this case to overcome mathematical difficulties, and to develop a relatively simple theory, which includes also transsonic and supersonic cases. The aforementioned functions $\mathbf{E}_\kappa$ can be expressed in this case in a closed form by hypergeometric functions of a suitably chosen variable, which is a combination of θ, λ and t. For example $\mathbf{E}_2=H\,\mathrm{F}[1/6, 5/6, 1/2, (it^2Z/-2\lambda)]$, where F is the hypergeometric function, and H is given by (2.14). Let us add that if instead of λ we introduce the variable $\mathrm{H}=c\lambda^{2/3}$, where c is a suitably chosen constant, then the compressibility equation, see (2.11b), assumes the form $-\mathrm{H}\psi_{\theta\theta}+\psi_{\mathrm{HH}}=0$.

Investigations in fluid dynamics require in addition to the study of regular solutions the investigation of singularities, for example, sources, sinks, vortices, doublets, and so on. Since, further, the image in the hodograph (or an allied) plane of a flow pattern is not necessarily schlicht, various aerodynamical problems lead to the study of branch points of functions satisfying the compressibility equation as well as to the study of the corresponding Riemann surfaces in the large.

As has been indicated in [9, §5] the operator(8) $\mathbf{P}_3$ transforms branch points of analytic functions of a complex variable into branch points of the same order of functions satisfying the linear differential equation, so that the operator $\mathbf{P}_3$ can be used successfully for generating and investigating these singularities.

On the other hand, the operator $\mathbf{P}_3$ when applied to analytic functions $g(Z)$ of a complex variable which possess a pole yields solutions of differential equations which are infinite of the same order as $g(Z)$ and at the same point, but which are *no longer single-valued*. If we apply the operator $\mathbf{P}_3$ to functions

(8) It should be indicated in this connection that by using the Chaplygin operator $\mathbf{P}_1$ it is possible to obtain various important types of singularities of supersonic flows, and various singularities at *stagnation points*. On the other hand certain singularities which can be obtained easily by using $\mathbf{P}_3$ and the methods indicated in §5 of the present paper cannot be directly obtained by Chaplygin's method. If we wish to represent these functions using Chaplygin's hypergeometric functions, we need several series developments, each of which represents the function in another part of the neighborhood of the singular point. It should be stressed that in considering the potentials, ϕ, and stream functions, ψ, in the large, there exists a basic difference between the incompressible and compressible fluid case: In the first case these functions are in general defined in the whole logarithmic plane, that is, for all values of λ and θ; while in the case of compressible fluids, it we limit ourselves to the subsonic case, then ϕ and ψ are defined only for $\lambda<0$. To the values $\lambda=0$ there corresponds $M=1$(M is the Mach number). For $M>1$ the flow becomes supersonic, the equations for ϕ and ψ become hyperbolic, and the functions have basically different properties. If one wishes to apply operators to the theory of flows which are partially subsonic and partially supersonic, it is useful to introduce as the class of functions to which the operator is to be applied functions $g(\zeta)$, $\zeta=\theta+i\lambda(M)$, defined in the (θ, M)-plane. For $M<1$, ζ is complex and for $M>1$ it becomes real.

with logarithmic singularity whose real part is single-valued, we obtain functions whose *real as well as imaginary parts are multi-valued.*

On the other hand, in connection with the transition to the physical plane the question of single-valuedness of at least one of the quantities $\phi^{(1)}$ or $\psi^{(1)}$ is of great importance, and therefore the question arises of defining procedures which generate *real* solutions of the compressibility equations which are logarithmic or infinite of the nth order and *single-valued.*

Developing the method of attack used in §17 of [3], §14 of [7] and in Appendix II of [10], we introduce and investigate in the present paper solutions of the compressibility equations, see (2.11a) and (2.11b), with singularities of the required type. As we shall see, they yield sinks, sources, vortices, and doublets at infinity in the physical plane.

In §2 we review some results obtained in previous papers and needed in the following, concerning the form which the compressibility equations assume in different planes, that is, when they are considered as functions of different arguments; in §3 we determine singularities in the logarithmic plane for an incompressible fluid, singularities which in the physical plane lead to sinks, sources, vortices, doublets, and so on, at infinity.

In §4 we discuss in more detail some properties (in the hodograph plane) of singularities in general as well as of special types of singularities in which we are mainly interested in the present paper. In §5 we determine these singularities. In §6 necessary and sufficient conditions are derived in order that a hodograph possessing the singularities under consideration leads to a flow in the physical plane around a closed contour, and in §7 we derive a formula (similar to the Cauchy integral formula) for expressing the values of the potential and stream function in a domain in terms of their values on the boundary curve.

In §8 we show that the transition from the physical to the hodograph (or an allied) plane represents in the subsonic case a quasi-conformal mapping. Finally, in §9 we indicate a system of equations in three variables which can be obtained from solutions of equations considered in the present paper.

I should like to take this opportunity to thank Dr. Bernard Epstein for his helpful advice and valuable assistance in connection with the present paper.

2. **Differential equations of the steady motion of a compressible fluid. Physical, hodograph, logarithmic, and pseudo-logarithmic planes.** In this section we shall describe in a more exact way certain notions mentioned in §1, such as physical, hodograph, logarithmic and pseudo-logarithmic planes, and indicate the partial differential equations which the potential and stream functions satisfy in each of these planes. Finally, we shall determine the function pair representing the mapping of the pseudo-logarithmic plane into the physical one.

Any actual flow of a fluid takes place in three-dimensional space. However, a large class of problems possesses the special character that the velocity vector has the same magnitude and direction at corresponding points of all planes parallel to some fixed plane(9). Evidently in this case it is sufficient to study the motion in one representative plane, denoted as the *physical plane*. Let the fluid motion be referred to a system of orthogonal cartesian coordinates in this plane, denoted by x, y, or using complex notation, by $z=x+iy$. The velocity vector $\mathbf{q}$ at any point (x, y) lies in this plane, and its components are denoted by u and v respectively. The magnitude of the velocity (the speed) is q; the angle between the positive x-axis and the direction of the velocity vector is called θ. The assumption of the law of conservation of matter leads in the case of a steady flow to the continuity equation

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0. \tag{2.1}$$

Here ρ is the density. The assumption that the flow is irrotational is expressed by the equation

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \tag{2.2}$$

It follows from (2.1) and (2.2) that for every flow we can introduce a potential ϕ and a streamfunction ψ, such that

$$u = \frac{\partial \phi}{\partial x} = \frac{1}{\rho}\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{1}{\rho}\frac{\partial \psi}{\partial x}. \tag{2.3}$$

REMARK 2.1. The potential and streamfunction, ϕ and ψ, will in the following be denoted as components of the flow. Henceforth we shall consider only adiabatic flows for which the thermodynamical equation of state may be expressed in the form

$$p = \sigma \rho^{\gamma} + \beta \tag{2.4}$$

where σ, β and γ are constants. By combining the Bernoulli equation $q^2/2+\int_{p_0}^{p} dp/\rho(p)=0$ (where p_0 is the pressure at a stagnation point) with the equation of state, we can eliminate the pressure, and we obtain

$$\rho = \left(1 - \frac{\gamma - 1}{2} q^2\right)^{1/(\gamma-1)} \tag{2.5}$$

where the units of mass and velocity are so chosen that at a stagnation point $\rho=1$, and $(dp/d\rho)=1$.

(9) Turbulent flows are evidently excluded by these assumptions. According to the von Mises hydraulic hypothesis [21], this is admissible for a rather large class of flows.

Since $q^2=\phi_x^2+\phi_y^2$, $\phi_x\equiv\partial\phi/\partial x$, $\cdots$ the equations (2.3) and (2.5) represent a system of three (nonlinear) partial differential equations for ϕ, ψ and ρ.

An important simplification in the study of the motion of a compressible fluid has been achieved by Molenbroek [18] and Chaplygin [13]. They showed that if ϕ and ψ are considered as functions of q and θ instead of x and y (see figs. 1 and 2, pp. 453, 454) they satisfy the system

$$\frac{\partial\phi}{\partial\theta}=\frac{q}{\rho}\frac{\partial\psi}{\partial q},\qquad \frac{\partial\phi}{\partial q}=-\frac{1-M^2}{\rho q}\frac{\partial\psi}{\partial\theta}\tag{2.6}$$

where $M=q/[1-(\gamma-1)q^2/2]^{1/2}$ is the Mach number. Since ρ is a known function of q—see (2.5)—equation (2.6) represents now a system of two *linear* partial differential equations.

In order to simplify the form of the equation (2.6) it is convenient to introduce in the subsonic case instead of q a new variable

$$\lambda=\frac{1}{2}\lg\left[\frac{1-(1-M^2)^{1/2}}{1+(1-M^2)^{1/2}}\left(\frac{1+h(1-M^2)^{1/2}}{1-h(1-M^2)^{1/2}}\right)^{1/h}\right],\qquad h=\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2},\quad \gamma>1.\tag{2.7}$$

The plane whose cartesian coordinates are θ and λ will be denoted as the *pseudo-logarithmic plane*(10).

In the pseudo-logarithmic plane equations (2.6) assume the form

$$\phi_\theta-\psi_\lambda l^{1/2}=0,\quad \phi_\lambda+\psi_\theta l^{1/2}=0,\quad \phi_\theta\equiv\partial\phi/\partial\theta,\ \cdots,\tag{2.8}$$

where

$$l=l(\lambda)=\frac{1-M^2}{\rho^2}.\tag{2.9}$$

It is convenient in the following to use the complex notation,

$$Z=\theta+i\lambda,\qquad \bar{Z}=\theta-i\lambda,$$
$$\frac{\partial}{\partial Z}=\frac{1}{2}\left(\frac{\partial}{\partial\theta}-i\frac{\partial}{\partial\lambda}\right),\qquad \frac{\partial}{\partial\bar{Z}}=\frac{1}{2}\left(\frac{\partial}{\partial\theta}+i\frac{\partial}{\partial\lambda}\right),\tag{2.10}$$

that is

(10) In the present paper the potential and the stream function, ϕ and ψ, as well as some other quantities, are considered in different planes; that is, they are considered as functions of different pairs of variables. In passing from one plane to another, new symbols should be introduced for ϕ and ψ, since they are different functions of their respective arguments. For instance, when passing from the physical plane to the (θ,λ)-plane, we should write $\phi^{(2)}(\theta,\lambda)\equiv\phi[x(\theta,\lambda),y(\theta,\lambda)]$, and so on. However, for the sake of brevity we omit the superscript and in the following always write ϕ, ψ, and so on, no matter in which plane the functions are considered.

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial Z} + \frac{\partial}{\partial \bar{Z}}, \qquad \frac{\partial}{\partial\lambda} = i\left(\frac{\partial}{\partial Z} - \frac{\partial}{\partial \bar{Z}}\right).$$

Equations (2.8) can now be written

$$\phi_Z - il^{1/2}\psi_Z = 0, \qquad \phi_{\bar{Z}} + il^{1/2}\psi_{\bar{Z}} = 0. \tag{2.11}$$

Eliminating ψ and ϕ respectively, we obtain for ϕ and ψ the equations

(2.12a) $\phi_{Z\bar{Z}} - iN(\phi_Z - \phi_{\bar{Z}}) = 0,$ (2.12b) $\psi_{Z\bar{Z}} + iN(\psi_Z - \psi_{\bar{Z}}) = 0,$

$$N = -\frac{\gamma+1}{8}\,\frac{M^4}{(1-M^2)^{3/2}}. \tag{2.12c}$$

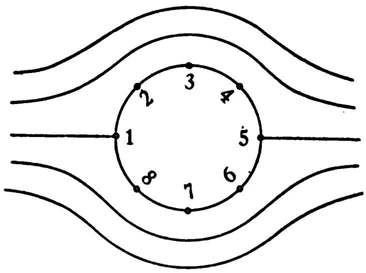

FIG. 3. A flow (in the physical plane) around a circle.

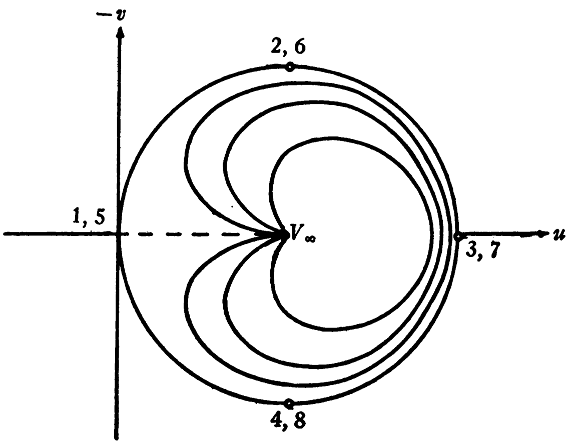

FIG. 4. The image in the hodograph plane of a flow around a circle.

REMARK 2.2. We note that in the pseudo-logarithmic plane the equations for the potential and stream functions appear in the cannonical form.

Instead of ϕ and ψ, it is often more convenient to consider the "reduced" potential and stream functions,

(2.13a) $$\phi^* = H\phi,$$ (2.13b) $$\psi^* = H^{-1}\psi,$$

where

$$H = \exp\left[-\int_{-\infty}^{(Z-\bar{Z})/i} N(\tau)d\tau\right] \tag{2.14}$$
$$= [1 - M^2]^{-1/4}[1 + (\gamma - 1)M^2/2]^{-1/2(\gamma-1)}.$$

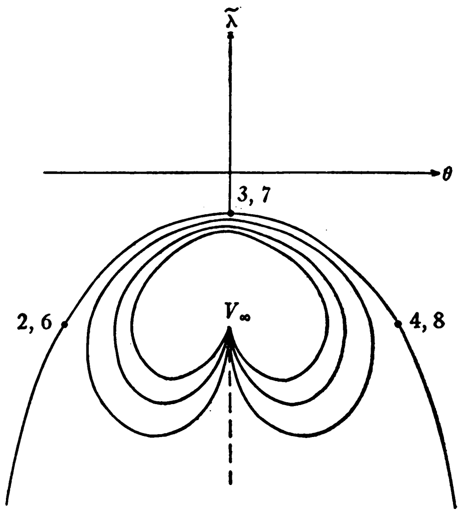

FIG. 5. The image in the logarithmic plane of a flow around a circle.

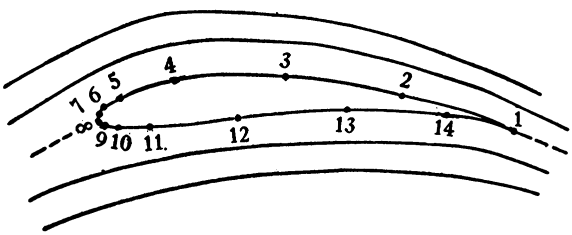

FIG. 6. A flow (in the physical plane) around a Joukowski profile.

