ON THE STRONG DIFFERENTIATION OF THE INDEFINITE INTEGRAL[1]

BY

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1. **Definitions** [1, p. 106][2]. A sequence of intervals[3] $\{I_n\}$ is said to be *regular* if there exists a positive number α such that, with a_n the larger and b_n the smaller side of I_n, we have $b_n/a_n > \alpha$ for every n.

We shall say that the sequence $\{I_n\}$ tends to the point (x, y) if[4] $\delta(I_n) \to 0$ as $n \to \infty$, and the point (x, y) belongs to all the intervals of the sequence.

Given a set function F defined for all intervals, we define $F^*(x, y)$, the *ordinary upper derivate*, as the upper bound of the numbers t such that there exists a regular sequence of intervals $\{I_n\}$ tending to (x, y), for which

$$\lim_n \frac{F(I_n)}{|I_n|} = t[5].$$

If we remove the condition of regularity of the sequences of intervals considered, we obtain the definition of *strong upper derivate* $F_s^*(x, y)$. Similarly we define the *lower* derivates $F_*(x, y)$ and $F_{*s}(x, y)$.

If $F^*(x, y) = F_*(x, y)$, we say that $F(I)$ has an *ordinary derivative* $F'(x, y) = F^*(x, y) = F_*(x, y)$. If $F_s^*(x, y) = F_{*s}(x, y)$, we say that $F(I)$ has a *strong derivative* $F_s'(x, y) = F_s^*(x, y) = F_{*s}(x, y)$. Obviously if $F_s'(x, y)$ exists, then also $F'(x, y)$ exists and equals $F_s'(x, y)$.

2. Suppose $f(x, y)$ defined and integrable on a set E of positive measure. We extend the definition of $f(x, y)$ on the whole plane P by putting $f(x, y) = 0$ for $(x, y) \in P - E$. We can therefore, as we shall in the following, consider only functions defined in the whole plane.

We form

$$(1) \qquad W(I) = \int\int_I f(x, y)dxdy.$$

Then we have the following theorems.

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[2] Numbers in brackets refer to the references cited at the end of the paper.

[3] By intervals we shall mean in the following rectangles with sides parallel to coordinate axes.

[4] $\delta(E)$ will mean the diameter of the set E.

[5] The symbol $|E|$ denotes throughout this paper the Lebesgue measure of the plane set E.

THEOREM 1 (Lebesgue). $W^*(x, y) = W_*(x, y) = f(x, y)$ *p.p.*

THEOREM 2. (*See* [2], [3]; *also part* II *of Theorem 6 of this paper.*) *There exist integrable functions for which* $W_s^*(x, y) = +\infty$ *on a set* E *of positive measure. And since* $W_s^*(x, y) \leq W_*(x, y) = W^*(x, y) = f(x, y) < +\infty$ *p.p., it follows that* $W_s^*(x, y) \neq W_{*s}(x, y)$ *on* E.

THEOREM 3. *If* $f(x, y)$ *is bounded, then* $W_s'(x, y)$ *exists and equals* $f(x, y)$ *p.p.*

The above theorem is an easy generalization of the following:

THEOREM 4([6]) (*density theorem, see* [3] *and* [4]). *With* $f(x, y)$ *the characteristic function of a set* E, $W_s'(x, y) = f(x, y)$ *p.p. where* $W(I)$ *is as in* (1).

3. THEOREM 5. *With* $f(x, y)$ *an integrable function, we form*

$$(2) \qquad F(I) = \iint_I |f(x, y)|\, dxdy, \qquad W(I) = \iint_I f(x, y) dxdy.$$

If $F(I)$ *is strongly differentiable p.p.,* $W(I)$ *is also strongly differentiable p.p.*

Proof ([7]). We write $f(x, y) = f_N(x, y) + r_N(x, y)$ where

$$f_N(x, y) = \begin{cases} f(x, y) & \text{for} \quad |f(x, y)| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $|f(x, y)| = |f_N(x, y)| + |r_N(x, y)|$. $f(x, y)$ is integrable. Hence given $\epsilon > 0$, there exists an N_0 and a plane set M of measure smaller than ϵ, such that $r_N(x, y) = 0$ for $(x, y) \notin M$, and every $N \geq N_0$.

