RECTILINEAR CONGRUENCES

BY

CHUAN-CHIH HSIUNG(1)

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Introduction. In his prize memoir [9](2), Wilczynski has established the theory of a rectilinear congruence in ordinary three-dimensional projective space by using a system of linear partial differential equations. However, his method of deriving the system of differential equations is not completely geometric. The author proposes to remedy this lack of geometric content in the present paper.

In §1 we introduce, by a purely geometric method, a completely integrable system of linear homogeneous partial differential equations which defines a rectilinear congruence in ordinary space except for a projective transformation. The integrability conditions of the system of differential equations are also calculated.

In §2 we study the effect, on the differential equations of §1, of a group of transformations which leave invariant the parametric focal nets N_y, N_z on the focal surfaces S_y, S_z of an integral rectilinear congruence $\mathcal{L}$ of these equations. Some invariants and covariants of these equations under this group of transformations are also obtained and listed.

In §3 we calculate for the focal surfaces S_y, S_z local power series expansions, each set of which expresses a local nonhomogeneous projective co-

Presented to the Society, June 19, 1948; received by the editors April 14, 1948, and, in revised form, May 13, 1948.

(1) The author is indebted to Professor V. G. Grove and the referee for their suggestions during the preparation and the revision of this paper.

(2) Numbers in brackets refer to the bibliography at the end of the paper.

ordinate of a point on one surface as a power series in the other two coordinates and represents the surface in the neighborhood of an ordinary point on it.

In §4 a canonical form of the system of differential equations of §1 is obtained by a geometric determination. We then reduce the power series expansions of the focal surfaces S_y, S_z in §3 to canonical forms, and find the loci of some osculants associated with the plane sections of the focal surfaces S_y, S_z made by a variable plane through a generator yz of the congruence $\mathcal{L}$.

In §5, by means of the quadrics of Moutard for the tangents to the curves of the focal nets N_y, N_z of the focal surfaces S_y, S_z we study some special kinds of the congruence $\mathcal{L}$ and determine geometrically the unit point of the coordinate system for a general congruence $\mathcal{L}$.

In §6, we find the equations of the curves of Darboux and the pencils of quadrics of Darboux at the focal points y, z of the focal surfaces S_y, S_z and geometrically characterize the congruence $\mathcal{L}$ when one or both of its focal nets are Segre-Darboux nets. The conditions for both surfaces S_y, S_z to be isothermally asymptotic at the same time are also deduced.

In §7, the Weingarten invariants and the tangential invariants of the focal nets N_y, N_z are derived and a simple geometrical characterization of a W congruence is given.

§8 contains the local power series expansions for the u-, v-curves of the parametric focal net N_y on the focal surface S_y. These expansions express two local nonhomogeneous projective coordinates of a point on each curve as power series in the other coordinate, and represent the curve in the neighborhood of an ordinary point on it. Quadrics having contact of different orders with the u-, v-curves and the surface S_y at the point y are considered.

By *a line* $l_1(l_1')$ we mean, as usual, any line through the point $y(z)$ of the surface $S_y(S_z)$ and not lying in the tangent plane of the surface $S_y(S_z)$ at $y(z)$; by *a line* $l_2(l_2')$ we mean, dually, any line in the tangent plane of the surface $S_y(S_z)$ at $y(z)$ but not passing through the point $y(z)$. In §9, we derive two correspondences between l_1 and l_2 and between l_1' and l_2', and present another new geometrical characterization of a W congruence.

§10 is concerned with the determination of the developables and focal surfaces of the axis congruences, and also of the ray congruences, of the focal nets N_y, N_z of the congruence $\mathcal{L}$. The condition for the focal net N_y or N_z to be harmonic is also obtained.

The last section is devoted to a study of some covariant congruences associated with the congruence $\mathcal{L}$ by methods similar to those used in §10.

1. **Differential equations and integrability conditions.** First of all, we consider in ordinary projective space a congruence with two distinct proper focal surfaces S_y, S_z generated respectively by the two focal points y, z of a generator yz of the congruence. Let the parametric curves u, v on the surfaces S_y, S_z be taken as the curves of the conjugate nets N_y, N_z in which the de-

velopables of the congruence touch the surfaces S_y, S_z; and let the u-tangent at the point y of the surface S_y and the v-tangent at the point z of the surface S_z coincide in the generator yz. If we select two points η, ζ respectively on the v-tangent at the point y of the surface S_y and the u-tangent at the point z of the surface S_z, and if we suppose that the coordinates η, ζ of the points η, ζ are functions of u, v; then it can be shown that the coordinates y, z, η, ζ of corresponding points y, z, η, ζ satisfy a system of linear homogeneous partial differential equations of the form

$$
\begin{aligned}
y_u &= \alpha y + \beta z, \\
z_v &= \gamma y + \delta z, \\
y_v &= a y + c\eta, \\
z_u &= b' z + c'\zeta, \\
y_{vv} &= m y + n z + p\eta + q\zeta, \\
z_{uu} &= m' y + n' z + p'\eta + q'\zeta \qquad (cc'p'q\beta\gamma \neq 0),
\end{aligned}
\tag{1.1}
$$

in which subscripts indicate partial differentiation and the coefficients are scalar functions of u, v.

The derivatives η_u, η_v may be written as linear combinations of y, z, η, ζ by using equations (1.1) and

$$
(y_u)_v = (y_v)_u, \qquad y_{vv} = (y_v)_v;
$$

the result is

$$
\begin{aligned}
\eta_u &= e y + f z + g\eta, \\
\eta_v &= r y + \frac{n}{c} z + s\eta + \frac{q}{c}\zeta,
\end{aligned}
\tag{1.2}
$$

in which the coefficients are defined by the following equations:

$$
\begin{aligned}
ce &= \alpha_v + \beta\gamma - a_u, \qquad & cf &= \beta_v + \beta\delta - a\beta, \\
cr &= m - a_v - a^2, & cs &= p - c_v - ac, \\
g &= \alpha - (\log c)_u.
\end{aligned}
\tag{1.3}
$$

Analogous expressions for ζ_u, ζ_v can be written by making the substitution

$$
\begin{pmatrix}
y & \eta & u & 1 & 3 & \alpha & \beta & a & c & e & f & g & m & n & p & q & r & s \\
z & \zeta & v & 2 & 4 & \delta & \gamma & b' & c' & f' & e' & g' & n' & m' & q' & p' & s' & r'
\end{pmatrix}.
\tag{1.4}
$$

The integrability conditions of equations (1.1) are found by the usual method from the equations

$$
(\eta_u)_v = (\eta_v)_u, \qquad (\zeta_u)_v = (\zeta_v)_u
$$

and the fact that the points y, z, η, ζ are linearly independent. These conditions are

$$
(1.5)\qquad
\begin{aligned}
e_v + ae + f\gamma + gr &= r_u + r\alpha + es + m'q/cc', \\
f_v + f\delta + gn/c &= r\beta + (n/c)_u + b'n/c + fs + qs'/c, \\
ce + g_v &= s_u + p'q/cc', \\
gq &= c'n + c(q/c)_u + qr',
\end{aligned}
$$

and the analogous ones obtainable therefrom by the substitution (1.4).