The functions ϕ^* and ψ^* satisfy the equations

(2.15a) $$\phi^*_{Z\bar{Z}} + P\phi^* = 0,$$ (2.15b) $$\psi^*_{Z\bar{Z}} + F\psi^* = 0$$

where

$$P = \frac{(\gamma + 1)M^4}{64}\left[\frac{(\gamma - 3)M^4 + (12 - 8\gamma)M^2 - 16}{(1 - M^2)^3}\right], \tag{2.16a}$$

$$F = \frac{(\gamma + 1)M^4}{64}\left[\frac{-(3\gamma - 1)M^4 - 4(3 - 2\gamma)M^2 + 16}{(1 - M^2)^3}\right]. \tag{2.16b}$$

The plane whose cartesian coordinates are the velocity components u and $-v$ (or polar coordinates q and $-\theta$) is denoted as the *hodograph plane*(11).

By the transformation

$$\tilde{Z} = i \log \bar{q}, \qquad \bar{q} = u - iv = qe^{-i\theta} \tag{2.17}$$

we pass from the hodograph plane to the *logarithmic plane*, whose cartesian coordinates are θ and

$$\tilde{\lambda} = \lg q. \tag{2.17a}$$

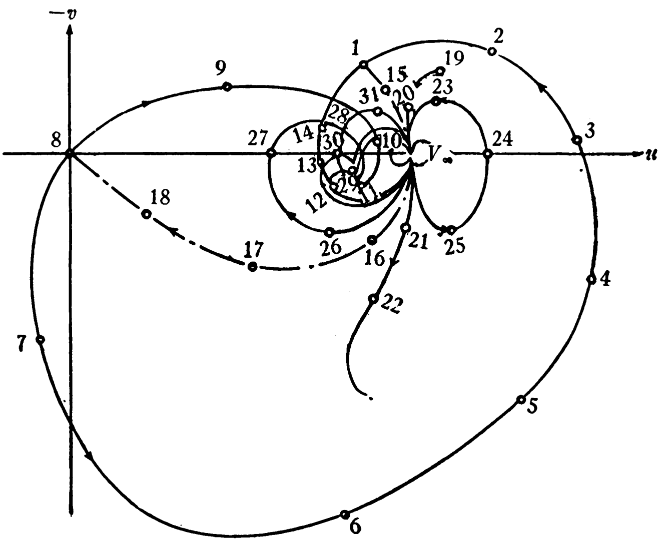

FIG. 7. The image in the hodograph plane of a flow around a Joukowski profile.

Examples. In figs. 3, 4, 5 a flow around a circle in the physical plane, and its images in the hodograph and logarithmic planes are indicated. In figs. 6, 7, 8 the corresponding images for a flow around a Joukowski profile are given. We note that figs. 4 and 5 are doubly covered: the images lie on two sheets, the points 2, 3, 4 lying on one, the points 6, 7, 8 lying on the other sheet. Figs. 7 and 8 are partially doubly covered. The point V_∞ corresponds to $z = \infty$.

(11) In figs. 1 and 2 a stream line in the physical plane and its image in the hodograph plane are indicated.

If the potential or the stream function is known in the pseudo-logarithmic plane, we can express it in the physical plane by the use of the following rela-

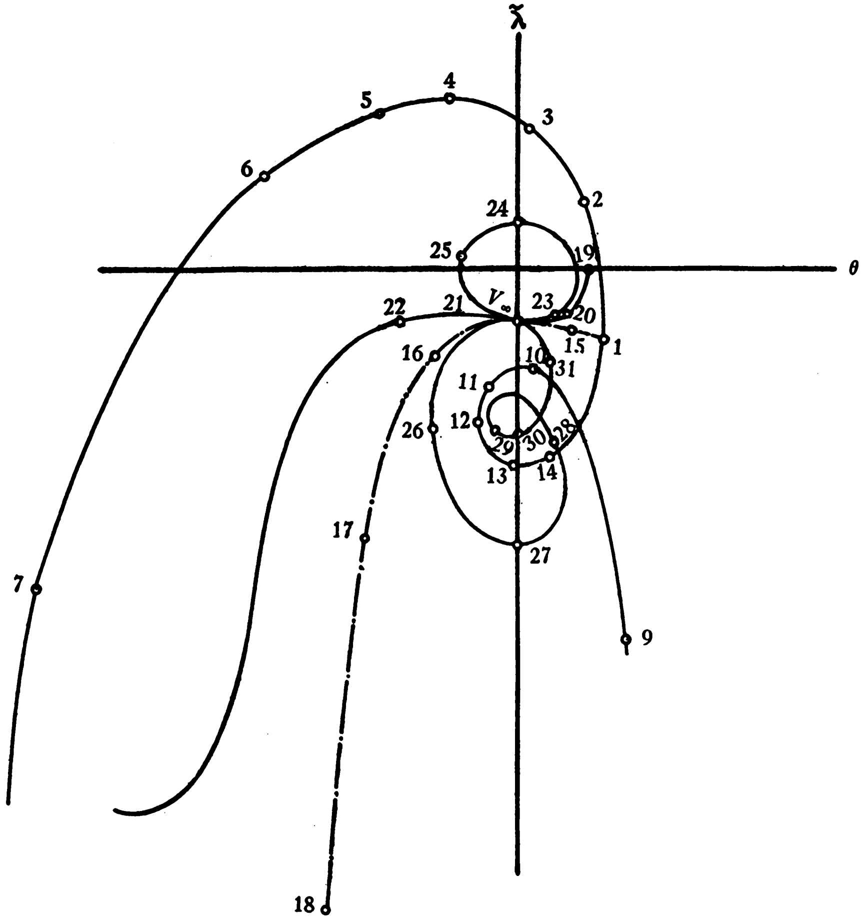

FIG. 8. The image in the logarithmic plane of a flow around a Joukowski profile.

tionships between the two pairs of independent variables:

$$x = \int \frac{1}{\rho}\left\{\left[-\frac{(1-M^2)^{1/2}\cos\theta}{q}\psi_\theta - \frac{\sin\theta}{q}\psi_\lambda\right]d\lambda + \left[\frac{(1-M^2)^{1/2}\cos\theta}{q}\psi_\lambda - \frac{\sin\theta}{q}\psi_\theta\right]d\theta\right\}$$

(2.18a)

$$= 2\,\mathrm{Re}\left\{\int\left[\frac{-\sin\theta + i(1-M^2)^{1/2}\cos\theta}{q\rho}\right]\psi_Z dZ\right\},$$

$$y = \int \frac{1}{\rho}\left\{\left[-\frac{(1-M^2)^{1/2}\sin\theta}{q}\psi_\theta + \frac{\cos\theta}{q}\psi_\lambda\right]d\lambda\right.$$

$$(2.18\text{b}) \qquad \left. + \left[\frac{(1-M^2)^{1/2}\sin\theta}{q}\psi_\lambda + \frac{\cos\theta}{q}\psi_\theta\right]d\theta\right\}$$

$$= 2\,\text{Re}\left\{\int\left[\frac{\cos\theta + i(1-M^2)^{1/2}\sin\theta}{q\rho}\right]\psi_Z dZ\right\}$$

(Re = Real part); or, replacing the derivatives of ψ by those of ϕ, see (2.8), by

$$x = \int \frac{1}{\rho l^{1/2}}\left\{\left[\frac{(1-M^2)^{1/2}\cos\theta}{q}\phi_\lambda - \frac{\sin\theta}{q}\phi_\theta\right]d\lambda\right.$$

$$(2.19\text{a}) \qquad \left. + \left[\frac{(1-M^2)^{1/2}\cos\theta}{q}\phi_\theta + \frac{\sin\theta}{q}\phi_\lambda\right]d\theta\right\}$$

$$= 2\,\text{Re}\left\{\int\left[\frac{(1-M^2)^{1/2}\cos\theta + i\sin\theta}{q\rho l^{1/2}}\right]\phi_Z dZ\right\},$$

$$y = \int \frac{1}{\rho l^{1/2}}\left\{\left[\frac{(1-M^2)^{1/2}\sin\theta}{q}\phi_\lambda + \frac{\cos\theta}{q}\phi_\theta\right]d\lambda\right.$$

$$(2.19\text{b}) \qquad \left. + \left[\frac{(1-M^2)^{1/2}\sin\theta}{q}\phi_\theta - \frac{\cos\theta}{q}\phi_\lambda\right]d\theta\right\}$$

$$= 2\,\text{Re}\int\left\{\left[\frac{(1-M^2)^{1/2}\sin\theta - i\cos\theta}{q\rho l^{1/2}}\right]\phi_Z dZ\right\}.$$

DEFINITION 2.1. A flow (a vector function) $\mathcal{J} = (\phi, \psi)$ is said to be regular in the domain $\mathfrak{D}$ of a plane $\mathfrak{P}$ if the quantities (ϕ, ψ) are functions of two real variables (representing the cartesian coordinates of $\mathfrak{P}$), which functions are regular in $\mathfrak{D}$.

3. The duality between the flows of an incompressible and a compressible fluid. In considering the two-dimensional flows of a perfect fluid, it is convenient to introduce a certain correspondence principle between subsonic flows of a compressible fluid and flows of an incompressible fluid. Naturally such a correspondence can be defined in various ways. On the other hand, one can formulate certain requirements which will simplify the form of this correspondence.

First, it is convenient to consider the flows in a plane where ϕ and ψ satisfy a linear homogeneous equation, since in this case the principle of superposition holds. Second it is natural to require that the equations for ϕ and ψ have the simplest form possible, which for linear equations is the canonical form.

In the case of a compressible fluid, these requirements lead to considering the potential and the stream function in the pseudo-logarithmic plane. In the case of an incompressible fluid, ϕ and ψ satisfy in various planes (that is, in the physical, hodograph, logarithmic planes) the same equation, namely Laplace's. Since, however, we consider the compressible fluid motion in the pseudo-logarithmic plane, it is natural to introduce the correspondence with incompressible fluid flows defined in the logarithmic plane, since in this case the equations for ϕ and ψ have the canonical form, and the transition from the pseudo-logarithmic plane to the logarithmic plane means only a stretching: one of the coordinates, θ, is the same, and for the second coordinate we take in the case of the incompressible fluid

$$\tilde{\lambda} = \lg q \tag{3.1}$$

and in the case of the compressible fluid the quantity λ, which is defined by (2.7) and which reduces to $\lg q$ as the compressibility effect goes to zero.

Each of the operators mentioned in §1 can be interpreted as a correspondence rule which associates with the complex potential of an incompressible fluid flow (defined in a domain $\mathfrak{B}$) a stream function of a compressible fluid flow which is defined in a domain $\mathfrak{B}'$. However, as we mentioned before, operators $\mathbf{P}_1$ and $\mathbf{P}_2$ act only on power series developments, and therefore can be applied only to functions which (in the hodograph plane) are defined in a circle with the center at the origin(12).

The operator $\mathbf{P}_{3,\kappa}$ ($\kappa=1$ or 2) generates solutions which are defined in any simply-connected domain which includes the origin. All three operators act primarily on functions which are regular in the domain in which they are considered, and they produce solutions which are regular in the domain in which they are defined. It is, however, possible to extend the operator $\mathbf{P}_1$ so that it produces functions which have singularities in the supersonic region and/or in the subsonic region at stagnation points. (For details see [15, 19].)

On the other hand, in applications we also need other types of singularities. As we shall see in §4, complex potentials of an incompressible fluid flow have branch points(13), poles and logarithmic singularities, around points other than the origin, in the hodograph plane.

(12) We note that we are often interested in considering stream functions in domains different from a circle. The methods for analytic continuation of solutions obtained in this manner are comparatively little studied. A method of this kind has been indicated in [7, §17] where a representation in a domain (of the hodograph plane) which is bounded by two arcs of concentric circles with center at the origin, and by two rays from the origin, is derived.

(13) For instance, the complex potential in the physical plane around a curve of oval shape is $w(z) = Uz - m \lg (z-a) + m \lg (z+a)$ where U (speed at infinity), m and a are real constants. We obtain for the corresponding potential $g(\tilde{Z}) = w[z(\tilde{Z})]$, $\tilde{Z} = \theta + i\tilde{\lambda}$, $\tilde{\lambda} = \lg q$, in the logarithmic plane $g(\tilde{Z}) = U[a^2 + 2am(U - \exp(-i\tilde{Z}))^{-1}]^{1/2} - m \lg \{[(a^2+2am(U-\exp(-i\tilde{Z}))^{-1}]^{1/2} - a\} + m \lg \{[(a^2+2am(U-\exp(-i\tilde{Z}))^{-1}]^{1/2} + a\}$. Examples of complex potentials in the logarithmic plane which lead to flows in the physical plane around curves of other shapes can be found in [8] and [10, Appendix IV].

Operators $P_{3,\kappa}$ can be easily extended so as to act on functions possessing these singularities; see [9, §5]. However, as we shall explain, it is convenient to use operators $P_{3,\kappa}$ only in the case of branch points, defining the correspondence in the case of poles and logarithmic singularities in a different way([14]).

It is natural to require that, insofar as possible, *the stream functions and the potential functions of compressible fluid flows behave in the pseudo-logarithmic plane at singular points in the same manner as the corresponding functions for incompressible fluids in the logarithmic plane*; that is, they become infinite of the same order, the geometrical structure of the stream lines and potential lines in a sufficiently small neighborhood of the singularity is essentially the same, and so on.

As has been shown in §5 of [9], operators $P_{3,\kappa}$, $\kappa=1, 2$, preserve the location of the singular point as well as the order of infinity. In the case of branch points the generated solution has a branch point of the same order; however, in the case of poles, instead of single-valued solutions we obtain solutions of which *both* components are many-valued. See, for example, (5.2) of [9]. In the case where f has a logarithmic singularity but one of the components is single-valued, *both* components of $P_{3,\kappa}(f)$ are many-valued.

Let us now explain why single-valuedness of at least one of the components (that is, of the potential or stream function) is of importance for our purposes.

Poles and logarithmic singularities in the logarithmic plane are, as a rule, images of doublets, vortices, sinks and sources at infinity in the physical plane. See §4. It is natural to require for compressible fluids that at infinity the speed is constant and that the motion has the same character as in the case of an incompressible fluid.

It follows from (2.18a), (2.18b) and (2.19a), (2.19b) that if the derivatives of at least one of the functions, ϕ or ψ, are single-valued, then, if we move around the singular point, the functions $x=x(u, v)$, $y=y(u, v)$ can each increase at most by a constant, say X, Y. In most cases we have at infinity a linear combination with constant coefficients of these singularities, and the constants have to be so chosen that the quantities X and Y vanish, and therefore $x(u, v)$ and $y(u, v)$ are single-valued. This is the case for an incompressible fluid, and if this property were not preserved, the flow would completely change its character: indeed suppose that $x(u, v)$ and $y(u, v)$ are multivalued; then the functions $u(x, y)$ and $v(x, y)$ are periodic, that is, in the

([14]) This means that if a complex potential g is given which possesses poles and logarithmic singularities, we decompose it, writing $g=g^*+\sum\alpha_\nu g^{(\nu)}$, where g^* possesses as its only singularities algebraic branch points; α_ν are constants and $g^{(\nu)}$ functions (normalized in a certain manner) each of which possesses a pole or a logarithmic singularity. The stream function ψ of a compressible fluid flow corresponding to g will be $\psi=\text{Im}\,[P_{3,\kappa}(g^*)]+\sum\alpha_\nu\psi^{(\nu)}$ where $\psi^{(\nu)}$ are certain stream functions with singularities introduced in §5.

physical plane every value of u, v is assumed infinitely often.

