ON FUNCTIONS ANALYTIC IN A REGION: APPROXIMATION IN THE SENSE OF LEAST pTH POWERS

BY

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Introduction. The authors have recently investigated(1) the following problem. Let S be a closed point set interior to the region R. To study approximation on S to a function $f(z)$ analytic on S but not analytic throughout R by functions $f_M(z)$ required to be analytic and in modulus not greater than M in R. For the functions $f_M(z)$ of best approximation to $f(z)$ on S, defined for every positive M, under suitable conditions the expression

$$\limsup_{M\to\infty} \left[\max \left| f(z) - f_M(z) \right|, z \text{ on } S\right]^{1/\log M}$$

has been evaluated, and there have been obtained results on the degree and regions of convergence of the functions $f_M(z)$. The purpose of the present paper is to continue the study of the same problem, modified now by using as norm in R not $[\text{l.u.b.} \; |f_M(z)|, z \text{ in } R]$ but an integral mean over the boundary of R, and by using as measure of approximation of $f_M(z)$ to $f(z)$ also an integral mean, here taken over the boundary of S. We investigate in especial detail approximation in the sense of least squares, when R and S are bounded by circles, obtaining explicitly the extremal functions and measures of approximation, and investigate also general existence, uniqueness, and continuity properties of the extremal functions.

1. **Properties of integral means.** It will be necessary to make use of certain fundamental properties of integral means of analytic functions. Those included here will be extensions of classical results for circles and annuli.

Let R be a finite region bounded by a finite sum C_1 of disjoint Jordan curves. Let S be a closed set interior to R whose boundary C_0 consists of a finite number of disjoint Jordan curves such that C_0 separates no point of $R-S$ from C_1. Let $\phi(z)$ be the function(2) harmonic in $R-S$, continuous on the corresponding closed set $\overline{R-S}$, equal to zero and unity on C_0 and C_1, respectively. Denote generically by C_ν, $0<\nu<1$, the locus $\phi(z)=\nu$; by R_ν denote the point set consisting of S together with points of $R-S$ where $0<\phi(z)<\nu$. Let $\psi(z)$ be a function conjugate to $\phi(z)$ in $R-S$. Then $\psi(z)$ is continuous in $\overline{R-S}$ and strictly monotone on C_0 and C_1. Let $\tau>0$ be the variation of

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(1) J. L. Walsh, Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 477–486. E. N. Nilson and J. L. Walsh, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 53–67. J. L. Walsh, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 572–579.

(2) H. Lebesgue, Rend. Circ. Mat. Palermo vol. 24 (1907) pp. 371–402.

$\psi(z)$ as z traces C_1 in the positive sense with respect to $R-S$; then $-\tau$ is the variation of $\psi(z)$ as z traces C_0 in this sense.

If $F(z)$ is analytic in R and such that the quantities(3)

$$(1)\qquad \mu_p(F, \nu) = \left\{\frac{1}{\tau}\int_{C_\nu} |F(z)|^p d\psi(z)\right\}^{1/p}, \qquad 0 < \nu < 1, 0 < p \leqq \infty,$$

are bounded for fixed p, then we shall show that *for all points* ζ *on* C_1, *with the possible exception of a null set*, $\lim_{z\to\zeta} F(z)$ *exists*, the approach being made along the level curves of the function $\psi(z)$. This generalization of the classical results of F. and M. Riesz(4) is to be expected, of course.

The problem at hand can be reduced to the consideration of a finite number of annular regions by taking the point set $R-\overline{R}_\epsilon$, $0<\epsilon<1$, with ϵ sufficiently close to unity. Let $T_1, T_2, \cdots, T_m$ represent these annular regions and let $(\psi_{k-1}), \psi_k)$ represent the range of ψ on that part of C_1 which belongs to the boundary of T_k. Then the function $\log \rho+i\theta=2\pi(\phi+i\psi)/\tau_k$, where $\tau_k=\psi_k-\psi_{k-1}$, maps the regions T_k into the annulus $e^{2\pi\epsilon/\tau_k}<|\rho e^{i\theta}| <e^{2\pi/\tau_k}$. If $\mu_p(F, \nu)\leqq M$ for each ν, $0<\nu<1$, we have

$$\int_0^{2\pi} |\Phi(\rho e^{i\theta})|^p d\theta \leqq \frac{2\pi\tau}{\tau_k} M^p,$$

where $\Phi(\rho e^{i\theta})\equiv F(z)$. Thus the problem is reduced to the case of a region bounded by concentric circles, and this case is essentially covered by the classical theorem, by considering the components of $F(z)$.

Similarly, it is seen that $F(z)$ *is of Lebesgue class* L^p *with respect to* ψ *on* C_1 when it is defined there in terms of the limiting values described above. Moreover

$$\lim_{\nu\to 1} \mu_p(F, \nu) = \mu_p(F, 1).$$

The Hardy Mean-Value Theorem(5) admits of a generalization to the present configuration.

THEOREM 1. *Suppose* $F(z)$ *is a function analytic in* $R-S$ *with integral means* $\mu_p(F, \nu)$ *uniformly bounded*, $0<\nu<1$. *Then* $\log \mu_p(F, \nu)$ *is a convex function of* ν *in* $0\leqq\nu\leqq 1$, *where* $F(z)$ *is defined on* C_0 *and* C_1 *by means of limiting values along level curves of* $\psi(z)$.

(3) In this notation, $\mu_\infty(F, \nu)=\{\text{l.u.b. } |F(z)|, z \text{ on } C_\nu\}$.

(4) The case $p\geqq 1$ is considered by F. and M. Riesz, Comptes Rendus du Quatrième Congrès (1916) des Mathématiciens Scandinaves, Uppsala, 1920, pp. 27–44. For the case $0<p<1$, see F. Riesz, Math. Zeit. vol. 18 (1923) pp. 87–95. For particularly elegant proofs of these theorems, see A. Zygmund, *Trigonometric series*, Warsaw, 1935 (Stechert, New York), pp. 160–162.

(5) G. H. Hardy, Proc. London. Math. Soc. (2) vol. 14 (1915) pp. 269–277.

The proof involves an adaptation of the Pólya-Szegö proof(6) of the Hardy theorem. Suppose, first, that $R-S$ is connected. Consider the functions.

$$e^{2\pi\alpha(\phi+i\psi)/\tau p}\Phi(\phi, \psi + k\tau/n), \qquad k = 1, 2, \cdots, n,$$

where α is real, the quantities $\psi+k\tau/n$ are reduced *modulo* τ, and $\Phi(\phi, \psi) \equiv F(z)$. These functions are analytic functions of $\phi+i\psi$ in the small, except possibly at the singularities of $z=z(\phi, \psi)$, and have single-valued moduli. The function

$$\sum_{k=1}^{n} \left| e^{2\pi\alpha(\phi+i\psi)/\tau p}\Phi(\phi, \psi + k\tau/n) \right|^p \tag{2}$$

is subharmonic, $0 \leqq \phi \leqq 1$ and $0 \leqq \psi \leqq \tau$, except at these singularities. The function (2), however, can have no absolute maximum for $0<\phi<1$. Choose α such that

$$\{\mu_p(F, 0)\}^p = e^{2\pi\alpha/\tau}\{\mu_p(F, 1)\}^p. \tag{3}$$

It follows that, for $0<\nu<1$, the expression

$$e^{2\pi\alpha\nu/\tau}\{\mu_p(F, \nu)\}^p = \lim_{n\to\infty} \frac{\tau}{n} \sum_{k=1}^{n} e^{2\pi\alpha\nu/\tau} \left| \Phi(\nu, \psi + k\tau/n) \right|^p$$

has as its least upper bound the quantity given in (3). The convexity of $\log \mu_p(F, \nu)$ is an immediate consequence.