Making use of equations (1.3), the third of equations (1.5), and the substitution (1.4) we obtain the equation

$$
(1.6)\qquad (a + g' + \delta + s)_u = (g + b' + \alpha + r')_v.
$$

It follows that there exists a function θ of u, v which is defined, except for an arbitrary additive constant, as a solution of the differential equations

$$
(1.7)\qquad \theta_u = g + b' + \alpha + r', \qquad \theta_v = a + g' + \delta + s.
$$

Accordingly, the following formula is valid:

$$
(1.8)\qquad (y, z, \eta, \zeta) = e^\theta,
$$

where a determinant is indicated by writing only a typical row within parentheses.

2. **Transformations, invariants, and covariants.** Let us consider the group of transformations on the coordinates y, z, η, ζ and the parameters u, v:

$$
(2.1)\qquad y = \lambda\bar{y}, \qquad z = \mu\bar{z}, \qquad \eta = \nu\bar{\eta}, \qquad \zeta = \tau\bar{\zeta} \qquad (\lambda\mu\nu\tau \neq 0),
$$

$$
(2.2)\qquad u = U(u), \qquad \bar{v} = V(v) \qquad (U'V' \neq 0),
$$

where λ, μ, ν, τ are scalar functions of u, v and the accent denotes differentiation with respect to the appropriate variable.

The effect of the transformation (2.1) on the system of equations (1.1) is to produce another system of equations of the same form whose coefficients, indicated by dashes, are given by the following formulas and the analogous ones:

$$
(2.3)\qquad
\begin{aligned}
&\bar{\alpha} = \alpha - \lambda_u/\lambda, \qquad \bar{\beta} = \beta\mu/\lambda, \qquad \bar{a} = a - \lambda_v/\lambda, \qquad \bar{c} = c\nu/\lambda, \\
&\bar{m} = \frac{1}{\lambda}(m\lambda - \lambda_{vv} - 2a\lambda_v + 2\lambda_v^2/\lambda), \qquad \bar{n} = n\mu/\lambda, \\
&\bar{p} = \frac{\nu}{\lambda}(p - 2c\lambda_v/\lambda), \qquad \bar{q} = q\tau/\lambda.
\end{aligned}
$$

The effect of the transformation (2.2) on the system of equations (1.1) is to produce another system of equations of the same form whose coefficients, indicated by stars, are given by the following formulas and the analogous ones:

$$(2.4)\qquad \begin{aligned} &\alpha^* = \alpha/U', \qquad \beta^* = \beta/U', \qquad a^* = a/V', \qquad c^* = c/V',\\ &m^* = (m - aV''/V')/V'^2, \qquad n^* = n/V'^2,\\ &p^* = (p - cV''/V')/V'^2, \qquad q^* = q/V'^2. \end{aligned}$$

From equations (1.3), (2.3), (2.4), and the substitution (1.4) we may obtain, after some calculation, the following functions which are absolute invariants under the transformation (2.1), and relative invariants under the transformation (2.2) of the system of equations (1.1):

$$(2.5)\qquad \begin{aligned} &A = cm'/p', && B = m'\beta,\\ &D = c'\beta/q, && G = m'n,\\ &H = \alpha_v + \beta\gamma - \delta_u - (\log\beta)uv, && K = \beta\gamma,\\ &\mathfrak{H} = p'q/cc', && \mathfrak{K} = c'f' + (c'n/q)_v,\\ &W_{(u)} = \mathfrak{H} - K, && W_{(v)} = \mathfrak{K} - H,\\ &I = cf/\beta, && J = ce,\\ &L = c's', && M = 3a + g' - 2p/c + q_v/q,\\ &N = \frac{n}{\beta}(2b' - q'/c' + \psi_u - c'n/q + qs'/n), && P = 2a - p/c + \phi_v, \end{aligned}$$

where

$$\phi = \log A, \qquad \psi = \log(c'n/q).$$

Analogous invariants can be written by using the substitutions (1.4) and

$$(2.6)\qquad \begin{pmatrix} A & B & D & G & H & K & \mathfrak{H} & \mathfrak{K} & W_{(u)} & W_{(v)} & I & J & L & M & N & P\\ A' & B' & D' & G & K' & H' & \mathfrak{K}' & \mathfrak{H}' & W'_{(v)} & W'_{(u)} & I' & J' & L' & M' & N' & P' \end{pmatrix}.$$

These invariants are obviously not all independent. Among them it is easy to obtain the following relations and the analogous ones:

$$(2.7)\qquad \begin{aligned} &AA' = G/\mathfrak{H}, \qquad BB' = GK, \qquad DD' = K/\mathfrak{H},\\ &J = H + I_u = \mathfrak{H}' - A_u, \qquad N = [L + A'(P' - A')]/D. \end{aligned}$$

In terms of the invariants, the second and the last of the integrability conditions (1.5) then can be written in the form

$$(2.8)\qquad I_v + I(I + P - \phi_v) = L' + N, \alpha - r' - q_u/q = A'.$$

In §4 we shall derive similar expressions for the other two conditions of (1.5).

From equations (1.1), by differentiation and substitution we may obtain the Laplace equation of the focal net N_y:

$$(2.9)\qquad y_{uv} = (\alpha_v + \beta\gamma - \alpha\delta - \alpha\beta_v/\beta)y + (\beta_v/\beta + \delta)y_u + \alpha y_v.$$

It is easy to show that the point invariants of Laplace-Darboux of the net N_y are H, K defined by equations (2.5). Moreover, the point z is the ray-point y_{-1} of the v-curve corresponding to the point y or the minus-first Laplace transformed point of y with respect to the net N_y, and the ray-point y_1 of the u-curve corresponding to the point y or the first Laplace transformed point of y with respect to the net N_y is defined by the equation

$$y_1 = -Iy + c\eta. \tag{2.10}$$

Similarly, the point invariants of Laplace-Darboux of the net N_z are H', K', the ray-point z_1 of the u-curve corresponding to the point z is the point y, and the ray-point z_{-1} of the v-curve corresponding to the point z is

$$z_{-1} = -I'z + c'\zeta. \tag{2.11}$$

3. **Power series expansions for the focal surfaces S_y, S_z.** From the system (1.1) and equations (1.2), (1.3) and the substitution (1.4) by differentiation and substitution, any derivative of y can be expressed as a linear combination of y, z, η, ζ. In particular, one obtains