REMARK 3.1. The question whether other types of singularities—in particular those for which both components are multi-valued—are of interest in hydrodynamical applications remains open. The first step in answering this question would consist of studying the behavior of flow patterns in the logarithmic and physical planes in the neighborhood of these singularities.

4. The behavior of certain types of flows in the case of an incompressible fluid. Following the line of our approach, that is, applying the principle of correspondence described in §3, we have at first to describe the behavior in the logarithmic plane of various types of flows of an incompressible fluid, and in particular those possessing singularities such as sinks, sources, vortices, and so on.

Let us consider the flow of an incompressible fluid around a closed curve, say an air wing profile (see fig. 6), and let us assume that neither the velocity vector nor the circulation vanish at infinity.

The complex potential in the physical plane (z-plane) may be expressed, for $|z|$ sufficiently large, in the form:

$$w(z) = \bar{q}_0 z + m \log z + \sum_{n=1}^{\infty} a_n z^{-n}, \qquad \bar{q}_0 \neq 0,\ m \neq 0, \tag{4.1}$$

where $q_0 = q_0 \exp(i\theta_0)$ is the velocity vector at infinity and the real constant $= m/i$ is the strength of the vortex at infinity.

For the velocity vector q we obtain by differentiation

$$\bar{q} = \bar{q}(z) = \bar{q}_0 + \frac{m}{z} - \sum_{n=1}^{\infty} n a_n z^{-n-1}, \qquad \bar{q} = q \exp(-i\theta). \tag{4.2}$$

Let $\zeta = 1/z$, so that for $|\zeta|$ sufficiently small:

$$w(z) = \frac{\bar{a}_0}{\zeta} - m \log \zeta + \sum_{n=1}^{\infty} a_n \zeta^n \tag{4.3}$$

and

$$\bar{q}(z) = \bar{\ }_0 + m\zeta - \sum_{n=1}^{\infty} n a_n \zeta^{n+1}. \tag{4.4}$$

Since $m \neq 0$, series (4.4) can be inverted, in some sufficiently small neighborhoods of the points $\zeta = 0$, and we obtain

$$\zeta = \frac{\bar{q} - \bar{a}_0}{m} + \sum_{n=1}^{\infty} \frac{b_n}{m^{n+1}} (\bar{q} - \bar{q}_0)^{n+1} \tag{4.5}$$

where the b_n may be expressed in terms of the a_n and m.

Inserting (4.5) into (4.3), there results:

$$w[z(\bar{q})] = \frac{m\bar{q}_0}{\bar{q}-\bar{q}_0} - m\log\left[(\bar{q}-\bar{q}_0) + \sum_{n=1}^{\infty}\frac{b_n}{m^n}(\bar{q}-\bar{q}_0)^{n+1}\right] \tag{4.6}$$
$$+ \text{Power series in } (\bar{q} - {}^{-}{}_0).$$

For $|q-q_0|$ sufficiently small, the second term in (4.6) may be written:

$$-m\log\left[(\bar{q}-\bar{q}_0) + \sum \frac{b_n}{m^n}(\bar{q}-\bar{q}_0)^{n+1}\right] = -m\log(\bar{q}-\bar{q}_0) \tag{4.7}$$
$$+ \text{Power series in } (\bar{q}-\bar{q}_0).$$

Therefore, for $|q-q_0|$ sufficiently small:

$$w[z(\bar{q})] = \frac{m\bar{q}_0}{\bar{q}-\bar{q}_0} - m\log(\bar{q}-\bar{q}_0) + \text{Power series in } (\bar{q}-\bar{q}_0). \tag{4.8}$$

As before let $\tilde{Z}=\theta+i\tilde{\lambda}$, $\tilde{\lambda}=\lg q$, $\tilde{Z}_0=\theta_0+i\tilde{\lambda}_0$, $\tilde{\lambda}_0=\lg q_0$. For sufficiently small $|\bar{q}-\bar{q}_0|$ and therefore for sufficiently small $|\tilde{Z}-\tilde{Z}_0|$, (4.8) becomes:

$$\begin{aligned} W_1(\tilde{Z}) = w[z(\tilde{Z})] &= \frac{m\exp(-i\tilde{Z}_0)}{\exp(-i\tilde{Z})-\exp(-i\tilde{Z}_0)} \\ &\quad - m\log(\exp(-i\tilde{Z})-\exp(-i\tilde{Z}_0)) \\ &\quad + \text{Power series in } \exp(-i\tilde{Z})-\exp(-i\tilde{Z}_0). \end{aligned} \tag{4.9}$$

Since $\exp(-i\tilde{Z})-\exp(-i\tilde{Z}_0)$ has a simple zero at $\tilde{Z}=\tilde{Z}_0$, (4.9) may be written:

$$W_1(\tilde{Z}) = \frac{mi}{\tilde{Z}-\tilde{Z}_0} - m\log(\tilde{Z}-\tilde{Z}_0) + \text{Power series in } (\tilde{Z}-\tilde{Z}_0). \tag{4.10}$$

This is the expression for the complex potential in the logarithmic plane. It is clear that the singular part of (4.10) is not influenced by the regular part of (4.1), but depends only on the velocity vector and vortex-strength at infinity.

Writing $m=\mu_0^{(1)}+i\mu_0^{(2)}$, we have for the potential and the stream function in a sufficiently small neighborhood of $\tilde{Z}=\tilde{Z}_0$ the expressions

$$\begin{aligned} \tilde{\phi} &= \frac{\mu_0^{(1)}(\theta-\theta_0)}{(\tilde{\lambda}-\tilde{\lambda}_0)^2+(\theta-\theta_0)^2} - \frac{\mu_0^{(2)}(\tilde{\lambda}-\tilde{\lambda}_0)}{(\tilde{\lambda}-\tilde{\lambda}_0)+(\theta-\theta_0)^2} \\ &\quad - im\arctan\left(\frac{\tilde{\lambda}-\tilde{\lambda}_0}{\theta-\theta_0}\right) + \cdots, \end{aligned} \tag{4.11a}$$

$$\psi^\dagger = \frac{\overset{(1)}{\mu_0}(\tilde\lambda - \tilde\lambda_0)}{(\tilde\lambda - \tilde\lambda_0)^2 + (\theta - \theta_0)^2} + \frac{\overset{(2)}{\mu_0}(\theta - \theta_0)}{(\tilde\lambda - \tilde\lambda_0)^2 + (\theta - \theta_0)^2} + \frac{im}{2} \lg [(\tilde\lambda - \tilde\lambda_0)^2 + (\theta - \theta_0)^2] + \cdots, \tag{4.11b}$$

$$\overset{(1)}{\mu_0} = 0, \qquad \mu = (m/i)\overset{(2)}{\mu_0} \tag{4.11c}$$

where dots indicate terms which go to zero as $\tilde{Z}\to\tilde{Z}_0$ and therefore can be neglected in considering the behavior of $\phi^\sim$ and $\psi^\dagger$ in a sufficiently small neighborhood of $\tilde{Z}=\tilde{Z}_0$.

It is possible to get the same result in a somewhat different way: namely we write $\phi^\sim$ and $\psi^\dagger$ in the forms (4.11a) and (4.11b), respectively, and then determine the constants $\mu_0^{(1)}$, $\mu_0^{(2)}$ so that $x=x(\theta, \lambda)=x+iy$ (which can be obtained using the formulas analogous to (2.18a), (2.18b)) is single-valued at the singular point θ_0, λ_0. A formal computation yields (4.11c). We shall use this last procedure in a compressible fluid case.

Following our principle of introducing (in the pseudo-logarithmic plane) for the compressible fluid case functions which have a behavior similar to that of the corresponding functions in the logarithmic plane for the incompressible fluid case, we shall consider flow patterns whose stream functions ψ possess, as singularity at $\mathbf{Z}=\mathbf{Z}_0$, a linear combination of a logarithmic singularity and two independent singularities of the first order.

DEFINITION 4.1. Suppose a flow $\mathcal{J}(\phi, \psi)$ (of a compressible or incompressible fluid) has a singularity at the point (θ_0, λ_0). If at least one of the functions, ϕ or ψ, is single-valued at this point, the singularity of $\mathcal{J}$ at (θ_0, λ_0) will be said to be of type $\mathfrak{S}$.

The flow $\tilde\phi=\log\ [(\theta-\theta_0)^2+(\tilde\lambda-\tilde\lambda_0)^2]/2$, $\psi=\arctan\ ((\theta-\theta_0)/(\tilde\lambda-\tilde\lambda))$ represents an example of a flow possessing a singularity of type $\mathfrak{S}$.

As we indicated in §3, the above property plays an essential role for our purposes. In the following sections we shall describe a method of generating required singularities of type $\mathfrak{S}$.

5. **Flows with singularities of type $\mathfrak{S}$.** Classical methods in the theory of partial differential equations yield almost immediately two independent flows with logarithmic singularities of type $\mathfrak{S}$.

Let us denote by $\psi^{(L,1)}$ and $\phi^{(L,2)}$ the fundamental solutions of (2.11b) and (2.11a), respectively.

We shall show that

$$\mathcal{J}^{(L,k)}(\phi^{(L,k)}, \psi^{(L,k)}), \qquad k = 1, 2, \tag{5.1}$$

define two linearly independent flows. (The superscript L symbolizes that either the stream function or the potential function has a logarithmic singularity.)

An analytical representation for $\psi^{(L,1)}$ and $\phi^{(L,2)}$ can be given immediately. We have

$$\psi^{(L,1)} = A(\log \zeta + \log \bar{\zeta})/2 + B, \tag{5.2a}$$

$$\phi^{(L,2)} = C(\log \zeta + \log \bar{\zeta})/2 + D, \tag{5.2b}$$

where $\zeta = Z - Z_0$, $\bar{\zeta} = \bar{Z} - \bar{Z}_0$, $Z = \theta + i\lambda$ (see (2.10)), $Z_0 = \theta_0 + i\lambda_0$, $-\infty < \lambda_0 < 0$,

$$\begin{aligned} A = H\Big[1 - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} F dZ_1 d\bar{Z}_1 \\ + \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} F\Big(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} F dZ_2 d\bar{Z}_2\Big) dZ_1 d\bar{Z}_1 \cdots\Big], \end{aligned} \tag{5.3a}$$

$$\begin{aligned} B = H\Big[\int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} G dZ_1 d\bar{Z}_1 \\ - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} F\Big(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} G dZ_2 d\bar{Z}_2\Big) dZ_1 d\bar{Z}_2 + \cdots\Big], \end{aligned} \tag{5.3b}$$

$$G = -\frac{1}{\bar{\zeta}}\frac{\partial(H^{-1}A)}{\partial Z} - \frac{1}{\zeta}\frac{\partial(H^{-1}A)}{\partial \bar{Z}}$$

(see (2.13)), and

$$\begin{aligned} C = H^{-1}\Big[1 - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} P dZ_1 d\bar{Z}_1 \\ + \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} P\Big(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} P dZ_2 d\bar{Z}_2\Big) dZ_1 d\bar{Z}_1 - \cdots\Big], \end{aligned} \tag{5.4a}$$

$$\begin{aligned} D = H^{-1}\Big[\int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} K dZ_1 d\bar{Z}_1 \\ - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} P\Big(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} K dZ_2 d\bar{Z}_2\Big) + \cdots\Big], \end{aligned} \tag{5.4b}$$

$$K = -\frac{1}{\bar{\zeta}}\frac{\partial(HC)}{\partial Z} - \frac{1}{\zeta}\frac{\partial(HC)}{\partial \bar{Z}}.$$

By (2.11) we obtain that the first component of $\mathcal{J}^{(L,1)}$ and the second component of $\mathcal{J}^{(L,2)}$ are given, respectively, by

$$\phi^{(L,1)} = i\int (l^{1/2}\psi_Z^{(L,1)} dZ - l^{1/2}\psi_{\bar{Z}}^{(L,1)} d\bar{Z}), \tag{5.5}$$

$$\psi^{(L,2)} = -i\int (l^{-1/2}\phi_Z^{(L,2)} dZ - l^{-1/2}\phi_{\bar{Z}}^{(L,2)} d\bar{Z}). \tag{5.6}$$

REMARK 5.1. It follows from (2.11) that the integrands of the right-hand sides of (5.5) and (5.6) are complete differentials. Clearly, $\mathcal{J}^{(L,k)}(\phi^{(L,k)}, \psi^{(L,k)})$, $k=1, 2$, are flows with singularities of type $\mathfrak{S}$. The coefficients of system (2.11) are independent of θ, and therefore if we differentiate the components of the $\mathcal{J}^{(L,k)}$ with respect to θ we obtain new flows

$$\mathcal{J}^{(n,k)}\left(\frac{\partial^n \phi^{(L,k)}}{\partial \theta^n}, \frac{\partial^n \psi^{(L,k)}}{\partial \theta^n}\right), \qquad k = 1, 2, \tag{5.7}$$

whose components are infinite of nth order at the point (θ_0, λ_0).

REMARK 5.2. We note that when the coefficients of the equation depend upon both variables, we obtain singularities of higher order by taking partial derivatives with respect to parameters λ_0 or θ_0.

Every $\mathcal{J}^{(n,k)}$, $k=1, 2$, has a singularity of type $\mathfrak{S}$, since at least one of the components (being a derivative of a single-valued function) is itself single-valued. In the case where l in (2.8) equals 1 and $\lambda = \tilde{\lambda}$, that is, when equations (2.8) reduce to the Cauchy-Riemann equations, the expressions

$$\phi^{(L,1)} = \psi^{(L,2)} = \arctan\left(\frac{\tilde{\lambda} - \tilde{\lambda}_0}{\theta - \theta_0}\right) \tag{5.8}$$

[see (4.4) and (4.5a)] are infinitely-many-valued functions, while

$$\phi^{(n,1)} = \mathrm{Re}\,[(\theta + i\lambda)^{-n}], \qquad \psi^{(n,2)} = \mathrm{Im}\,[i(\theta + i\lambda)^{-n}], n = 1, 2, 3, \cdots, \tag{5.9}$$

are single-valued. (Therefore in this case *both* components of the $\mathcal{J}^{(n,k)}$'s are single-valued.) We shall show that a similar situation holds in the general case, considered in the present paper.

We prove at first the following lemma.

LEMMA 5.1. *Let (as before) $\zeta = Z - Z_0$, $\bar{\zeta} = \bar{Z} - \bar{Z}_0$, and $\mathfrak{l}_\epsilon$ be a circle of radius ϵ and center at (θ_0, λ_0). Then*

$$\lim_{\epsilon \to 0} \int_{\mathfrak{l}_\epsilon} \zeta^m \bar{\zeta}^n d\zeta = 0 \qquad (\textit{for } m - n + 1 \neq 0 \textit{ or } m + n + 1 > 0) \tag{5.10}$$

$$= 2\pi i \qquad (\textit{for } m = -1 \textit{ and } n = 0). \tag{5.11}$$

$$\lim_{\epsilon \to 0} \int_{\mathfrak{l}_\epsilon} \zeta^m \bar{\zeta}^n \lg \zeta d\zeta = \lim_{\epsilon \to 0} \int_{\mathfrak{l}_\epsilon} \zeta^m \bar{\zeta}^n \lg \bar{\zeta} d\zeta = 0 \quad (\textit{for } m + n + 1 > 0) \tag{5.12}$$

where m and n are supposed to be integers.