If $R-S$ is not connected, let $T_1, T_2, \cdots, T_m$ be the component regions of $R-S$. Let the subinterval (ψ_{k-1}, ψ_k) of the interval $(0, \tau)$ be the range of ψ on that part of C_1 which belongs to the boundary of T_k. Then by what we have just shown and by an application of the Hölder inequality for sums, we obtain

$$\begin{aligned}
\frac{1}{\tau}\int_0^\tau |\Phi(\nu, \psi)|^p d\psi &= \sum_{k=1}^{m} \frac{1}{\tau}\int_{\psi_{k-1}}^{\psi_k} |\Phi(\nu, \psi)|^p d\psi \\
&\leqq \sum_{k=1}^{m} \left\{\frac{1}{\tau}\int_{\psi_{k-1}}^{\psi_k} |\Phi(0, \psi)|^p d\psi\right\}^{1-\nu} \left\{\frac{1}{\tau}\int_{\psi_{k-1}}^{\psi_k} |\Phi(1, \psi)|^p d\psi\right\}^{\nu} \\
&\leqq \left\{\sum_{k=1}^{m} \frac{1}{\tau}\int_{\psi_{k-1}}^{\psi_k} |\Phi(0, \psi)|^p d\psi\right\}^{1-\nu} \left\{\sum_{k=1}^{m} \frac{1}{\tau}\int_{\psi_{k-1}}^{\psi_k} |\Phi(1, \psi)|^p d\psi\right\}^{\nu},
\end{aligned}$$

which completes the proof.

The theorem may also be proved, of course, by Hardy's original method. Similarly, the generalization of the Hardy theorem to subharmonic functions by F. Riesz(7) may be extended to the present configuration, affording another

(6) G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis* I, Berlin, 1925, §310. G. Julia, *Principes géometriques d'analyse* I, Paris, 1930, p. 100.

(7) F. Riesz, Acta Math. vol. 48 (1926) pp. 329–343.

method of proof.

If $\{F_\alpha(z)\}$ is an infinite family of functions analytic in R such that for fixed p the quantities $\mu_p(F_\alpha, \nu)$, $0<\nu<1$, are uniformly bounded, then the family is normal in R. For suppose D is any closed point set interior to R. Then there is some $\nu<1$ such that D is contained in R_ν and R_ν has the same connectivity as R. If $h_\nu(z, t)$ is a conjugate of Green's function for R_ν with pole at z, running coördinate t, then

$$\begin{aligned} |F_\alpha(z)|^p &\leqq -\frac{1}{2\pi}\int_{C_\nu} |F_\alpha(t)|^p dh_\nu(z, t) \\ &\leqq \frac{1}{2\pi}\int_{C_\nu} |F_\alpha(t)|^p \left|\frac{\partial h_\nu}{\partial \psi}\right| d\psi, \qquad z \text{ on } D. \end{aligned} \tag{4}$$

The function $|\partial h_\nu/\partial\psi|$ is bounded for z on D, so that the moduli of the functions $F_\alpha(z)$ are uniformly bounded in D.

The family of functions $\{F_\alpha(z)\}$ is therefore normal in R: from any infinite sequence of functions of the family can be extracted a subsequence converging uniformly on each closed subset of R to a function $F(z)$ analytic in R. Necessarily, the least upper bound of the integral means $\mu_p(F, \nu)$ is not greater than the least upper bound of the quantities $\mu_p(F_\alpha, \nu)$.

It is to be noted that the disjointness provision on the component curves of C_0 and C_1 is made for convenience only. If the integrals concerned in this section are properly interpreted, this condition may, of course, be omitted.

2. **Functions of best approximation. Functions of minimum norm.** Suppose that the function $f(z)$ is analytic upon the closed set S interior to the region R, but coincides on S with no function analytic throughout R. For a given positive quantity M and for each positive p and q, we may infer from §1 that there exists an extremal function $\mathcal{J}_M(z)$, analytic in R, of class L^q on C_1, having $\mu_q(\mathcal{J}_M, 1)\leqq M$, with bounded integral means on the C_ν of order q, and such that the quantity $m_M=\mu_p(f-\mathcal{J}_M, 0)$ is least. Let m_M be the greatest lower bound of the expressions $\mu_p(f-F, 0)$ for functions $F(z)$ analytic in R with $\mu_q(F, \nu)\leqq M$, $0<\nu<1$. Choose a sequence $\{F_n(z)\}$ of these functions such that $\mu_p(f-F_n, 0)$ approaches m_M. This sequence forms a normal family and from it can be extracted a subsequence $\{F_{n_i}(z)\}$ converging uniformly on each closed subset of R to a function $\mathcal{J}_M(z)$ analytic in R. Now

$$\{\mu_p(f-\mathcal{J}_M, 0)\}^t \leqq \{\mu_p(f-F_{n_i}, 0)\}^t + \{\mu_p(\mathcal{J}_M - F_{n_i}, 0)\}^t,$$

where $t=1$ if $p\geqq 1$ (Minkowski's inequality) and $t=p$ if $0<p<1$[8]. Thus $\mu_p(f-\mathcal{J}_M, 0)\leqq m_M$. On the other hand, for $0<\nu<1$,

$$\{\mu_q(\mathcal{J}_M, \nu)\}^t \leqq \{\mu_q(F_{n_i}, \nu)\}^t + \{\mu_q(F_{n_i} - \mathcal{J}_M, \nu)\}^t,$$

so that $\mu_q(\mathcal{J}_M, \nu)\leqq M$. It follows that $\mathcal{J}_M$ is defined almost everywhere on C_1,

[8] Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge, 1934, Theorem 27.

is of Lebesgue class L^q there, and $\mu_q(\mathcal{J}_M, 1) \leqq M$. Hence $\mu_p(f-\mathcal{J}_M, 0) \geqq m_M$, and the equality $\mu_p(f-\mathcal{J}_M, 0) = m_M$ follows.

This function of "best approximation"(9), $\mathcal{J}_M(z)$, is unique for $1 \leqq p < \infty$ and $1 \leqq q \leqq \infty$. For if $\mathcal{J}_M^*$ were another function analytic in R, of class L^q on C_1, of bounded integral means, having $\mu_q(\mathcal{J}_M^*, 1) \leqq M$ and $\mu_p(f-\mathcal{J}_M^*, 0) \leqq m_M$, it would follow from the Minkowski inequality for any λ, $0<\lambda<1$, that $\mu_q(\lambda\mathcal{J}_M+(1-\lambda)\mathcal{J}_M^*, 1) \leqq M$, and that the measure of approximation, $\mu_p(f-\{\lambda\mathcal{J}_M+(1-\lambda)\mathcal{J}_M\}, 0)$, was actually less than m_M. (The strong inequality results from the assumption that $\mathcal{J}_M$ and $\mathcal{J}_M^*$ are distinct.) Thus a contradiction is reached.

There is a related extremal function $\mathcal{G}_m(z)$ which is defined as follows: Let m be given. Then $\mathcal{G}_m(z)$ is analytic throughout R, is of class L^q on C_1, of bounded integral means, has $\mu_p(f-\mathcal{G}_m, 0) \leqq m$, and is such that the quantity $M_m = \mu_q(\mathcal{G}_m, 1)$ is a minimum. That such a function should exist is a consequence of Theorem 2 in §3 and of properties of normal families of functions. The function $\mathcal{G}_m(z)$ of minimum norm is unique for $1 \leqq p \leqq \infty$ and $1 \leqq q < \infty$.

For $1 \leqq p \leqq \infty$, the functions $\mathcal{J}_M(z)$ have the property that $\mu_q(\mathcal{J}_M, 1)$ is actually equal to M. The proof is indirect: suppose for some $M>0$ that $\mu_q(\mathcal{J}_M, 1) = M' < M$. We reach a contradiction by exhibiting a function $F(z)$ having $\mu_p(f-F, 0)$ less than $m_M = \mu_p(f-\mathcal{J}_M, 0)$ but having $\mu_q(F, 1) < M$. By Theorem 2 below it will be possible to find M'' such that $m'' = \mu_p(f-\mathcal{J}_{M''}, 0) < m$. Take $F(z)$ to be $(1-\lambda)\mathcal{J}_M(z)+\lambda\mathcal{J}_{M''}(z)$, where λ, $0<\lambda<1$, is to be chosen subsequently. Clearly $\mu_p(f-F, 0) \leqq (1-\lambda)m+\lambda m'' < m$. Also $\mu_q(F, 1) \leqq (1-\lambda)M'+\lambda M''$ if $1 \leqq q \leqq \infty$ and $\{\mu_q(F, 1)\}^q \leqq (1-\lambda)^q M'^q + \lambda^q M''^q$ if $0<q<1$. In either case, $\mu_q(F, 1)$ can be made less than M by choosing λ sufficiently small.

