MAPPING BY ANALYTIC FUNCTIONS. PART I. CONFORMAL MAPPING OF MULTIPLY-CONNECTED DOMAINS

BY

LEONARD GREENSTONE

INTRODUCTION

1. **Methods employed.** In the theory of analytic functions of one complex variable it is possible to obtain many results of far-reaching importance by means of relatively few general theorems, for example, Riemann's mapping theorem and the lemma of Schwarz.

The existence of a non-Euclidean metric which is invariant with respect to conformal mapping follows from Riemann's mapping theorem and is in many instances equivalent to it. In the case of a simply-connected domain B with more than one boundary point, the distance (in this sense) between two points a and b of B is defined as the hyperbolic distance between the images of a and b in the unit circle. By means of this metric it is possible to give to Schwarz's lemma a new and useful formulation, the lemma of Schwarz-Pick.

Completely avoiding Riemann's theorem, Bergman succeeded in showing that an invariant Hermitian metric can be derived from a quite different approach and can be generalized to the case of several complex variables. This is important, since Poincaré has shown that it is not in general possible to map one domain in $2n$-dimensional space on another by means of n analytic functions of n complex variables (pseudo-conformal mapping), so that an immediate generalization of classical function theoretic methods is not possible. Further, in the case of mappings into *schlicht* domains, Bergman has obtained results which include the lemma of Schwarz-Pick but which can also be applied to the case of several complex variables.

However, if those methods originally employed to provide results in pseudo-conformal mapping are specialized to the theory of analytic functions of one complex variable, they do more than merely provide known results: they constitute methods for the investigation of many questions in conformal mapping of multiply-connected domains. Further, theorems that we may obtain in the case of one variable by these methods suggest analogous theorems for pseudo-conformal mapping. It is this dual role which makes both the theorems obtained and the methods employed take on added importance (1).

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(1) (Added in proof.) The general character of the present approach has been clarified still more by the recently established fact (see Duke Math. J. vol. 14 (1947) pp. 609–638, in particular (66′); see also Schiffer [1]) that there exists quite a simple connection between the kernel functions on one side and Green's and Neumann's functions on the other side so that the present method can be considered as a direct generalization to the case of functions of several complex variables of classical procedures used in the case of one variable.

2. **Results obtained.** In §I we prove that if $\{B_n\}$ is a sequence of bounded domains, subject to certain restrictions, which converges in the sense of Carathéodory to a bounded domain B, then the curvature of the invariant metric of B_n converges uniformly to the curvature of the metric of B; in §II, bounds are obtained for the distortion, under conformal mapping by rather general classes of functions, of the Euclidean length of an arc; in §III we obtain a mean value theorem for the curvature; and in §IV we prove a generalized Poisson integral theorem for multiply-connected domains.

In Part II generalization of the results obtained in §II to pseudo-conformal mapping is discussed as an illustration of the dual nature of these theorems mentioned above.

3. **Previously obtained results.** In the following we shall frequently employ the notation

$$(0.01)\qquad \frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right),\qquad \frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right),$$
$$\frac{\partial^2}{\partial z\partial\bar z}=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=\frac{\Delta}{4},$$

where $z=x+iy$, $\bar z=x-iy$, x, y real; further, we employ the abbreviation

$$(0.02)\qquad f_{10}=f_z=\frac{\partial f}{\partial z},\qquad f_{01}=f_{\bar z}=\frac{\partial f}{\partial \bar z},\qquad \text{and in general,}$$
$$f_{ij}=\frac{\partial^{i+j}f(z,\bar z)}{\partial z^i\partial\bar z^j}\cdot$$

If B is a domain in the z-plane whose closure we denote by B^+ and whose area shall be assumed to be finite unless the contrary be specified, then $f\in\mathcal{L}^2(B)$ shall be taken to mean that:

$$\int_B |f|^2 d\omega<\infty$$

where the integral is taken in the Lebesgue sense;

(2) f is an analytic function which is regular in B.

Note that to avoid possible ambiguity $\int_B\cdot d\omega$ will occasionally be written $\int_B\cdot d\omega_z$; the subscript z has been added to indicate the plane of integration.

Let $\{\phi_n\}$ be a sequence of functions orthonormal with respect to the domain B; that is,

$$(0.03)\qquad \int_B \phi_m\bar\phi_n d\omega=\delta_{mn}\equiv\text{the Kronecker delta.}$$

We shall indicate this property by writing $\{\phi_n\}_B$, that is, by adding the subscript B to the brace.

The "kernel function" associated with the domain B is defined as

$$K_B(z, \bar{t}) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(t)}, \tag{0.04}$$

where $\{\phi_n\}_B$ is closed with respect to $\mathcal{L}^2(B)$ (see §IV). The kernel function is defined for every domain B and is an analytic function of z and $\bar{t}$ for $z\in B$, $t\in B$. It further possesses the property that if $z=z(z^*)$ is a conformal mapping of B into B^*,

$$K_{B^*}(z^*, \bar{z}^*) \mid dz^* \mid^2 = K_B(z, \bar{z}) \mid dz \mid^2. \tag{0.05}$$

The kernel function depends only on the domain with respect to which it is defined and is independent of the particular choice of sequence $\{\phi_n\}_B$ employed in definition (0.04) provided that $\{\phi_n\}_B$ is closed with respect to $\mathcal{L}^2(B)$; however compare (2.11) with Bergman [1; equation (6.3), p. 43](2).

For the circle $|z| < r$, such a sequence of orthonormal functions is

$$\left\{\left(\frac{n}{\pi}\right)^{1/2} \frac{z^{n-1}}{r^n}\right\}, \qquad n = 1, 2, \cdots, \tag{0.06}$$

and the kernel function is given by

$$K(z, \bar{t}) = \frac{r^2}{\pi(r^2 - z\bar{t})^2}. \tag{0.07}$$

A non-Euclidean metric for B may be introduced by setting

$$ds_B^2 = K_B(z, \bar{z}) \mid dz \mid^2, \tag{0.08}$$

and consequently by defining the length of a rectifiable Jordan curve $z=z(t)$, $t_0 \leqq t \leqq t_1$, situated in the domain B, by

$$\int_{t_0}^{t_1} ds_B(z(t)) = \int_{t_0}^{t_1} (K_B(z(t), \overline{z(t)}))^{1/2} \mid dz(t) \mid; \tag{0.09}$$

this length will be invariant under conformal transformation and is defined for any multiply-connected domain, as is the "Gaussian" curvature, $2\mathfrak{J}_B$, given by

$$\begin{aligned} 2\mathfrak{J}_B(z, \bar{z}) &= -\frac{2}{K_B(z, \bar{z})} \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \log K_B(z, \bar{z}) \\ &= -\frac{2}{K_{00}^3} \begin{vmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{vmatrix}, \qquad K = K_B. \end{aligned} \tag{0.10}$$

The results stated here are only those which will be employed in the body

(2) The numbers in square brackets preceding the semicolon refer to the bibliography, q.v.

of the paper. For a complete exposition, we refer to Bergman [1, 2], especially Chapters VI–IX of the former; in both of these books there are extensive bibliographies as well as detailed references to original papers.

