

ON GEÖCZE'S PROBLEM FOR NON-PARAMETRIC SURFACES

BY

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1. **Introduction.** Suppose that P stands for a (closed) polygonal region in the (x, y) -plane, and that we consider the portion

$$(1.1) \quad S(f, P): \quad z = f(x, y) \quad [(x, y) \in P]$$

of the surface $z=f(x, y)$, where

(1.2) $f(x, y)$ is everywhere a continuous and one-valued function of (x, y) .

Now let $\{S(f_n, P)\}$ be a sequence of polyhedra inscribed to $S(f, P)$ such that

(1.3) $f_n(x, y)$ is a linear function of x and y on every triangle in some triangulation⁽¹⁾ $\mathfrak{T}_n(P)$ of P , and $f_n(x, y)=f(x, y)$ at every vertex of $\mathfrak{T}_n(P)$;

(1.4) the greatest diameter of any triangle of $\mathfrak{T}_n(P)$ tends to 0 as $n \rightarrow \infty$.

Denoting by $E(f_n, P)$ the area of $S(f_n, P)$ (in the elementary sense), we define

$$(1.5) \quad A^*(f, P) = \inf \liminf E(f_n, P) \quad (n \rightarrow \infty),$$

the infimum being for all sequences $\{f_n(x, y)\}$ satisfying (1.3) and (1.4); and we denote by $A(f, P)$ the Lebesgue area of $S(f, P)$. The following theorem, settling Geöcze's problem⁽²⁾ for non-parametric surfaces, has recently been established⁽³⁾: the object of the present paper is to give a new and comparatively short proof of it by following up an approach devised by Radó (cf. [8, V. 3.50–53, pp. 545–549]) in earlier work on this problem. The reader interested in the historical development of some of the ideas involved in this approach may read the papers by L. C. Young [11], H. D. Huskey [2], and T. Radó [7] listed at the end.

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(¹) By a triangulation of P we shall understand a subdivision of P into a finite number of nonoverlapping triangles such that no vertex of a triangle is an internal point of a side of another.

(²) An account of this problem and a restricted solution of it are given by T. Radó in *Length and area* [8, V. 3.50–57, pp. 545–550, and V. 4.10, pp. 560, 561]. N. B.: Radó's definition of $A^*(f, P)$ differs slightly from that above, since (1.4) is more stringent than his requirement of uniform convergence on P of $f_n(x, y)$ to $f(x, y)$: however, it may easily be verified that this difference does not disturb the appeals to his results on $A^*(f, P)$ made below (at (1.7), (1.9), and the end of §5). Figures in square brackets refer to the references at the end of the paper.

(³) For the theorem below I have given a proof in a forthcoming paper [5]: a somewhat less sharp result, not requiring the inscribed polyhedra to be of the form $z=f_n(x, y)$, was obtained independently by A. Mambriani [3], whose main result covers a wider class of surfaces. In both papers the proofs are independent of previous work on Geöcze's problem.

THEOREM. *If $f(x, y)$ is one-valued and continuous on the unit square $Q_0: 0 \leq x \leq 1, 0 \leq y \leq 1$, then (cf. (1.5) above) the Lebesgue area*

$$(1.6) \quad A(f, Q_0) = A^*(f, Q_0).$$

Without loss of generality we may suppose (cf. [8, V.3.22, p. 525, and V.3.46, p. 543]) that $f(x, y)$ satisfies (1.2) and that

$$(1.7) \quad A(f, Q_0) < \infty.$$

LEMMA 1. *Let $f(x, y)$ satisfy (1.2) and R be an oriented (closed) rectangle in the (x, y) -plane. Let $2^{-1}hk\Phi(x, y; h, k; f)$ stand for the area of the triangle whose vertices are $(x, y, f(x, y)), (x+h, y, f(x+h, y)), (x, y+k, f(x, y+k))$, and let*

$$(1.8) \quad I(h, k; R; f) = \iint_R \Phi(x, y; h, k; f) dx dy.$$

Then (cf. (1.5)), if q is any given nonzero constant,

$$(1.9) \quad A^*(f, R) \leq I^*(R, f, q) = \limsup I(h, k; R; f) \quad (h, k \rightarrow 0, h/k = q).$$

This differs from Radó's inequality [8, V. 3.53, (7), p. 549] by the restriction $h/k = q$. If R is the rectangle $[a, b; c, d]$, Radó's proof actually covers the case in which $q(d-c)/(b-a)$ is positive and rational⁽⁴⁾. The modifications required for the remaining cases are obvious and so need not be stated here.