On the other hand, the measure of approximation of the functions $\mathcal{G}_m(z)$ is always known precisely: for all p and q, $0<p \leqq \infty$ and $0<q \leqq \infty$, we have $\mu_p(f-\mathcal{G}_m, 0) = m$. The proof is again indirect: assume $\mu_p(f-\mathcal{G}_m, 0) = m' < m$ for some $m>0$. Take $F(z) = \mathcal{G}_m(z)/(1+\epsilon)$, where ϵ will be restricted immediately. Clearly, $\mu_q(F, 1) < M_m$. But $\mu_p(f-F, 0) \leqq \mu_p(f-\mathcal{G}_m, 0) + \mu_p(\mathcal{G}_m - F, 0) < m' + M_m\epsilon/(1+\epsilon)$ if $1 \leqq p \leqq \infty$, and $\{\mu_p(f-F, 0)\}^p < m'^p + \{M_m\epsilon/(1+\epsilon)\}^p$ if $0<p<1$. In either case, the measure of approximation can be made less than m if ϵ is chosen sufficiently small.

It is now possible to state the connection between these two types of extremal functions, $\mathcal{J}_M(z)$ and $\mathcal{G}_m(z)$. If $1 \leqq p \leqq \infty$ and if $m = m_M = \mu_p(f-\mathcal{J}_M, 0)$, then $\mathcal{J}_M(z)$ is also a function of minimum norm $\mathcal{G}_m(z)$. Of course, if $1 \leqq p \leqq \infty$ and $1 \leqq q < \infty$, then $\mathcal{J}_M(z)$ is the unique function $\mathcal{G}_m(z)$. In contrast, the functions $\mathcal{G}_m(z)$ are functions of best approximation for all p and q: if $M = M_m = \mu_q(\mathcal{G}_m, 1)$, then $\mathcal{G}_m$ is a function of best approximation $\mathcal{J}_M(z)$.

3. **Asymptotic relations on integral means.** It will be necessary to make

(9) Cf. J. L. Walsh, Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 477–486, especially pp. 477, 478. The situation there considered is the case $p=q=\infty$.

frequent application of the following lemma, an extension of a recent theorem by Walsh[10].

LEMMA. *Suppose $f(z)$ is a function analytic throughout R_ρ, $0<\rho<1$, but coincides on S with no function analytic throughout $R_{\rho'}$ for $\rho'>\rho$. If $\{f_n(z)\}$ is a sequence of functions analytic in R, of class L^q on C_1, of bounded integral means, such that*

$$\limsup_{n\to\infty} \{\mu_q(f_n, 1)\}^{1/n} \leqq e^{\alpha}, \qquad 0<q\leqq\infty, \tag{5}$$

and

$$\limsup_{n\to\infty} \{\mu_p(f-f_n, 0)\}^{1/n} \leqq e^{\beta}, \qquad 0<p\leqq\infty, \tag{6}$$

then

$$\alpha\rho+\beta-\beta\rho \geqq 0. \tag{7}$$

Note that α is necessarily positive, and that there is no loss in generality in assuming β to be negative. Let t be the smaller of p and q. Then for arbitrary positive ϵ and all n sufficiently large, we have

$$\mu_t(f_{n+1}-f_n, 1) \leqq Te^{(\alpha+\epsilon)(n+1)}$$

and

$$\mu_t(f_{n+1}-f_n, 0) \leqq Te^{(\beta+\epsilon)n},$$

where $T=2$ if $1\leqq t\leqq\infty$ and $T=2^{1/t}$ for $0<t<1$. Thus, by Theorem 1, we have

$$\mu_t(f_{n+1}-f_n, \nu) \leqq [Te^{(\alpha+\epsilon)(n+1)}]^{\nu}[Te^{(\beta+\epsilon)n}]^{1-\nu}, \qquad 0<\nu<1,$$

for all n sufficiently large. It follows (compare (4)) for z on any closed subset of R_ν that

$$|f_{n+1}(z)-f_n(z)| \leqq T'e^{(\alpha\nu+\beta-\beta\nu+\epsilon)n}, \tag{8}$$

where T' is independent of n. Thus, if $\alpha\nu+\beta-\beta\nu$ were negative, ϵ could be chosen such that the exponent in (8) would be negative, and the sequence $\{f_n(z)\}$ would converge uniformly on $\overline{R}_{\rho'}$ for some $\rho'>\rho$. The sequence, however, converges to $f(z)$ on S, and $f(z)$ is not analytic throughout $R_{\rho'}$. A contradiction is reached, therefore; hence, $\alpha\rho+\beta-\beta\rho\geqq 0$. It is to be noted that the remaining parts of the earlier theorem by Walsh are open to similar extension.

THEOREM 2. *Let $f(z)$ be analytic throughout R_ρ, $0<\rho<1$, but let $f(z)$ coincide on S with no function analytic throughout $R_{\rho'}$ for any $\rho'>\rho$. For fixed p and q, $0<p\leqq\infty$ and $0<q\leqq\infty$, the extremal functions $\{\mathcal{J}_M(z)\}$ converge uniformly to $f(z)$ on S as $M\to\infty$, and we have*

([10]) J. L. Walsh, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 293–304, Theorem 1.

$$\limsup_{M\to\infty} \{\mu_t(f - \mathcal{J}_M, \sigma)\}^{1/\log M} = e^{(\sigma-\rho)/(1-\rho)} \tag{9}$$

for each σ, $0 \leqq \sigma < \rho$, *and each* t, $0 < t \leqq \infty$. *We have also*

$$\limsup_{M\to\infty} \{\mu_t(\mathcal{J}_M, \sigma)\}^{1/\log M} = e^{(\sigma-\rho)/(1-\rho)} \tag{10}$$

for all t, $0 < t \leqq \infty$ *if* $\rho \leqq \sigma < 1$ *and for* $0 < t \leqq q$ *if* $\sigma = 1$.

For a given positive M, the function $f_M(z)$ considered in our earlier paper[11] has the property that $\mu_q(f_M, 1) \leqq M$ and $\mu_p(f - f_M, 0) \leqq \exp\{(-\rho+\epsilon)\log M/(1-\rho)\}$ provided M is sufficiently large, the quantity $\epsilon > 0$ arbitrary. *A fortiori*, it can be said for all such M that $\mu_p(f - \mathcal{J}_M, 0) \leqq \exp\{(-\rho+\epsilon)\log M/(1-\rho)\}$, and we obtain

$$\limsup_{M\to\infty} \{\mu_p(f - \mathcal{J}_M, 0)\}^{1/\log M} \leqq e^{-\rho/(1-\rho)},$$

while

$$\limsup_{M\to\infty} \{\mu_q(\mathcal{J}_M, 1)\}^{1/\log M} \leqq e.$$

Thus

$$\mu_t(f_M - \mathcal{J}_M, 0) \leqq Te^{(-\rho+\epsilon)\log M/(1-\rho)}$$

and

$$\mu_t(f_M - \mathcal{J}_M, 1) \leqq Te^{\log M},$$

where $t = \min(p, q)$ and $T = \max(2^{1/t}, 2)$. By Theorem 1 we have

$$\begin{aligned} \mu_t(f_M - \mathcal{J}_M, \nu) &\leqq \left[Te^{(-\rho+\epsilon)\log M/(1-\rho)}\right]^{1-\nu}\left[Te^{\log M}\right]^{\nu} \\ &\leqq Te^{(\nu-\rho+\epsilon)\log M/(1-\rho)}, \qquad 0 < \nu < 1, \end{aligned}$$

for $0 < \nu < 1$. By the properties of $f_M(z)$, we can infer for M sufficiently large that

$$\mu_t(f - f_M, \nu) \leqq e^{(\nu-\rho+\epsilon)\log M/(1-\rho)}, \qquad 0 \leqq \nu < \rho,$$

and

$$\mu_t(f_M, \nu) \leqq e^{(\nu-\rho+\epsilon)\log M/(1-\rho)}, \qquad \rho \leqq \nu \leqq 1.$$

We conclude, therefore, that

$$\limsup_{M\to\infty} \{\mu_t(f - \mathcal{J}_M, \nu)\}^{1/\log M} \leqq e^{(\nu-\rho)/(1-\rho)}, \qquad 0 \leqq \nu < \rho, \tag{11}$$

and

$$\limsup_{M\to\infty} \{\mu_t(\mathcal{J}_M, \nu)\}^{1/\log M} \leqq e^{(\nu-\rho)/(1-\rho)}, \qquad \rho \leqq \nu \leqq 1. \tag{12}$$

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(11) E. N. Nilson and J. L. Walsh, loc. cit. p. 57, Theorem 4.