Proof. (5.10) follows immediately from

$$\int_{\mathfrak{l}_\epsilon} \zeta^m \bar{\zeta}^n d\zeta = i\epsilon^{m+n+1} \int_{\phi=0}^{2\pi} e^{i(m-n+1)\phi} d\phi, \qquad \zeta = \epsilon e^{i\phi} \tag{5.13}$$

(5.12) can be derived analogously.

LEMMA 5.2. *The increments* $\Delta\phi^{(L,1)}$ *and* $\Delta\psi^{(L,2)}$ *if we move along a simple closed curve, counterclockwise, surrounding* (θ_0, λ_0) *are*

$$\Delta\phi^{(L,1)} = -2\pi[l(\lambda_0)]^{1/2}H(2\lambda_0) \tag{5.14}$$

and

$$\Delta\psi^{(L,2)} = -2\pi[l(\lambda_0)]^{1/2}/H(2\lambda_0) \tag{5.15}$$

respectively, where l *and* H *are the functions defined in* (2.9) *and* (2.13) *respectively.*

Proof. We have by (2.8)

$$\Delta\phi^{(L,1)} = i\oint l^{1/2}(\psi_\zeta^{(L,1)}d\zeta - \psi_{\bar\zeta}^{(L,1)}d\bar\zeta) = -2\,\mathrm{Im}\left[\oint l^{1/2}\psi_\zeta^{(L,1)}d\zeta\right], \tag{5.16}$$

$$\Delta\psi^{(L,2)} = -i\oint l^{1/2}(\phi_\zeta^{(L,2)}d\zeta - \phi^{(L,2)}d\bar\zeta) = 2\,\mathrm{Im}\left[\oint l^{-1/2}\phi_\zeta^{(L,2)}d\zeta\right] \tag{5.17}$$

(Im = imaginary part of), where $\oint$ means integration in the counterclockwise sense along a smooth simple closed curve which surrounds $\zeta=0$, that is, the point (θ_0, λ_0).

REMARK 5.3. The integrands of (5.17) and (5.18) are, naturally, complete differentials. Note that these integrands are single-valued in the schlicht $\theta\lambda$ plane.

Since $l^{1/2}$ and A are regular at (θ_0, λ_0), we can develop $l^{1/2}\psi_\zeta^{(L,1)}$ in the neighborhood of $\zeta=0$ in a series of ζ and $\bar\zeta$. By (5.2a), (5.3a), (2.9), (2.13) we obtain

$$l^{1/2} = l_0 + \frac{l_1}{2i}\zeta - \frac{l_1}{2i}\bar\zeta - \frac{l_2}{4}\zeta^2 + \frac{l_2}{2}\zeta\bar\zeta - \frac{l_2}{4}\bar\zeta^2 + \cdots, \tag{5.18}$$

$$l^{-1/2} = \frac{1}{l_0} + \frac{l_{-1}}{2i}\zeta - \frac{l_{-1}}{2i}\bar\zeta - \frac{l_{-2}}{4}\zeta^2 + \frac{l_{-2}}{2}\zeta\bar\zeta - \frac{l_{-2}}{4}\bar\zeta^2 + \cdots, \tag{5.19}$$

$$A = h_0 + \frac{h_1}{2i}\zeta - \frac{h_1}{2i}\bar\zeta - \frac{h_2}{4}\zeta^2 + \left(\frac{h_2}{2} + h_0A_{11}\right)\zeta\bar\zeta - \frac{h_2}{4}\bar\zeta^2 + \cdots, \tag{5.20}$$

$$\begin{aligned} C = {} & \frac{1}{h_0} + \frac{h_{-1}}{2i}\zeta - \frac{h_{-1}}{2i}\bar\zeta - \frac{h_{-2}}{4}\zeta^2 + \left(\frac{h_{-2}}{2} + \frac{C_{11}}{h_0}\right)\zeta\bar\zeta \\ & - \frac{h_{-2}}{4}\bar\zeta^2 + \cdots, \end{aligned} \tag{5.21}$$

where[15] l_n and h_n are *real* constants which depend upon λ_0, and A_{mn} and

[15] Note that the power series development of the expression in the bracket on the right-hand side of (5.3a) has the form $1+C_{11}\zeta\bar\zeta+\cdots$.

C_{mn} are *real* constants which depend upon θ_0 and λ_0.

According to (5.2a)

$$\operatorname{Im}\left[\oint l^{1/2}\psi_\zeta^{(L,1)}d\zeta\right] = \operatorname{Im}\left\{\oint l^{1/2}\left[\frac{1}{2}\frac{A}{\zeta}+\frac{1}{2}A_\zeta(\log\zeta+\log\bar{\zeta})+B_\zeta\right]d\zeta.\right. \tag{5.22}$$

In a sufficiently small neighborhood $\mathfrak{N}$ of $\zeta=0$, the expressions $l^{1/2}A$, $l^{1/2}A_\zeta$, $l^{1/2}B_\zeta$, and so on, can be represented in the form of uniformly convergent series in ζ and $\bar{\zeta}$.

If we choose $\mathfrak{l}_\epsilon$ for the integration curve and assume that ϵ is so small that $\mathfrak{l}_\epsilon\subset\mathfrak{N}$, then the integration can be carried out termwise. After carrying out the integration, we pass to the limit $\epsilon\to 0$.

By (5.18) and (5.20)

$$l^{1/2}\left[\frac{1}{2}\frac{A}{\zeta}+\frac{1}{2}A_\zeta(\lg\zeta+\lg\bar{\zeta})+B_\zeta\right]=\frac{h_0l_0}{2\zeta}+\cdots, \tag{5.23}$$

where dots represent terms of the form $\gamma_{mn}\zeta^m\bar{\zeta}^n$ or $\gamma_{mn}\zeta^m\bar{\zeta}^n(\lg\zeta+\lg\bar{\zeta})$ for which $m+n+1>0$ and which according to Lemma 5.1 (after the integration is carried out and $\epsilon\to 0$) converge to 0, and therefore can be neglected.

According to (5.12)

$$-2\operatorname{Im}\left[\oint\frac{1}{2}\frac{l_0h_0}{\zeta}d\zeta\right]=-2\pi l_0h_0 \tag{5.24}$$

which yields the first relation of (5.14), since $l_0=[l(\lambda_0)]^{1/2}$, $h_0=H(2\lambda_0)$.

The second relation, (5.15), can be proved analogously.

LEMMA 5.3. *$\phi^{(1,1)}$ is a single-valued function (in the schlicht $\theta\lambda$ plane.)*

PROOF. In order to prove the above statement, we shall compute the increment $\Delta\phi^{(1,1)}$ in a manner similar to that used in Lemma 5.2, and show that this increment equals 0. By (2.8)

$$\Delta\phi^{(1,1)}=-2\operatorname{Im}\left[\oint l^{1/2}\psi_\zeta^{(1,1)}d\zeta\right]. \tag{5.25}$$

By definition, see (5.7), (2.10),

$$\psi^{(1,1)}=\frac{\partial\psi^{(L,1)}}{\partial\theta}=\frac{1}{2}\frac{A}{\zeta}+\frac{1}{2}\frac{A}{\bar{\zeta}}+\frac{1}{2}(A_\zeta+A_{\bar{\zeta}})(\lg\zeta+\lg\bar{\zeta}) + B_\zeta+B_{\bar{\zeta}}, \tag{5.26}$$

and

$$\psi_\zeta^{(1,1)} = \frac{A_\zeta}{\zeta} - \frac{1}{2}\frac{A}{\zeta^2} + \frac{1}{2}\frac{A_{\bar\zeta}}{\zeta} + \frac{1}{2}\frac{A_\zeta}{\bar\zeta} + \frac{1}{2}(A_{\zeta\zeta} + A_{\zeta\bar\zeta})(\lg \zeta + \lg \bar\zeta) \tag{5.27}$$
$$+ B_{\zeta\zeta} + B_{\zeta\bar\zeta}.$$

Now we proceed as before, that is, develop the above expressions in power series, integrate termwise along $\mathfrak{l}_\epsilon$, and then pass to the limit $\epsilon = 0$. It is clear that only the first three terms of the right-hand side of (5.27) can contribute terms which do not vanish. According to (5.18) and (5.20)

$$l^{1/2}\left(-\frac{1}{2}\frac{A}{\zeta^2} + \frac{A_\zeta}{\zeta} + \frac{1}{2}\frac{A_{\bar\zeta}}{\zeta} + \cdots\right) = -\frac{l_0 h_0}{2\zeta^2} - \frac{h_0 l_1}{4i}\frac{1}{\zeta} + \cdots, \tag{5.28}$$

where dots again mean terms whose contribution will be 0. According to Lemma 5.1

$$-2\,\mathrm{Im}\left[\oint\left(-\frac{l_0 h_0}{2\zeta^2} + \frac{h_0 l_1}{4i}\frac{1}{\zeta} + \cdots\right)d\zeta\right] = 0, \tag{5.29}$$

which means by (5.25) that $\Delta\phi^{(1,1)} = 0$, that is, that $\phi^{(1,1)}$ is a single-valued function in the schlicht $\theta\lambda$ plane.

Lemma 5.4. *$\psi^{(1,2)}$ is a single-valued function (in the schlicht $\theta\lambda$ plane).*

The proof of Lemma 5.4 proceeds exactly in the same manner as that of Lemma 5.3. As before,

$$\Delta\psi^{(1,2)} = -i\oint l^{-1/2}(\phi_\zeta^{(1,2)}d\zeta - \phi_{\bar\zeta}^{(1,2)}d\bar\zeta); \tag{5.30}$$

for $l^{-1/2}\phi_\zeta^{(1,2)}$ we obtain a development similar to that of $l^{1/2}\psi_\zeta^{(1,1)}$ with the difference that l_0 and h_0 have to be replaced by $1/l_0$ and $1/h_0$, and l_n and h_n, $n>0$, by l_{-n} and h_{-n}. As before we obtain finally

$$\Delta\psi^{(1,2)} = 0 \tag{5.31}$$

which means that $\psi^{(1,2)}$ is single-valued. The $\phi^{(n,1)}$'s and $\psi^{(n,2)}$'s, $n = 2, 3, \cdots$, being derivatives of single-valued functions, see (5.7), are also single-valued, so that both components of the flows $\mathcal{J}^{(n,k)}$, $n = 1, 2, \cdots$, $k = 1, 2$, are single-valued.

Remark 5.4. We note that Lemmas (5.3) and (5.4) can be obtained as follows: since by going around the singularity $\psi^{(L,1)}$, $\phi^{(L,2)}$ increase by constants, their derivatives are single-valued at the singular point.

By adding to $\phi^{(n,k)}$ or $\psi^{(n,k)}$ solutions of (2.12a) or (2.12b) which are regular in a given domain $\mathfrak{B}$ (situated in the region $\mathfrak{L} = \mathrm{E}[\lambda < 0]$), we obtain flows with the above singularities and satisfying the given boundary conditions. For instance, by an appropriate choice of ψ_1, we obtain a flow for which

$$\psi = \psi^{(L,1)} + \psi_1 \tag{5.32}$$

vanishes on the boundary b of the given domain $\mathfrak{B}$ and which has at (θ_0, λ_0) a logarithmic singularity (b is supposed to be sufficiently smooth). Similarly we can determine a function

$$\phi = \phi^{(L,1)} + \phi_1 \tag{5.33}$$

such that $\partial\phi/\partial n = 0$ on b.

The existence of ψ_1 and ϕ_1 follows easily from classical theorems concerning the existence of regular solutions with prescribed boundary values of linear partial differential equations.

The flows $\mathcal{J}^{(L,1)}$ and $\mathcal{J}^{(L,2)}$ are defined by the equations (5.2a), (5.2b), (5.5) and (5.6). The flows $\mathcal{J}^{(n,1)}$ and $\mathcal{J}^{(n,2)}$ for $n = 1, 2, 3, \cdots$ are obtained, as indicated in (5.7), by differentiating n times, with respect to θ, the functions $\psi^{(L,1)}$, $\phi^{(L,1)}$ and $\psi^{(L,2)}$, $\phi^{(L,2)}$ respectively. The question may arise as to whether for some index n (and hence of all higher indices) the two flows $\mathcal{J}^{(n,1)}$ and $\mathcal{J}^{(n,2)}$ are not essentially the same—that is, whether functions $\psi^{(n,1)}$ and $\psi^{(n,2)}$ (and likewise $\phi^{(n,1)}$ and $\phi^{(n,2)}$) are not linearly dependent.

To show(16) that the flows $\mathcal{J}^{(n,1)}$ and $\mathcal{J}^{(n,2)}$ are always distinct, we may proceed as follows: The flows $\mathcal{J}^{(L,1)}$ and $\mathcal{J}^{(L,2)}$ are surely distinct, for in one case the stream function is single-valued and in the other case multi-valued (and likewise for the potential). Now, the potential functions of the flows $\mathcal{J}^{(1,1)}$ and $\mathcal{J}^{(1,2)}$ are given as follows:

$$\begin{aligned}\phi^{(1,1)} &= \phi_\theta^{(L,1)} = \phi_\zeta^{(L,1)} + \phi_{\bar\zeta}^{(L,1)} = il^{1/2}(\psi_\zeta^{(L,1)} - \psi_{\bar\zeta}^{(L,1)}) \\ &= il^{1/2}\left[\frac{1}{2}A\left(\frac{1}{\zeta} - \frac{1}{\bar\zeta}\right) + \frac{1}{2}(A_\zeta - A_{\bar\zeta})\lg(\zeta\bar\zeta) + (B_\zeta - B_{\bar\zeta})\right],\end{aligned} \tag{5.34}$$

$$\begin{aligned}\phi^{(1,2)} &= \phi_\theta^{(L,2)} = \phi_\zeta^{(L,2)} + \phi_{\bar\zeta}^{(L,2)} \\ &= \frac{1}{2}C\left(\frac{1}{\zeta} + \frac{1}{\bar\zeta}\right) + \frac{1}{2}(C_\zeta + C_{\bar\zeta})\lg(\zeta\bar\zeta) + (D_\zeta + D^{\cdot}).\end{aligned} \tag{5.35}$$

If these two flows, or any of the higher-order flows obtained from them by the aforementioned differentiation procedure, were linearly dependent, it is clear that the most strongly singular terms would have to be proportional; that is, the following equality would have to hold:

$$\frac{il^{1/2}A(\zeta^{-1} - \bar\zeta^{-1})}{C(\zeta^{-1} + \bar\zeta^{-1})} = \text{constant}. \tag{5.36}$$

Taking account of the definitions of ζ and $\bar\zeta$, this may be written:

(16) The author's original proof has been simplified by Dr. Bernard Epstein.

$$\frac{il^{1/2}A(-2i(\lambda-\lambda_0))}{C(2(\theta-\theta_0))} = \text{constant.} \tag{5.37}$$

Now, since l, A, and C each approach limits ($\neq 0$), it would follow that $\lim_{\theta\to\theta_0,\lambda\to\lambda_0} [(\lambda-\lambda_0)/(\theta-\theta_0)]$ exists with λ and θ approaching their respective limits *independently*. Since such a limit does *not* exist, it follows that the assumption that the potentials $\phi^{(n,1)}$ and $\phi^{(n,2)}$ are linearly dependent is incorrect. Similarly, the flows $\mathcal{J}^{(n,1)}$ and $\mathcal{J}^{(n,2)}$ are distinct for all n.