2. An upper bound for $I^*(R, f, q) - A(f, R)$ in terms of a Tonelli integral.

LEMMA 2. *Let $f(x, y)$ satisfy (1.2) and (1.7), R be a closed rectangle in the (x, y) -plane oriented relative to the axes of x and y , $O\xi, O\eta$ be any axes of rectangular coordinates in this plane, and m_η be the slope of $O\eta$ relative to the x -axis. Let $V_\xi(R, \eta, f)$ denote the total variation of $f(x, y)$ on R for fixed η and varying ξ (or 0 if η does not occur on R), and let*

$$(2.1) \quad T_\xi(R, f) = \int_{-\infty}^{\infty} V_\xi(R, \eta, f) d\eta.$$

Then, taking $q = m_\eta$ when $0 < |m_\eta| < \infty$, and $q = 1$ otherwise, we have (cf. (1.9))

$$(2.2) \quad I^*(R, f, q) - A(f, R) \leq 2T_\xi(R, f).$$

Firstly, let $|m_\eta| = \infty$, and, for definiteness, suppose $\xi = x, \eta = y$. Putting $\Delta_x f(x, y) = f(x+h, y) - f(x, y), \Delta_y f(x, y) = f(x, y+k) - f(x, y)$, we get (cf. 1.8)

$$(2.3) \quad \begin{aligned} \Phi(x, y; h, k; f) &= [(\Delta_x f(x, y)/h)^2 + (\Delta_y f(x, y)/k)^2 + 1]^{1/2} \\ &\leq |\Delta_x f(x, y)/h| + [(\Delta_y f(x, y)/k)^2 + 1]^{1/2} \end{aligned}$$

⁽⁴⁾ N.B.: the use of an enclosing rectangle R^{δ} in Radó's proof, and the consequent appeal in its final step to the elaborate analysis in [8, V. 3.25-36, pp. 527-536], can be avoided if the terms with either $i = m - 1$ or $j = n - 1$ are omitted from the summation in [8, V. 3.50, (1), p. 545], and the requisite adjustments made in the ensuing argument.

by elementary calculations. Suppose that R is the rectangle $[a, b; c, d]$: then, from the known fact⁽⁵⁾ that as $h \rightarrow 0$ the integral from a to b of $|\Delta_x f(x, y)/h|$ tends to $V_x(R, y, f)$, it can be deduced⁽⁶⁾ that

$$(2.4) \quad \lim_{h \rightarrow 0} \iint_R |\Delta_x f(x, y)/h| dx dy = \int_c^d V_x(R, y, f) dy = T_x(R, f).$$

Further, from the known fact (cf. [9, chap. V, (8.10), p. 184]) that as $k \rightarrow 0$ the integral from c to d of $[(\Delta_y f(x, y)/k)^2 + 1]^{1/2}$ tends to the length $L_y(R, x, f)$ of the curve $z = f(x, y)$ ($x = \text{constant}$, $c \leq y \leq d$) it can be deduced similarly that

$$(2.5) \quad \lim_{k \rightarrow 0} \iint_R [(\Delta_y f(x, y)/k)^2 + 1]^{1/2} dx dy = \int_a^b L_y(R, x, f) dx = S_y(R, f),$$

say. From (2.3), on integrating $\Phi(x, y; h, k; f)$ over R and taking upper limits as $h, k \rightarrow 0$ ($h/k = q = 1$), we obtain (cf. (1.8), (1.9), (2.4), (2.5)) the inequality $I^*(R, f, 1) \leq T_x(R, f) + S_y(R, f)$. But it was proved by Tonelli [10, §1, p. 635], that $S_y(R, f) \leq A(f, R)$: thus

$$(2.6) \quad I^*(R, f, 1) - A(f, R) \leq T_x(R, f).$$

This implies (2.2) in the case where m_η is infinite; the case $m_\eta = 0$ is treated similarly, with x and y interchanged. Lastly, suppose $0 < |m_\eta| < \infty$ and so $h/k = q = m_\eta$. For brevity, we shall put $\alpha = h^2/(h^2 + k^2)^{1/2}$, $\beta = k^2/(h^2 + k^2)^{1/2}$, $\gamma = (\alpha\beta)^{1/2}$; then for the points whose old coordinates are (x, y) , $(x+h, y)$, $(x, y+k)$, and $(x+hk^2/(h^2+k^2), y+h^2k/(h^2+k^2))$ we find easily the new coordinates (ξ, η) , $(\xi + \alpha, \eta + \gamma)$, $(\xi - \beta, \eta + \gamma)$, and $(\xi, \eta + \gamma)$ respectively. We write $\phi(\xi, \eta)$, and so on, for $f(x, y)$, and so on, and make the following abbreviations:

$$(2.7) \quad \begin{aligned} \phi(\xi + \alpha, \eta + \gamma) - \phi(\xi, \eta + \gamma) &= \lambda\alpha, \\ \phi(\xi - \beta, \eta + \gamma) - \phi(\xi, \eta + \gamma) &= \mu\beta, \\ \phi(\xi, \eta) - \phi(\xi, \eta + \gamma) &= \nu\gamma. \end{aligned}$$