These inequalities for the remaining values of t are obtained, by the device already employed several times (compare (4)), from the monotone character of the integral means.

The remainder of the proof involves only the elimination of the strong inequality in (11) and (12). This is accomplished by Lemma 1 in precisely the same way as it was done in our earlier paper. The uniform convergence of the functions $\{\mathcal{J}_M(z)\}$ to $f(z)$ on any closed subset of R_ρ is an immediate consequence of (11).

The theorem has several further implications, some of which are included in the following corollaries:

COROLLARY 1. *For any family of functions* $\{F_M(z)\}$ *(defined for each* $M>0$*) such that* $F_M(z)$ *is analytic in* R*, of class* L^q *on* C_1*, of bounded integral means, with*

$$\limsup_{M\to\infty} \{\mu_q(F_M, 1)\}^{1/\log M} \leqq e,$$

we have for each σ, $0 \leqq \sigma < \rho$, *and each* t, $0 < t \leqq \infty$, *the inequality*

$$\limsup_{M\to\infty} \mu_t(f - F_M, \sigma)^{1/\log M} \geqq e^{(\sigma-\rho)/(1-\rho)}.$$

COROLLARY 2. *Let* $\{M_n\}$ *be a monotone sequence of positive quantities such that*

$$(13) \qquad 1 < \liminf_{n\to\infty} \frac{M_{n+1}}{M_n} \leqq \limsup_{n\to\infty} \frac{M_{n+1}}{M_n} < \infty.$$

Then for $0 \leqq \sigma < \rho$ *and* $0 < t \leqq \infty$ *we have*

$$\limsup_{n\to\infty} \{\mu_t(f - \mathcal{J}_{M_n}, \sigma)\}^{1/\log M_n} = e^{(\sigma-\rho)/(1-\rho)}.$$

We have also

$$\limsup_{n\to\infty} \{\mu_t(\mathcal{J}_{M_n}, \sigma)\}^{1/\log M_n} = e^{(\sigma-\rho)/(1-\rho)}$$

for $0 < t \leqq \infty$ *if* $f \leqq \sigma < 1$ *and for* $0 < t \leqq q$ *if* $\sigma = 1$.

COROLLARY 3. *Let* $\{M_n\}$ *be a monotone sequence of positive quantities satisfying* (13) *and let* $\{F_{M_n}(z)\}$ *be a sequence of functions analytic in* R*, of class* L^q *on* C_1*, of bounded integral means, with*

$$\limsup_{n\to\infty} \{\mu_q(F_{M_n}, 1)\}^{1/\log M_n} \leqq e;$$

then for each σ, $0 \leqq \sigma < \rho$, *and each* t, $0 < t \leqq \infty$, *we have*

$$\limsup_{n\to\infty} \{\mu_t(f - F_{M_n}, \sigma)\}^{1/\log M_n} \geqq e^{(\sigma-\rho)/(1-\rho)}.$$

The situation in which $f(z)$ is analytic throughout R but not $\bar{R}$ may be considered as a limiting case of the situation in Theorem 2. In fact, it follows immediately from that theorem (by letting $\rho\to 1$) that for $0\leqq\sigma<1$ and $0<t\leqq\infty$ we have

$$\lim_{M\to\infty} \{\mu_t(f-\mathcal{J}_M,\sigma)\}^{1/\log M}=0$$

and

$$\limsup_{M\to\infty} \{\mu_t(\mathcal{J}_M,\sigma)\}^{1/\log M}\leqq 1. \tag{14}$$

Equality must hold in (14), however. For if the strong inequality were valid for any σ, $0<\sigma<1$, the set of functions $\{\mathcal{J}_M(z)\}$ would converge uniformly to zero on any closed subset of R_σ. The case $\sigma=0$ is established by an application of Theorem 1. Finally,

$$\limsup_{M\to\infty} \{\mu_t(\mathcal{J}_M,1)\}^{1/\log M}\leqq e, \qquad 0<t\leqq q.$$

If the sharp inequality should hold for $t=q$, it can be shown that $f(z)$ is of class L^q on C_1, of bounded integral means, and $\mathcal{J}_M(z)\equiv f(z)$ for $M\geqq\mu_q(f,1)$.

It has been demonstrated that the functions $\mathcal{G}_m(z)$ of minimum norm are actually functions of best approximation. As an immediate consequence of this fact we have the following theorem.

THEOREM 3. *Let $f(z)$ be analytic throughout R_ρ, $0<\rho<1$, but let $f(z)$ coincide on S with no function analytic throughout $R_{\rho'}$ for any $\rho'>\rho$. Let the extremal functions $\mathcal{G}_m(z)$ be defined for fixed p and q, $0<p\leqq\infty$ and $0<q\leqq\infty$, and for $m>0$. Then as $m\to 0$, the functions $\{\mathcal{G}_m(z)\}$ converge to $f(z)$ on S; and for arbitrary σ, $0\leqq\sigma<\rho$, we have*

$$\limsup_{m\to 0} \{\mu_t(f-\mathcal{G}_m,\sigma)\}^{-1/\log m}=e^{(\sigma-\rho)/\rho}$$

for each t, $0<t\leqq\infty$. If $\rho\leqq\sigma\leqq 1$, then

$$\limsup_{m\to 0} \{\mu_t(\mathcal{G}_m,\sigma)\}^{-1/\log m}=e^{(\sigma-\rho)/\rho}$$

for $0<t\leqq\infty$ if $\sigma<1$ and for $0<t\leqq q$ if $\sigma=1$.

The corollaries of Theorem 2 have their analogues here:

COROLLARY 1. *For any family of functions $\{G_m(z)\}$ (defined for each $m>0$) such that $G_m(z)$ is analytic in R, of class L^q on C_1, of bounded integral means, with*

$$\limsup_{m\to 0} \{\mu_p(f-G_m,0)\}^{-1/\log m}\leqq e^{-1},$$

we have

$$\limsup_{m\to 0} \{\mu_t(G_m, \sigma)\}^{-1/\log m} \geqq e^{(\sigma-\rho)/\rho}$$

for $0<t\leqq\infty$ *if* $\rho\leqq\sigma<1$ *and for* $0<t\leqq q$ *if* $\sigma=1$.

Corollary 2. *Let* $\{m_n\}$ *be a monotone sequence of positive quantities such that*

$$1 < \liminf_{n\to\infty} \frac{m_n}{m_{n+1}} \leqq \limsup_{n\to\infty} \frac{m_n}{m_{n+1}} < \infty.$$

Then for $0\leqq\sigma<\rho$ *and* $0<t\leqq\infty$ *we have*

$$\limsup_{n\to\infty} \{\mu_t(f - \mathcal{G}_{m_n}, \sigma)\}^{-1/\log m_n} = e^{(\sigma-\rho)/\rho}.$$

We have, moreover,

$$\limsup_{n\to\infty} \{\mu_t(\mathcal{G}_{m_n}, \sigma)\}^{-1/\log m_n} = e^{(\sigma-\rho)/\rho}$$

for $0<t\leqq\infty$ *if* $\rho\leqq\sigma<1$ *and for* $0<t\leqq q$ *if* $\sigma=1$.

Corollary 3. *Let* $\{m_n\}$ *satisfy the requirements of Corollary* 2, *and let* $\{G_{m_n}\}$ *be a sequence of functions analytic in* R, *of class* L^q *on* C_1, *of bounded integral means, with*

$$\limsup_{n\to\infty} \{\mu_p(f - G_{m_n}, 0)\}^{-1/\log m_n} \leqq e^{-1}.$$

Then we have

$$\limsup_{n\to\infty} \{\mu_t(G_{m_n}, \sigma)\}^{-1/\log m_n} \geqq e^{(\sigma-\rho)/\rho}$$

for $0<t\leqq\infty$ *if* $\rho\leqq\sigma<1$ *and for* $0<t\leqq q$ *if* $\sigma=1$.