$$\begin{aligned}
y_{uu} &= (\alpha_u + \alpha^2)y + (\beta_u + \alpha\beta + b'\beta)z + c'\beta\zeta,\\
y_{uv} &= (\alpha_v + a\alpha + \beta\gamma)y + (\beta_v + \beta\delta)z + c\alpha\eta,\\
y_{uuu} &= (\alpha_{uu} + 3\alpha\alpha_u + \alpha^3 + m'\beta)y\\
&\quad + (\beta_{uu} + 2\alpha_u\beta + \alpha\beta_u + 2b'\beta_u + \alpha^2\beta + b'\alpha\beta + n'\beta)z\\
&\quad + p'\beta\eta + (2c'\beta_u + c'\alpha\beta + q'\beta)\zeta,\\
y_{uuv} &= (\alpha_{uv} + a\alpha_u + 2\alpha\alpha_v + \beta_u\gamma + \beta\gamma_u + 2\alpha\beta\gamma + a\alpha^2)y\\
&\quad + (\beta_{uv} + \alpha_v\beta + \beta_u\delta + \alpha\beta_v + b'\beta_v + \beta\delta_u + \alpha\beta\delta + b'\beta\delta + \beta^2\gamma)z\\
&\quad + c(\alpha_u + \alpha^2)\eta + c'(\beta_v + \beta\delta)\zeta,\\
y_{uvv} &= (m_u + m\alpha + ep + m'q/c')y + (n_u + m\beta + b'n + fp + qs')z\\
&\quad + (p_u + gp + p'q/c')\eta + (q_u + c'n + qr')\zeta,\\
y_{vvv} &= (m_v + am + n\gamma + e'q + pr)y + (n_v + n\delta + f'q + np/c)z\\
&\quad + (p_v + cm + ps)\eta + (q_v + g'q + pq/c)\zeta,\\
y_{uuuu} &= (*)y + (*)z + (3p'\beta_u + p_u'\beta + 2p'\alpha\beta - c_up'\beta/c + p'q'\beta/c')\eta\\
&\quad + (3c'\beta_{uu} + 3c'\alpha_u\beta + 2c'\alpha\beta_u + 3q'\beta_u + q_u'\beta + c'\alpha^2\beta\\
&\quad + c'n'\beta + q'\alpha\beta + q'r'\beta)\zeta,\\
y_{vvvv} &= (*)y + (*)\eta + [n_{vv} + 2n_v\delta + n\delta_v + 2f'q_v + f_v'q + np_v/c\\
&\quad + (np/c)_v + mn + n\delta^2 + np\delta/c + f'q\delta + f'g'q + f'pq/c\\
&\quad + nps/c]z + (q_{vv} + 2g'q_v + g_v'q + pq_v/c + 2p_vq/c\\
&\quad - c_vpq/c^2 + g'pq/c + pqs/c + mq + g'^2q)\zeta,
\end{aligned} \tag{3.1}$$

where (*) denotes terms immaterial for our purpose.

The coordinates Y, where $Y=Y(u+\Delta u,\ v+\Delta v)$, of any point Y near the point y on the focal surface S_y are given by the Taylor's series

$$
\begin{aligned}
(3.2)\qquad Y &= y + y_u\Delta u + y_v\Delta v \\
&\quad + 2^{-1}(y_{uu}\Delta u^2 + 2y_{uv}\Delta u\Delta v + y_{vv}\Delta v^2) + \cdots ,
\end{aligned}
$$

in which the increments Δu and Δv correspond to displacement on the surface S_y from the point y to the point Y.

If the points y, z, η, ζ are used as the vertices of the tetrahedron of reference, with unit point suitably chosen, then any point given by an expression of the form

$$
(3.3)\qquad x_1 y + x_2 z + x_3\eta + x_4\zeta
$$

has local coordinates proportional to $x_1, \cdots, x_4$. Substitution of the expressions (3.1) leads to the following power series expansions of the local coordinates of the point Y:

$$
\begin{aligned}
x_1 &= 1 + \alpha\Delta u + a\Delta v + 2^{-1}(\alpha_u + \alpha^2)\Delta u^2 + (\alpha_v + a\alpha + \beta\gamma)\Delta u\Delta v \\
&\quad + 2^{-1}m\Delta v^2 + \cdots , \\
x_2 &= \beta\Delta u + 2^{-1}(\beta_u + \alpha\beta + b'\beta)\Delta u^2 + (\beta_v + \beta\delta)\Delta u\Delta v + 2^{-1}n\Delta v^2 \\
&\quad + 6^{-1}(\beta_{uu} + 2\alpha_u\beta + \alpha\beta_u + 2b'\beta_u + \alpha^2\beta + b'\alpha\beta + n'\beta)\Delta u^3 \\
&\quad + 2^{-1}(\beta_{uv} + \alpha_v\beta + \beta_u\delta + \alpha\beta_v + b'\beta_v + \beta\delta_u + \alpha\beta\delta + \beta^2\gamma \\
&\quad + b'\beta\delta)\Delta u^2\Delta v + 2^{-1}(n_u + m\beta + b'n + fp + qs')\Delta u\Delta v^2 \\
&\quad + 6^{-1}(n_v + n\delta + f'q + np/c)\Delta v^3 + \cdots + (24)^{-1}[n_{vv} + 2n_v\delta + n\delta_v \\
&\quad + 2f'q_v + f_v'q + np_v/c + (np/c)_v + mn + n\delta^2 + np\delta/c + f'q\delta \\
(3.4)\qquad &\quad + f'g'q + f'pq/c + nps/c]\Delta v^4 + \cdots , \\
x_3 &= c\Delta v + c\alpha\Delta u\Delta v + 2^{-1}p\Delta v^2 + 6^{-1}p'\beta\Delta u^3 + \cdots \\
&\quad + 6^{-1}(p_v + cm + ps)\Delta v^3 + (24)^{-1}(3p'\beta_u + p_u'\beta + 2p'\alpha\beta \\
&\quad - c_up'\beta/c + p'q'\beta/c')\Delta u^4 + \cdots , \\
x_4 &= 2^{-1}c'\beta\Delta u^2 + 2^{-1}q\Delta v^2 + 6^{-1}(2c'\beta_u + c'\alpha\beta + q'\beta)\Delta u^3 \\
&\quad + 2^{-1}c'(\beta_v + \beta\delta)\Delta u^2\Delta v + 2^{-1}(q_u + c'n + qr')\Delta u\Delta v^2 \\
&\quad + 6^{-1}(q_v + g'q + pq/c)\Delta v^3 + (24)^{-1}(3c'\beta_{uu} + 3c'\alpha_u\beta + 2c'\alpha\beta_u \\
&\quad + 3q'\beta_u + q_u'\beta + c'\alpha^2\beta + c'n'\beta + q'\alpha\beta + q'r'\beta)\Delta u^4 + \cdots \\
&\quad + (24)^{-1}(q_{vv} + 2g'q_v + g_v'q + pq_v/c + 2p_vq/c - c_vpq/c^2 + g'pq/c \\
&\quad + pqs/c + mq + g'^2q)\Delta v^4 + \cdots .
\end{aligned}
$$

From the expansion (3.4) it is possible to calculate to as many terms as desired an expansion for x_4/x_1 as a power series in x_2/x_1, x_3/x_1. The result to terms of the fourth degree is found, by the use of equations (1.3), (2.5), (2.8) and the substitutions (1.4), (2.6), to be

$$
(3.5)\qquad \begin{aligned} \frac{x_4}{x_1} = {} & \frac{c'}{2\beta}\left(\frac{x_2}{x_1}\right)^2 + \frac{q}{2c^2}\left(\frac{x_3}{x_1}\right)^2 + B_1\left(\frac{x_2}{x_1}\right)^3 - \frac{c'I}{2c\beta}\left(\frac{x_2}{x_1}\right)^2\left(\frac{x_3}{x_1}\right) \\ & - \frac{qA'}{2c^2\beta}\left(\frac{x_2}{x_1}\right)\left(\frac{x_3}{x_1}\right)^2 + \frac{qM}{6c^3}\left(\frac{x_3}{x_1}\right)^3 + C_1\left(\frac{x_2}{x_1}\right)^4 + \cdots \\ & + C_5\left(\frac{x_3}{x_1}\right)^4 + \cdots, \end{aligned}
$$

where

$$
(3.6)\qquad B_1 = \frac{1}{6\beta^3}\left(q'\beta + c'\alpha\beta - 3b'c'\beta - c'\beta_u\right),
$$

and the coefficients C_1, C_5 will be simplified in the next section.