Then, for example, $\Delta_x f(x, y) = \lambda\alpha - \nu\gamma$, and thus, from the first line of (2.3),

$$\begin{aligned} \Phi(x, y; h, k; f) &= [(\lambda\alpha - \nu\gamma)^2(\alpha^2 + \gamma^2)^{-1} + (\mu\beta - \nu\gamma)^2(\beta^2 + \gamma^2)^{-1} + 1]^{1/2} \\ &\leq \left\{ \frac{\lambda^2\alpha^2}{\alpha^2 + \gamma^2} + \frac{\mu^2\beta^2}{\beta^2 + \gamma^2} \right\}^{1/2} + \left\{ \frac{\nu^2\gamma^2}{\alpha^2 + \gamma^2} + \frac{\nu^2\gamma^2}{\beta^2 + \gamma^2} + 1 \right\}^{1/2}, \end{aligned}$$

by the triangle inequality. Since $\gamma^2 = \alpha\beta$ the last expression reduces to

$$(\lambda^2\alpha + \mu^2\beta)^{1/2}(\alpha + \beta)^{-1/2} + (\nu^2 + 1)^{1/2} \leq |\lambda| + |\mu| + (\nu^2 + 1)^{1/2}.$$

Hence, recalling the definition of $I^*(R, f, q)$ in Lemma 1, we have

⁽⁵⁾ Cf. [8, III. 2.42, p. 207, and V. 3.26, p. 528]; see also [4, (12.2), p. 309]. Note that $T_x(R, f)$ is one of Tonelli's integrals, cf. [8, III. 2.49–51, pp. 210–212].

$$(2.8) \quad I^*(R, f, q) \leq \lim \sup \left\{ \iint_R |\lambda| \, d\xi d\eta + \iint_R |\mu| \, d\xi d\eta + \iint_R (\nu^2 + 1)^{1/2} d\xi d\eta \right\},$$

as $\alpha \rightarrow 0$ with $\beta = \alpha/q^2$ and $\gamma = (\alpha\beta)^{1/2}$. But, in virtue of (2.7) the last integral in (2.8) has essentially the same form⁽⁶⁾ as the first in (2.5), and the other two integrals in (2.8) can be reduced to essentially the same form⁽⁶⁾ as the first in (2.4) by change of origin to $\xi = 0, \eta = -\gamma$. The deduction of the required inequality (2.2) can thus be completed in the same way as that of (2.6) above.

3. Use of W. H. Young's vector-areas to orient the auxiliary axes $O\xi, O\eta$. Let R be any rectangle (oriented or not) in the plane $z = 0$. Then (cf. (1.1)), for the vector-area (cf. [12]) of $S(f, R)$ —or rather, of its boundary-curve $C(R)$ —we may write

$$(3.1) \quad \mathbf{Y}(R, f) = \frac{1}{2} \int_{C(R)} \mathbf{r} \times d\mathbf{r}, \quad Y(R, f) = |\mathbf{Y}(R, f)|.$$

If we write $c(R)$ for the boundary of R , described positively, then for the components of \mathbf{Y} we have $Y_z(R, f) = Y_z(R) = |R|$, that is, the area of R ,

$$(3.2) \quad Y_x(R, f) = - \int_{c(R)} z dy = - \int dy \int d_x f(x, y) \quad [(x, y) \in R],$$

and similarly for $Y_y(R, f)$. Since the inner integral does not exceed $V_x(R, y, f)$ in absolute value we have (cf. (2.1) and (2.4))

$$(3.3) \quad |Y_x(R, f)| \leq T_x(R, f), \quad |Y_y(R, f)| \leq T_y(R, f).$$

We put $T_z(R) = |R| = Y_z(R)$ and (suppressing arguments) $T = (T_x^2 + T_y^2 + T_z^2)^{1/2}$. Using (3.3) and the known fact⁽⁷⁾ that T does not exceed the area A , we get

$$T_x^2 = T^2 - T_y^2 - T_z^2 \leq T^2 - Y_y^2 - Y_z^2 = T^2 - Y^2 + Y_x^2 \leq A^2 - Y^2 + Y_x^2.$$

Similarly if we work with auxiliary coordinate axes $O\xi, O\eta$, we have

$$(3.4) \quad [T_\xi(R, f)]^2 \leq [A(f, R)]^2 - [Y(R, f)]^2 + [Y_\xi(R, f)]^2.$$

⁽⁶⁾ That R is not oriented relative to the axes of ξ and η causes no trouble, since the theorems needed in the subsequent deduction are obtainable by trivial adaptations from those for oriented rectangles: cf. [4, (12.2), p. 309]. The change of origin translates R through a distance γ , but it can easily be verified that this makes no difference in the limit, when $\gamma \rightarrow 0$, since $T_\xi(R, f)$ is a continuous function of R (cf. [8, III. 2.51, pp. 210–212]).