Letting $\rho\to 1$ in Theorem 3 gives results for the limiting situation in which $f(z)$ is analytic throughout R but not $\bar{R}$. Here we have, for $0<t\leqq\infty$,

$$\limsup_{m\to 0} \{\mu_t(f - \mathcal{G}_m, 0)\}^{-1/\log m} = e^{-1},$$

while, if $0<\sigma<1$,

$$\limsup_{m\to 0} \{\mu_t(f - \mathcal{G}_m, \sigma)\}^{-1/\log m} \leqq e^{\sigma-1}.$$

Moreover, for $0<t\leqq q$, we have

$$\limsup_{m\to 0} \mu_t(\mathcal{G}_m, 1)\}^{-1/\log m} = 1.$$

It is to be remarked that the case $t=\infty$ for Theorems 2 and 3 with their associated corollaries are the corresponding results on maximum moduli of our earlier paper.

4. **Circular regions and $p=q=2$.** The case in which $R-S$ is a circular annulus and $p=q=2$ is naturally of special interest. Here it is possible to exhibit the extremal functions $\mathcal{J}_M(z)\equiv\mathcal{G}_{m_M}(z)$ and to determine the precise measures of approximation.

Suppose that R is the interior of the circle $|z|=r_1>1$ and S is the closed point set $|z|\leqq r_0$, where $0<r_0<1$. Assume that the series representation

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

has unit radius of convergence. Given $M>0$, there exists a unique function

$$\mathcal{J}_M(z) = \sum_{k=0}^{\infty} b_k z^k$$

of best approximation. Indeed,

$$\mu_2(\mathcal{J}_M, 1) = \left\{ \sum_{k=0}^{\infty} |b_k|^2 r_1^{2k} \right\}^{1/2} = M, \tag{15}$$

while

$$m_M = \mu_2(f - \mathcal{J}_M, 0) = \left\{ \sum_{k=0}^{\infty} |a_k - b_k|^2 r_0^{2k} \right\}^{1/2} \tag{16}$$

is a minimum.

For each positive integer N, we can find the (unique) polynomial $p_N(z)$ of degree N such that $\mu_2(p_N, 1)\leqq M$ but such that $\mu_2(f-p_N, 0)$ is least. There exists a subsequence of these polynomials, $\{p_{N_i}(z)\}$, converging uniformly on each closed subset of R to a function $F_M(z)$ analytic in R, of norm not greater than M, but with $\mu_2(f-F_M, 0)=\{\text{g.l.b.}_{N\to\infty}\,\mu_2(f-p_N, 0)\}$. Necessarily, we have $m_M\leqq\mu_2(f-F_M, 0)$. On the other hand, $m_M\geqq\mu_2(f-P_N, 0)$; indeed, we have $\mu_2(\sum_{k=0}^N b_k z^k, 1)\leqq M$ so that, *a fortiori*, $\mu_2(f-p_N, 0)\leqq\mu_2(f-\sum_{k=0}^N b_k z^k, 0)$. It follows that $m_M\geqq\mu_2(f-F_M, 0)$ and hence that $m_M=\mu_2(f-F_M, 0)$. By the uniqueness of the extremal function, we have $F_M(z)\equiv\mathcal{J}_M(z)$.

Let $p_N(z)=A_0+A_1z+\cdots+A_Nz^N$. Thus we minimize

$$\sum_{k=0}^{N} |A_k - a_k|^2 r_0^{2k} + \sum_{k=N+1}^{\infty} |a_k|^2 r_0^{2k}, \tag{17}$$

while satisfying the condition

$$\sum_{k=0}^{N} |A_k|^2 r_1^{2k} \leqq M^2. \tag{18}$$

If N is taken sufficiently large, $\sum_{k=0}^{N} |a_k|^2 r_1^{2k} > M^2$ and it follows that equality must hold in (18). Write $a_k = \alpha_k + i\beta_k$ and $A_k = \alpha_k^* + i\beta_k^*$. Using Lagrange's multipliers λ_N, we find that

$$2(\alpha_k^* - \alpha_k) r_0^{2k} + 2\lambda_N \alpha_k^* r_1^{2k} = 0,$$

$$2(\beta_k^* - \beta_k) r_0^{2k} + 2\lambda_N \beta_k^* r_1^{2k} = 0, \qquad k = 0, 1, \cdots, N.$$

Thus, for the extremal polynomial, we have

$$A_k = \frac{a_k r_0^{2k}}{r_0^{2k} + \lambda_N r_1^{2k}};$$

the extremal polynomial $p_N(z)$ is the extremal function for the partial sums of $f(z)$. It is clear from (17) and (18) that $\arg A_k = \arg a_k$. Thus λ_N is necessarily positive.

Since each subsequence $\{p_{N_i}(z)\}$ of the extremal polynomials contains a new subsequence which converges to the extremal function $\mathcal{J}_M(z)$, the sequence $p_N(z)$ converges to $\mathcal{J}_M(z)$, each coefficient A_k will approach a limit as $N \to \infty$, so λ_N approaches some limit λ, and we have

$$\mathcal{J}_M(z) = \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} \frac{a_k r_0^{2k} z^k}{r_0^{2k} + \lambda r_1^{2k}} \tag{19}$$

where

$$\sum_{k=0}^{\infty} \left(\frac{|a_k| r_0^{2k}}{r_0^{2k} + \lambda r_1^{2k}} \right)^2 r_1^{2k} = M^2 \tag{20}$$

and

$$m_M^2 = \lambda^2 \sum_{k=0}^{\infty} \left(\frac{|a_k| r_1^{2k}}{r_0^{2k} + \lambda r_1^{2k}} \right)^2 r_0^{2k}. \tag{21}$$

Equation (20) determines λ uniquely as a function of M; λ is necessarily positive, but λ takes on every positive value as M varies from 0 to ∞.

There are several immediate consequences of this series representation of the extremal function. The function $\mathcal{J}_M(z)$ is actually analytic throughout $|z| < r_1^2/r_0^2$ rather than merely throughout $|z| < r_1$. From (19) it follows for each k that

$$a_k - b_k = \lambda b_k r_1^{2k} / r_0^{2k} \tag{22}$$

so that

$$f(z) = \mathcal{J}_M(z) + \lambda \mathcal{J}_M(z \cdot r_1^2/r_0^2). \tag{23}$$

Thus the extremal function $\mathcal{J}_M(z)$ is of[12] class H^2 in $|z| < r_1^2/r_0^2$ if and only if $f(z)$ has this property in $|z| < 1$: indeed, if $e^{i\theta}$ is a singular point of $f(z)$, then $r_1^2 e^{i\theta}/r_0^2$ is a singularity of the same type for $\mathcal{J}_M(z)$. From (23), moreover, we obtain

$$\lambda^n \left\{ \mathcal{J}_M(z) + \sum_{k=1}^{n} (-1)^k \frac{f(r_0^{2k} z/r_1^{2k})}{\lambda^k} \right\} = (-1)^n \mathcal{J}_M(r_0^{2n} z/r_1^{2n}). \tag{24}$$

Thus, if $f(z)$ is meromorphic in a region R containing the interior of the circle $|z| = 1$, then $\mathcal{J}_M(z)$ is meromorphic in the enlarged region obtained by stretching R in the ratio r_1^2/r_0^2 from the origin. Poles, however, may be increased in number along a radial line from the origin. For example, if $f(z) = 1/(1-z)$, the function $\mathcal{J}_M(z)$ is analytic in the entire (finite) plane except for simple poles at $z = (r_1^2/r_0^2)^k$, $k = 1, 2, \cdots$.

For $\lambda > 1$, the relation (24) yields an alternative representation for the function of best approximation:

$$\mathcal{J}_M(z) = \mathcal{J}_M(z; f) = \sum_{k=1}^{\infty} (-1)^{k+1} \lambda^{-k} f(r_0^{2k} z/r_1^{2k}). \tag{25}$$

This representation, as well as equation (23), is independent of the requirement that $|z| = 1$ be the circle of convergence of $\sum a_k z^k = f(z)$. The condition $\lambda > 1$ may be relaxed somewhat in certain cases: for example, if $a_0 = 0$, the series converges for $\lambda > r_0^2/r_1^2$. Equation (25) could also have been derived directly from (23) if that difference equation were solved by the usual method of successive approximations.