An analogous power series expansion for the surface S_z in the neighborhood of the point z can be obtained in a way similar to the foregoing, or else can be written immediately by making the substitutions (1.4), (2.6).

4. Canonical form of the differential equations and loci of some osculants. For the purpose of choosing for the points η, ζ two particular covariant points respectively on the v-tangent at the point y of the surface S_y and the u-tangent at the point z of the surface S_z, we recall the definitions [5, p. 541] of two osculants associated with two ordinary points of the second kind of two plane curves. Suppose that O_1, O_2 are two ordinary points of the second kind of two plane curves C_1, C_2 respectively, so that O_1O_2 is the common tangent. Let K_1, K_2 be any four-point conics of the curves C_1, C_2 at the points O_1, O_2, and l_1, l_2 the polar lines of the points O_2, O_1 with respect to the conics K_1, K_2, respectively; then the intersection of the polar lines l_1, l_2 is called *the principal point* associated with the points O_1, O_2 of the curves C_1, C_2. Moreover, among the pencils of four-point conics of the curves C_1, C_2 at the points O_1, O_2, there are *principal conics* which pass through the principal point.

Let us consider a general plane π through the generator yz:

$$
(4.1)\qquad x_4 - \lambda x_3 = 0 \qquad (\lambda \neq 0).
$$

It is obvious that the plane π intersects the surfaces S_y, S_z in two curves C_y, C_z having y, z as two ordinary points of the second kind. By virtue of equations (3.5), (4.1) we may obtain the power series expansion for the projection of the section C_y in the plane $x_4=0$, namely,

$$
(4.2)\qquad \frac{x_3}{x_1} = \frac{1}{\lambda}\left[\frac{c'}{2\beta}\left(\frac{x_2}{x_1}\right)^2 + B_1\left(\frac{x_2}{x_1}\right)^3 + \cdots\right].
$$

The equations of any four-point conic of the section C_y at the point y are found, from equation (4.2), to be (4.1) and

$$x_1x_3 - \frac{c'}{2\beta\lambda}x_2^2 - \frac{2\beta B_1}{c'}x_2x_3 + kx_3^2 = 0, \tag{4.3}$$

where k is a parameter. The polar line of the point z with respect to this conic is given by equation (4.1) and

$$c'^2x_2 + 2\beta^2B_1\lambda x_3 = 0. \tag{4.4}$$

Elimination of λ between equations (4.1), (4.4) shows that as the plane π revolves about the line yz, the locus of this polar line is a plane through the line $y\eta$:

$$c'^2x_2 + 2\beta^2B_1x_4 = 0. \tag{4.5}$$

Similarly, as the plane π revolves about the line yz, the locus of the polar line of the point y with respect to any four-point conic of the section C_z at the point z is a plane through the line $z\zeta$:

$$c^2x_1 + 2\gamma^2B_1'x_3 = 0, \tag{4.6}$$

where B_1' is given by an expression similar to (3.6). Thus we reach the following conclusion:

As the plane π revolves about the line yz, the locus of the principal point associated with the points y, z of the plane sections C_y, C_z describes a line l, whose equations are (4.5), (4.6).

The line l intersects the lines $x_2=x_4=0$ and $x_1=x_3=0$ in two points. If we choose these two points respectively for the points η and ζ, then

$$p = c(\gamma_v/\gamma + 3a - \delta), \qquad q' = c'(\beta_u/\beta + 3b' - \alpha). \tag{4.7}$$

Hereafter it will be supposed that the differential equations (1.1) are in *the canonical form* for which the conditions (4.7) are satisfied. Accordingly, by means of equations (1.3), (2.5), (4.7) and the substitutions (1.4), (2.6) we can easily compute the coefficients C_1, C_5 of the power series expansion (3.5), and thus we arrive at the required power series expansion for the surface S_y in the neighborhood of the point y, namely,

$$\begin{aligned}\frac{x_4}{x_1} = {} & \frac{c'}{2\beta}\left(\frac{x_2}{x_1}\right)^2 + \frac{q}{2c^2}\left(\frac{x_3}{x_1}\right)^2 - \frac{c'I}{2c\beta}\left(\frac{x_2}{x_1}\right)^2\left(\frac{x_3}{x_1}\right) \\ & - \frac{qA'}{2c^2\beta}\left(\frac{x_2}{x_1}\right)\left(\frac{x_3}{x_1}\right)^2 + \frac{qM}{6c^3}\left(\frac{x_3}{x_1}\right)^3 - \frac{c'L}{8\beta^3}\left(\frac{x_2}{x_1}\right)^4 + \cdots \\ & + \frac{q}{24c^4}\left[M_v + M^2 + (P - \phi_v)M + 3L' + 3A'^2/D\right]\left(\frac{x_3}{x_1}\right)^4 \\ & + \cdots .\end{aligned} \tag{4.8}$$

Now we can also express the first and the third of the integrability conditions (1.5) in the form

$$
\begin{aligned}
J_v + J(P - \phi_v) + IK &= L_u' + A\mathfrak{H}, \\
K' - \mathfrak{H} &= 3(H' - J).
\end{aligned} \tag{4.9}
$$

From equations (1.1), it follows immediately that *the point ζ is in the osculating plane of the v-curve of the net N_y at the point y if, and only if,* $n=0$. A similar argument holds for the u-curve of the net N_z at the point z. Moreover, from equation (2.10), *the point y_1 coincides with the point η in case* $I=0$; and similarly, *the point z_{-1} coincides with the point ζ if, and only if,* $I'=0$.

Buzano [3] and Bompiani [2] have shown the existence of a projective invariant, together with metric and projective characterizations, determined by the neighborhood of the second order of two surfaces σ, σ^* at two ordinary points O, O^* in ordinary space under the conditions that the tangent planes of the surfaces σ, σ^* at the points O, O^* be distinct and have OO^* for the common line. For the focal surfaces S_y, S_z at the points y, z, this invariant is easily found, from Bompiani's note [2, p. 237], the expansion (4.8), and the similar one, to be $\mathfrak{H}/(16K)$.