6. **The transition from the pseudo-logarithmic to the physical plane. Conditions that $\psi(\theta, \lambda)$ defines in the physical plane a flow around a closed curve.** As has been indicated in §5, the introduction of singularities of type $\mathfrak{S}$ enables us to define flow patterns, see (5.34), in the pseudo-logarithmic plane, which have in this plane a behavior similar (in the sense indicated in §3) to that of the flow patterns of an incompressible fluid in the logarithmic plane.

As we have also stressed, the problem arises to determine the conditions that the flow defined in this manner in the pseudo-logarithmic plane will represent a flow around a closed curve in the physical plane.

In the following discussion we shall derive necessary and sufficient conditions to assure this.

LEMMA 6.1. *Let the flow $\mathcal{J}$ be regular in a bounded simply-connected domain $\overline{\mathfrak{B}}$, $\overline{\mathfrak{B}}\subset\mathfrak{L}=E[\lambda<0]$. The image $\mathfrak{P}(\mathfrak{b})$ of the boundary curve $\mathfrak{b}$ of $\mathfrak{B}$ in the physical plane is a closed curve.*

The statement of the lemma follows immediately from the fact that the integrands of (2.18a) and (2.18b) are exact differentials; see §15 of [7].

If a flow has a singularity of type $\mathfrak{S}$ at (θ_0, λ_0) then either ϕ or ψ is a single-valued function and, according to (2.19) or (2.18), x and y increase by constants if we go around the singular point (θ_0, λ_0).

DEFINITION 6.1. The constants X, Y by which the functions $x(\theta, \lambda)$ and $y(\theta, \lambda)$—see (2.19a), (2.19b) or (2.18a), (2.18b)—increase if we move once counterclockwise around a singularity of type $\mathfrak{S}$ of a flow $\mathcal{J}$ will be denoted as *T-periods*. (The letter T is suggested by the expression "transition to the physical plane.")

We proceed now to the determination of the T-periods of the singularities of the flows $\mathcal{J}^{(L,1)}$, $\mathcal{J}^{(1,k)}$, $k=1, 2$, introduced in §5. Since the integrands of (2.19a) and (2.19b) and of (2.18a), (2.18b) are complete differentials and therefore the integration can be carried out along an arbitrary simple closed curve which surrounds (θ_0, λ_0) and has the desired orientation, we choose for the integration curve again the circle $\mathfrak{l}_\epsilon$ with center at (θ_0, λ_0) and radius ϵ; in the course of our further discussion we shall assume that $\epsilon\to 0$. Thus our considerations will represent a repetition of the method applied in §5, with the difference that the factor $l^{1/2}$ of (5.16) has to be replaced by $(-\sin\theta+i(1-M^2)^{1/2}\cos\theta)/q\rho$ or similar factors, and that we have to take the real part of the integral $\mathcal{J}$ instead of its imaginary part. Let

$$(6.1)\quad \frac{-\sin\theta + i(1-M^2)^{1/2}\cos\theta}{q\rho} = (\alpha_{00}+i\beta_{00}) + (\alpha_{11}+i\beta_{11})\zeta + (\alpha_{12}+i\beta_{12})\bar{\zeta} + \cdots,$$

$$(6.2)\quad \frac{\cos\theta + i(1-M^2)^{1/2}\sin\theta}{q\rho} = (\overset{*}{\alpha}_{00}+i\overset{*}{\beta}_{00}) + (\overset{*}{\alpha}_{11}+i\overset{*}{\beta}_{11})\zeta + (\overset{*}{\alpha}_{12}+i\overset{*}{\beta}_{12})\bar{\zeta} + \cdots$$

be the series developments of the above functions; α_{mn}, β_{mn}, $\overset{*}{\alpha}_{mn}$, $\overset{*}{\beta}_{mn}$ are supposed to be *real* constants which depend on θ_0 and λ_0.

LEMMA 6.2. *The T-periods of the flow* $\mathcal{J}^{(L,1)}$ *are*

$$(6.3)\quad X^{(L,1)} = -2\pi h_0\beta_{00}, \qquad Y^{(L,1)} = -2\pi h_0\overset{*}{\beta}_{00}$$

where $h_0 = H(2\lambda_0)$ *is the quantity introduced in* (5.20) *and* β_{00} *and* $\overset{*}{\beta}_{00}$ *are defined in* (6.1) *and* (6.2).

Proof. As we have indicated, we have to compute the power series development of the integrands of (2.17a) and (2.17b), assuming that $\psi_Z^{(L,1)}$ is substituted for ψ_Z.

According to (5.2a) and (5.20)

$$(6.4)\quad \psi_Z^{(L,1)} = \frac{1}{2}\frac{A}{\zeta} + \frac{1}{2}A_Z(\lg\zeta + \lg\bar{\zeta}) + B_Z = \left(\frac{1}{2}\frac{h_0}{\zeta} + \cdots\right)$$

and therefore

$$(6.5)\quad \frac{-\sin\theta + i(1-M^2)^{1/2}\cos\theta}{q\rho}\psi_Z^{(L,1)} = \frac{1}{2}\frac{h_0(\alpha_{00}+i\beta_{00})}{\zeta} + \cdots.$$

By (2.17a)

$$(6.6)\quad X^{(L,1)} = 2\,\mathrm{Re}\left\{\lim_{\rho\to 0}\int_{I_\rho}\left[\frac{1}{2}\frac{h_0(\alpha_{00}+i\beta_{00})}{\zeta} + \cdots\right]d\zeta\right\} = -2\pi h_0\beta_{00},$$

that is, the first formula of (6.3). The second formula can be derived in an analogous manner.

LEMMA 6.3. *The T-periods of the flow* $\mathcal{J}^{(1,1)}$ *are*

$$(6.7)\quad X^{(1,1)} = 2\pi h_0\beta_{11}, \qquad Y^{(1,1)} = 2\pi h_0\overset{*}{\beta}_{11}.$$

Proof. The determination of the periods proceeds as before, except that instead of $\psi^{(L,1)}$ we now substitute $\psi^{(1,1)}$; see (5.26). According to (5.27)

$$\begin{aligned} \psi_Z^{(1,1)} &= -\frac{A}{2\zeta^2} + \frac{2A_\zeta}{2\zeta} + \frac{A_\zeta}{2\bar{\zeta}} + \frac{A_{\bar{\zeta}}}{2\zeta} + \cdots \\ &= -\frac{h_0}{2\zeta^2} + \frac{h_1}{4i}\cdot\frac{1}{\bar{\zeta}} - \frac{h_1 i}{4}\frac{\bar{\zeta}}{\zeta^2} + \cdots . \end{aligned} \tag{6.8}$$

Therefore

$$\begin{aligned} \frac{-\sin\theta + i(1-M^2)^{1/2}\cos\theta}{q\rho}\psi_Z^{(1,1)} = &\frac{-h_0(\alpha_{00}+i\beta_{00})}{2\zeta^2} \\ &- \frac{h_0(\alpha_{11}+i\beta_{11})}{2\zeta} + \cdots . \end{aligned} \tag{6.9}$$

Applying Lemma 5.1, we obtain the first formula of (6.7). The second formula is derived analogously.

LEMMA 6.4. *The T-periods of the flow $\mathcal{J}^{(1,2)}$ are*

$$X^{(1,2)} = -\pi \overset{-1}{h_0}[l_{-1}\beta_{00} + 2\alpha_{11}\overset{-1}{l_0}], \quad Y^{(1,2)} = -\pi \overset{-1}{h_0}[l_{-1}\overset{*}{\beta}_{00} + 2\overset{*}{\alpha}_{11}\overset{-1}{l_0}]. \tag{6.10}$$

Proof. We proceed as before, using (2.19a) and (2.19b) instead of (2.18a) and (2.18b). From (5.4a) we obtain

$$\phi^{(1,2)} = \frac{1}{2}\left[\frac{C}{\zeta} + \frac{C}{\bar{\zeta}} + (C_\zeta + C_{\bar{\zeta}})(\lg\zeta + \lg\bar{\zeta}) + \cdots\right] \tag{6.11}$$

and by (5.21)

$$\phi_Z^{(1,2)} = \frac{1}{2}\left[2\frac{C_\zeta}{\zeta} - \frac{C}{\zeta^2} + \frac{C_{\bar{\zeta}}}{\zeta} + \cdots\right] = \frac{1}{2}\left[-\frac{\overset{-1}{h_0}}{\zeta^2} + \cdots\right]. \tag{6.12}$$

Thus by (5.19)

$$\begin{aligned} \frac{(1-M^2)^{1/2}\cos\theta + i\sin\theta}{q\rho l^{1/2}}\phi_Z^{(1,2)} = &\frac{1}{2}(-i)\left(\frac{1}{l_0} + \frac{l_{-1}}{2i}\zeta + \cdots\right) \\ &\cdot[(\alpha_{00}+i\beta_{00}) + (\alpha_{11}+i\beta_{11})\zeta + \cdots]\cdot\left[-\frac{\overset{-1}{h_0}}{\zeta^2} + \cdots\right] \end{aligned} \tag{6.13}$$

which by Lemma 5.1 yields the first relations of (6.10). The second relation can be obtained analogously.

The T-periods of the singularities of higher order can be obtained analogously.

Following the principle of correspondence explained in §3, we shall introduce stream functions which in the pseudo-logarithmic plane have a behavior similar to that of stream functions of an incompressible fluid in the logarithmic plane.

In this connection it will be necessary to investigate the behavior of solutions of equation (2.12b) at branch points.

Suppose that $Z=\alpha$, $Z=\theta+i\lambda$, is a branch-point of the Riemann surface of the function $\psi(Z)$ such that by the transformation

$$\zeta = (Z-\alpha)^{1/\kappa} \tag{6.14}$$

(κ a positive integer) a neighborhood of α will be transformed into a *schlicht* neighborhood of the origin.

We now introduce a representation of solutions ψ of (2.12b) at such a branch point which can be considered as an analogue of the representation $\sum_{n=0}^{\infty} [\mathrm{Im}\ (\alpha_n(Z-\alpha)^{n/\kappa})]$ of a harmonic function at a branch point.

DEFINITION 6.2. If a solution ψ of (2.11b) can be represented in a neighborhood of a branch-point $Z=\alpha$ of the type described above in the form of a uniformly convergent series

$$\psi = \sum_{n=0}^{\infty} \mathrm{Im}\ [a_n \mathrm{P}_{3,1}((Z-\alpha)^{n/\kappa})] \tag{6.15}$$

(κ a positive integer), the function ψ will be said to be κ-regular at this branch point.

LEMMA 6.5. *Let* $Z=\alpha$ *be* κ*-regular at the point* $Z=\alpha$, *and let* $\mathfrak{l}_\epsilon = \mathrm{E}[Z=\alpha+\epsilon e^{i\phi}$, $0\leq\phi\leq 2\kappa\pi]$ *be a simple closed curve on the Riemann surface of the function* ψ, ϵ *being sufficiently small so that the curve* $\mathfrak{l}_\epsilon$ (*a* κ*-fold covered circle*) *encloses no other singularities of* ψ.

Then the T-periods of the flow defined by the stream function ψ *vanish when the integrals* (2.18)([17]) *are evaluated along any curve which is* (*on the Riemann surface*) *isomorphic to* $\mathfrak{l}_\epsilon$.

Proof. Since the integrands of (2.18) satisfy, except at the singular points of ψ, the integrability conditions, it follows that the path of integration may be shrunk continuously in any manner, providing only that no singular points of ψ are crossed. Thus from the hypotheses, it suffices to consider the integrals (2.18) taken around $\mathfrak{l}_\epsilon$. Since by hypothesis series (6.15) is uniformly convergent for ϵ (the radius of $\mathfrak{l}_\epsilon$) sufficiently small, it is permissible to invert the order of summation and integration, and it therefore suffices to show that each term of the series (6.15) contributes zero to the transition periods. Now, for $|Z-\alpha|$ sufficiently small, the expansion (1.4) of $\mathrm{P}_3[(Z-\alpha)^{n/\kappa}]$ is uniformly convergent, so that it is once again permitted to invert the order of summation and integration. Therefore, it suffices to show that each term of

([17]) Here and hereafter we write (2.18) instead of (2.18a) and (2.18b).

the expansion $\mathbf{P}_3[(Z-\alpha)^{n/\kappa}]$ yields 0 when inserted into equations (2.18) in place of ψ.

From (1.4) it is easily seen that each such term contains $(Z-\alpha)$ only to a non-negative power; therefore, ψ_Z, which appears as a factor in the integrands of (2.18a) and (2.18b), contains $(Z-\alpha)$ only to powers *greater* than -1. Since the term ψ_Z is multiplied in the integrands of (2.18) by functions which are continuous at $Z=\alpha$, and since the length of the path of integration is $2\kappa\pi\epsilon$, it follows that each integral can be made less in absolute value than any preassigned quantity by choosing ϵ sufficiently small.

The image of a flow of an incompressible fluid in the logarithmic plane and that of a compressible fluid in the pseudo-logarithmic plane extend in general to infinity, since at a stagnation point $q=0$ and therefore $\tilde{\lambda}=\lg q$ and λ (see 2.7) become $-\infty$.

In the following we shall consider only flows satisfying the following conditions:

(1) The stagnation point (or points), if any, lies on the boundary (but not in the interior) of the flow.

(2) The flow (that is, the stream and potential functions) possesses only a finite number of singularities (including branch points), so that there exists a finite number R such that for $|Z|\geqq R$ the stream function is regular at every point of $\mathfrak{P}$.

(3) The stream function ψ as well as ψ_Z and $\psi_{\bar{Z}}$ exist and are continuous on the boundary except perhaps at a finite number of points, say β_ν.

The integration over (2.18) will be carried out in the following along the boundary curve $\mathfrak{p}$. In this connection it is necessary to make certain conventions.

(1) The integration in the neighborhood of a point β_ν will be understood as follows: we draw a circle of a sufficiently small radius ϵ around β_ν, denote by $\mathfrak{C}_\epsilon$ the part of this circle (assumed to consist of a single arc) which lies in $\mathfrak{P}$, and replace the arc $p_1\beta_\nu p_2$ of $\mathfrak{p}$ by $\mathfrak{C}_\epsilon$, p_1 and p_2 being the intersections of $\mathfrak{p}$ with $\mathfrak{C}_\epsilon$.

(2) As we mentioned before, the boundary curve $\mathfrak{p}$ of $\mathfrak{P}$ extends to infinity. Let $\mathfrak{l}$ denote a part of it which is situated in $|Z|\geqq R$ and which together with a corresponding arc of $|Z|=R$ bounds a simply-connected part, say $\mathfrak{P}_1$, of $\mathfrak{P}$, which extends to infinity.