From the boundedness of $f_N(x, y)$ and Theorem 3, we conclude that with $F_N(I) = \iint_I |f_N(x, y)| dxdy$, $\lim_{\{I_n\} \to (x,y)} F_N(I_n)/|I_n|$ exists and equals $f_N(x, y)$ p.p. But, $\lim_{\{I_n\} \to (x,y)} F(I_n)/|I_n|$ also exists (by assumption), therefore, with $R_N(I) = \iint_I r_N(x, y) dxdy$, $\lim_{\{I_n\} \to (x,y)} R_N(I_n)/|I_n|$ exists p.p. and equals $|r_N(x, y)|$.

Similarly with $W_N(I) = \iint_I f_N(x, y) dxdy$, $\lim_{I_n} W_N(I_n)/|I_n| = f_N(x, y)$ p.p. But

$$(3) \qquad \limsup_{I_n} \frac{\left| \iint_{I_n} r_N(x, y) dxdy \right|}{|I_n|} \leq \limsup_{I_n} \frac{\iint_{I_n} |r_N(x, y)| dxdy}{|I_n|}$$

$$= \lim_{I_n} \frac{R_N(I_N)}{|I_N|} = |r_N(x, y)| \text{ p.p.,}$$

([6]) This theorem is not true if the intervals I_n (see definition) are replaced by any rectangles (see (3) and (6)).

([7]) The following simple proof is due to Professor B. Jessen and was kindly communicated to the author by Professor A. Zygmund.

and since $r_N(x, y) = 0$ for $(x, y) \notin M$, it follows that

$$\lim \frac{\iint_{I_n} r_N(x, y)dxdy}{|I_n|} = 0 \quad \text{p.p.} \quad \text{for} \quad (x, y) \notin M.$$

Hence

$$\lim_{I_n} \frac{W(I_n)}{|I_n|} = \lim_{I_n} \frac{W_N(I_n)}{|I_n|} + \lim_{I_n} \frac{\iint_{I_n} r_N(x, y)dxdy}{|I_n|}$$

$$= f_N(x, y) + r_N(x, y) = f(x, y) \quad \text{for} \quad (x, y) \notin M$$

and, since ϵ is arbitrary, Theorem 5 is true.

4. THEOREM 6. *There exist integrable functions $f(x, y)$ for which $W(I)$ is strongly differentiable p.p. but $F_s'(x, y)$ exists nowhere on a set E of positive measure; in fact $F_s^*(x, y) = + \infty$ for every point (x, y) of E.*

Proof. *Part* I (from a construction by H. Bohr on the Vitali covering theorem, see [5, pp. 689–692]). We subdivide the interval

$$I: \begin{array}{l} a < x < a + h, \\ b < y < b + k \end{array}$$

into the intervals

$$I_p^{(1)}: \quad a < x < a + \frac{ph}{N}, \qquad b < y < b + \frac{k}{p}, \qquad 1 \le p \le N;$$

then for every p we have $|I_p^{(1)}| = hk/N$ and with $V^{(1)} = \sum_{p=1}^N I_p^{(1)}$,

$$(4) \qquad |V^{(1)}| = \frac{hk}{N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) = |I_p^{(1)}| \mu(N)$$

where $\mu(N) = 1 + 1/2 + \cdots + 1/N$.

Through the vertices of the intervals $I_p^{(1)}$ we draw lines parallel to the y-axis; thus the point-set $I - V^{(1)}$ is subdivided into $(N-1)$ intervals. We perform the above construction on each of these intervals using the same N as before. We so obtain $(N-1)$ point-sets, $V^{(2)}, \cdots, V^{(N)}$, of a staircase form, having no points in common. With

$$(5) \qquad \frac{1}{N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) = 1 - \theta(N)$$

we have from (4)

$$\left| V^{(1)} \right| = (1 - \theta(N)) \left| I \right|, \qquad \left| I - V^{(1)} \right| = \theta(N) \left| I \right| ;$$