As the plane π (4.1) revolves about the line yz the locus of the principal conic having four-point contact with the section C_y at the point y is a quadric cone with vertex at the point η, whose equation is found, by means of equations (4.1), (4.3), and the conditions (4.7), to be

$$
2\beta x_1 x_4 - c' x_2^2 = 0. \tag{4.10}
$$

Similarly, the locus of the principal conic having four-point contact with the section C_z at the point z is a quadric cone with vertex at the point ζ:

$$
2\gamma x_2 x_3 - c x_1^2 = 0. \tag{4.11}
$$

From equations (2.10), (2.11), we find the equations of the line $y_1 z_{-1}$ to be

$$
c x_1 + I x_3 = 0, \qquad c' x_2 + I' x_4 = 0, \tag{4.12}
$$

which intersects the plane (4.1) in a point with the coordinates

$$
(-I/c,\ -I'\lambda/c',\ 1,\ \lambda). \tag{4.13}
$$

If a four-point conic (4.3) at the point y of the section C_y of the surface S_y by the plane (4.1) passes through the point (4.13), then as the plane (4.1) revolves about the line yz the locus of this conic is a quadric cone with vertex at the point y_1, whose equation is easily found to be

$$
x_1 x_4 - \frac{c'}{2\beta} x_2^2 + \frac{I}{c} x_3 x_4 + \frac{I'^2}{2c'\beta} x_4^2 = 0. \tag{4.14}
$$

Similarly, we may obtain the quadric cone with vertex at the point z_{-1},

$$x_2x_3 - \frac{c}{2\gamma} x_1^2 + \frac{I^2}{2c\gamma} x_3^2 + \frac{I'}{c'} x_3x_4 = 0. \tag{4.15}$$

The line $\eta\zeta$ intersects the quadric cones (4.14), (4.15) in the points η, ζ and two other points with the coordinates

$$(0,\ 0,\ cI'^2,\ -2c'\beta I), \tag{4.16}$$

$$(0,\ 0,\ 2c\gamma I',\ -c'I^2). \tag{4.17}$$

The two points (4.16), (4.17) coincide in neither η nor ζ if, and only if,

$$II' = 4K, \tag{4.18}$$

and they are separated harmonically by the points η, ζ in case

$$II' + 4K = 0. \tag{4.19}$$

5. **Moutard quadrics of the focal surfaces S_y, S_z.** It is easy to obtain the equations of the quadrics, $Q_y^{(u)}$, $Q_y^{(v)}$ of Moutard for the tangents yz, $y\eta$ of the surface S_y, namely,

$$x_1x_4 - \frac{c'}{2\beta} x_2^2 - \frac{q}{2c^2} x_3^2 + \frac{I}{c} x_3x_4 + \frac{L}{2c'\beta} x_4^2 = 0, \tag{5.1}$$

$$\begin{aligned} &x_1x_4 - \frac{c'}{2\beta} x_2^2 - \frac{q}{2c^2} x_3^2 + \frac{A'}{\beta} x_2x_4 - \frac{M}{3c} x_3x_4 \\ &\quad + \frac{1}{18q} [M^2 - 3M_v - 3(P - \phi_v)M - 9L' - 9A'^2/D]x_4^2 = 0. \end{aligned} \tag{5.2}$$

The equations of the quadrics $Q_y^{(u)}$, $Q_z^{(v)}$ of Moutard for the tangents yz, $z\zeta$ can be written out immediately by making use of the substitutions (1.4), (2.6).

It is obvious that if the quadrics $Q_y^{(u)}$, $Q_z^{(v)}$ pass through the points ζ, η, then the focal nets N_y, N_z are restricted, respectively, by the conditions $L=0$, $L'=0$. The polar planes of the points η, ζ with respect to the quadric $Q_y^{(u)}$ and with respect to the quadric $Q_z^{(v)}$ are, respectively,

$$qx_3 - cIx_4 = 0, \tag{5.3}$$

$$x_1 + \frac{I}{c} x_3 + \frac{L}{c'\beta} x_4 = 0, \tag{5.4}$$

$$x_2 + \frac{L'}{c\gamma} x_3 + \frac{I'}{c'} x_4 = 0, \tag{5.5}$$

$$c'I'x_3 - p'x_4 = 0. \tag{5.6}$$

If the planes (5.3), (5.6) pass through the points ζ, η, then $I=0$, $I'=0$, respectively. Furthermore, these two planes coincide in case the focal nets N_y, N_z are restricted by the condition

$$II' = \mathfrak{H}. \tag{5.7}$$

The line $\eta\zeta$ intersects the two planes (5.4), (5.5) in two points with the coordinates

$$(0,\ 0,\ cL,\ -c'\beta I), \qquad (0,\ 0,\ c\gamma I',\ -c'L'). \tag{5.8}$$

The points (5.8) coincide in neither η nor ζ if, and only if,

$$LL' = II'K, \tag{5.9}$$

and they are separated harmonically by the points η, ζ in case

$$LL' + II'K = 0. \tag{5.10}$$

On the other hand, the polar planes of the points z, η, ζ with respect to the quadric $Q_y^{(v)}$ have the equations, respectively,

$$c'x_2 - A'x_4 = 0, \tag{5.11}$$

$$3qx_3 + cMx_4 = 0, \tag{5.12}$$

$$x_1 + \frac{A'}{\beta}x_2 - \frac{M}{3c}x_3 + \frac{1}{9q}[M^2 - 3M_v - 3(P - \phi_v)M - 9L' - 9A'^2/D]x_4 = 0. \tag{5.13}$$

If the planes (5.11), (5.12) pass through the point ζ, then $n=0$, $M=0$, respectively; and a similar argument can be made with regard to the quadric $Q_z^{(u)}$. Moreover, the plane (5.12) coincides with the polar plane of the point ζ with respect to the quadric $Q_z^{(u)}$ in case

$$MM' = 9\mathfrak{H}. \tag{5.14}$$

The line $y\eta$ intersects the planes (5.4), (5.13) in two points with the coordinates

$$(I,\ 0,\ -c,\ 0), \qquad (M,\ 0,\ 3c,\ 0). \tag{5.15}$$

These two points coincide in neither y nor η if, and only if,

$$3I + M = 0. \tag{5.16}$$

Further, if the points (5.15) are separated harmonically by the points y, η, then

$$3I = M. \tag{5.17}$$

Likewise, we can discuss the conditions similar to (5.16), (5.17).

Finally, if the planes (5.3), (5.6) coincide, and if the points (4.16), (4.17) are coincident or separated harmonically by the points η, ζ, then the focal nets N_y, N_z are restricted by the conditions (5.7) and

$$\mathfrak{H} = \pm 4K. \tag{5.18}$$

Now we are in a position to determine geometrically the unit point of the coordinate system. To this end, we observe that the locus of a moving point, whose polar planes with respect to the quadrics $Q_y^{(v)}$, $Q_z^{(u)}$ intersect the generator yz at the same point, is a quadric, its equation can easily be found to be

$$cc'x_1x_2 - cA'x_1x_4 - c'Ax_2x_3 + (AA' - \beta\gamma)x_3x_4 = 0. \tag{5.19}$$

This quadric cuts the cubic curve of intersection, besides the line $\eta\zeta$, of the two cones (4.10), (4.11) in two points. It is easily seen that *if m' and n do not vanish at the same time for the congruence yz, then we may take one of these two points as the unit point of the coordinate system, and therefore we have*

$$c = 2\gamma, \qquad c' = 2\beta, \qquad 3p'q + 4m'n = 4(m'q + np'). \tag{5.20}$$