I. A CONVERGENCE THEOREM FOR THE CURVATURE

1. **Definitions and previous results.** Given a domain(3) B, let

$$\overset{1}{\lambda}_B(t) = \min \|f\|_B^2 < \infty, \qquad f \in \mathcal{L}^2(B), \tag{1.01}$$

where $\|f\|_B^2 = \int_B |f|^2 d\omega$ and f has been normalized so that for one fixed $t \in B$, $f(t) = 1$.

The validity of this definition follows from the fact that, as Bergman [1; p. 60] has shown, there exists a unique minimizing function f_0, $f_0(z) = 1 + a_1(z-t) + \cdots$, possessing the property that

$$\overset{1}{\lambda}_B(t) = \|f_0\|_B^2 \leqq \|f\|_B^2, \tag{1.02}$$

for all $f \in \mathcal{L}^2(B)$ which possess the development $f(z) = 1 + A_1(z-t) + \cdots$.

Further, define

$$\overset{0X_1\cdots X_n}{\lambda_B}(t) = \min \|f\|_B^2, \qquad f \in \mathcal{L}^2(B), \tag{1.03}$$

where f has been normalized so that for one fixed $t \in B$, $f(t) = 0$, and

$$\left.\frac{d^j f(z)}{dz^j}\right|_{z=t} = X_j, \qquad j = 1, 2, \cdots, n,$$

X_j real or complex numbers.

In fact, there exists a unique minimizing function f_0, $f_0(z) = X_1(z-t) + X_2(z-t)^2 + \cdots + X_n(z-t)^n + b_1(z-t)^{n+1} + b_2(z-t)^{n+2} + \cdots$ possessing the property that

$$\overset{0X_1\cdots X_n}{\lambda_B} = \|f_0\|_B^2 \leqq \|f\|_B^2$$

for all $f \in \mathcal{L}^2(B)$ which possess the development

$$f(z) = X_1(z-t) + X_2(z-t)^2 + \cdots + X_n(z-t)^n + B_1(z-t)^{n+1} + B_2(z-t)^{n+2} + \cdots.$$

(See Bergman [1; p. 65].)

The minima defined in (1.01), (1.02) possess the property that if G is a domain such that $G \subset B$, then

$$\overset{1}{\lambda}_G(t) \leqq \overset{1}{\lambda}_B(t), \qquad \overset{0X_1\cdots X_n}{\lambda_G}(t) \leqq \overset{0X_1\cdots X_n}{\lambda_B}(t), \qquad t \in G. \tag{1.04}$$

The inequalities of (1.04) are the generalization of the lemma of Schwarz-

(3) Unless the contrary be specified, all domains are supposed *schlicht*.

Pick obtained by Bergman cited in the introduction. See Bergman [1; chap. IX].

It may be shown that

$$\overset{1}{\lambda}_B(t) = \frac{1}{K_B(t, \bar{t})}, \tag{1.05}$$

which proves the assertion made earlier that for closed orthonormal systems $\{\phi_n\}_B$ the kernel function depends only on the domain and not on the choice of the particular sequence $\{\phi_n\}_B$.

The other minima $\lambda_B^{0X_1\cdots X_n}$(4) may be obtained similarly in terms of the kernel function and its derivatives; we shall write them down later as they are needed, referring the reader to Bergman [1] for the general formula. However, we note that

$$\overset{01}{\lambda}_B = \frac{K_{00}}{\begin{vmatrix} K_{00} & K_{10} \\ K_{01} & K_{11} \end{vmatrix}}, \qquad K = K_B, \tag{1.06}$$

(see (0.02)).

The Gaussian curvature $2\mathfrak{J}_B$, given by (0.10), may then be written in the form

$$2\mathfrak{J}_B = \frac{-2(\overset{1}{\lambda}_B)^2}{\overset{01}{\lambda}_B}. \tag{1.07}$$

As $\overset{1}{\lambda}_B$, $\overset{01}{\lambda}_B$ are always positive, it follows that $\mathfrak{J}_B$ is always negative, and therefore, as may be seen from the definition of $\mathfrak{J}_B$ or from a result of Beckenbach and Radó [1], K_B is a function whose logarithm is subharmonic in B, so that K_B is also subharmonic in B.

2. **Statement of the theorem.** The main result of this section is:

THEOREM 1. *Given a sequence of domains $\{B_n\}_{n=1}^{\infty}, B_\infty = B$, such that* (1) *$B_n \subset B$, for all n, B bounded,* (2) *B_n converges in the sense of Carathéodory to the limit domain B*(5), *and* (3) *every B_n satisfies the condition*(6) (1.09), *then in every closed subdomain of B the curvature $2\mathfrak{J}_{B_n}$ converges uniformly to $2\mathfrak{J}_B$.*

Let f_n, $n = 1, 2, \cdots$, be the unique function defined in (1.01) with respect

(4) When there is no danger of ambiguity, we shall omit the argument of the functions $\overset{1}{\lambda}_B, \overset{01}{\lambda}_B, \cdots, K_B$, and so on.

(5) Given a sequence of domains $\{G_n\}$, all lying in the same plane and all possessing at least the point a in common, then the sequence $\{G_n\}$ is said to converge in the sense of Carathéodory to G if G is the greatest domain such that every closed subset of G_n, $n \geqq N$, lies in G. If there does not exist some neighborhood of a common to all G_n, $n \geqq N$, N sufficiently large, then G is just the point a.

(6) It is necessary to assume this only for some subsequence $\{B_{n_i}\}$; however, for the sake of simplicity we shall retain the more restrictive assumption.

to the domain B_n; f_0 is the minimum function with respect to B. It is supposed that f_n, $n=0, 1, 2, \cdots$, have all been normalized with respect to the same point t, $t \in B$.