reasoning similarly on $V^{(2)}, \cdots, V^{(N)}$ we obtain

$$\left| I - (V^{(1)} + V^{(2)} + \cdots + V^{(N)}) \right| = \theta(N) \left| I - V^{(1)} \right| = \theta^2(N) \left| I \right|.$$
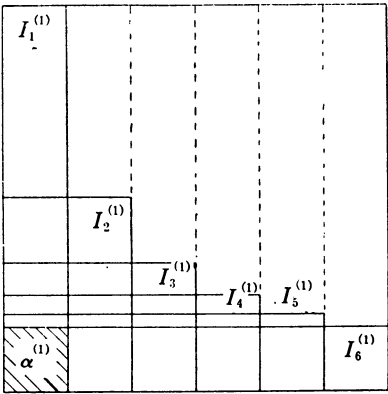

FIG. 1

5. We repeat this process $(k-1)$ times. After the $(k-1)$th iteration we have constructed $(1+(N-1)+ \cdots +(N-1)^k)$ sets $V^{(i)}$ of a staircase form; the remaining set $R = I - \sum V^{(i)}$, which we shall call the "rest" set, consists of $(N-1)$: separate intervals of total area,

$$(6) \qquad \left| R \right| = \theta^k(N) \left| I \right|.$$

The intervals $I_p^{(i)}$, forming the set $V^{(i)}$, have a nonempty common part: the shaded interval $a^{(i)}$. From our construction it follows that:

$$(7) \qquad \left| a^{(i)} \right| = \frac{\left| I_p^{(i)} \right|}{N} = \frac{\left| V^{(i)} \right|}{N\mu(N)},$$

hence with $A = \sum a^{(i)}$, the sum of all the shaded parts of I,

$$(8) \qquad \left| A \right| < \frac{\left| I \right|}{N\mu(N)}.$$

6. The above construction performed on an interval I will be called in the following the operation B_N^k, N as in §4, k as in §5. We shall mean by $\{i\}$ the set of all the intervals $I_p^{(i)}$ of our construction, and by $\{a\}$ the set of the shaded intervals of I. Then $A = \sum_{a \in \{a\}} a$. If $a \in \{a\}$ is included in an $i \in \{i\}$, that is, if $a = i \cdot A$ (a point-set product), then $\left| a \right| = \left| i \right| / N$.

Part II. The following construction gives a proof of Theorem 2. It will

be used in Part III, with minor modifications, for the proof of Theorem 6; in addition it will familiarize the reader with the method to be followed.

We shall give the construction of an integrable function $f(x, y)$, whose indefinite integral has no finite strong derivative at any point of a set of positive measure.

First step: On the unit square S_1 we perform the operation B_N^k by taking $N = N_1 = 2^{2^2}$ and k such that $R = R_1 < \epsilon/2$. We thus obtain the sets $\{i_1\}$ and $\{a_1\}$. And with $A_1 = \sum_{a \in \{a_i\}} a$, $|A_1| < 1/N_1\mu(N_1)$, and if $i \in \{i_1\}$, $a = A_1 \cdot i$, then

$$(9) \qquad\qquad |a| = \frac{|i|}{N_1}, \quad \delta(i) < 1.$$

We further define the function

$$(10) \qquad f_1(x, y) = \begin{cases} N_1(\log N_1)^{1/2} = 2^{2^2} \cdot 2 & \text{for} \quad x \in A_1, \\ 0 & \text{elsewhere in } S_1. \end{cases}$$

(In the following, the logarithms will be taken with base 2.) With

$$W_1(I) = \int\int_I f_1(x, y)dxdy$$

we get from (8) and (10)

$$W_1(S_1) = |A_1| f_1(x, y) < \frac{N_1(\log N_1)^{1/2}}{N_1\mu(N_1)} = \frac{(\log N_1)^{1/2}}{\mu(N_1)},$$

$$\frac{W_1(i)}{|i|} = \frac{N_1(\log N_1)^{1/2}|A_1 \cdot i|}{|i|} = (\log N_1)^{1/2}.$$