6. **Segre-Darboux nets on the focal surfaces S_y, S_z.** Let us first consider the quadrics having second order contact with the surface S_y at the point y. From the series (3.4), the equation of a general one of these quadrics can be written in the form

$$x_1x_4 - \frac{c'}{4\beta}x_2^2 - \frac{q}{2c^2}x_3^2 + (k_2x_2 + k_3x_3 + k_4x_4)x_4 = 0, \tag{6.1}$$

where k_2, k_3, k_4 are parameters. Each quadric (6.1) cuts the surface S_y in a curve with a triple point at y, whose tangents are in the directions satisfying the equation

$$\begin{aligned} 3c'\beta^2k_2du^3 + 3c'\beta(ck_3 - I)du^2dv + 3q(\beta k_2 - A')dudv^2 \\ + q(3ck_3 + M)dv^3 = 0. \end{aligned} \tag{6.2}$$

It is not difficult to verify that if the binary cubic form that appears in equation (6.2) is a perfect cube of a linear form, then

$$k_2 = A'/4\beta, \qquad k_3 = (I - M)/4c, \tag{6.3}$$

and therefore the equations of the curves and the pencil of quadrics of Darboux at the point y of the surface S_y are, respectively,

$$3c'\beta A'du^3 - 3c'\beta(3I + M)du^2dv - 9qA'dudv^2 + q(3I + M)dv^3 = 0, \tag{6.4}$$

$$4c^2\beta x_1x_4 - 2c^2c'x_2^2 - 2q\beta x_3^2 + c^2A'x_2x_4 + c\beta(I - M)x_3x_4 + k_4x_4^2 = 0. \tag{6.5}$$

The polar line of the line $y\zeta$ with respect to the pencil (6.5) of quadrics of Darboux at the point y of the surface S_y is

$$x_4 = 4c\beta x_1 + cA'x_2 + \beta(I - M)x_3 = 0, \tag{6.6}$$

which intersects the line yz in the point

$$(-A', 4\beta, 0, 0), \tag{6.7}$$

and passes through the point z or η in case $n=0$ or $I=M$. If the point (6.7) and the similar one are coincident or separated harmonically by the points y, z, then the focal nets N_y, N_z are restricted by one of the conditions

$$AA' = \pm 16K. \tag{6.8}$$

From equation (6.4) and the similar one it is easily seen that the curves of Darboux correspond on the two surfaces S_y, S_z if, and only if,

$$W_{(u)} = 0, \qquad (3I + M)(3I' + M') = 9AA'. \tag{6.9}$$

It is known [4, p. 283] that if the curves of Darboux correspond on the two focal surfaces of a congruence, the congruence is a W congruence and both surfaces have the property of being isothermally asymptotic. Thus *the conditions for both surfaces S_y, S_z to be isothermally asymptotic at the same time can be reduced to the form* (6.9).

On the other hand, a necessary and sufficient condition for the u-curves or the v-curves of the surface S_y to be curves of Darboux is $n=0$ or (5.16). It is known that if the curves of one family of a conjugate net are curves of Darboux, then the curves of the other family must be the corresponding curves of Segre. Such a net is called *a Segre-Darboux net.* Combining the above results and the ones in §§4, 5 we obtain the following theorem:

The focal net N_y is a Segre-Darboux net with the u-curves as curves of Darboux if, and only if, the point ζ is in the osculating plane of the v-curve of the net N_y at the point y. The focal net N_y is a Segre-Darboux net with the v-curves as curves of Darboux if, and only if, the line $y\eta$ intersects, in the same point, the polar planes of the point ζ with respect to the quadrics $Q_y^{(u)}$, $Q_y^{(v)}$ of Moutard at the point y of the surface S_y.

We shall call a congruence *a Segre-Darboux congruence* when its two focal nets both are Segre-Darboux nets. Noticing the theorem concerning the other focal net N_z and similar to the above one, we may obtain a geometric interpretation for a Segre-Darboux congruence yz.

7. W **congruences.** By means of equations (1.1), (3.1), it is easy to find the differential equation of the asymptotic curves on the surface S_y, namely,

$$c'\beta du^2 + q dv^2 = 0. \tag{7.1}$$

Similarly, the differential equation of the asymptotic curves on the surface S_z is

$$p' du^2 + c\gamma dv^2 = 0. \tag{7.2}$$

Equations (7.1), (7.2) show that *the u-tangents of the net N_y or the v-tangents* *of the net N_z form a W congruence in case the Weingarten invariant $W_{(u)}$ or $W'_{(v)}$ vanishes.*

From equations (1.1), (1.2), (1.3), (2.5), (2.10) and the substitution (1.4), by differentiation and substitution any derivative of y_1 can be expressed as a linear combination of y, z, η, ζ. In particular, one obtains

$$
\begin{aligned}
y_{1u} &= (H - cf\alpha/\beta)y + c\alpha\eta,\\
y_{1v} &= [cr - acf/\beta - (cf/\beta)_v]y + nz + (cs + c_v - c^2f/\beta)\eta + q\zeta,\\
y_{1uu} &= (^*)y + \beta Hz + (^*)\eta,\\
y_{1uv} &= (^*)y + n\alpha z + (^*)\eta + q\alpha\zeta,\\
y_{1vv} &= (^*)y + (n_v - n\beta_v/\beta + np/c + f'q)z + (^*)\eta\\
&\quad + q(p/c - c'_v/c' - \beta_v/\beta + q_v/q)\zeta.
\end{aligned}
\tag{7.3}
$$

Making use of equations (7.3), the differential equation of the asymptotic curves on the surface Sy_1 sustaining the first Laplace transformed net N_{y_1} of the net N_y is found to be

$$c'\beta H du^2 + q\mathfrak{K} dv^2 = 0. \tag{7.4}$$

Thus *the v-tangents of the net N_y form a W congruence in case the Weingarten invariant $W_{(v)}$ vanishes.*

Similarly, *the u-tangents of the net N_z form a W congruence if, and only if, the Weingarten invariant $W'_{(u)}$ vanishes.*

From the relation between the Weingarten and the Laplace-Darboux invariants of a conjugate net, we know immediately that $\mathfrak{H}$, $\mathfrak{K}$ and $\mathfrak{H}'$, $\mathfrak{K}'$ are respectively *the tangential invariants of the focal nets N_y and N_z.*

Now we proceed to give a simple geometric interpretation for the condition for the u-tangents or the v-tangents of the net N_y to form a W congruence.

The equation of any quadric having the lines yz, $y\eta$, $z\zeta$ as generators can be written in the form

$$x_1x_4 + k_1x_2x_3 + k_2x_3x_4 = 0, \tag{7.5}$$

where k_1, k_2 are parameters. A general quadric (7.5) intersects the surface S_y in a curve with a double point at y, whose tangents are found, from the series (4.8), to be

$$x_4 = \frac{c'}{\beta}x_2^2 + 2k_1x_2x_3 + \frac{q}{c^2}x_3^2 = 0. \tag{7.6}$$

If these two tangents coincide, then

$$k_1^2 = \frac{c'q}{c^2\beta}\,. \tag{7.7}$$

Similarly, the quadric (7.5) cuts the surface S_z in a curve with a cusp at z if, and only if,

$$k_1^2 = \frac{c'^2\gamma}{cp'}. \tag{7.8}$$

The conditions (7.7), (7.8) hold simultaneously in case $W_{(u)}=0$. Thus we arrive at the following theorem:

A necessary and sufficient condition for the congruence yz to be a W congruence is that there exists a quadric (and therefore one-parameter family of such quadrics), which has yz, $y\eta$, $z\zeta$ as generators and whose curves of intersection with the focal surfaces S_y, S_z have cusps at the points y, z respectively.