As we wish to deal with a set of increasing domains, instead of employing $\{B_n\}$, we shall use the sequence $\{B_n^*\}_{n=1}^{\infty}$, where

$$B_m^* = B \cdot \left(\sum_{n=1}^{m} B_n \right), \qquad m = 1, 2, \cdots. \tag{1.08}$$

We suppose that the sets $\{B_n\}$ are so restricted that:
Given an integer p, then for all n sufficiently great

$$B_p^* \subset B_n. \tag{1.09}$$

Note the following properties of $\{B_n^*\}$:

$$B_m^* \subset B_{m+1}^* \subset B; \tag{1.10}$$

(1.11) $\{B_n^*\}$ possesses the same limit domain B as does $\{B_n\}$.

It follows from (1.07) that it will be sufficient to prove that $\lambda_{B_n}^1$, $\lambda_{B_n}^{01}$ converge uniformly to λ_B^1, λ_B^{01}, respectively, for $\{\lambda_{B_n}^{01}(t)\}$ cannot be a null sequence unless t is a boundary point of B.

3. **Convergence of λ^1.** In the remainder of the proof we shall suppose that, given B_m^*, n is any integer sufficiently large so that $B_m^* \subset B_j$, $j=n, n+1, \cdots$ (see (1.09)).

LEMMA 1

$$\|f_n\|_{B_m^*}^2 \leqq \|f_n\|_{B_n}^2 \leqq \|f_0\|_{B_n}^2 \leqq \|f_0\|_{B}^2. \tag{1.12}$$

The first inequality is immediate, since, by hypothesis, $B_m^* \subset B_n$ and therefore

$$\int_{B_m^*} |f_n|^2 d\omega \leqq \int_{B_n} |f_n|^2 d\omega. \tag{1.13}$$

The second inequality follows from the fact that f_n is the unique function among all $f \in \mathcal{L}^2(B_n)$, $f(t)=1$, $t \in B_n$, which minimizes $\|f\|_{B_n}^2$; but since $B_n \subset B$, $f_0 \in \mathcal{L}^2(B_n)$, therefore $\|f_n\|_{B_n}^2 \leqq \|f_0\|_{B_n}^2$.

The third inequality holds by the argument that validated the first.

LEMMA 2. *$\{f_p\}_{p=n}^{\infty}$ forms a normal family in the sense of Montel in B_m^*. That is, in every partial set $\{f_{p_\nu}\}_{\nu=1}^{\infty}$ there exists a subsequence which converges uniformly in every closed subdomain of B_m^*.*

(We note from Lemma 1 that $\{\|f_p\|_{B_m^*}^2\}_{p=n}^{\infty}$ is uniformly bounded in B_m^*.) A proof of this lemma is given in Bergman [3].

LEMMA 3. *$\{f_n\}_{n=1}^{\infty}$ forms a normal family in B.*

Let $\{f_{p_\nu}\}_{\nu=1}^{\infty}$ be a subsequence of $\{f_n\}_{n=1}^{\infty}$. Since $\{f_p\}_{p=n}^{\infty}$ forms a normal family in B_m^*, we can select a subsequence of $\{f_{p_\nu}\}_{\nu=1}^{\infty}$ which converges uniformly in every closed subdomain of B_m^*. Call this subsequence for B_m^*, $\{f_{m,i}\}_{i=0}^{\infty}$. Delete from $\{f_{m,i}\}_{i=0}^{\infty}$ all f_p which are not regular in B_{m+1}^*. (Since $B_{m+1}^* \subset B_r$ for all r sufficiently great (see (1.09)), and since, from (1.10), $B_m^* \subset B_{m+1}^*$, it follows that there are infinitely many f_p regular in B_{m+1}^*.) From this "deleted" subsequence of $\{f_{m,i}\}_{i=0}^{\infty}$ we can then pick a subsequence which converges uniformly in every closed subdomain of B_{m+1}^*. This procedure is then repeated *ad. inf.*

Construct the diagonal sequence $\{f_{n+p,p}\}_{n=1,\,p=0}^{n=\infty,p=\infty}$, which will then converge uniformly in every closed subdomain of B.

LEMMA 4. *In every closed subdomain of B, every uniformly convergent subsequence of $\{f_n\}$ converges to f_0.*

Let f_p, $p \geqq m$, be a member of some subsequence described in the hypothesis, and let f^* be the limit of this subsequence.

By (1.12):

$$\|f_p\|^2_{B_m^*} \leqq \|f_0\|^2_B.$$

Let $p \to \infty$; then

$$\|f^*\|^2_{B_m^*} \leqq \|f_0\|^2_B.$$

Now let $m \to \infty$; hence

$$\|f^*\|^2_B \leqq \|f_0\|^2_B, \tag{1.14}$$

since the measure of $B - B_m^*$ converges to zero and therefore $\|f^*\|^2 B_m^*$ converges to $\|f^*\|^2_B$; for, by a classical result of Lebesgue theory, if f is an L-integrable function and $\{E_m\}$ a sequence of sets whose measure tends to zero, then the integral of f over E_m tends to zero.

But f_0 minimizes $\|f\|^2_B$, for all $f \in \mathfrak{L}^2(B)$, normalized so that for some $t \in B$, $f(t) = 1$; clearly f^* possesses all these properties, so that

$$\|f_0\|^2_B \leqq \|f^*\|^2_B. \tag{1.15}$$

Comparing (1.14) and (1.15), we see that

$$\|f_0\|^2_B = \|f^*\|^2_B;$$

however, we know that there exists a unique function f_0 minimizing $\|f\|^2_B$ so that $f_0 = f^*$.

LEMMA 5. *f_n converges uniformly to f_0 in every closed subdomain of B.*

Every subsequence of $\{f_n\}$ will contain a uniformly convergent subsequence, and thus $\{f_n\}$ will be uniformly convergent, since, by Lemma 4, the limit will be the same for all uniformly convergent subsequences.