We no longer require $f(z)$ to have a singularity on $|z| = 1$. Suppose now that $f(z) = 1/(t-z)$ where t is fixed, $r_0 < |t| < r_1$. Let M be given. By (25) we have

$$\mathcal{J}_M(z;\ 1/(t-z)) = \sum_{k=1}^{\infty} (-1)^{k+1} \lambda^{-k} (t - r_0^{2k} z/r_1^{2k})^{-1}.$$

By the Cauchy Integral Formula, we have

$$f(r_0^{2k} z/r_1^{2k}) = \frac{1}{2\pi i} \int_{|t|=r} \frac{f(t)dt}{t - r_0^{2k} z/r_1^{2k}},$$

provided $r_0^{2k}|z|/r_1^{2k} < r < 1$. Thus, for $|z| < r_1^2 r/r_0^2$, we have

$$\frac{1}{2\pi i} \int_{|t|=r} \mathcal{J}_M(z;\ 1/(t-z)) f(t)dt = \sum_{k=1}^{\infty} (-1)^{k+1} \lambda^{-k} f(r_0^{2k} z/r_1^{2k}) = \mathcal{J}_{M'}(z; f),$$

where

[12] For this notation see J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1935, Chap. 6.

$$\sum_{k=0}^{\infty} \frac{|a_k|^2 r_0^{4k} r_1^{2k}}{(r_0^{2k} + \lambda r_1^{2k})^2} = M'^2, \qquad \sum_{k=0}^{\infty} \frac{r^{-2k-2} r_0^{4k} r_1^{2k}}{(r_0^{2k} + \lambda r_1^{2k})^2} = M^2.$$

The relation (22) has certain consequences involving orthogonality. If $\sum_{k=0}^{\infty} c_k z^k$ is analytic in $|z| < r_1$, then by (22), for $0 < r < 1$, we have

$$\sum_{k=0}^{\infty} c_k(\bar{a}_k - \bar{b}_k) r^{2k} = \lambda \sum_{k=0}^{\infty} c_k \bar{b}_k r_1^{2k} r^{2k} / r_0^{2k}.$$

Thus, if $\sum c_k z^k$ and $f(z) - \mathcal{J}_M(z)$ are orthogonal on $|z| = r < 1$, then $\sum c_k z^k$ and $\mathcal{J}_M(z)$ are orthogonal on $|z| = r_1 r / r_0$, and conversely. The cases $r = r_0$ and $r = r_0/r_1$ are especially interesting. Moreover, if we choose $r = r_0^{1/2}/r_1^{1/2}$, the circles $|z| = r$ and $|z| = r_1 r / r_0$ are mutually inverse in $|z| = 1$. This orthogonality may be considered from another point of view: $\sum c_k z^k$ is orthogonal to $f(z)$ on $|z| = r < 1$ if $\sum c_k z^k$ is simultaneously orthogonal to $\mathcal{J}_M(z)$ on $|z| = r$ and on $|z| = r_1 r / r_0$. If $M^2 < \{\text{l.u.b.}_{r<r_1} \mu_2(f, |z| = r)\}$, we have $\mathcal{J}_M(z) = f(z)/(1+\lambda)$ so that $\sum c_k z^k$ is orthogonal on $|z| = r < r_1$ to $\mathcal{J}_M(z)$ if it is orthogonal there to $f(z)$.

From (23) there follow the functional relationships

$$\mu_2(f - \mathcal{J}_M;\ |z| = r) = \lambda \mu_2(\mathcal{J}_M;\ |z| = r_1^2 r / r_0^2)$$

and

$$\{\mu_2(f - \mathcal{J}_M;\ |z| = r)\}^2 = \{\mu_2(f;\ |z| = r)\}^2 - \{\mu_2(\mathcal{J}_M;\ |z| = r)\}^2 - 2\lambda\{\mu_2(\mathcal{J}_M;\ |z| = r_1 r / r_0)\}^2,$$

where $\mu_2(F; |z| = r)$ denotes the integral mean of $|F|^2$ over the circle $|z| = r$. The second of these identities has, as a special case, the relation

$$m_M^2 = \{\mu_2(f;\ |z| = r_0)\}^2 - \{\mu_2(\mathcal{J}_M,\ |z| = r_0)\}^2 - 2\lambda M^2.$$

To these we may add a final one,

$$\frac{dm_M^2}{d\lambda} = -\lambda \frac{dM^2}{d\lambda}.$$

It is seen from (23) that every zero of $\mathcal{J}_M(r_1^2 z / r_0^2)$ is a point of interpolation of $\mathcal{J}_M(z)$ to $f(z)$. Since $\mathcal{J}_M(z)$ converges to $f(z)$ uniformly on every closed set in $|z| < 1$, to every sequence $\{M_n\}$, $M_n \to \infty$, and z_0, $|z_0| < 1$, corresponds a sequence of points $\{z_n\}$ within the unit circle approaching z_0 such that $\mathcal{J}_{M_n}(z_n) = f(z_0)$. Thus, if $f(z_0) = 0$, we have $f(z_n r_0^2/r_1^2) - \mathcal{J}_{M_n}(z_n r_0^2 / r_1^2) = 0$ and each $z_n r_0^2 / r_1^2$ is a point of interpolation. If $f(z) \neq 0$ in $|z| < 1$, then on each closed subset of $|z| < 1$, $\mathcal{J}_M(z) \neq 0$ for M sufficiently large, and there will be no points of interpolation in $|z| < r_0^2 / r_1^2$. Conversely, if $\{z_n\}$ is a sequence of points of interpolation of $\mathcal{J}_{M_n}(z)$ to $f(z)$ having a limit point z_0 interior to $|z| < r_0^2/r_1^2$, then $f(r_1^2 z_0/r_1^2) = 0$; and if there exists no such infinite set of points $\{z_n\}$, $f(z) \neq 0$ in $|z| < 1$.

The totality of functions $\mathcal{J}_M(z) = \sum_{k=0}^{\infty} a_k r_0^{2k} z^k/(r_0^{2k} + \lambda r_1^{2k})$ for a given $f(z)$ depends only upon r_1/r_0 rather than upon r_1 and r_0 separately. Moreover, *every* function analytic in $|z| < r_1^2/r_0^2$ but analytic in no larger circle whose center is the origin is a function of best approximation $\mathcal{J}_M(z)$ for some function $f(z)$ of the type under discussion (analytic in $|z| < 1$ with a non-removable singularity on the unit circle).

5. **Overconvergence and lacunary structure.** Under suitable circumstances, as first shown by M. B. Porter, certain subsequences of the partial sums of the Taylor series for $f(z)$ will converge uniformly in a region containing in its interior points of the circle of convergence. The relation of this phenomenon of "overconvergence" to the structure of the Taylor series is well known(13). Ostrowski has shown that if the series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has an infinite numbers of gaps—if $a_k = 0$ for $\mu_n < k \leqq \nu_n$ $(n = 1, 2, \cdots)$, $\mu_n < \delta \nu_n$ where $\delta < 1$ is fixed—then the partial sums $\left\{ \sum_{k=0}^{\mu_n} a_k z^k \right\}$ converge uniformly throughout the neighborhood of every point on the circle of convergence where the function $f(z)$ is analytic. Conversely, any Taylor series which contains an overconverging subsequence of partial sums can be expressed as the sum of a series having gaps of the above type and a series with a larger radius of convergence. A function which can be so expressed is said to be of *lacunary structure.*

The authors have recently(14) proved the following result connecting the property of lacunary structure with the degree of convergence of Tchebycheff polynomials of best approximation:

THEOREM. *Let the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ have unit radius of convergence and let $\Pi_n(z)$ denote the polynomial of degree n of best approximation to $f(z)$ in the closed region $|z| \leqq \rho < 1$ in the sense of Tchebycheff. A necessary and sufficient condition that $f(z)$ be of lacunary structure is that*

$$\liminf_{n\to\infty} \left[\max |f(z) - \Pi_n(z)|, \ |z| \leqq \rho\right]^{1/n} < \limsup_{n\to\infty} \left[\max |f(z) - \Pi_n(z)|, \ |z| \leqq \rho\right]^{1/n}.$$

As may be expected from this result, the existence of the limit $\lim_{M\to\infty} m_M^{1/\log M}$ is also connected with the structure of $f(z)$. In order to investigate this relationship, we shall prove the following lemma concerning the behavior of λ as a function of M. For the case of circular regions and $p = q = 2$ just considered in §4, we have:

(13) G. Bourion, *L'ultraconvergence dans les séries de Taylor*, Actualités Scientifiques et Industrielles, no. 472, 1937.