The integrals (2.18) taken over $\mathfrak{l}$ are understood again in the improper sense, namely, we connect by a curve $\mathfrak{C}$ two points, say p_1 and p_2, which lie on two different branches of $\mathfrak{l}$, which branches meet at infinity. The integration along $p_1\infty p_2$ is replaced by integration along the curve $\mathfrak{C}$.

REMARK 6.1. Note that the value of the integrals are independent of the choice of $\mathfrak{C}$, since, by assumption, no singularities of ψ are situated in the domain bounded by $p_1\mathfrak{C}p_2p_2'\mathfrak{C}'p_1'p$.

In analogy to (4.11b) we assume that the stream function ψ of $\mathcal{J}$ is defined in a (not necessarily schlicht) domain $\mathfrak{P}$ of the pseudo-logarithmic plane and can be represented there by

$$(6.16) \qquad \psi = \overset{(1)}{\mu_0} \psi^{(1,1)} + \overset{(2)}{\mu_0} \psi^{(1,2)} + \mu \psi^{(L,1)} + \psi_1$$

where ψ_1 is m-regular in $\mathfrak{P}$, and satisfies hypotheses (1), (2), (3). $\psi^{(1,1)}$, $\psi^{(1,2)}$, $\psi^{(L,1)}$ are stream functions possessing at the point θ_0, λ_0 singularities described in §5. θ_0, λ_0 is the point which corresponds to $z=\infty$.

THEOREM 6.1. *The necessary and sufficient condition in order that the stream function* (6.16) *defined in a simply-connected domain* $\mathfrak{P}$ *represents a flow in the physical plane around a closed curve is that*

$$(6.17) \qquad \begin{aligned} \overset{(1)}{\mu_0} X^{(1,1)} + \overset{(2)}{\mu_0} X^{(1,2)} + \mu X^{(L,1)} &= 0, \\ \overset{(1)}{\mu_0} Y^{(1,1)} + \overset{(2)}{\mu_0} Y^{(1,2)} + \mu Y^{(L,1)} &= 0, \end{aligned}$$

$(X^{(1,\kappa)}, Y^{(1,\kappa)})$ *being the T-periods of* $\mathcal{J}^{(1,\kappa)}$, $\kappa=1, 2$, $(X^{(L,1)}, Y^{(L,1)})$ *those of* $\mathcal{J}^{(L,1)}$[18].

Proof. Let

$$(6.18) \qquad x = x(\theta, \lambda), \qquad y = y(\theta, \lambda)$$

(see (2.18)) represent the function pair which maps the domain $\mathfrak{P}$ into the domain in which the flow is defined in the physical plane. These integrals have been defined initially along curves which lie within $\mathfrak{P}$; however, since ψ_1 satisfies the hypotheses (1), (2), (3), the integration can be carried out along the boundary curve $\mathfrak{p}$ (understood with the conventions stated above). The image in the physical plane of the boundary $\mathfrak{p}$ of $\mathfrak{P}$ will be a closed curve if and only if the functions $x(\theta, \lambda)$, $y(\theta, \lambda)$ return to their initial values when we move once along $\mathfrak{p}$. According to our hypotheses (see in particular (3)) and our conventions, we can replace the integration curve $\mathfrak{p}$ by another curve $\mathfrak{p}^*$ which (except perhaps at $Z=-\infty$) differs sufficiently little from $\mathfrak{p}$, so that the values of the integrals (2.18) over $\mathfrak{p}$ and those over $\mathfrak{p}^*$ differ by a quantity ϵ, which can be made arbitrarily small. $\mathfrak{p}^*$ is assumed to lie *inside* $\mathfrak{P}$.

Since the function ψ_1 is m-regular in $\mathfrak{P}$, and $\psi^{(1,1)}$, $\psi^{(1,2)}$, $\psi^{(L,1)}$ are regular except at the point (θ_0, λ_0), the integrals (2.18) taken along $\mathfrak{p}^*$ are equal to the same integrals taken over a sufficiently small circle with center at (θ_0, λ_0). According to Lemmas 6.1, 6.2 and 6.3, the values of these integrals equal the left-hand sides of (6.17), respectively. Thus these integrals will vanish if and only if the left-hand sides of (6.17) vanish.

A generalization of the Blasius formula. As is well known, see [23, pp. 36

(18) It should be remarked that the image of $\mathfrak{P}$ in the physical plane is not necessarily schlicht, so that the question whether the flow given by (6.16) in the $\theta\lambda$-plane has a physical significance requires a separate investigation.

ff.], the behavior of the flow at infinity determines the force and the moment acting upon a profile immersed in the flow.

As we shall show, the results of the present section enable us to obtain for the force $\overline{F}$ and the moment M_{z_0} about a given point $z_0=(x_0, y_0)$ formulas analogous to those holding in the incompressible case. For an incompressible fluid these quantities are given by the formulas of Blasius [23, p. 35]:

$$F = \frac{i\rho_0}{2}\int_{\mathfrak{C}}\left(\frac{dw}{dz}\right)^2 dz; \tag{6.19a}$$

$$M_{z_0} = -\operatorname{Re}\left[\frac{\rho_0}{2}\int_{\mathfrak{C}}(z - z_0)\left(\frac{dw}{dz}\right)^2 dz\right]. \tag{6.19b}$$

Here ρ_0 is the density of the fluid and w is the complex potential of the flow, $\mathfrak{C}$ a control contour. The above formulas can be expressed in terms of the coordinates of θ, λ of the logarithmic plane. We proceed to the derivation of formula (6.28), a generalization to the compressible fluid case of the formula which one obtains from (6.19a), using the classicial considerations. According to [23, p. 23 (29)] in the case of a steady two-dimensional flow in the absence of external forces, we have:

$$F = -\int_{\mathfrak{C}}\rho q q_n ds - \int_{\mathfrak{C}} p n ds \tag{6.20}$$

where $\mathfrak{C}$ (the control contour) is any simple closed curve which surrounds the immersed profile. Here n is the unit outward normal to $\mathfrak{C}$, q_n the component along n of q, and ds the line element of $\mathfrak{C}$. We have

$$\begin{aligned} q &= qe^{i\theta}, \qquad n = \frac{dy - idx}{ds} = -\frac{idz}{ds}, \\ q_n &= \frac{(u, v)\cdot(dy, -dx)}{ds} = \frac{udy - vdx}{ds}. \end{aligned} \tag{6.21}$$

Thus

$$\begin{aligned} F &= -\int_{\mathfrak{C}}\rho q e^{i\theta}[udy - vdx] + i\int_{\mathfrak{C}} pdz \\ &= -\int_{\mathfrak{C}} qe^{i\theta}\left[\frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx\right] + i\int_{\mathfrak{C}} pdz. \end{aligned} \tag{6.22}$$

Since, see (2.3), (2.5),

$$\rho u = \frac{\partial\psi}{\partial y}, \quad \rho v = -\frac{\partial\psi}{\partial x}, \quad p = \sigma\left[1 - \frac{(\gamma - 1)}{2}q^2\right]^{\gamma/(\gamma-1)} \tag{6.23}$$

(here it is assumed that in (2.4), $\beta=0$).

$$F = -\int_{\mathfrak{C}} q e^{i\theta} d\psi + i\sigma \int_{\mathfrak{C}} \left[1 - \frac{(\gamma-1)}{2} q^2\right]^{\gamma/(\gamma-1)} dz. \tag{6.24}$$

According to [7, formula (137), p. 50],

$$dz = \frac{1}{\rho} e^{i\theta} \left[\left(-\frac{1-M^2}{q^2}\psi_\theta \frac{dq}{d\lambda} + \frac{i\psi_\lambda}{q}\right) d\lambda + \left(\psi_\lambda \frac{d\lambda}{dq} + \frac{i\psi_\theta}{q}\right) d\theta\right). \tag{6.25}$$

Since $\rho = [1-2^{-1}(\gamma-1)q^2]^{1/(\gamma-1)}$, we obtain by using the last formula of (6.23),

$$\begin{aligned} i\int_{\mathfrak{C}} p\,dz = \sigma i \int_{\mathfrak{C}} &\left(1 - \frac{(\gamma-1)}{2} q^2\right) e^{i\theta} \\ &\times \left[\left(-\frac{1-M^2}{q^2}\psi_\theta \frac{dq}{d\lambda} + \frac{i\psi_\lambda}{q}\right) d\lambda + \left(\psi_\lambda \frac{d\lambda}{dq} + \frac{i\psi_\theta}{q}\right) d\theta\right]. \end{aligned} \tag{6.26}$$

Since ψ is single-valued at infinity, the first integral on the right-hand side of (6.24) can be transformed, by integration by parts, as follows:

$$-\int_{\mathfrak{C}} q e^{i\theta} d\psi = \int_{\mathfrak{C}} \psi \left(e^{i\theta} \frac{dq}{d\lambda} d\lambda + i q e^{i\theta} d\theta\right). \tag{6.27}$$

Now, according to (6.16):

$$\psi = \overset{(1)}{\mu_0} \psi^{(1,1)} + \overset{(2)}{\mu_0} \psi^{(1,2)} + \mu \psi^{(L,1)} + \psi_1,$$

where ψ_1 is regular at the point (θ_0, λ_0) corresponding to $z=\infty$. If we choose, therefore, for $\mathfrak{C}$ a sufficiently small circle $\mathfrak{l}_\epsilon$ of radius ϵ and center at (θ_0, λ_0), that is, if we write $\mathfrak{l}_\epsilon = E(\theta=\theta_0+\epsilon\cos\phi,\ \lambda=\lambda_0+\epsilon\sin\phi,\ 0\leqq\phi\leqq 2\pi)$ and then proceed to the limit, $\epsilon\rightarrow 0$, then we can neglect terms corresponding to ψ_1, so that we obtain.

$$F = \overset{(1)}{\mu_0} F^{(1,1)}(\theta_0, \lambda_0) + \overset{(2)}{\mu_0} F^{(1,2)}(\theta_0, \lambda_0) + \mu F^{(L,1)}(\theta_0, \lambda_0) \tag{6.28}$$

where $F^{(1,1)}(\theta_0, \lambda_0)$, $F^{(1,2)}(\theta_0, \lambda_0)$, $F^{(L,1)}(\theta_0, \lambda_0)$ are expressions (6.24) corresponding to $\psi = \psi^{(1,1)}$, $\psi^{(1,2)}$, $\psi^{(L,1)}$, respectively. These quantities can be computed, using (5.2a), (5.2b) and so on, and the relations,

$$d\lambda = -\epsilon\sin\phi\,d\phi, \qquad d\theta = \epsilon\cos\phi\,d\phi,$$

$$q = q_0 + \left(\frac{dq}{d\lambda}\right)_0 (\lambda-\lambda_0) + \frac{1}{2!}\left(\frac{d^2q}{d\lambda^2}\right)_0 (\lambda-\lambda_0)^2 + \cdots,$$

$$\frac{dq}{d\lambda} = \left(\frac{dq}{d\lambda}\right)_0 + \left(\frac{d^2q}{d\lambda^2}\right)_0 (\lambda-\lambda_0) + \cdots,$$

where

$$\left(\frac{d_\kappa q}{d\lambda_\kappa}\right)_0 \equiv \left(\frac{d_\kappa q(\lambda)}{d\lambda_\kappa}\right)_{\lambda=\lambda_0}.$$

In a similar manner the moment M_{z_0} can be determined.

7. **An integral formula representing a subsonic flow inside a domain in terms of its values on the boundary.** The Cauchy integral formula is one of the most powerful tools in the study of analytic functions of a complex variable. As we shall show, there exists in the subsonic case an analogue of this formula which enables us to express the values of ϕ and ψ inside a domain $\mathfrak{B}$ of the (θ, λ)-plane in terms of the values of ϕ and ψ on the boundary of $\mathfrak{B}$.

THEOREM 7.1. *Let $\mathcal{J}(\phi, \psi)$ be a flow which is regular in the bounded domain $\mathfrak{B}$ (lying entirely within the subsonic region) whose boundary $\mathfrak{C}$ is a simple closed curve. Then at every point (θ_0, λ_0) of*[19] $\mathfrak{B}$:

$$(7.1)\quad 2\pi\phi(\theta_0, \lambda_0) = [H(2\lambda_0)]^{-1}\int_{\mathfrak{C}}\left[-H^2\phi\frac{\partial\phi^{(L,2)}}{\partial n} + \psi\frac{\partial(H^2l^{1/2}\phi^{(L,2)})}{\partial s}\right]ds,$$

$$(7.2)\quad 2\pi\psi(\theta_0, \lambda_0) = H(2\lambda_0)\int_{\mathfrak{C}}\left[-H^{-2}\psi\frac{\partial\psi^{(L,1)}}{\partial n} + \phi\frac{\partial(H^{-2}l^{-1/2}\psi^{(L,1)})}{\partial s}\right]ds$$

where $\partial/\partial n$ indicates differentiation in the direction of the inward normal, and $\phi^{(L,2)}$ and $\psi^{(L,1)}$ are the fundamental solutions introduced in (5.2b) *and* (5.2a) *respectively, and H is given by* (2.14).

Proof. Applying Green's formula to the domain obtained by deleting from $\mathfrak{B}$ a small circle with center at (θ_0, λ_0), and taking account of equations (2.12), (2.13), (2.14), (5.2), (5.3), and (5.4), we have:

$$(7.3)\qquad 2\pi H(2\lambda_0)\phi(\theta_0, \lambda_0) = -\int_{\mathfrak{C}}H^2\phi\frac{\partial\phi^{(L,2)}}{\partial n}ds + \int_{\mathfrak{C}}H^2\phi^{(L,2)}\frac{\partial\phi}{\partial n}ds.$$

Now $\partial\phi/\partial n = (\partial\phi/\partial\lambda)(d\lambda/dn) + (\partial\phi/\partial\theta)(d\theta/dn) = -l^{1/2}\partial\psi/\partial s$ (see equations (2.8)). Inserting this into the last term of equation (7.3), we have:

$$(7.4)\qquad 2\pi H(2\lambda_0)\phi(\theta_0, \lambda_0) = -\int_{\mathfrak{C}}H^2\phi\frac{\partial\phi^{(L,2)}}{\partial n}ds - \int_{\mathfrak{C}}H^2l^{1/2}\phi^{(L,2)}\frac{\partial\psi}{\partial s}ds.$$

Performing integration by parts on the last term and multiplying both sides of the resulting equation by $[H(2\lambda_0)]^{-1}$, we obtain equation (7.1). Equation (7.2) is obtained by a completely analogous procedure.