LEMMA 6. *$\lambda^1_{B_n}$ converges uniformly to λ^1_B in every closed subdomain of B.*

By (1.03) and the definition of f_0, f_n:

$$\lambda^1_B - \lambda^1_{B_n} = \|f_0\|^2_B - \|f_n\|^2_{B_n}. \tag{1.16}$$

Suppose $B_m^* \subset B_n$; then by the first inequality of (1.12)

$$\begin{aligned} \lambda^1_B - \lambda^1_{B_n} &\leqq \|f_0\|^2_B - \|f_n\|^2_{B_m^*} \\ &\leqq \|f_0\|^2_{B-B_m^*} + (\|f_0\|^2_{B_m^*} - \|f_n\|^2_{B_m^*}) \\ &\leqq \|f_0\|^2_{B-B_m^*} + \left\{\int_{B_m^*} |f_0^2 - f_n^2| \, d\omega\right\}. \end{aligned} \tag{1.17}$$

Applying Schwarz' inequality to the term in braces in (1.17), and using the fact that $B_m^* \subset B_n$, we have

$$\lambda^1_B - \lambda^1_{B_n} \leqq \|f_0\|^2_{B-B_m^*} + \{\|f_0 + f_n\|_{B_n} \cdot \|f_0 - f_n\|_{B_m^*}\}. \tag{1.18}$$

The successive use of Minkowski's inequality, the second and third inequalities of (1.12), yields

$$\|f_0 + f_n\|_{B_n} \leqq \|f_0\|_{B_n} + \|f_n\|_{B_n} \leqq 2\|f_0\|_{B_n} \leqq 2\|f_0\|_B = 2C, \quad C^2 = \|f_0\|^2_B. \tag{1.19}$$

Substituting (1.19) into (1.18), we have

$$\lambda^1_B - \lambda^1_{B_n} \leqq \|f_0\|^2_{B-B_m^*} + 2C\|f_0 - f_n\|_{B_m^*}. \tag{1.20}$$

We recall that the measure of $B - B_m^*$ converges to zero as m tends to infinity and therefore, as in Lemma 4, we may choose m so large that $\|f_0\|^2_{B-B_m^*} \leqq \epsilon/2$. Clearly, by Lemma 5, it is now possible to choose n so large that $n \geqq m$, $\|f_0 - f_n\|_{B_m^*} \leqq \epsilon/4C$; hence, by (1.20), $\lambda^1_B - \lambda^1_{B_n} \leqq \epsilon$.

However, from (1.04) and the hypothesis of Theorem 1, we know that $\lambda^1_B - \lambda^1_{B_n^*}$ is positive; Lemma 6 follows.

4. Convergence of the curvature.

LEMMA 7. *$\lambda^{0X_1\cdots X_m}_{B_n}$ converges uniformly to $\lambda^{0X_1\cdots X_m}_B$ in every closed subdomain of B.*

The proof is exactly the same as for Lemma 6, once the necessary changes in the definition of f_0, f_n, $n = 1, 2, \cdots$; , have been made.

We then construct the sequence $\{-(\lambda^1_{B_n})^2/\lambda^{01}_{B_n}\}$; from Lemmas 6 and 7 and the remarks following the statement of the theorem, it follows that this sequence will converge uniformly to $-(\lambda_B^1)^2/\lambda_B^{01}=\mathfrak{J}_B$, and hence the theorem.

5. **Generalization to the case of two complex variables.** As we stressed in the introduction, the main interest of our proof consists in its generality: indeed we make nowhere the assumption that the f are functions of *one* complex variable. Therefore, Theorem 1 can be immediately generalized to the case of two complex variables and will yield there similar results concerning any quantity which can be represented as a combination of finitely many $\lambda^{\cdots}$. In particular, in the case of two complex variables there appear two invariants, $\boldsymbol{R}_B$ and $\boldsymbol{J}_B$, which are connected in a simple manner with the curvature tensor of a metric invariant with respect to pseudo-conformal mappings. In analogy to (1.07), $\boldsymbol{J}_B$ can be represented in the form $\lambda_B^{01}\lambda_B^{001}/(\lambda_B^1)^3$ where $\lambda_B^{\cdot\cdot}$ are minimum values of $\int_B|f|^2d\omega$ under suitably chosen conditions; a similar representation holds for $\boldsymbol{R}_B$. (See, for details, Bergman [2; p. 55].) Therefore, Theorem 1 holds if B, B_n mean domains of four-dimensional space, and $\mathfrak{J}_{B_n}$, $\mathfrak{J}_B$ are replaced by $\boldsymbol{J}_{B_n}$, $\boldsymbol{R}_{B_n}$ and $\boldsymbol{J}_B$, $\boldsymbol{R}_B$, respectively.

6. **An application of the results of paragraph 3.** As an application of Lemma 6 it would, for example, be possible to prove the following well known theorem (see Bieberbach [1; vol. 2, p. 12]) by making use of (0.05) and (1.05).

THEOREM 2. *Consider a sequence $\{A_n\}$ of schlicht domains*(7) *in the z-plane, all of which contain $z=0$; let $w=f_n(z)$, where $f_n(0)=0$ and $f_n'(0)=1$, map A_n into a schlicht domain*(7) *B_n in the w-plane. If A_n converges to a limit domain A in the sense of Carathéodory, then the necessary and sufficient condition that B_n converge (Carathéodory) to a limit domain B is that f_n converge uniformly to f in every closed subdomain of A. f is the schlicht mapping of A on B.*

II. DISTORTION THEOREM

1. **Introduction.** In this section we establish the following inequalities for the distortion of the Euclidean length of an arc subject to conformal transformation.

Let the arc a lie entirely in some finitely connected domain G, G fixed, and let G be mapped conformally on a domain B so that a is mapped into $p\subset B$. Suppose that the outer boundary b_1 of B is sufficiently smooth so that a circle C^+ of radius τ, exterior to B, can roll freely along b_1 for some segment $b^*\subset b_1$ (see condition (ii) below), and that no point of p is further from b^* than η (see condition (i) below); then there exists a constant c independent of B such that if $L(p)$ is the Euclidean length of p, $L(p)\leqq 2\pi^{1/2}\eta c(1+\eta/2\tau)$; in particular, if the tangent at each point $q\in b^*$ is a line of support for B and intersects b^* in no point other than q (unless $q\in b^{**}\subset b^*$ and b^{**} is a segment of a straight line, in which case the tangent's only intersection with b^* is b^{**}), then $L(p)\leqq 2\pi^{1/2}\eta c$.

(7) Subject to the hypotheses of Theorem 1.