Second step: We divide S_1 into 4 congruent squares S_2 and perform in each of them the operation B_N^k by taking $N = N_2 = 2^{2^{2 \cdot 2}}$ and k such that, with R_2 the sum of all the "rests" (§5) in the 4 squares S_2, we have $|R_2| < \epsilon/2^2$. By $\{i_2\}$ we now mean the set of intervals i included in all 4 squares S_2 and by $\{a_2\}$ the set of the shaded intervals whose sum we call A_2. Again

$$(11) \qquad |A_2| < \frac{1}{N_2\mu(N_2)} \text{ and if } i \in \{i_2\}, a = A_2 \cdot i, \text{ then}$$

$$|a| = \frac{|i|}{N_2}, \quad \delta(i) < \frac{1}{2}.$$

We define:

$$(12) \qquad f_2(x, y) = \begin{cases} N_2(\log N_2)^{1/2} = 2^{2^2} \cdot 2^2 & \text{for} \quad x \in A_2, \\ 0 & \text{elsewhere in } S_1, \end{cases}$$

and with

$$W_2(I) = \int\int_I f_2(x, y)dxdy$$

we get from (11) and (12)

$$W_2(S_1) = |A_2| f_2(x, y) < \frac{(\log N_2)^{1/2}}{\mu(N_2)},$$

$$\frac{W_2^{(i)}}{|i|} = (\log N_2)^{1/2}.$$

Similarly we continue.

nth step: We divide S_1 into 4^{n-1} congruent squares S_n and perform in each of them operation B_N^k by taking

(13) $$N = N_n = 2^{2^{2n}} \quad \text{and } k \text{ such that} \quad |R_n| < \frac{\epsilon}{2^n}.$$

As before:

(13a) $$|A_n| < \frac{1}{N_n \mu(N_n)}, \quad \text{and if} \quad i \in \{i_n\}, \quad a = A_n \cdot i, \quad \text{then}$$

$$|a| = \frac{|i|}{N_n}, \quad \delta(i) < \frac{1}{2^{n-1}},$$

(14) $$f_n(x, y) = \begin{cases} N_n(\log N_n)^{1/2} = 2^{2^{2n}} \cdot 2^n & \text{for} \quad x \in A_n, \\ 0 & \text{elsewhere in } S_1, \end{cases}$$

$$W_n(I) = \int\int_I f_n(x, y)dxdy,$$

(15) $$W_n(S_1) = |A_n| f_n(x) < \frac{(\log N_n)^{1/2}}{\mu(N_n)},$$

$$\frac{W_n(i)}{|i|} = (\log N_n)^{1/2} = 2^{2^{2n}}$$

and so forth.

7. Form $f(x, y) = \sum_{n=1}^{\infty} f_n(x, y)$. We shall show that $f(x, y)$ satisfies our requirements. Indeed, from (15) we have

$$\sum_{n=1}^{\infty} W_n(S_1) < \sum_{n=1}^{\infty} \frac{2^n}{\mu(2^{2^{2n}})}$$

but

$$\mu(N) = 1 + 1/2 + \cdots + \frac{1}{N} \sim \log N$$

hence

$$\mu(N) > \frac{\log N}{2} \quad \text{for} \quad N > N_0$$

or $\mu(2^{2^{2n}})2^{2n}/2$ for $n > n_0$; hence

(16) $$\sum_{n=n_0}^{\infty} W_n(S_1) < 2 \sum_{n=n_0}^{\infty} \frac{2^n}{2^{2n}} = \frac{4}{2^{n_0}} \cdot$$

Therefore $\sum_{n=1}^{\infty} W_n(S_1)$ converges and

(17) $$W(I) = \iint_I f(x, y) dx dy < + \infty.$$

We shall now show that $W_s^*(x) = + \infty$ for every $x \in E = S_1 - \sum_{n=1}^{\infty} R_n$ where $|E| > 1 - \epsilon$ because of (13).

   x is contained in one of the intervals of $\{i_n\}$ for every n. Call i_n the interval of $\{i_n\}$ containing x; then $i_1, i_2, \cdots, i_n, \cdots$ is a sequence of intervals containing x and with diameter tending to zero (13a), and since $W(i_n)/|i_n| > W_n(i_n)/|i_n| = (\log N_n)^{1/2}$ (see (14)) and since $(\log N_n)^{1/2} \to \infty$ with $n \to \infty$, we conclude that $W_s^*(x) = + \infty$.

   REMARKS. (1) The sequence $\{i_n\}$ is not regular.