REMARK 7.1. $\phi(\theta_0, \lambda_0)$ and $\psi(\theta_0, \lambda_0)$ can be represented in a slightly different form which is of interest for some purposes. The first term in the integrand of (7.1) may be expressed as follows:

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(19) The relation (7.2) in a slightly different form has been indicated in [3, pp. 16–18].

$$(7.5)\qquad -\int_{\mathfrak{C}} H^2\phi \frac{\partial\phi^{(L,2)}}{\partial n}\,ds = \int_{\mathfrak{C}} H^2\phi l^{1/2}\frac{\partial\psi^{(L,2)}}{\partial s}\,ds.$$

If the right-hand side of (7.5) is integrated by parts, beginning and ending at a point P on the curve $\mathfrak{C}$, the result obtained is[20]:

$$(7.6)\qquad \begin{aligned}\int_{\mathfrak{C}} H^2\phi l^{1/2}\frac{\partial\psi^{(L,2)}}{\partial s}\,ds = \Delta\psi^{(L,2)}(\theta_0,\lambda_0)\left[H^2l^{1/2}\phi\right]_P \\ -\int_{\mathfrak{C}}\psi^{(L,2)}\frac{\partial(H^2l^{1/2}\phi)}{\partial s}\,ds.\end{aligned}$$

The reason that the integrated part does not vanish is, of course, that $\psi^{(L,2)}$ is multiple-valued, and after describing the curve $\mathfrak{C}$ once (in the positive sense), beginning at any point, the function $\psi^{(L,2)}$ increases by the amount $\Delta\psi^{(L,2)}(\theta_0, \lambda_0)$. In the integral on the right-hand side of (7.6) it is also necessary to take account of the multiple-valuedness of $\psi^{(L,2)}$. It is apparent that a concept quite analogous to that of a Riemann surface may be employed here. The function $\psi^{(L,2)}(\theta, \lambda; \theta_0, \lambda_0)$ is to be considered defined on a multi-sheeted surface with branch points at (θ_0, λ_0) and ∞, and then $\psi^{(L,2)}$ is rendered single-valued by a cut joining (θ_0, λ_0) to ∞ and cutting the curve $\mathfrak{C}$ at the point P; now the curve $\mathfrak{b}$, although closed in the "schlicht" (θ, λ)-plane, is open on the Riemann surface, beginning at P in one sheet and ending at P in the next sheet.

Taking account of equations (7.5) and (7.6) we may write (7.1) as follows:

$$(7.7)\qquad \begin{aligned}2\pi\phi(\theta_0,\lambda_0) = \left[H(2\lambda_0)\right]^{-1}\Bigg\{\Delta\psi^{(L,2)}(\theta_0,\lambda_0)\left[H^2l^{1/2}\phi\right]_P \\ +\int_{\mathfrak{C}}\left[\psi\frac{\partial(H^2l^{1/2}\phi^{(L,2)})}{\partial s} - \psi^{(L,2)}\frac{\partial(H^2l^{1/2}\phi)}{\partial s}\right]ds\Bigg\}.\end{aligned}$$

A completely analogous expression may be found for $2\pi\psi(\theta_0, \lambda_0)$, involving the period of $\phi^{(L,1)}$.

As was mentioned in §1, the approach developed in the present paper may be considered as a generalization of methods used in the theory of analytic functions of a complex variable, in particular in the theory of integrals of algebraic functions. It will perhaps be of interest to discuss analogues in this direction.

An analytic function $f=\phi+i\psi$ of a complex variable can be interpreted as a flow of an incompressible fluid, the flow being in either the physical (x, y)-plane or the logarithmic (θ, λ)-plane, $\lambda=\lg q$.

As we have seen in §2, functions which are defined on Riemann surfaces arise in a quite natural manner, representing flows in the physical plane which

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[20] $[\ ]_P$ means the value of the expression in brackets at the point P.

have physical significance. If we attempt to develop in the (θ, λ)-plane a theory for compressible flows analogous to the theory of analytic functions of a complex variable, and in particular to develop an analogue to the theory of algebraic functions and their integrals, there arises a fundamental difference. In the first case (that is, incompressible flows) the functions can be defined for all values of θ and $\bar{\lambda}$, while in the second case (compressible flows), the functions are defined only for $\lambda<0$, if we restrict ourselves to subsonic flows.

On the other hand, in analogy to the integrals of the second and third kinds, we can introduce flows which will be characterized by their singularities and their behavior on the boundary of the domain. For instance, we can consider flows $\mathcal{J}_{\mathfrak{P}}^{(L,k)}=\mathcal{J}^{(L,k)}+\mathcal{J}_k$, $k=1, 2$ (see (5.1)), where $\mathcal{J}_k$ are flows which are m-regular in $\mathfrak{P}$ (see definition 6.1) and determined in such a way that the stream function $\psi^{(L,1)}$ or the potential function $\phi^{(L,2)}$ vanish on the boundary of([21]) $\mathfrak{P}$. $\mathcal{J}_{\mathfrak{P}}^{(L,k)}$ can be considered as analogues of integrals of the third kind.

Similarly, normalized flows $\mathcal{J}_{\mathfrak{P}}^{(1,k)}=\mathcal{J}^{(1,k)}+\mathcal{J}_k$ can be considered as analogues of integrals of the second kind.

We note that the above singularities admit a simple physical interpretation. $\mathcal{J}^{(L,1)}$ yields in the physical plane a vortex at infinity $z=\infty$, $\mathcal{J}^{(L,2)}$ a source or sink at $z=\infty$, $\mathcal{J}^{(1,k)}$, $k=1, 2$, yield doublets at $z=\infty$.

REMARK 7.2. An essential role in the theory of integrals of algebraic functions is played by the relations which exist between the periods of these integrals. Applying the procedure in the proof of Theorem 7.1 and in Remark 7.1, we can obtain formulas which can be considered as a kind of generalization of results in the classical theory, concerning the relations which exist between the periods of normal integrals of the first, second and third kinds and the values of these integrals at singular points of integrals of the third kind.

8. **A property of the transition from the physical plane to the hodograph and allied planes.** The theory of two-dimensional steady motion of an incompressible fluid, and in particular the study of relations which exist between the flow in the physical plane and its images in the hodograph and logarithmic planes, is an interesting field for application of the theory of schlicht functions and that of value distribution of analytic functions of a complex variable. Most of the abstract notions in these theories admit a natural physical interpretation, so that many mathematical theorems may be formulated as relations between quantities which have a physical significance.

It is of importance that these relations can be generalized to the case of subsonic flows, since, as we shall show, the transformation of a flow pattern in the physical plane into its image in the hodograph or pseudo-logarithmic plane is a quasi-conformal mapping. As is well known, this latter mapping has many properties in common with conformal transformation. See [1, 2, 4].

([21]) $\psi_{\mathfrak{P}}^{(L,1)}$ and $\phi_{\mathfrak{P}}^{(L,2)}$ will be Green's functions, with respect to the domain $\mathfrak{P}$, of equations (2.11b) and (2.11a) respectively.

REMARK 8.1. ϕ and ψ in the hodograph and pseudo-logarithmic planes are solutions of a system of linear partial differential equations. See (2.6) and (2.8). These equations represent a generalization of those of Cauchy-Riemann, and one can expect that by using the theory of operators and some other tools of investigation, it will be possible to obtain for the function-pair $[\phi(\theta, \lambda), \psi(\theta, \lambda)]$ various theorems similar to those found in the theory of analytic functions. Using, then, the above-mentioned correspondence between the images of the flow in the pseudo-logarithmic and in the physical planes, that is, the correspondence between

$$[\phi(\theta, \lambda), \psi(\theta, \lambda)] \rightarrow [\phi^{(1)}(x, y), \psi^{(1)}(x, y)],$$
$$\phi^{(1)}(x, y) \equiv \phi[\theta(x, y), \lambda(x, y)], \qquad \psi^{(1)}(x, y) = \psi[\theta(x, y), \lambda(x, y)],$$

we may expect to obtain theorems for $\phi^{(1)}$ and $\psi^{(1)}$, that is, for solutions of a system (2.3), (2.5) of *nonlinear* equations(22).

The first part of the present section will be devoted to the proof that the transition from the physical to the hodograph or allied (that is, logarithmic and pseudo-logarithmic) planes represents a quasi-conformal mapping. In the second part, Theorem 8.2 will be proved, which by the use of well-established methods yields a certain result which follows from the pseudo-conformality of the transformation. We shall return in a later publication to a more systematic exploitation of Theorem 8.1.

DEFINITION 8.1. Let M be a one-to-one mapping of a (closed) domain $\overline{\mathfrak{B}}$ into $\overline{\mathfrak{B}}^*$ such that if P^* denotes the image of P, an infinitesimal circle of radius ds around P goes into an infinitesimal ellipse

$$G_{11}(P^*)dx^{*2} + 2G_{12}(P^*)dx^*dy^* + G_{22}(P^*)dy^{*2} = ds^2 \tag{8.1}$$

around P^*. If there exists a fixed constant K, $1 \leq K < \infty$, which is independent of P^*, such that for all $P^* \in \overline{\mathfrak{B}}^*$

$$S(P^*) \leq 2KD^{1/2} \tag{8.2}$$

where

$$S(P^*) = G_{11}(P^*) + G_{22}(P^*), \qquad D(P^*) = G_{11}(P^*)G_{22}(P^*) - G_{12}^2(P^*), \tag{8.3}$$

then the mapping M is said to be quasi-conformal in $\overline{\mathfrak{B}}$.

LEMMA 8.1. *Let $\phi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ be two continuously differentiable functions defined in a (closed) domain $\overline{\mathfrak{S}}$ of the (α, β)-plane, whose derivatives are connected by the relations*

$$\phi_\alpha = A(\alpha, \beta)\psi_\beta, \qquad \phi_\beta = -B(\alpha, \beta)\psi_\alpha \tag{8.4}$$

(22) Despite the fact that (2.3), (2.5) represent a very special system of equations, these results are of considerable theoretical interest as the few examples of theorems in the theory of nonlinear partial differential equations, about which comparatively little is known.

where

$$0 < a \leqq A(\alpha, \beta) \leqq b < \infty, \qquad 0 < a \leqq B(\alpha, \beta) \leqq b < \infty, \tag{8.5}$$

a and b being conveniently chosen constants.

If at every point P of $\overline{\mathfrak{S}}$ *the inequality*

$$|\psi_\alpha| + |\psi_\beta| > 0 \tag{8.6}$$

holds, then the transformation M,

$$\phi = \phi(\alpha, \beta), \qquad \psi = \psi(\alpha, \beta), \tag{8.7}$$

is quasi-conformal in $\mathfrak{S}$.

Proof. Let ds be the radius of an infinitesimal circle in the $\alpha\beta$-plane with center at P^*. A formal computation yields that

$$\begin{aligned} ds^2 &= d\phi^2 + d\psi^2 = (\phi_\alpha d\alpha + \phi_\beta d\beta)^2 + (\psi_\alpha\, d\alpha + \psi_\beta d\beta)^2 \\ &= (A\psi_\beta d\alpha - B\psi_\alpha d\beta)^2 + (\psi_\alpha d\alpha + \psi_\lambda d\beta)^2 \\ &= (A^2\psi_\beta^2 + \psi_\alpha^2)d\alpha^2 + 2(1 - AB)\psi_\alpha\psi_\beta d\alpha d\beta + (B^2\psi_\alpha^2 + \psi_\beta^2)d\beta^2 \end{aligned}$$

and

$$\frac{S}{2D^{1/2}} = \frac{(A^2 + 1)\psi_\beta^2 + (B^2 + 1)\psi_\alpha^2}{2(A\psi_\beta^2 + B\psi_\alpha^2)}. \tag{8.8}$$

ψ_α and ψ_β are continuous and therefore bounded in the closed domain $\overline{\mathfrak{S}}$, say $\psi_\alpha \leqq M$, $\psi_\beta \leqq M$, M being a conveniently chosen constant. Since $|\psi_\alpha| + |\psi_\beta|$ is positive and continuous in a closed domain, there exists a constant, say m, such that $|\psi_\alpha| + |\psi_\beta| \geqq m > 0$ in $\overline{\mathfrak{S}}$. Therefore $\psi_\alpha^2 + \psi_\beta^2 \geqq m^2/3$. (Were $\psi_\alpha^2 + \psi_\beta^2 < m^2/3$, then we would have

$$(|\psi_\alpha| + |\psi_\beta|)^2 \leqq (\psi_\alpha^2 + \psi_\beta^2 + 2|\psi_\alpha\psi_\beta|) < 3(\psi_\alpha^2 + \psi_\beta^2) < m^2).$$

Therefore

$$\frac{S}{2D^{1/2}} \leqq 2\frac{(b^2 + 1)M^2}{am^2} \equiv K, \tag{8.9}$$

which proves our assertion.

The simplest procedure to show that the transformations above-mentioned are quasi-conformal is to show that the transformation of the physical plane into the potential plane (that is, the plane whose cartesian coordinates are ϕ and ψ), as well as of the hodograph plane into the potential plane, is quasi-conformal. Since any one-to-one mapping inverse to a quasi-conformal transformation, as well as a combination of two quasi-conformal transformations, is again quasi-conformal, we obtain in this manner the desired results.

Lemma 8.2. *Let $\mathcal{J}$ be a flow-pattern defined in a bounded domain $\overline{\mathfrak{G}}$ of the physical plane, and $\mathfrak{P}$ its image in the potential plane. Assuming that the domain in which $\mathcal{J}$ is defined does not include any stagnation point, the mapping of $\overline{\mathfrak{G}}$ into $\overline{\mathfrak{P}}$, defined by the correspondence*

$$[x, y] \to [\phi(x, y), \psi(x, y)], \tag{8.10}$$

is quasi-conformal.

Proof. (ϕ, ψ) and (x, y) are connected by the relations (2.3). On the other hand, if we replace α, β by x, y, and choose $A = B = 1/\rho > 0$, the system (8.4) becomes (2.3). Our assertion follows, therefore, from Lemma 8.1.

Lemma 8.3. *Let $\overline{\mathfrak{S}}$ be the image of $\overline{\mathfrak{G}}$ in the pseudo-logarithmic plane. Assuming that $\overline{\mathfrak{S}}$ is bounded and schlicht, that no stagnation points lie in $\overline{\mathfrak{S}}$, and that the flow $\mathcal{J}$ is subsonic, the mapping of $\overline{\mathfrak{S}}$ onto $\overline{\mathfrak{P}}$, defined by the correspondence*

$$[\theta, \lambda] \to [\phi(\theta, \lambda), \psi(\theta, \lambda)], \tag{8.11}$$

is quasi-conformal.

Proof. (ϕ, ψ) and (θ, λ) are connected by relation (2.8). Replacing α, β by θ, λ and substituting $A = B = l^{1/2}$, (8.4) becomes (2.8). In the subsonic case

$$l(\lambda) = (1 - M^2)/\rho^2 > 0, \tag{8.12}$$

and therefore the inequalities (8.5) hold. A fortiori, the assertion of Lemma 8.3 is a direct consequence of Lemma 8.1.

The transition from the pseudo-logarithmic plane (whose cartesian coordinates are θ, λ) to the logarithmic plane (whose cartesian coordinates are θ, $\tilde{\lambda}$) means a stretching of the λ-axis. See (2.7) and (2.7.a). Since for $0 < \lambda_1 \geqq \lambda \geqq \lambda_0 > 0$,

$$0 < \frac{1}{a} \leqq \frac{d\lambda}{d\tilde{\lambda}} = (1 - M^2)^{1/2} \leqq a < \infty \tag{8.13}$$

(see (45) of [7]), this mapping is also quasi-conformal. As previously remarked, the combination of a number of quasi-conformal mappings (in particular that of a quasi-conformal and a conformal mapping), as well as the inverse of a quasi-conformal mapping, is again quasi-conformal.