If, instead of the first of these conditions, we merely assume that for at least one point of b^* (see condition (iii) below) it is possible to place a circle C^+ of radius τ tangent to b_1 which lies outside B, then, if h is the diameter of p, $L(p) \leqq 2\pi^{1/2}c(\eta+h+((\eta+h)^2+h^2)/2\tau)$.

2. **Inequalities for the case of a multiply-connected domain.** With the notation of the introduction in force, we suppose that the arc a is rectifiable, and that the non-Euclidean length of a with respect to G, which is invariant under conformal transformation, equals

$$c = S_G(a) = \int_a (K_G(z, \bar{z}))^{1/2} | dz | . \tag{2.01}$$

(See (0.09).)

Map G into some domain B of the same connectivity and let p be the image in B of a, b is the boundary of B and b_1 the outer boundary of B. This mapping will then be supposed to satisfy the following conditions:

(i) *The distance of any point of p from b^* (see below) is less than η.*

(ii) *At every point q of some connected subset b^* of b_1 (to be described presently), it is possible to construct a circle C^+ of radius of at least τ, tangent at q to b_1 and situated entirely outside B^+-q; that is, there is no intersection of C^+ with b_1 other than the point of tangency q.*

Choose the coordinate frame so that the y-axis is the line of tangency of C^+ with b_1 and the x-axis is the extension of the radius perpendicular to this line. The positive directions of the x- and y-axes are chosen along the direction of the inner normal at q and that part of the tangent line which forms an angle of $+\pi/2$ with it, respectively. *We suppose further that the x-axis intersects p, say, in the point z_1; and also that the part of the axis lying between z_1 and q lies entirely within B.*

Let z_1 range over p; then to each z_1 there will correspond some point q of b_1, described above, although, perhaps, several points of p may correspond to the same $q \in b_1$. Call one of the connected sets of all such q, b^*.

Combining relations (1.04) and (1.05) and calling the complement of C^+ with respect to the whole plane T, we note that as $B \subset T$,

$$K_B(z_1, \bar{z}_1) \geqq K_T(z_1, \bar{z}_1), \qquad z_1 \in B; \tag{2.02}$$

while from (0.05) and (0.06) it follows that

$$K_T = \frac{1}{\pi} \frac{1}{(z + \bar{z} + z\bar{z}/\tau)^2} . \tag{2.03}$$

Now, from (0.04) we know that the non-Euclidean length, (0.09), is invariant with respect to conformal transformation. Hence if we denote the non-Euclidean length of p with respect to B by $S_B(p)$, we have

$$S_{C^{(n)}}(a) = S_B(p) = c. \tag{2.04}$$

From (2.02) and (2.03),

$$
\begin{aligned}
S_B(p) &= \int_p (K_B(z, \bar{z}))^{1/2} | dz | \geqq \int_p (K_T(z, \bar{z}))^{1/2} | dz | \\
&= \frac{1}{\pi^{1/2}} \int_p \frac{| dz |}{(z + \bar{z} + z\bar{z}/\tau)} .
\end{aligned} \tag{2.05}
$$

Now, for each point $z_1 \in p$, we choose T, so that z_1 always lies on the corresponding x-axis. Consequently we may then employ condition (i) to obtain

$$
S_B(p) \geqq \left(\int_p | dz | \right) \frac{1}{\pi^{1/2}(2\eta + \eta^2/\tau)} . \tag{2.06}
$$

Denoting by $L(p)$ the Euclidean length of p and employing (2.04), we then obtain the following bound for $L(p)$:

$$
L(p) \leqq 2\pi^{1/2}\eta c \left(1 + \frac{\eta}{2\tau} \right), \tag{2.07}
$$

where c is a constant independent of B, defined by (2.01).

Now, let $\tau \rightarrow \infty$; that is, suppose that there exists a sufficiently large segment of b_1, b^{**}, which is a smooth convex curve containing b^*; then the following inequality holds for $L(p)$:

$$
L(p) \leqq 2\pi^{1/2}\eta c. \tag{2.08}
$$

However, suppose, instead of condition (ii), we assume:

(iii) *At one point of b^* it is possible to place a circle C^+ with the properties described in* (i) (*all other requirements and conventions of* (i) *holding*).

Then, calling the diameter of p, h, we have from (2.05)

$$
S_B(p) \geqq \frac{L(p)}{\pi^{1/2}(2(\eta + h) + (h^2 + (\eta + h)^2)/\tau)},
$$

or

$$
L(p) \leqq 2\pi^{1/2}c(\eta + h + ((\eta + h)^2 + h^2)/2\tau). \tag{2.09}
$$

Summing up, we obtain the following theorem.

THEOREM 3. *Given the n-tuply-connected domain G, and an arc $a \subset G$; let G be mapped conformally on a domain B and a be mapped into $p \subset B$; suppose that this mapping is such that p, B^+ satisfy conditions* (i), (ii) (*above*), *then if $L(p)$ is the Euclidean length of p,* (2.07) *holds where c is a constant independent of B, defined by* (2.01).

If only conditions (i), (iii) *hold on* p, B^+, *then, if* h *is the diameter of* p, (2.09) *holds.*

Again it is possible to formulate the above theorem in the case of two complex variables, considering, instead of the length of a curve, the B-area of a segment of a surface (see Bergman [1, pp. S III, 4 and 5]), and using instead of a circle the complement to the product of the exteriors of two suitably chosen circles or certain other auxiliary domains. This will be carried out in detail in Part II.

3. **Inequalities for the case of a ring.** The sequence

$$\left\{\left(\frac{n+1}{\pi}\right)^{1/2} \frac{z^n}{(1-r^{2(n+1)})^{1/2}}\right\}, \qquad n = \cdots ; -3, -2, 0, 1, 2, \cdots, \tag{2.10}$$

is a set of functions which are orthonormal with respect to the ring $C^{(2)} \equiv R(r)$.

Inserting this value for ϕ_n in formula (0.04), we obtain the kernel function for $R(r)$, $K_{R(r)}$, in closed form in terms of elliptic functions (see Zarankiewicz [1]):

$$\begin{aligned} K_{R(r)}(z, \bar{t}) &= \frac{1}{\pi z \bar{t}} \sum_{n=1}^{\infty} \frac{n r^n}{1 - r^{2n}} \left(\frac{z^n \bar{t}^n}{r^n} + \frac{r^n}{z^n \bar{t}^n} \right) \\ &= -\frac{1}{\pi z \bar{t}} \cdot \frac{\omega}{8\pi^2} \left[\frac{\eta_1}{\omega} + \wp(u + \omega'; \omega, \omega') \right], \end{aligned} \tag{2.11}$$

where

$$\exp\left(\frac{i\pi u}{\omega}\right) = \frac{z\bar{t}}{r}. \tag{2.12}$$

$\wp$ is the Weierstrass $\wp$-function with semi-periods ω, ω', and η_1 is the "period" of the elliptic integral of the second kind corresponding to ω.