   (2) $f(x, y) \cdot \log |f(x, y)|$ is not finitely integrable, in agreement with the theorem of Jessen, Marcinkiewicz, Zygmund.

   Part III. We shall make the following modifications in the construction of Part II. The operation B_N^k of the first step will remain the same. With m_1 the smallest side of the shaded intervals $\{a_1\}$, we divide S_1 in the second step, not into 4 squares, but into such a number of congruent squares $\{S_2\}$ that their side is smaller than m_1/N_1^2. With m_2 the smallest side of the intervals $\{a_2\}$, we divide S_1 in the third step into such a number of congruent squares $\{S_3\}$ that their side is smaller than m_2/N_2^2; and so we continue.

   From the way the squares S_n have been constructed it follows that:

(18) $$|a_n| > |S_{n+1}| N_n^4.$$

As before (13) and (13a) hold.

   We now subdivide each rectangle of the set $\{a_n\}$ into N_n^4 congruent rectangles, half of them we call *black* and the remaining *white* (as on a chessboard). We thus obtain the point-set B_n of the black rectangles of A_n, and the point-set W_n of the white rectangles of A_n. With W one of the rectangles of the set W_n, we obtain from (18)

(19)
$$| W | = \frac{| a_n |}{N_n^4} > | a_{n+1} |$$

because a_{n+1} is included in one of the squares of $\{S_{n+1}\}$ whose side is smaller than $1/N_n^2$ times the smaller side of a_n.

We define

(20) $f_n^{(+)}(x, y) = \begin{cases} N_n(\log N_n)^{1/2} & \text{for} \quad x \in W_n, \\ 0 & \text{elsewhere,} \end{cases}$ $f^{(+)}(x, y) = \sum_{n=1}^{\infty} f_n^{(+)}(x, y),$

(21) $f_n^{(-)}(x, y) = \begin{cases} N_n(\log N_n)^{1/2} & \text{for} \quad x \in B_n, \\ 0 & \text{elsewhere,} \end{cases}$ $f^{(-)}(x, y) = \sum_{n=1}^{\infty} f_n^{(-)}(x, y),$

$$f_n(x, y) = f_n^{(+)}(x, y) - f_n^{(-)}(x, y), \qquad f(x, y) = f^{(+)}(x, y) - f^{(-)}(x, y).$$

With similar reasoning as in Part II, we conclude that $f^{(+)}(x, y), f^{(-)}(x, y)$, and $f(x, y)$ are integrable, with $F^{(+)}(I)$, $F^{(-)}(I)$, and $W(I) = F^{(+)}(I) - F^{(-)}(I)$ their respective integrals over I. We put

(22)
$$F(I) = \int \int_I | f(x, y) | \, dx dy.$$

8. Consider the set ΔA_n consisting of all rectangles similar to a_n, with the same center and with sides twice the sides of a_n. Clearly from (13a) we have

(23)
$$| \Delta A_n | = 4 | A_n | < \frac{4}{N_n \mu(N_n)} < \frac{4}{16^n}$$

hence

$$\sum_{n=1}^{\infty} | \Delta A_n | < \frac{4}{15}$$

and with

$$E = S_1 - \sum_{n=1}^{\infty} \Delta A_n - \sum_{n=1}^{\infty} R_n$$

we obtain from (13) and (23)

(24)
$$| E | > 1 - \frac{4}{15} - \epsilon.$$

Consider the rectangle $a \in \{a_n\}$; it is subdivided into $N_n^4/2$ black and $N_n^4/2$ white rectangles of area $| a |/N_n^4$. With L any rectangle, we can easily see that the difference D of the areas of the black and white parts of a, which are included in L, is not greater than the area of one of the N_n^4 rectangles into which a has been subidivded; that is

$$(25) \qquad\qquad D < \frac{|a|}{N_n^4} \cdot$$

If therefore L is such that $|L| > |a|$, then

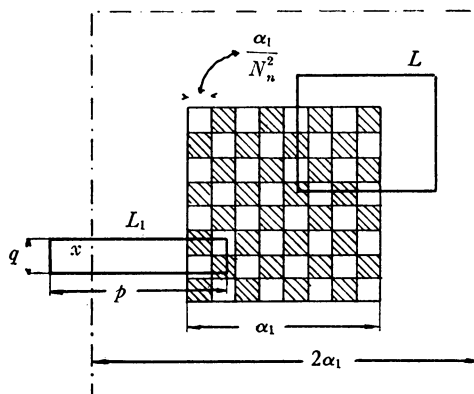$$(26) \qquad\qquad D < \frac{|L|}{N_n^4} \cdot$$

