

SPECTRAL THEORY FOR SEMI-GROUPS OF LINEAR OPERATORS

BY

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1. **Introduction.** In this paper various aspects of spectral theory for semi-groups of linear transformations are investigated by means of an associated Banach algebra containing the semi-group of operators. This approach seems to be quite suited to the problems considered, and once the machinery has been set up the theorems follow a natural pattern.

The definitive work in this field is Hille's *Functional analysis and semi-groups* [6]⁽¹⁾. By utilizing the theory of Banach algebras we have been able to extend many of Hille's results to a point which does not seem to be accessible by Laplace transform methods. We have also been able to obtain a new and direct proof of the Stone Theorem [11; 1; 9] for locally compact abelian groups of unitary transformations. This proof makes extensive use of the representation theorem for commutative self-adjoint rings of operators on a Hilbert space and thus is able to avoid the use of positive definite functions.

We first sketch the theory of a Banach algebra \mathfrak{S} consisting in essence of functions of bounded variation α on $(0, \infty)$ with convolution as their product. We then consider a commutative Banach algebra \mathfrak{R} of linear bounded transformations containing the given semi-group of operators $T(t)$. By means of the mapping

$$\Theta(\alpha) = \int_0^\infty T(t) d\alpha$$

we are able to study the structure of the ring \mathfrak{R} in terms of \mathfrak{S} . In particular the behavior of the multiplicative linear functionals on $\Theta(\alpha)$ is given in terms of the corresponding functionals on α .

In section four we are concerned with semi-groups $T(t)$ uniformly continuous at the origin. Such a semi-group is completely characterized by the fact that it has a representation as a continuous function in the space of all maximal ideals $\mathfrak{M}' = [m']$ which is of the form $\exp [a(m')t]$. This in turn enables us to characterize the semi-groups with unbounded infinitesimal generators in terms of the form of their representation.

If we now define

$$f_\alpha(\lambda) = \int_0^\infty \exp (\lambda t) d\alpha$$

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⁽¹⁾ Numbers in brackets refer to the references cited at the end of the paper.

and denote the spectrum of an operator B by $\Sigma(B)$, then for semi-groups with unbounded infinitesimal generator one obtains directly that $\Sigma[\Theta(\alpha)] = f_\alpha[\Sigma(A)] \cup 0$ for α absolutely continuous. If we assume that $T(t)$ is uniformly continuous for $t \geq c > 0$, then for any $\alpha \in \mathfrak{S}$ we can show that $\Sigma[\Theta(\alpha)] = f_\alpha[\Sigma(A)] \cup \alpha(0)$. Even more, this correspondence holds for the respective point, residual, and continuous parts of the spectra of $\Theta(\alpha)$ and A .

In section six we show that the indicator

$$\sigma(\phi) = \lim_{r \rightarrow \infty} r^{-1} \log \|T[r \exp(i\phi)]\| \quad (\phi_1 < \phi < \phi_2)$$

is the function of support for the convex closed extension of the complex conjugate points to $\Sigma(A)$. Here we assume only that $T(\zeta)$ is analytic sufficiently far out in the sector (ϕ_1, ϕ_2) . Finally section seven is devoted to a new proof of the Stone Theorem and because of its independent interest this section is essentially self-contained.

In conclusion it should be pointed out that we have been unable to obtain any new results by these methods for the spectral theory of semi-groups of operators $T(t)$ which are merely strongly continuous for $t > 0$.

2. The Banach algebra $\mathfrak{S}(\omega)$. It will be the purpose of this section to introduce a base space upon which we shall construct an operational calculus for the infinitesimal generator of any given semi-group of linear transformations. This space differs from the familiar Banach algebra of functions of bounded variation on $[0, \infty)$ with a convolution product only because of a weighting factor. Both A. Beurling [2] and I. Gelfand [5] have considered such algebras with continuous weighting functions on $(-\infty, \infty)$. It will be necessary for us to extend these results to Baire measurable weighting functions on $[0, \infty)$. Because of the close similarity with the continuous case we shall be very sketchy in our exposition.

We consider weighting functions $\exp[\omega(t)]$ with the following properties:

- (a) $\omega(t)$ is a real-valued Baire measurable function defined on $[0, \infty)$,
- (1) (b) $\omega(0) = 0$; $\int_0^1 \exp[\omega(t)] dt < \infty$,
- (c) $\omega(t)$ is subadditive: $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$.