If t is taken equal to z and ρ^2 set equal to $z\bar{z}$, then it is clear that $K_{R(r)}$ is a function of ρ alone and hence that it is constant along every circle $|z| = \rho_0$, $r < \rho_0 < 1$; thus if a is some arc of the circle $|z| = \rho_0$ with Euclidean length $2\pi\rho_0\gamma$, $0 < \gamma < 1$, then replacing G in (2.01) by $R(r)$,

$$c = S_{R(r)}(a) = 2\pi\rho_0\gamma(K_{R(r)}(\rho_0))^{1/2}. \tag{2.13}$$

If (2.11)–(2.13) are then substituted into (2.07)–(2.09), explicit inequalities are obtained.

III. A mean value theorem for the curvature

Hitherto the only inequality which had been obtained for the curvature $\mathfrak{Z}_B$ of the metric of a domain B was the following which was given by Bergman [1; p. 83]. Employing the formula for $\mathfrak{Z}_B$ given in (1.07) and making use of (1.04), we see that if I and A are domains such that $I \subset B \subset A$,

$$(3.01) \qquad \frac{(\overset{1}{\lambda}_I(x, y))^2}{\overset{01}{\lambda}_A(x, y)} \leqq -\mathfrak{Z}_B(x, y) \leqq \frac{(\overset{1}{\lambda}_A(x, y))^2}{\overset{01}{\lambda}_I(x, y)}.$$

We shall now establish the following theorem on the values of $\mathfrak{Z}_B$ at the center of a circle $C^+ \subset B$ in terms of the values both it and the kernel function K_B assume on the boundary c of C.

THEOREM 4. *Let I, B, and A be domains of finite connectivity such that $I \subset B \subset A$; suppose further that the circle C^+, whose center is (x_0, y_0), lies entirely in B, then*

$$(3.02a) \quad \mathfrak{Z}_B(x_0, y_0) \leqq \left(\int_c \overset{3}{K}_B ds\right)^{-1}\left[\int_c \overset{3}{K}_B \mathfrak{Z}_B \frac{\partial g_C}{\partial n} ds + 4\int_C \frac{\overset{010}{\lambda}_A}{\overset{1}{\lambda}_I \overset{01}{\lambda}_I \overset{001}{\lambda}_I} g_C d\omega\right],$$

where g_C is the Green's function for C, n the inner normal and s the arc length.

Let

$$(3.03) \qquad U = -\overset{3}{K}_B \mathfrak{Z}_B;$$

then from (1.05) and (1.06),

$$(3.04) \qquad U = \begin{vmatrix} K_{00} & K_{10} \\ K_{01} & K_{11} \end{vmatrix}; \qquad K = K_B(^8).$$

$\Delta U/4$, (see (0.01)), will then be given by

$$(3.05) \qquad U_{z\bar{z}} = \begin{vmatrix} K_{00} & K_{20} \\ K_{02} & K_{22} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & K_{00} & K_{10} & K_{20} \\ 1 & K_{01} & K_{11} & K_{21} \\ 0 & K_{02} & K_{12} & K_{22} \end{vmatrix}.$$

Now, from Bergman [1; (4.6), p. 64], we have

$$(3.06) \qquad \overset{010}{\lambda}_B = -\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & K_{00} & K_{10} & K_{20} \\ 1 & K_{01} & K_{11} & K_{21} \\ 0 & K_{02} & K_{12} & K_{22} \end{vmatrix} \div \Delta_B,$$

where

$$(3.07) \qquad \Delta_B = \begin{vmatrix} K_{00} & K_{10} & K_{20} \\ K_{01} & K_{11} & K_{21} \\ K_{02} & K_{12} & K_{22} \end{vmatrix};$$

(8) In the remainder of this section we shall occasionally omit the subscript in K_B, writing only K.

also

$$\lambda_B^{001} = -\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & K_{00} & K_{10} & K_{20} \\ 0 & K_{01} & K_{11} & K_{21} \\ 1 & K_{02} & K_{12} & K_{22} \end{vmatrix} \div \Delta_B. \tag{3.08}$$

Employing (1.05) and (1.06) again, we see that

$$U_{z\bar{z}} = \frac{\lambda_B^{010}}{\lambda_B^{1}\lambda_B^{01}\lambda_B^{001}} = U\frac{\lambda_B^{010}}{\lambda_B^{001}} = S; \tag{3.09}$$

$S>0$ since U and all $\lambda_B^{0X_1\cdots X_n}$ are positive.

Further, let g_C be the Green's function (for Laplace's equation) for the circle C, then by classical results of the theory of elliptic equations we have, since U satisfies the equation $\Delta U/4 - S = 0$,

$$U(x_0, y_0) = \frac{1}{2\pi}\int_c U\frac{\partial g_C}{\partial n}\,ds - \frac{2}{\pi}\int_C S g_C d\omega. \tag{3.10}$$

Let A, I be domains of finite connectivity such that $I \subset B \subset A$; the structure of I and A can be specified later and will vary depending on the type of results one wishes to obtain.