(14) J. L. Walsh and E. N. Nilson, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 116–117.

LEMMA. *The inequality*

$$\liminf_{M\to\infty} m_M^{1/\log M} < \limsup_{M\to\infty} m_M^{1/\log M} \tag{26}$$

implies

$$\liminf_{\lambda\to 0} \frac{\log M^2}{-\log \lambda} < \alpha, \tag{27}$$

where $\alpha = \log r_1/\log (r_1/r_0)$.

By hypothesis, there exists a sequence $\{M_k\}$ and $\epsilon > 0$ such that

$$\limsup_{k\to\infty} \frac{\log m_{M_k}}{\log M_k} < \frac{\log r_0}{\log r_1} - \epsilon,$$

since the right-hand member of (26) is equal to $r_0^{1/\log r_1}$. It follows by means of the Hölder inequality for sums that, for $t>1$,

$$M_k^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2 r_0^{4n} r_1^{2n}}{(r_0^{2n} + \lambda_k r_1^{2n})^2}$$

$$\leqq \left\{\sum_{n=0}^{\infty} \frac{|a_n|^2 r_1^{4n} r_0^{2n}}{(r_0^{2n} + \lambda_k r_1^{2n})^2}\right\}^{1/t} \left\{\sum_{n=0}^{\infty} \frac{|a_n|^2 r_0^{4n}}{(r_0^{2n} + \lambda_k r_1^{2n})^2} (r_1^{(t-2)/(t-1)} r_0^{1/(t-1)})^{2n}\right\}^{(t-1)/t},$$

where λ_k is the value of λ associated with $M = M_k$. The second expression in braces converges uniformly with respect to λ_k, $0 \leqq \lambda_k < \infty$, provided

$$r_1^{(t-2)/(t-1)} r_0^{1/(t-1)} < 1;$$

that is, provided $t < 1 + 1/\alpha$. Now we have

$$m_{M_k}^2 = \lambda_k^2 \sum_{n=0}^{\infty} \frac{|a_n|^2 r_1^{4n} r_0^{2n}}{(r_0^{2n} + \lambda_k r_1^{2n})^2}.$$

Thus, by the definition of the sequence $\{M_k\}$,

$$\limsup_{\lambda_k\to 0} \frac{\log M_k^2}{-\log \lambda_k} \leqq \frac{2}{t - 1 + 1/\alpha + \epsilon}. \tag{28}$$

Since the left-hand member of (28) is independent of $t < 1 + 1/\alpha$, inequality (27) follows. Actually, inequality (27) is a sufficient as well as necessary condition for the validity of (26); but the sufficiency proof, which is direct but rather cumbersome, is not essential to the development here.

THEOREM 4. *A necessary and sufficient condition that $f(z)$ be of lacunary structure is that*

$$\liminf_{M\to\infty} m_M^{1/\log M} < \limsup_{M\to\infty} m_M^{1/\log M}. \tag{29}$$

By Bourion[15] it follows, if $f(z)$ is of lacunary structure, that there exists a subsequence $\{s_{n_k}(z)\}$ of the partial sums of the Taylor series for $f(z)$ about $z=0$ such that for some $\epsilon>0$ and all n_k sufficiently large

$$|f(z) - s_{n_k}(z)| \leqq r_0^{(1+\epsilon)n_k}, \qquad |z| \leqq r_0.$$

Moreover, it is true for all partial sums that

$$|s_{n_k}(z)| \leqq r_1^{(1+\delta)n}, \qquad |z| \leqq r_1,$$

for arbitrary $\delta>0$ and all n_k sufficiently large. Choose $M_k = r_1^{(1+\delta)n_k}$. Then $\mu_2(s_{n_k}, 1) \leqq M_k$ while $\mu_2(f-s_{n_k}, 0) \leqq r_0^{(1+\epsilon)n_k}$. From the extremal character of the functions $\mathcal{J}_M(z)$ it follows that

$$\limsup_{M_k\to\infty} m_{M_k}^{1/\log M_k} < r_0^{1/\log r_1} = \limsup_{M\to\infty} m_M^{1/\log M}.$$

In order to prove the sufficiency, a sequence of polynomials $\{q_{n_k}(z)\}$ will be exhibited satisfying, for some positive $\rho < r_0$, the inequality

$$\limsup_{k\to\infty} \{\max |f(z) - q_{n_k}(z)|, |z| \leqq \rho\}^{1/n_k} < \rho. \tag{30}$$

It will follow that (30) is satisfied by the Tchebycheff polynomials $\{\Pi_{n_k}(z)\}$ of best approximation to $f(z)$ on $|z| \leqq \rho$ and hence, by the theorem of the authors quoted above, that $f(z)$ is of lacunary structure.

Assume, then, that (29) is valid: there exists a sequence $\{M_k\}$ of values of M such that

$$\limsup_{k\to\infty} m_{M_k}^{1/\log M_k} < \limsup_{M\to\infty} m_M^{1/\log M} = r_0^{1/\log r_1}.$$

Take

$$q_{n_k}(z) = \sum_{n=0}^{n_k} \frac{a_n r_0^{2n} z^n}{r_0^{2n} + \lambda_k r_1^{2n}},$$

where $n_k = [\log M_k/\log r_1]$. By the Minkowski inequality,

$$\mu_2(f - q_{n_k}, 0) \leqq \mu_2(f - \mathcal{J}_{M_k}, 0) + \mu_2(\mathcal{J}_{M_k} - q_{n_k}, 0).$$

In view of (29),

$$\limsup_{k\to\infty} \mu_2(f - \mathcal{J}_{M_k}, 0)^{1/n_k} < r_0;$$

thus

[15] G. Bourion op. cit. pp. 9 ff.

(31) $$\limsup_{k\to\infty} \mu_2(f - q_{n_k}, 0)^{1/n_k} < r_0$$

provided only that

(32) $$\limsup_{k\to\infty} \mu_2(\mathcal{J}_{M_k} - q_{n_k}, 0)^{1/n_k} < r_0.$$

Now

$$\left\{\mu_2(\mathcal{J}_{M_k} - q_{n_k}, 0)\right\}^2 = \sum_{n=n_k+1}^{\infty} \frac{|a_n|^2 r_0^{6n}}{(r_0^{2n} + \lambda_k r_1^{2n})^2} \leqq \int_{n_k}^{\infty} \frac{r_0^{2(1-\epsilon)x}\, dx}{(1 + \lambda_k r_1^{2x}/r_0^{2x})^2}$$

for arbitrary $\epsilon > 0$ provided n_k is sufficiently large. Set $y = \lambda_k r_1^{2x}/r_0^{2x}$ and $(r_1/r_0)^{\beta} = r_0^{1-\epsilon}$. Then

$$\left\{\mu_2(\mathcal{J}_{M_k} - q_{n_k}, 0)\right\}^2 \leqq \frac{\lambda_k^{-\beta}}{2 \log (r_1/r_0)} \int_{\lambda_k r_1^{2n}/r_0^{2n}}^{\infty} \frac{y^{\beta-1}\, dy}{(1 + y)^2}.$$

From the lemma it follows that for some $\epsilon_1 > 0$ and all n_k sufficiently large,

$$2n_k \leqq \frac{(1 - \epsilon_1)\alpha(-\log \lambda_k)}{\log r_1}.$$

Thus

$$\limsup_{k\to\infty} \frac{1}{n_k} \log \mu_2(\mathcal{J}_{M_k} - q_{n_k}, 0) \leqq \frac{1 - \epsilon}{1 - \epsilon_1} \log r_0,$$

which is less than $\log r_0 < 0$ if ϵ is sufficiently small, and (32) results.