FIG. 2

Suppose now that L is any rectangle with sides p and q, containing a point x of E. Clearly (see Fig. 2)

$$p > \frac{\alpha_1}{2} \quad \text{and} \quad D < \frac{\alpha_1 q}{N_n^2},$$

hence

$$(27) \qquad\qquad D < \frac{2pq}{N_n^2} = \frac{2|L|}{N_n} \cdot$$

9. We shall first show[7] that $W(I)$ has a strong derivative p.p. on E. We write

$$W(I) = W_m(I) + R_m(I) \quad \text{where} \quad W_m(I) = \sum_{n=1}^{m} \int\int_I f_n(x, y) dx dy$$

with $x \in E$, consider an arbitrary interval L containing the point x; we have

$$\frac{R_n(I)}{L} < \frac{\sum_{n=m}^{\infty} \left| \int\int_I f_n(x, y) dx dy \right|}{L},$$

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[7] The proof of this paragraph is based on a suggestion by Professor A. Zygmund. Originally, it was proved by using Ward's theorem.

but $\left| \int\int_I f_n(x, y)dxdy \right|$ equals $N_n(\log N_n)^{1/2}$ times the difference D_n of the areas of the black and white parts of A_n which are included in L. Hence from (27) we have

$$\frac{R_m(I)}{|L|} < \sum_{n=m}^{\infty} \frac{D_n}{|L|} N_n(\log N_n)^{1/2} < 2\sum_{n=m}^{\infty} \frac{(\log N_n)^{1/2}}{N_n}$$

which tends to zero with $m \to \infty$ (see 16)), and since $W_m(I)$ has a strong derivative p.p. $(W_m)'_s$ (Theorem 3), we conclude that given $\epsilon > 0$ we can find m_0 such that

$$(W_m)'_s(x, y) - \epsilon < W_{*s}(x, y) \leqq W_s^*(x, y) < (W_{m0})'_s + \epsilon;$$

hence with $\epsilon \to 0$ we must have $W_{*s}(x, y) = W_s^*(x, y)$, that is, $W(I)$ is strongly differentiable p.p.

10. It remains to show that $F_s^*(x, y) = +\infty$ for $x \in E$.

We first see by an easy induction that with $\rho(n) = N_n(\log N_n)^{1/2} = 2^{2^{2n}} \cdot 2^n$,

$$(28) \qquad \rho(n-1) + \rho(n-2) + \cdots + \rho(1) < \rho(n)/2 \cdot$$

Hence for every $(x, y) \in W_n$ we have (see (20))

$$\sum_{k=1}^{n-1} |f_k(x, y)| < \frac{f_n^{(+)}(x, y)}{2} = \frac{f_n(x, y)}{2}$$

(since $f_k(x, y) = 0$ or $\rho(k)$ for $k \leqq n-1$), therefore

$$(29) \quad \sum_{k=1}^{n} f_k(x, y) > f_n(x, y) - \sum_{k=1}^{n-1} |f_k(x, y)| > \frac{f_n(x, y)}{2} = \frac{f_n^{(+)}(x, y)}{2} \geqq 0.$$