⁽⁷⁾ Cf. [10, §§5 and 7, pp. 447, 448]; for non-oriented R cf. [1, Lemmas 5.4 and 5.5, pp. 427, 428].

Now from (3.1), $Y(R, f)$ is independent of the choice of coordinate axes; hence (3.4) yields, in particular, the following lemma.

LEMMA 3. *Let $f(x, y)$ and R be as in Lemma 2, and let us choose axes $O\xi, O\eta$ of rectangular coordinates in the (x, y) -plane such that $O\xi$ is perpendicular to $Y(R, f)$ (cf. (3.1)). Then (cf. (2.1) and §1, below (1.5))*

$$(3.5) \quad [T_\xi(R, f)]^2 \leq [A(f, R)]^2 - [Y(R, f)]^2 \leq 2A(f, R)D(f, R),$$

where $D(f, R)$ stands for $A(f, R) - Y(R, f)$.

COROLLARY TO LEMMAS 1, 2, AND 3. *With $f(x, y)$ and R as in Lemma 2 we have*

$$(3.6) \quad A^*(f, R) - A(f, R) \leq [8A(f, R)D(f, R)]^{1/2}.$$

4. In general, $A^*(f, R)$ increases (broadly) by subdivision.

LEMMA 4. *Let $f(x, y)$ and R be as in Lemma 1, and let R be divided into two oriented rectangles R_1 and R_2 by a line segment l . Then (cf. (1.5))*

$$(4.1) \quad A^*(f, R) \leq A^*(f, R_1) + A^*(f, R_2),$$

provided that the vertical plane section through l of $S(f, R)$ is rectifiable.

Suppose, for definiteness, that R is the rectangle $[a, b; c, d]$, that l is the line $x = x', c \leq y \leq d$, where $a < x' < b$, and that $x \leq x'$ in $R_1, x \geq x'$ in R_2 . From (1.5) it follows that there are sequences $\{f_n^i(x, y)\}, i = 1, 2$, such that $f_n^i(x, y)$ satisfies on R_i the conditions (1.3) and (1.4) and also

$$(4.2) \quad E(f_n^i, R_i) \rightarrow A^*(f, R_i) \quad \text{as } n \rightarrow \infty \quad (i = 1, 2);$$

$$(4.3) \quad \tau_n = \max \text{diam } \Delta \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (\Delta \in \mathfrak{T}_n^i(R_i), i = 1, 2),$$

where $\mathfrak{T}_n^i(R_i)$ is the triangulation of R_i for $f_n^i(x, y)$ (cf. (1.3) and (1.4)).

Now suppose that Δ is any triangle of $\mathfrak{T}_n^1(R_1)$ that has two vertices P_0 and P'_0 on l and the third C_0 thus in $R_1 - l$. Then we subdivide Δ , when possible, by lines joining C_0 to each vertex of $\mathfrak{T}_n^2(R_2)$ lying on l strictly between P_0 and P'_0 . Let $\mathfrak{T}_n(R_1)$ be the triangulation obtained from $\mathfrak{T}_n^1(R_1)$ after all such subdivisions, and let $\mathfrak{T}_n(R_2)$ be constructed similarly: then we can combine $\mathfrak{T}_n(R_1)$ and $\mathfrak{T}_n(R_2)$ into a single triangulation $\mathfrak{T}_n(R)$ of R , since $\mathfrak{T}_n(R_1)$ and $\mathfrak{T}_n(R_2)$ have the same vertices on l . Clearly $\mathfrak{T}_n(R)$ satisfies (1.4), and determines a function $f_n(x, y)$ on R in accordance with (1.3).