Inequality (30) is now an immediate consequence of (31) and of the Cauchy Integral Formula; if ρ is arbitrary, $0 < \rho < r_0$, we have for $|z| \leqq \rho < r_0$,

$$\begin{aligned} |f(z) - q_{n_k}(z)| &= \left| \frac{1}{2\pi i} \int_{|t|=r_0} \frac{f(t) - q_{n_k}(t)}{t - z}\, dt \right| \\ &\leqq \frac{r_0^{1/2}}{r_0 - \rho} \left\{ \frac{1}{2\pi r_0} \int_{|t|=r_0} |f(t) - q_{n_k}(t)|^2 |dt| \right\}^{1/2}. \end{aligned}$$

This completes the proof of the theorem.

COROLLARY 1. *If there exists a sequence of extremal functions* $\{\mathcal{J}_{M_k}(z)\}$ *which overconverges, then* (29) *follows. Conversely, if* (29) *holds, there exists an overconverging sequence* $\{\mathcal{J}_{M_k}(z)\}$ *provided* $f(z)$ *is analytic at some point on* $|z| = 1$.

COROLLARY 2. *A necessary and sufficient condition that* $f(z)$ *be of lacunary structure is the validity of* (29) *for any particular values of* p *and* q, $0 < p \leqq \infty$ *and* $0 < q \leqq \infty$.

By taking a suitably chosen $r_1^* > r_1$ and considering the functions $\mathcal{J}_M^*(z)$ of best approximation to $f(z)$ on $S(|z| \leqq r_0)$ with R the point set $|z| < r_1^*$, it can be shown directly that the validity of (29) for $p = q = 2$ and this configuration implies its validity for arbitrary p and q, $0 < p \leqq \infty$ and $0 < q \leqq \infty$, with the original configuration (S: $|z| \leqq r_0$ and R: $|z| < r_1$). In a similar manner the sufficiency is proved.

6. Norm functions. Functional properties of extremal functions. Suppose that the norm function $n_0(\psi)$ and $n_1(\psi)$ are positive and continuous. For given positive M, we wish to consider those functions $F(z)$ analytic in R with bounded integral means on the C_ν of order q, and

$$\left\{\frac{1}{\tau}\int_{C_1} |F|^q n_1(\psi)d\psi\right\}^{1/q} \leqq M. \tag{33}$$

The measure of approximation of $F(z)$ to $f(z)$ is taken as

$$\left\{\frac{1}{\tau}\int_{C_0} |f - F|^p n_0(\psi)d\psi\right\}^{1/p}. \tag{34}$$

Compactness results for the set of functions satisfying (33), and there exists a function of best approximation in the sense of (34). The analogue of Theorem 2 carries over without serious difficulty to this situation. The uniqueness of the extremal function for $1 \leqq p < \infty$ and $1 \leqq q \leqq \infty$ is also immediate.

It is worth while pointing out here that if C_0 and C_1 are smooth, we may choose $n_1(\psi) = ds/d\psi$ on C_1 and $n_0(\psi) = ds/d\psi$ on C_0. Then the integrals over C_0 and C_1 become ordinary line integrals.

The measure of best approximation, m_M, as defined in §2 is a continuous functional of $f(z)$ and M. For let $\mathcal{J}_{M'}^*$ be a function of best approximation to $f^*(z)$ and $m_{M'}^* = \mu_p(f^* - \mathcal{J}_{M'}^*, 0)$. Then $\mu_k(M\mathcal{J}_{M'}^*/M', 1) \leqq M$ so that

$$\overset{t}{m}_M \leqq \{\mu_p(f - M\mathcal{J}_{M'}^*/M', 0)\}^t \leqq \{\mu_p(f - f^*, 0)\}^t + \{\mu_p(f^* - \mathcal{J}_{M'}^*, 0)\}^t + \{\mu_p(\mathcal{J}_{M'}^* - M\mathcal{J}_{M'}^*/M', 0)\}^t,$$

where $t = 1$ or p according as $p \geqq 1$ or $0 < p < 1$. Thus

$$\overset{t}{m}_M - \overset{*t}{m}_{M'} \leqq \{\mu_p(f - f^*, 0)\}^t + |M - M'|^t.$$

Similarly,

$$\overset{*t}{m}_{M'} - \overset{t}{m}_M \leqq \{\mu_p(f - f^*, 0)\}^t + |M' - M|^t;$$

hence, $m_{M'}^* \to m_M$ if $M' \to M$ and $\mu_p(f - f^*, 0) \to 0$.

An immediate consequence of the continuity of m_M is the fact that

$\mu_q(\mathcal{J}^*_{M'}-\mathcal{J}_M,\ 1)\to 0$ as $M'\to M$ and $\mu_p(f-f^*,\ 0)\to 0$, provided $\mathcal{J}_M$ is unique. Assume the contrary: that there exist $\{M_k\}$ and $\{f^*_k\}$, $M_k\to M$ and $\mu_p(f-f^*_k,\ 0)\to 0$, such that $\mu_q(\mathcal{J}^*_{M_k}-\mathcal{J}_M,\ 1)>\epsilon$ for some $\epsilon>0$. The functions $\{\mathcal{J}^*_{M_k}\}$ form a normal family in R; hence a subsequence can be found converging uniformly on each closed subset of R to a function F_M analytic in R, of norm M, and having measure of approximation m_M to $f(z)$: with t defined as previously,

$$m_M^t \leqq \{\mu_p(f - F_M,\ 0)\}^t \leqq \{\mu_p(f - f^*_k,\ 0)\}^t + \{\mu_p(f^*_k - \mathcal{J}^*_{M_k},\ 0)\}^t + \{\mu_p(\mathcal{J}^*_{M_k} - F_M,\ 0)\}^t.$$

But $\{\mu_p(f^*_k-\mathcal{J}^*_{M_k},\ 0)\}^t=m^{*t}_{M_k}\to m^t_M$, so that $\mu_p(f-F_M,\ 0)=m_M$. A contradiction is reached whenever $\mathcal{J}_M$ is unique, for

$$0 < \epsilon < \{\mu_q(\mathcal{J}_M - \mathcal{J}^*_{M_k},\ 1)\}^t \leqq \{\mu_q(\mathcal{J}_M - F_M,\ 1)\}^t + \{\mu_q(F_M - \mathcal{J}^*_{M_k},\ 1)\}^t.$$

Similarly, it can be shown that M_m is a continuous functional of $f(z)$ and m if $1\leqq p\leqq\infty$. The proof is again indirect. Assume that there exist $\{m_k\}$ and $\{f^*_k\}$, $m_k\to m$ and $\mu_p(f-f^*_k,\ 0)\to 0$, such that $|M^*_{m_k}-M_m|>\epsilon$ for some $\epsilon>0$ where $M^*_{m_k}=\mu_q(\mathcal{G}^*_{m_k},\ 1)$. From the extremal character of M_m, it follows immediately that $M^*_{m_k}>M_m+\epsilon$. Define $M=M_m+\epsilon/2$, and let $\mathcal{J}_M$ be the function of norm M of best approximation to $f(z)$. From §2 it is seen that $m_{M_m}=m$ always, while $M_{m_M}=M$ if $1\leqq p\leqq\infty$. Thus $m<m'$ implies $M_m>M_{m'}$ always, while $M>M'$ implies $m_M<m_{M'}$ if $1\leqq p\leqq\infty$. Thus $m_M=\mu_p(f-\mathcal{J}_M,\ 0)<m$. In the sequences $\{m_k\}$ and $\{f^*_k\}$ choose m_n and f^*_n such that $|m_n-m|<\delta/2$ and $\mu_p(f-f^*_n,\ 0)<\delta/2$, where $\delta=m-m_M$. Then

$$\mu_p(f^*_n - \mathcal{J}_M,\ 0) \leqq \mu_p(f - \mathcal{J}_M,\ 0) + \mu_p(f - f^*_n,\ 0) < m_M + \delta/2 < m_n,$$

and a contradiction is reached with the extremal nature of $\mathcal{G}^*_{m_k}$.

Again, $\mu_q(\mathcal{G}_m-\mathcal{G}^*_{m'},\ 1)\to 0$ as $m'\to m$ and $\mu_p(f-f^*,\ 0)\to 0$, provided M_m is continuous and $\mathcal{G}_m$ is unique. The proof is similar to that used above for functions of best approximation.

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