Similar statements can be obtained for the congruences $y\eta$ and $z\zeta$.

A conjugate net whose Weingarten invariants both vanish is called [8, p. 1078] an R net; each family of curves of an R net has tangents that form a W congruence. From the above theorem, we may also interpret geometrically a congruence when either or both of its focal nets are R nets.

8. **Curves of the focal nets N_y, N_z.** By putting Δv equal to zero in the expansions (3.4), it is not difficult to obtain *the power series expansions for the u-curve of the focal net N_y in the neighborhood of the point y, namely,*

$$\begin{aligned} \frac{x_3}{x_1} &= \frac{p'}{6\beta^2}\left(\frac{x_2}{x_1}\right)^3 + \frac{p'M'}{24\beta^3}\left(\frac{x_2}{x_1}\right)^4 + \cdots, \\ \frac{x_4}{x_1} &= \frac{c'}{2\beta}\left(\frac{x_2}{x_1}\right)^2 - \frac{c'L}{8\beta^3}\left(\frac{x_2}{x_1}\right)^4 + \cdots. \end{aligned} \tag{8.1}$$

Similarly, by putting Δu equal to zero in the expansions (3.4), we reach *the following power series expansions for the v-curve of the focal net N_y in the neighborhood of the point y:*

$$\begin{aligned} \frac{x_2}{x_1} &= \frac{n}{2c^2}\left(\frac{x_3}{x_1}\right)^2 + \frac{n}{6c^3}(M + \mathfrak{R}/A')\left(\frac{x_3}{x_1}\right)^3 \\ &\quad + \frac{n}{24c^4}[M_v + M^2 + 3L' + (\mathfrak{R}_v + 2M\mathfrak{R})/A' \\ &\quad + (P - \phi_v)(M + \mathfrak{R}/A')]\left(\frac{x_3}{x_1}\right)^4 + \cdots, \\ \frac{x_4}{x_1} &= \frac{q}{2c^2}\left(\frac{x_3}{x_1}\right)^2 + \frac{qM}{6c^3}\left(\frac{x_3}{x_1}\right)^3 \\ &\quad + \frac{q}{24c^4}[M_v + M^2 + (P - \phi_v)M + 3L']\left(\frac{x_3}{x_1}\right)^4 + \cdots. \end{aligned} \tag{8.2}$$

Analogous expansions for the u-, v-curves at the point z of the focal net N_z

can be written by making the substitutions (1.4), (2.6).

The first of the expansions (8.2) and the expansion (4.8) demonstrate that *the projection of the v-curve at the point y of the focal net N_y from the point z onto the plane $y\eta\zeta$ has third order contact at the point y with the section of the focal surface S_y by the plane $y\eta\zeta$.*

From the expansions (8.1), (8.2) and equation (6.1), we find that *any quadric of the pencil*

$$x_1x_4 - \frac{c'}{2\beta}x_2^2 - \frac{q}{2c^2}x_3^2 - \frac{M}{3c}x_3x_4 + k_4x_4^2 = 0, \tag{8.3}$$

where k_4 is arbitrary, has third order contact with the parametric curves of the surface S_y at the point y. The polar plane of the point η with respect to any quadric of the pencil (8.3) is the plane (5.12).

Among the quadrics (6.1) there is a pencil of quadrics having fourth order contact with the u-curve (8.1) of the surface S_y at the point y. For this pencil we find

$$k_2 = 0, \qquad k_4 = \frac{L}{2c'\beta}, \tag{8.4}$$

with k_3 arbitrary. The quadric $Q_y^{(u)}$ (5.1) of Moutard for the u-tangent of the surface S_y is a unique quadric of this pencil [7, p. 698], and for this quadric we have $k_3 = I/c$.

Among the quadrics (6.1) there is also a pencil of quadrics having fourth order contact with the v-curve (8.2) of the surface S_y at the point y. For this pencil we find

$$\begin{aligned} k_3 &= -\frac{M}{3c}, \\ k_4 &= \frac{1}{18q}\left[M^2 - 3M_v - 3(P - \phi_v)M - 9L' + \frac{9nA'}{\beta} - 18nk_2\right], \end{aligned} \tag{8.5}$$

with k_2 arbitrary. The quadric $Q_y^{(v)}$ (5.2) of Moutard for the v-tangent of the surface S_y is a unique quadric of this pencil, and for this quadric we have $k_2 = A'/\beta$.

9. **Correspondences associated with the focal nets N_y, N_z.** Let us consider a curve C_λ passing through the point y and belonging to the family defined on the focal surface S_y by the differential equation

$$dv - \lambda du = 0; \tag{9.1}$$

and let the parametric representation of the curve C_λ be

$$u = u(w), \qquad v = v(w). \tag{9.2}$$

The parametric u-, v-tangents at points of the curve C_λ generate two nondevelopable ruled surfaces $R_y^{(u)}$, $R_y^{(v)}$ respectively. The points

$$T = ty - z, \qquad \bar{T} = \bar{t}y - \eta, \tag{9.3}$$

where t, $\bar{t}$ are functions of u, v, lie on the parametric u-, v-tangents through the point y respectively, and the line l_2 determined by them lies in the tangent plane $x_4=0$ of the focal surface S_y at the point y.

From the system (1.1) and equations (1.2), one easily obtains

$$\begin{aligned} T_u &= (t_u + t\alpha)y + (t\beta - b')z - c'\zeta, \\ T_v &= (t_v + at - \gamma)y + ct\eta - \delta z, \\ \bar{T}_u &= (\bar{t}_u + \bar{t}\alpha - e)y + (\bar{t}\beta - f)z - g\eta, \\ \bar{T}_v &= (\bar{t}_v + a\bar{t} - r)y - \frac{n}{c}z + (c\bar{t} - s)\eta - \frac{q}{c}\zeta. \end{aligned} \tag{9.4}$$

The tangent planes to the ruled surface $R_y^{(u)}$ at the point T and to the ruled surface $R_y^{(v)}$ at the point $\bar{T}$ intersect in a line l_1 which passes through the point y. There must be a point on the line l_1 given by an expression of the form

$$X = \zeta + k_1 z + k_2 \eta.$$

The plane of the points y, X, T is to be tangent to the ruled surface $R_y^{(u)}$ at T, and therefore must contain the tangent to the curve traced by T as the point y moves along the curve C_λ. It is clear from the first two of equations (9.4) that a point on this tangent is given by

$$T_w = (*)y + (*)z + ctv_w\eta - c'u_w\zeta;$$

that is, it is to lie in the plane yXT, which can occur if and only if $k_2 = -ct\lambda/c'$. Similarly, the plane $yX\bar{T}$ is tangent to the ruled surface $R_y^{(v)}$ at $\bar{T}$ if and only if

$$k_1 = -\frac{c}{q\lambda}\left(\bar{t}\beta - f - \frac{n}{c}\lambda\right).$$

Thus we obtain a correspondence(3) between the line l_2 in the plane $x_4=0$:

$$x_4 = x_1 + tx_2 + \bar{t}x_3 = 0, \tag{9.5}$$

and the line l_1 through the point y:

$$x_3 + \frac{ct\lambda}{c'}x_4 = 0, \qquad x_2 + \frac{c}{q\lambda}\left(\bar{t}\beta - f - \frac{n}{c}\lambda\right)x_4 = 0. \tag{9.6}$$

This correspondence becomes a polarity with respect to a quadric (and therefore

(3) This correspondence has been introduced by the author [6, pp. 381–382]. The line l_1 (9.6) is the analogue for a conjugate parametric net of the R'_λ-associate of the reciprocal of l_2 (9.5) defined by Bell [1, pp. 390–391].