With $(x, y) \in E$, we denote by i_n the interval of the set $\{i_n\}$, which contains the point (x, y); then since $f^{(+)}(x, y) \geqq 0$, we have

$$\int\int_{i_n} f^{(+)}(x, y)dxdy \geqq \int\int_{i_n W_n} f^{(+)}(x, y)dxdy.$$

Denote by W one of the white rectangles of the set W_n, and by D_{n+1} the remaining part of $W \cdot B_{n+1}$, after subtracting from it a set of area equal to the area of $W \cdot W_{n+1}$; thus

$$(30) \qquad |W \cdot B_{n+1}| \leqq |D_{n+1}| + |W \cdot W_{n+1}|.$$

From (19) we conclude that $|W|$ is greater than the area of any one of the rectangles of the set $\{a_{n+1}\}$, hence (see (26)) we must have

$$(31) \qquad |D_{n+1}| < \frac{|W|}{N_{n+1}^4} \cdot$$

But $f_{n+1}(x, y)$ increases $\sum_{k=1}^{n} f_k(x, y)$ (which is positive as we can see from (29) in all points of $(W-D_{n+1})W_{n+1})$ by $N_{n+1}(\log N_{n+1})^{1/2}$, and it decreases it by the same amount in all points of $(W-D_{n+1})B_{n+1}$, and

$$\left| (W - D_{n+1})B_{n+1} \right| \leqq \left| (W - D_{n+1})W_{n+1} \right|$$

(see (30). Therefore

$$\iint_W \left| \sum_{k=1}^{n+1} f_k(x, y) \right| dxdy \geqq \iint_{W-D_{n+1}} \left| \sum_{k=1}^{n+1} f_k(x, y) \right| dxdy,$$

but

$$\iint_{W-D_{n+1}} \left| \sum_{k=1}^{n+1} f_k(x, y) \right| dxdy \geqq \iint_{W-D_{n+1}}^{n} \left| \sum f_k(x, y) \right| dxdy;$$

hence

$$\iint_W \left| \sum_{k=1}^{n+1} f_k(x, y) \right| dxdy \geqq \iint_{W-D_{n+1}} \left| \sum_{k=1}^{n} f_k(x, y) \right| dxdy$$

(32)
$$\geqq \frac{1}{2} \iint_{W-D_{n+1}} \left| f_n(x, y) \right| dxdy$$

$$= \frac{1}{2} \left| W - D_{n+1} \right| N_n(\log N_n)^{1/2}$$

$$\geqq \frac{1}{2} \left| W \right| \left(1 - \frac{1}{N_{n+1}^4} \right) N_n(\log N_n)^{1/2}$$

(because of (31)). Similarly we obtain

$$\iint_W \left| \sum_{k=1}^{n+r} f_k(x, y) \right| dxdy > \frac{1}{2} \left| W \right| \left(1 - \frac{1}{N_{n+1}^4} \right)$$

$$\cdots \left(1 - \frac{1}{N_{n+r}^4} \right) N_n(\log N_n)^{1/2};$$

but

$$\sum_{r=1}^{\infty} \frac{1}{N_{n+r}^4} < 10^{-4},$$

hence with $r \to \infty$ we obtain

$$\iint_W \left| \sum_{k=1}^{\infty} f_k(x, y) \right| dxdy > \frac{1}{2} \left| W \right| (1 - 10^{-4}) N_n(\log N_n)^{1/2}$$

or

(33) $$\iint_W |f(x, y)| \, dxdy > \frac{1}{2}|W|(1 - 10^{-4})N_n(\log N_n)^{1/2}.$$

We now sum over all the W's which are included in the set i_n. From (13a) we have $|i_nW_n| = |i_nA_n|/2 = |i_n|/2N_n$. Therefore

$$\iint_{i_nW_n} |f(x, y)| \, dxdy > \frac{|i_n|}{4N_n}(1 - 10^{-4})N_n(\log N_n)^{1/2},$$

hence

$$F(i_n) = \iint_{i_n} |f(x, y)| \, dxdy > \frac{|i_n|}{4}(1 - 10^{-4})(\log N_n)^{1/2}$$

or

$$\frac{F(i_n)}{|i_n|} > \frac{1}{4}(1 - 10^{-4})(\log N_n)^{1/2} = \frac{2^n}{4}(1 - 10^{-4}) \to \infty$$

with n; and since (see (13a)) $\delta i_n \to 0$, we conclude that $F_s^*(x, y) = +\infty$ and the theorem is proved.

REMARK. Even if we assume that not only $f(x, y)$, but also $\sigma(|f|)|f|\log|f|$, is integrable where $\sigma(t)$ is an arbitrary function such that $\lim \inf \sigma(t) = 0$ with $t \to \infty$, Theorem 6 is still true (see [2]).

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