From (1.04)

$$\frac{\lambda_I^{010}}{\lambda_A^{001}\lambda_A^{01}\lambda_A^{1}} \leqq \frac{\lambda_B^{010}}{\lambda_B^{001}\lambda_B^{01}\lambda_B^{1}} \leqq \frac{\lambda_A^{010}}{\lambda_I^{001}\lambda_I^{01}\lambda_I^{1}}, \tag{3.11}$$

so that from (3.09)

$$\frac{\lambda_I^{010}}{\lambda_A^{001}\lambda_A^{01}\lambda_A^{1}} \leqq S \leqq \frac{\lambda_A^{010}}{\lambda_I^{001}\lambda_I^{01}\lambda_I^{1}}. \tag{3.12}$$

The integrand for each of the integrals of (3.10) is positive; hence, employing (3.10), (3.11), and (3.12), we can write the inequalities

$$\begin{aligned} &\left\{\frac{1}{2\pi}\int_c U\frac{\partial g_C}{\partial n}\,ds - \frac{2}{\pi}\int_C\left(\frac{\lambda_A^{010}}{\lambda_I^{1}\lambda_I^{01}\lambda_I^{001}}\right)g_C d\omega\right\} \\ &\leqq U(x_0, y_0) \leqq \left\{\frac{1}{2\pi}\int_c U\frac{\partial g_C}{\partial n}\,ds - \frac{2}{\pi}\int_C\left(\frac{\lambda_I^{010}}{\lambda_A^{1}\lambda_A^{01}\lambda_A^{001}}\right)g_C d\omega\right\}. \end{aligned} \tag{3.13}$$

But, as has been remarked in §1, K is a function whose logarithm is subharmonic, so that K as well as all positive powers of K are also subharmonic, therefore

$$K_B^3(x_0, y_0) \leqq \frac{1}{2\pi}\int_c K_B^3 ds. \tag{3.14}$$

Using (3.14) together with the left-hand side of (3.13), we have (3.02).

IV. A GENERALIZED POISSON INTEGRAL THEOREM FOR MULTIPLY-CONNECTED DOMAINS

1. **Formulation of the problem.** In this section we shall derive an integral representation for an analytic function for interior points of a domain in terms of the values its real part assumes on the boundary of that domain. This is obtained as a consequence of a result of M. Schiffer [1] which identifies the kernel function with the second derivative of the Green's function of the domain (up to a factor of $-2/\pi$) for interior points of the domain.

Consider, at first, the case of the unit circle $|z| \leqq 1$. Suppose a function f, which is regular in the unit circle, has as its real part on $|z| = 1$ the value F; then f may be represented in the unit circle, within an additive purely imaginary constant, by the classical Poisson integral formulae:

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-iu}}{1 - ze^{-iu}} F(u)du \\ &= C_0 + \frac{1}{\pi}\int_0^{2\pi} \frac{1}{1 - ze^{-iu}} F(u)du, \qquad C_0 = -\frac{1}{2\pi}\int_0^{2\pi} F(u)du. \end{aligned} \tag{4.01}$$

Recalling that the kernel function $K_C(z, \bar{t})$ of the unit circle has the form $K_C(z, \bar{t}) = (\pi^{1/2}(1 - z\bar{t}))^{-2}$ (see (0.06)), we see that except for an arbitrary additive constant (4.01) may be written in the form

$$f(z) = i\int_0^{2\pi} F(u)\int^z K_C(z_1, e^{-iu})dz_1 d(e^{-iu}), \tag{4.02}$$

where $\int^z u(z_1)dz_1$ denotes an indefinite integral of u.

We shall now show that a suitable generalization of this formula is true for multiply-connected domains (K_C replaced by the appropriate kernel function), although we shall reserve an exact formulation of the theorem to be proved until the end of the next paragraph.

2. **Proof of the theorem.** Let $\{\phi_n'\}_B$ be a sequence of functions, orthonormal with respect to B, each of which is regular and Lebesgue square integrable in B^*, where $B^+ \subset B^*$. Here B is a domain of finite connectivity whose boundary we denote by b.

Given the function g, $g \in \mathcal{L}^2(B)$; then g may be written in the form of the "Fourier" series,

$$g(z) = \sum_{m=1}^{\infty} g_m \phi_m'(z), \qquad z \in B, \tag{4.03}$$

where

$$g_n = \int_B g\bar{\phi}_n' \, d\omega, \tag{4.04}$$

provided that $\{\phi_n'\}_B$ is closed with respect to $\mathcal{L}^2(B)$; that is, if $g \in \mathcal{L}^2(B)$, then g can be expanded in the series (4.03).

Set

$$\phi_n'(Z) = \frac{d\phi_n(Z)}{dZ} \quad \text{and} \quad f'(z) = \frac{df(z)}{dz}, \qquad f'(z) \in \mathcal{L}^2(B); \tag{4.05}$$

$f'(z)$ then possesses the following "Fourier" expansion in terms of $\{\phi_n'\}_B$ in B:

$$f'(z) = \sum_{m=1}^{\infty} f_m' \phi_m'(z) = \sum_{m=1}^{\infty} \left(\int_B f' \bar{\phi}_m' d\omega \right) \phi_m'(z), \tag{4.06}$$

where f_m' is obtained by setting $g = f'$ in (4.04). Now

$$K_B(z, \overline{Z}) = \sum_{m=1}^{\infty} \phi_m'(z) \bar{\phi}_m'(\overline{Z}) \tag{4.07}$$

converges uniformly in any closed subdomain $D^+ \subset B$, and is an analytic function of z and $\overline{Z}$ in D^+; consequently, by a proof entirely analogous to one given by Bergman [1; pp. 47–50] and [4; p. 25], we may interchange the order of summation and integration in (4.06) to obtain:

$$f'(z) = \int_B \left[f'(Z) \sum_{m=1}^{\infty} \phi_m'(z) \overline{\phi_m'(Z)} \right] d\omega_Z,$$

or

$$f(z) = \int_B f'(Z) \left[\int^z K_B(z_1, \overline{Z}) dz_1 \right] d\omega_Z. \tag{4.08}$$

Let us further suppose that $f = f_1 + if_2$ is a function which belongs to $\mathcal{L}^2(B)$; note that its derivative f' and its real part f_1 are single-valued in B.

Let

$$L = \int^{\overline{Z}} \int^{z} K_B(z_1, \overline{Z}_1) d\overline{Z}_1 dz_1 = L_1 - iL_2; \tag{4.09}$$

now let B be a domain such that the derivative of the function mapping it on a characteristic domain of type $C^{(n)}$(9) does not become infinite on the boundary, then define

(9) A characteristic domain of type $C^{(n)}$ is an n-tuply connected domain with $(n-2)$ slits along arcs of circles concentric with the bounding circles $|z| = r$, $|z| = 1$.

$$\frac{\partial^* L(z, \overline{\zeta})}{\partial \overline{\zeta}} = \lim_{\overline{Z} \to \overline{\zeta}} \frac{\partial L(z, \overline{Z})}{\partial \overline{Z}} = \frac{\partial^*}{\partial \overline{\zeta}}(L_1 - iL_2) = \int^z K_B^*(z_1, \overline{\zeta}) dz_1, \quad \overline{\zeta} \in b, \tag{4.10}$$

where $\overline{Z} \to \overline{\zeta}$ along the inner normal. This limit exists, since Schiffer [1] has shown that

$$L(z, \overline{Z}) = -2\pi^{-1} g(z, \overline{Z}), \qquad z \in B, Z \in B, \tag{4.11}$$

where g is the Green's function for B. Since the boundary of B, b, is composed of circular arcs (or can be mapped into such a domain without the derivative of the mapping function becoming infinite), it follows that $\partial g/\partial \overline{Z}$ exists for $\overline{Z} = \overline{\zeta} \in b$ and hence $\partial^* L/\partial \overline{\zeta}$ exists.