With Δ as above, let $\mathfrak{T}_n(\Delta)$ be the triangulation of Δ under $\mathfrak{T}_n(R)$, let $A_0B_0C_0$ be a triangle δ of $\mathfrak{T}_n(\Delta)$, and let ABC be the corresponding face $S(f_n, \delta)$ (cf. (1.1), (1.3)). Then $E(f_n, \delta) = 2^{-1}bc \sin A \leq 2^{-1}bc$, and so

$$(4.4) \quad E(f_n, \Delta) = \sum E(f_n, \delta) \leq 2^{-1} \sum bc \leq 2^{-1} \max b \cdot \sum c \quad [\delta \in \mathfrak{T}_n(\Delta)] \\ \leq 2^{-1} \max b \cdot L_y(\Delta, x', f),$$

since $\sum c = L_y(\Delta, x', f_n)$, where L_y is defined as before (2.5). Next, if (x, y) and (u, v) are two points varying on R , we write $\omega(\lambda)$ for $\max |f(x, y) - f(u, v)|$ when $(x-u)^2 + (y-v)^2 \leq \lambda^2$. But $\max A_0 C_0 \leq \text{diam } \Delta \leq \tau_n$, by (4.3); and so we get successively

$$(4.5) \quad \max b = \max AC \leq \sigma_n = [\tau_n^2 + \omega(\tau_n)^2]^{1/2},$$

$$(4.6) \quad E(f_n, \Delta) - E(f_n^1, \Delta) \leq 2^{-1} \sigma_n L_y(\Delta, x', f)$$

by (4.4), since $E(f_n^1, \Delta) > 0$. Further, any triangle Δ of $\mathcal{T}_n^1(R_1)$ not adjacent to l is not subdivided, and hence it satisfies (4.6) trivially since both sides vanish. Thus, summing (4.6) over all triangles Δ of $\mathcal{T}_n^1(R_1)$, we get

$$\begin{aligned} E(f_n, R_1) - E(f_n^1, R_1) &= \sum \{E(f_n, \Delta) - E(f_n^1, \Delta)\} \quad [\Delta \in \mathcal{T}_n^1(R_1)] \\ &\leq 2^{-1} \sigma_n \sum L_y(\Delta, x', f) = 2^{-1} \sigma_n L_y(R, x', f). \end{aligned}$$

Similarly for R_2 ; and so, since $E(f_n, R) = E(f_n, R_1) + E(f_n, R_2)$, we have

$$(4.7) \quad E(f_n, R) - E(f_n^1, R_1) - E(f_n^2, R_2) \leq \sigma_n L_y(R, x', f).$$

On taking upper limits as $n \rightarrow \infty$ we obtain from this, using (4.2),

$$(4.8) \quad \limsup E(f_n, R) - A^*(f, R_1) - A^*(f, R_2) \leq \limsup \sigma_n L_y(R, x', f) = 0;$$

for, using (4.3) and (4.5), we see that (as $n \rightarrow \infty$) $\omega(\tau_n) \rightarrow 0$ because $f(x, y)$ is continuous on R , and so $\sigma_n \rightarrow 0$; moreover, $L_y(R, x', f)$ is finite by hypothesis. But the desired inequality (4.1) follows from (4.8) and (1.5).

5. Proof of the theorem. As noted in §1, we may assume (1.2) and (1.7): further, we employ the notations of §1 and Lemma 3, but with the argument f suppressed. Then, following Radó [6, §1.6, p. 362, §2.4, p. 364, and §3.8 including footnote 14, p. 370] we note that there is a sequence of subdivisions $\{\mathcal{D}_n\}$ of Q_0 into oriented rectangles such that (i) the division-lines run from side to side of Q_0 and are the projections of rectifiable vertical sections of $S(f, Q_0)$, (ii) we have

$$(5.1) \quad \sum D(R) = \sum A(R) - \sum Y(R) = A(Q_0) - \sum Y(R) \rightarrow 0 \quad (R \in \mathcal{D}_n, n \rightarrow \infty),$$

the second equality holding because $A(R)$ is additive (cf. [8, V. 3.18, p. 524]). Now, for a fixed \mathcal{D}_n , summation of (3.6) and use of Cauchy's inequality yield

$$(5.2) \quad \sum A^*(R) - \sum A(R) \leq \sum 1[8A(R)D(R)]^{1/2} \leq [8 \sum A(R) \sum D(R)]^{1/2}.$$

But $A^*(Q_0) = A^*(\sum R) \leq \sum A^*(R)$ by repeated application of Lemma 4, $\sum A(R) = A(Q_0) < \infty$ by (1.7), and $\sum D(R) \rightarrow 0$ as $n \rightarrow \infty$ by (5.1); hence (5.2) implies that $A^*(Q_0) - A(Q_0) \leq 0$. Since $A^*(Q_0) \geq A(Q_0)$ trivially (cf. [8, V. 3.46, (1), p. 543]) we now have the desired equality (1.6).

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