From Lemmas 8.1, 8.2, and 8.3, therefore, there follows the theorem:

Theorem 8.1. *Let $\mathcal{J}$ be a subsonic flow defined in a bounded and schlicht domain $\overline{\mathfrak{G}}$, and let $\overline{\mathfrak{S}}_1$, $\overline{\mathfrak{S}}_2$, $\overline{\mathfrak{S}}_3$ be the images of $\overline{\mathfrak{G}}$ in the hodograph, logarithmic, and pseudo-logarithmic planes. If $\overline{\mathfrak{G}}$ does not include any stagnation point and if $\overline{\mathfrak{S}}_k$, $k = 1, 2, 3$, is bounded and schlicht, then the mapping of $\overline{\mathfrak{G}}$ into $\overline{\mathfrak{S}}_k$ is quasi-conformal.*

As we mentioned before, we shall now give a simple application of Theorem 8.1.

THEOREM 8.2. *Let*

$$\mathfrak{G}_1 = \mathrm{E}[\vartheta_1 \leqq \vartheta \leqq \vartheta_2, 0 < r_1 \leqq r \leqq r_2 < \infty] \tag{8.14}$$

be a domain in the physical plane in which a subsonic flow $\mathfrak{F}$ *is defined,* r, ϑ *being polar coordinates. We assume that no stagnation points lie in* $\overline{\mathfrak{G}}_1$.

Let $\overline{\mathfrak{G}}_2$, *the image of* $\overline{\mathfrak{G}}_1$ *in the hodograph plane, be bounded and schlicht. Let us further assume that on each of the arcs*

$$\alpha_k = \mathrm{E}[\vartheta_1 \leqq \vartheta \leqq \vartheta_2, r = r_k], \qquad k = 1, 2, \tag{8.15}$$

the speed q *varies in the range*

$$0 < q_k \leqq q \leqq Q_k < \infty, \qquad Q_1 < q_2, \tag{8.16}$$

and that in the whole domain $\overline{\mathfrak{G}}_1$ *the angle which the velocity vector* $q(x, y) = qe^{i\theta}$ *forms with the positive x-axis varies in the range*

$$\mathrm{E}[0 \leqq \theta \leqq \theta_1], \qquad \theta_1 \leqq 2\pi. \tag{8.17}$$

Then for the above quantities the inequality

$$\frac{(\lg q_2 - \lg Q_1)^2}{\lg Q_2 - \lg q_1} \leqq \frac{2K\theta_1}{\vartheta_2 - \vartheta_1} (\lg r_2 - \lg r_1) \tag{8.18}$$

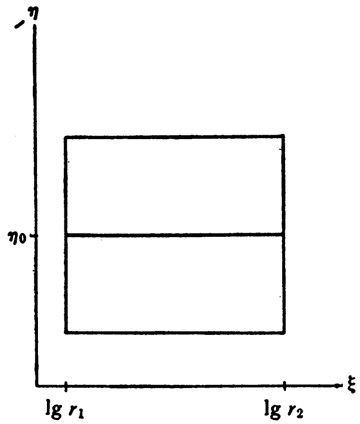

FIG. 9. The domain $\mathfrak{G}_1^+$.

holds. (*Here* K *is the constant characterizing the "quasi-conformality"* [*see* (8.2)] *of the mapping of* $\overline{\mathfrak{G}}_1^+$ *into* $\overline{\mathfrak{G}}_2^+$, *domains to be defined later.*)

REMARKS 8.2. A bound for K can be obtained by use of (8.9).

Proof. Inequality (8.18) follows immediately by well known considerations. See [3], in particular (8.9) and papers cited there. Let

$$\overline{\mathfrak{G}}_1^+ = \mathrm{E}[\lg r_1 \leqq \xi \leqq \lg r_2, \vartheta_1 \leqq \eta \leqq \vartheta_2] \tag{8.19}$$

be the domain obtained from $\overline{\mathfrak{G}}_1$ by the logarithmic transformation

$$\zeta = \lg z, \qquad \zeta = \xi + i\eta, \qquad z = x + iy = re^{i\theta}, \tag{8.20}$$

and let $\overline{\mathfrak{G}}_2^+$ be the domain obtained by the logarithmic transformation

$$Z^* = \lg q, \qquad q = qe^{-i\theta}, \qquad Z^* = \Xi + i\mathrm{H} \tag{8.21}$$

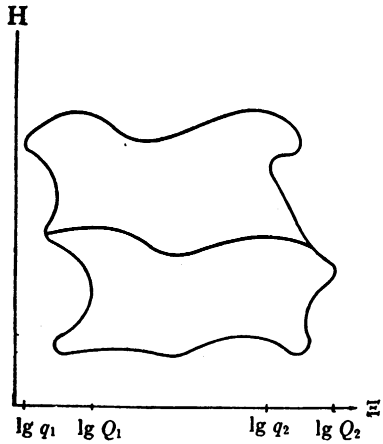

FIG. 10. The domain $\overline{\mathfrak{G}}_2^+$.

from the hodograph $\overline{\mathfrak{G}}_2$ of $\mathcal{J}$, that is, of the image of $\overline{\mathfrak{G}}_1$ in the hodograph plane. Since $\overline{\mathfrak{G}}_1$ does not include any stagnation point and $\overline{\mathfrak{G}}_2$ is assumed to be schlicht, $\overline{\mathfrak{G}}_2^+$ is bounded and schlicht.

According to Theorem 8.1 and previous remarks on combination of quasi-conformal transformations, the mapping

$$Z^* = g(\zeta) \tag{8.22}$$

which maps $\overline{\mathfrak{G}}_1^+$ onto $\overline{\mathfrak{G}}_2^+$ is quasi-conformal. Let $L(\eta_0)$ be the length of the image in the Z^*-plane of the line segment $\mathrm{E}[\lg r_1 \leqq \xi \leqq \lg r_2, \eta = \eta_0]$. Since the end points of the above segments lie on the images (in the ζ-plane) of arcs $\mathfrak{a}_1$ and $\mathfrak{a}_2$,

$$(\lg q_2 - \lg Q_1) \leqq L(\eta). \tag{8.23}$$

Integrating the left- and right-hand sides of (8.23) from $\eta = \vartheta_1$ to $\eta = \vartheta_2$ and applying the inequalities of Schwarz-Bouniakowsky, we obtain

$$(8.24)\qquad \begin{aligned}[(\lg q_2 - \lg Q_1)(\vartheta_2 - \vartheta_1)]^2 &\leqq \left[\int_{\vartheta_1}^{\vartheta_2} L(\eta)d\eta\right]^2 \\ &= \left[\int_{\vartheta_1}^{\vartheta_2}\int_{\lg r_1}^{\lg r_2} G_{11}^{1/2} d\xi d\eta\right]^2\end{aligned}$$

where G_{11} is the first coefficient of the quadratic form for the line element

$$(8.25)\qquad d\sigma^2 \equiv d\Xi^2 + dH^2 = G_{11}d\xi^2 + 2G_{12}d\xi d\eta + G_{22}d\eta^2.$$

Using the notation explained in (8.3), the inequality of Schwarz-Bouniakowsky, and (8.2), we obtain

$$(8.26)\qquad \begin{aligned}\left[\int_{\vartheta_1}^{\vartheta_2}\int_{\lg r_1}^{\lg r_2} G_{11}^{1/2} d\xi d\eta\right]^2 &\leqq \left[\int_{\vartheta_1}^{\vartheta_2}\int_{\lg r_1}^{\lg r_2} S^{1/2} d\xi d\eta\right]^2 \\ &\leqq \int_{\vartheta_1}^{\vartheta_2}\int_{\lg r_1}^{\lg r_2} S d\xi d\eta \cdot \int_{\vartheta_1}^{\vartheta_2}\int_{\lg r_1}^{\lg r_2} d\xi d\eta \\ &\leqq 2K\int_{\vartheta_1}^{\vartheta_2}\int_{\lg r_1}^{\lg r_2} D^{1/2} d\xi d\eta \cdot \int_{\vartheta_1}^{\vartheta_2}\int_{\lg r_1}^{\lg r_2} d\xi d\eta \\ &= 2K\mathrm{A}(\mathfrak{G}_1)\cdot\mathrm{A}(\mathfrak{G}_2) \\ &\leqq 2K(\lg r_2 - \lg r_1)(\vartheta_2 - \vartheta_1)(\lg Q_2 - \lg q_1)\theta_1\end{aligned}$$

where $\mathrm{A}(\mathfrak{G}_k)$ denotes the area of $\mathfrak{G}_k$. (8.24) and (8.26) imply (8.18).

9. **Pseudo-harmonic vectors**(23). As has been indicated in [6] and in papers cited there, it is possible by means of suitably chosen operators to generate from analytic functions of a complex variable (that is, from a pair of real functions connected by the Cauchy-Riemann equations) harmonic vectors, $\boldsymbol{H}$, whose components are connected by relations

$$(9.1)\qquad \nabla\cdot\boldsymbol{H} = 0, \qquad \nabla\times\boldsymbol{H} = 0.$$

These vectors possess the property that

$$(9.2)\qquad \int_{\mathfrak{s}} \boldsymbol{H}\cdot d\boldsymbol{X} = 0, \qquad \iint_{\mathfrak{B}} \boldsymbol{H}\cdot do = 0$$

where

$$\mathfrak{s} = \mathrm{E}[X = X(u), Y = Y(u), Z = Z(u), 0 \leqq u \leqq 1],$$
$$\mathfrak{B} = \mathrm{E}[X = X(u, v), Y = Y(u, v), Z = Z(u, v), 0 \leqq u \leqq 1, 0 \leqq v \leqq 1]$$

represent any closed sufficiently smooth curve and surface, respectively, which can be reduced to a point by a continuous one-to-one transformation

(23) Bull. Amer. Math. Soc. Abstract 51-9-155.

in the regularity domain of $\boldsymbol{H}$. Here

$$d\boldsymbol{X} = \left[\frac{\partial X}{\partial u}\boldsymbol{i}_1 + \frac{\partial Y}{\partial u}\boldsymbol{i}_2 + \frac{\partial Z}{\partial u}\boldsymbol{i}_3\right]du,$$

$$d\boldsymbol{o} = \left[\frac{\partial(Y,Z)}{\partial(u,v)}\boldsymbol{i}_1 + \frac{\partial(Z,X)}{\partial(u,v)}\boldsymbol{i}_2 + \frac{\partial(Y,Z)}{\partial(u,v)}\boldsymbol{i}_3\right]dudv.$$

As we shall show, the same operator generates from a pair of functions which are connected by the generalized Cauchy-Riemann equations (2.8) vectors $\boldsymbol{S}$ which possess similar properties.

LEMMA 9.1. *Let $f=f(x, iy, t)=\phi(x, y, t)+i\psi(x, y, t)$ be an analytic function of two real variables x and y, and a continuous function of x, y and t (t real), whose components are connected (identically in t) by the relations*

$$\phi_x = \psi_y, \qquad \psi_x = -l(x)\phi_y. \tag{2.8}$$

Now let

$$\begin{aligned} \boldsymbol{S}(X, Y, Z) &= \boldsymbol{S}^{(1)}(X, Y, Z) + i\boldsymbol{S}^{(2)}(X, Y, Z) \\ &= \mathrm{P}(f)\boldsymbol{i}_1 + \mathrm{P}(if\cos t)\boldsymbol{i}_2 + \mathrm{P}(if\sin t)\boldsymbol{i}_3, \end{aligned} \tag{9.3}$$

where

$$\mathrm{P}(g) \equiv \frac{1}{2\pi}\int_{t=0}^{2\pi} g(X, iY\cos t + iZ\sin t, t)dt \tag{9.4}$$

represents a pair of (real) vectors in the three-dimensional (XYZ)-space.

Then the components $S^{(n,k)}$, $n=1, 2, 3$, of the vectors $\boldsymbol{S}^{(k)}$, $k=1, 2$, are connected by the relations

$$S_X^{(2,1)} = l(X)S_Y^{(1,1)}, \quad S_X^{(3,1)} = l(X)S_Z^{(1,1)}, \quad S_Y^{(3,1)} = S_Z^{(2,1)}, \quad \nabla\cdot\boldsymbol{S}^{(1)} = 0, \tag{9.5}$$

$$\nabla\times\boldsymbol{S}^{(2)} = 0, \quad S_X^{(1,2)} + l(X)[S_Y^{(2,2)} + S_Z^{(3,2)}] = 0. \tag{9.6}$$

Proof. We have

$$\begin{aligned} &S^{(1,1)} = (2\pi)^{-1}\int\phi dt, \qquad S^{(2,1)} = -(2\pi)^{-1}\int\psi\cos t\,dt, \\ &S^{(3,1)} = -(2\pi)^{-1}\int\psi\sin t\,dt, \end{aligned} \tag{9.7}$$

$$\begin{aligned} &S^{(1,2)} = (2\pi)^{-1}\int\psi dt, \qquad S^{(2,2)} = (2\pi)^{-1}\int\phi\cos t\,dt, \\ &S^{(3,2)} = (2\pi)^{-1}\int\phi\sin t\,dt, \qquad \int \equiv \int_{t=0}^{2\pi}, \end{aligned} \tag{9.8}$$

and therefore if we write

$$
(9.9) \qquad \begin{aligned} S^{*(1)} &= l(X)S^{(1,1)}i_1 + S^{(2,1)}i_2 + S^{(3,1)}i_3, \\ S^{*(2)} &= S^{(1,2)}i_1 + l(X)S^{(2,2)}i_2 + l(X)S^{(3,2)}i_3 \end{aligned}
$$

then

$$
(9.10) \qquad \begin{aligned} \nabla \times S^{*(1)} = -(2\pi)^{-1}\int \{ [\psi_x - l(x)\phi_y] \cos t i_1 + [\psi_x - l(x)\phi_y] \sin t i_2 \\ + [\psi_y - \psi_y] \sin t \cos t \, dt i_3 \} dt = 0, \end{aligned}
$$

$$
(9.11) \qquad \nabla \cdot S^{(1)} = (2\pi)^{-1}\int [\phi_x - \psi_y \cos^2 t - \psi_y \sin^2 t] dt = 0,
$$

$$
(9.12) \qquad \begin{aligned} \nabla \times S^{(2)} = (2\pi)^{-1}\int \{ [\phi_x - \psi_y] \cos t i_1 + [\phi_x - \psi_y] \sin t i_2 \\ - [\phi_y - \phi_y] \sin t \cos t i_3 \} dt = 0, \end{aligned}
$$

$$
(9.13) \qquad \nabla \cdot S^{*(2)} = (2\pi)^{-1}\int [\psi_x + l(x)\phi_y \cos^2 t + l(x)\phi_y \sin^2 t] dt = 0,
$$

which yields (9.5) and (9.6).

COROLLARY 9.1. *It follows from* (9.7) *and* (9.8) *that*

$$
(9.14) \qquad \int_{\mathfrak{s}} S^{*(1)} \cdot dX = 0, \qquad \iint_{\mathfrak{B}} S^{(1)} \cdot do = 0,
$$

$$
(9.15) \qquad \int_{\mathfrak{s}} S^{(2)} \cdot dX = 0, \qquad \iint_{\mathfrak{B}} S^{*(2)} \cdot do = 0.
$$

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