∞^1 *quadrics) if and only if* λ *satisfies*

$$\lambda^2 = c'\beta/q, \tag{9.7}$$

that is, *if and only if the tangent of the curve* C_λ *at the point* y *is an associate conjugate tangent of the parametric conjugate net* N_y. The equation of any one of these quadrics is then of the form

$$cD^{1/2}x_1x_4 - c'x_2x_3 + (A' + ID^{1/2})x_3x_4 + k_4x_4^2 = 0, \tag{9.8}$$

where k_4 is a parameter. The polar plane of any point $(x_1^*, x_2^*, 0, 0)$ on the line yz, except the points y, z, with respect to any quadric (9.8) is

$$c'\overset{*}{x_2}x_3 - cD^{1/2}\overset{*}{x_1}x_4 = 0. \tag{9.9}$$

Similarly, we can obtain the plane

$$c'(D')^{1/2}\overset{*}{x_2}x_3 - c\overset{*}{x_1}x_4 = 0, \tag{9.10}$$

which coincides with the plane (9.9) if and only if $K = \mathfrak{H}$. Thus we obtain another geometric interpretation of a W congruence:

A necessary and sufficient condition for the congruence yz *to be a* W *congruence is that the polar planes of any point on the generator* yz*, except the points* y, z, *with respect to any quadric* (9.8) *and a similar one be coincident.*

10. **Axis congruences and ray congruences.** The axis at the point y of the net N_y is defined to be the line of intersection of the osculating planes of the u-, v-curves of the net N_y at the point y. It is easy to show that *the focal points of the axis at the point* y *of the net* N_y *are given by*

$$k_1y + nz + q\zeta, \qquad k_2y + nz + q\zeta, \tag{10.1}$$

where k_1, k_2 *are the two roots of the equation*

$$c'\beta k^2 + c'\beta Nk - q\mathfrak{H}\mathfrak{K} = 0. \tag{10.2}$$

Moreover, *the differential equation of the axis curves of the net* N_y *is*

$$c'\beta\mathfrak{H}du^2 - c'\beta N du dv - q\mathfrak{K}dv^2 = 0. \tag{10.3}$$

Since at each point of a harmonic conjugate net the axis tangents separate the tangents of the net harmonically [11, p. 215], it follows immediately that the net N_y is harmonic in case $N = 0$. Similarly, the net N_z is harmonic if, and only if, $N' = 0$. From equations (7.1), (10.3) and the analogous ones, we may also verify the known result that the axis curves of the nets N_y, N_z form conjugate nets in case $\mathfrak{H} = \mathfrak{K}$, $\mathfrak{H}' = \mathfrak{K}'$, respectively.

We shall next determine the developables and focal surfaces of the ray congruence of the net N_y. Making use of equations (1.1), (1.2), (1.3), (2.5), (2.10) and the second and the fourth of the integrability conditions (1.5), *the focal points of the ray at the point* y *of the net* N_y are found, in a way similar to the foregoing, to be

$$(10.4)\qquad y_1 + k_1 z, \qquad y_1 + k_2 z,$$

where k_1, k_2 are the two roots of the equation

$$(10.5)\qquad c'\gamma k^2 - c'Nk - qH = 0.$$

The differential equation of the ray curves of the net N_y is

$$(10.6)\qquad c'Hdu^2 - c'Ndudv - q\gamma dv^2 = 0.$$

The ray tangents of the net N_y separate the tangents of the net N_y harmonically in case $N=0$, that is, in case the net is harmonic [11, p. 215]. The ray curves of the net N_y form a conjugate net in case $H=K$ [10, p. 319].

In a similar way, we can discuss the ray congruence of the net N_z.

11. **The congruences $z\eta$, $y\zeta$ and the principal congruence $\eta\zeta$.** As the point y varies on the surface S_y, the line $z\eta$ describes a congruence. *The focal points of the line $z\eta$ are* given by

$$(11.1)\qquad \eta + k_1 z, \qquad \eta + k_2 z,$$

where k_1, k_2 are the two roots of the equation

$$(11.2)\qquad c^2c'\gamma k^2 + cc'L'k - qJ = 0.$$

The differential equation of the curves, in which the developables of the congruence $z\eta$ of the net N_y intersect the surface S_y, is

$$(11.3)\qquad c'Jdu^2 + c'L'dudv - q\gamma dv^2 = 0.$$

The tangents of the curves (11.3) *separate the tangents of the net N_y harmonically in case $L'=0$. Moreover, the curves* (11.3) *form a conjugate net if, and only if,* $J=K$. Finally, from equations (2.7), (2.10) it follows that *if the point y_1 coincides with the point η and if the curves* (11.3) *form a conjugate net, then the conjugate net N_y has equal Laplace-Darboux invariants.*

Similarly, we can discuss the congruence $y\zeta$.

Finally, we shall call the congruence $\eta\zeta$ *the principal congruence* of the congruence yz. The focal points of the line $\eta\zeta$ are

$$(11.4)\qquad \eta + k_1\zeta, \qquad \eta + k_2\zeta,$$

where k_1, k_2 are the two roots of the equation

$$(11.5)\qquad \begin{aligned} c^2m'(B'I'L - GJ')k^2 + cc'G(II'K + LL' - JJ' - G)k \\ + c'^2n(BIL' - GJ) = 0. \end{aligned}$$

The developables of the principal congruence $\eta\zeta$ intersect the surfaces S_y, S_z in the curves respectively represented on S_y, S_z by the differential equation

$$(11.6)\qquad (BI - JL)du^2 + (II'K - LL' - JJ' + G)dudv + (B'I' - J'L')dv^2 = 0.$$

We shall call these curves *the principal curves* of the nets N_y, N_z. The tangents of these principal curves will be called *the principal tangents* of the nets N_y, N_z.

The principal tangents of one net separate the tangents of the same net harmonically in case

$$G + II'K = JJ' + LL'. \tag{11.7}$$

Moreover, necessary and sufficient conditions for the principal curves to form conjugate nets on the surfaces S_y, S_z are, respectively,

$$D(B'I' - J'L') = JL - BI, \tag{11.8}$$

$$D'(BI - JL) = J'L' - B'I'. \tag{11.9}$$

Finally, the principal curves form conjugate nets on both surfaces S_y, S_z at the same time if the congruence yz is restricted by the conditions

$$BI = JL, \qquad B'I' = J'L', \tag{11.10}$$

or if it is a W congruence and restricted by one of the two conditions (11.8), (11.9).

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Michigan State College,
East Lansing, Mich.