Clearly, if B is multiply-connected, then both L and $\partial L/\partial \overline{Z}$ and $K_B(z, \overline{Z})$ will be multi-valued; to avoid ambiguity we introduce the simply-connected domain $B^\blacktriangle$ obtained from B by $n-1$ conveniently chosen cuts.

From (4.08)

$$\begin{aligned} f(z) &= \int_B f'(Z) \left(\int^z K_B(z_1, \overline{Z}) dz_1 \right) d\omega_Z \\ &= \int_{B^\blacktriangle} f'(Z) \left(\int^z K_B(z_1, \overline{Z}) dz_1 \right) d\omega_Z \\ &= \int_{B^\blacktriangle} f' \frac{\partial \overline{L}}{\partial \overline{Z}} d\omega \qquad\qquad (\overline{Z} = X - iY) \\ &= \int_b f_1 \left[-\frac{\partial^* L_1}{\partial n} ds + i \frac{\partial^* L_2}{\partial n} ds \right] \\ &\quad + \sum_{\nu=1}^{2n-2} \int_{e_\nu} f_1 \left[-\frac{\partial^* L_1}{\partial n} ds + i \frac{\partial^* L_2}{\partial n} ds \right], \end{aligned} \tag{4.12}$$

where $e_{2\nu}, e_{2\nu-1}, \nu = 1, 2, \cdots, n-1$, are the edges of the cross cuts which cut B into a simply-connected domain $B^\blacktriangle$, and n and ds are the interior normal and line element respectively. The integrals over e_ν will vanish, since we have supposed that the functions ϕ_n' are defined in B, in which case they will be single-valued in $B^\blacktriangle$ as will be $K_B(z, \overline{Z})$; hence (4.12) equals

$$\int_b f_1 [d^* L_2 + i d^* L_1] = i \int_b f_1(\xi, \eta) \int^z K_B^*(z_1, \overline{\zeta}) dz_1 d\overline{\zeta}, \qquad \overline{\zeta} = \xi - i\eta. \tag{4.13}$$

Upon setting the left-hand side of (4.12) equal to the right-hand side of (4.13) and integrating, we obtain

$$f(z) = i \int_b f_1(\xi, \eta) \left(\int^z K_B^*(z_1, \overline{\zeta}) dz_1 \right) d\overline{\zeta} + C_0^*, \qquad z \in B^\blacktriangle, \tag{4.14}$$

which is the desired generalization of (4.02).

THEOREM 5. *Let $f=f_1+if_2$ be a function which is regular in a bounded multiply-connected domain, B, whose boundary b satisfies the condition* (4.09) *ff.*; *suppose further, that the derivative f' and the real part f_1 of f are single-valued in B^+, while f itself is supposed Lebesgue square integrable in B; then f can be represented in $B^\blacktriangle$ in the form* (4.14).

3. **Villat's formula for a ring.** As a further illustration, we shall prove Villat's extension of Poisson's formula to the case of a ring, $R(r)$. The $\int_b$ in (4.14) can then be written as the difference of two integrals, one taken around $|z|=1$, on which boundary the real part of f is to assume the value F_1, and one around $|z|=r$, on which boundary the real part of f is to be F_2.

Giving K_B its value when $B=R(r)$, see (2.11), and taking account of the above, we obtain for (4.14) (as here $\partial^*/\partial\zeta=\partial/\partial\zeta$, that is, the interchange of limits is permissible),

$$f(z)=i\int_0^{2\pi}F_1(\theta)\left(\int^z\frac{1}{\pi z_1e^{-i\theta}}\sum_{n=1}^{\infty}\frac{nr^n}{1-r^{2n}}\left(\frac{z_1^ne^{-in\theta}}{r^n}+\frac{r^n}{z_1^ne^{-in\theta}}\right)dz_1\right)d(e^{-i\theta})$$

$$(4.15)\qquad -i\int_0^{2\pi}F_2(\theta)\left(\int^z\frac{1}{\pi z_1re^{-i\theta}}\sum_{n=1}^{\infty}\frac{nr^n}{1-r^{2n}}\right)\frac{z_1^nr^ne^{-in\theta}}{r^n}$$

$$+\frac{r^n}{z_1^nr^ne^{-in\theta}}\left(dz_1\right)\Bigg]d(re^{-i\theta})+C_0^*.$$

If the indicated indefinite integration be performed, then (4.15) assumes the form

$$(4.16)\qquad \begin{aligned} f(z) &= \frac{1}{\pi}\int_0^{2\pi}F_1(\theta)\sum_{n=1}^{\infty}\frac{r^n}{1-r^{2n}}\left(\frac{z^ne^{-in\theta}}{r^n}-\frac{r^n}{z^ne^{-in\theta}}\right)d\theta \\ &\quad -\frac{1}{\pi}\int_0^{2\pi}F_2(\theta)\sum_{n=1}^{\infty}\frac{r^n}{1-r^{2n}}(z^ne^{-in\theta}-z^{-n}e^{in\theta})d\theta+C_0^*. \end{aligned}$$

Now (4.16) coincides with Villat [1; equation (11), p. 14], so that we are led finally to the formula

$$(4.17)\qquad \begin{aligned} f(z) &= C_0^{**}+\frac{i\omega}{\pi^2}\int_0^{2\pi}F_1(\theta)\zeta\left(\frac{\omega}{i\pi}\log z-\frac{\omega}{\pi}\theta\right)d\theta \\ &\quad -\frac{i\omega}{\pi^2}\int_0^{2\pi}F_2(\theta)\zeta_3\left(\frac{\omega}{i\pi}\log z-\frac{\omega}{\pi}\theta\right)d\theta, \end{aligned}$$

where ζ, ζ_3 are the Weierstrass ζ functions.

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University of California at Los Angeles,
Los Angeles, Calif.