It can be shown (see [6, chap. 6]) that $\omega(t)$ is bounded in every finite subinterval $[\delta, 1/\delta]$ and that

$$(2) \quad \omega_0 = \inf_{t > 0} \omega(t)/t = \lim_{t \rightarrow \infty} \omega(t)/t.$$

We now define $\mathfrak{S}(\omega)$ to be the set of all completely additive complex-valued set functions $\alpha(\sigma)$ on the sigma-field of Baire measurable subsets \mathfrak{B} of $[0, \infty)$ such that $\int_0^\infty \exp[\omega(t)] |d\alpha| < \infty$. The norm is given by

$$(3) \quad \|\alpha\| = \int_0^\infty \exp [\omega(t)] |d\alpha|.$$

It is clear that $\mathfrak{S}(\omega)$ is a Banach space. In order to define the product $\gamma = \alpha\beta$ of two elements in $\mathfrak{S}(\omega)$, we consider the product measure $\tilde{\gamma}$ defined in the smallest sigma-field \mathfrak{B}_2 generated by $\mathfrak{B} \times \mathfrak{B}$. For any $\sigma \in \mathfrak{B}$ we set

$$\gamma(\sigma) \equiv \tilde{\gamma}[(u, v) | u + v \in \sigma; u, v \geq 0].$$

This product is clearly commutative; with the help of the Fubini Theorem it becomes

$$(4) \quad \gamma(\sigma) = \int_0^\infty \alpha(\sigma - u) d_u \beta = \int_0^\infty \beta(\sigma - u) d_u \alpha.$$

It also follows from the Fubini Theorem that

$$\begin{aligned} \|\gamma\| &= \int_0^\infty \exp [\omega(s)] |d_s \gamma| = \int_0^\infty \int_0^\infty \exp [\omega(u + v)] |d_u \alpha| |d_v \beta| \\ &\leq \int_0^\infty \int_0^\infty \exp [\omega(u)] \exp [\omega(v)] |d_u \alpha| |d_v \beta| = \|\alpha\| \cdot \|\beta\|. \end{aligned}$$

Finally we set

$$(5) \quad e_t(\sigma) = \begin{cases} 1 & \text{if } t \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then e_0 is the unit of our algebra and $e_t \alpha = \alpha(\sigma - t)$ is the translate of α (t units to the right); $\|e_t\| = \exp [\omega(t)]$. As defined, $\mathfrak{S}(\omega)$ is a commutative Banach algebra with unit.

Now let $\mathfrak{I}(\omega)$ be the absolutely continuous (relative to Lebesgue measure) elements of $\mathfrak{S}(\omega)$. Then $\mathfrak{I}(\omega)$ is an ideal in $\mathfrak{S}(\omega)$ and for each $\alpha \in \mathfrak{I}(\omega)$, $e_t \alpha$ is a continuous function of t (in the norm sense). Hence for $\alpha \in \mathfrak{I}(\omega)$ we may interpret the convolution (4) as an abstract Lebesgue-Stieltjes integral of a continuous function in $\mathfrak{S}(\omega)$, namely,

$$(6) \quad \gamma = \int_0^\infty (\alpha e_t) d_t \beta.$$

According to the Gelfand theory [3] of Banach algebras, each maximal ideal m corresponds to a unique multiplicative linear functional μ_m such that

- (a) $\mu_m(\alpha) = 0$ if and only if $\alpha \in m$,
- (7) (b) $\mu_m(e_0) = 1$,
- (c) $\mu_m(\alpha + \beta) = \mu_m(\alpha) + \mu_m(\beta)$; $\mu_m(c\alpha) = c\mu_m(\alpha)$; $\mu_m(\alpha\beta) = \mu_m(\alpha) \cdot \mu_m(\beta)$.

The set of all maximal ideals \mathfrak{M} can be topologized to form a compact Hausdorff space with a general neighborhood of m_0 defined by a finite set of α 's and $\epsilon > 0$ to be $[m \mid |\alpha_i(m) - \alpha_i(m_0)| < \epsilon; i = 1, \dots, n]$. It is clear that in this topology the representation $\alpha(m) = \mu_m(\alpha)$ is a continuous function on \mathfrak{M} . Another consequence of the theory is that an $\alpha \in \mathfrak{S}(\omega)$ has an inverse in $\mathfrak{S}(\omega)$ if and only if α does not belong to any $m \in \mathfrak{M}$ (that is, if $\alpha(m)$ is never zero).

We shall distinguish between three types of maximal ideals:

- (a) \mathfrak{B} = all maximal ideals m such that $\mathfrak{V}(\omega)$ is not contained in m .
- (8) (b) \mathfrak{Z} = the maximal ideal consisting of all α such that $\alpha(0) = 0$.
- (c) $\mathfrak{U} = \mathfrak{M} - (\mathfrak{B} \cup \mathfrak{Z})$.

THEOREM 2.1 \mathfrak{B} is homeomorphic with the complex numbers $\mathfrak{C}_0 \equiv [\lambda \mid \mathfrak{R}(\lambda) \leq \omega_0]$. Under this correspondence $\mu_m(\alpha) = \int_0^\infty \exp(\lambda t) d\alpha$. \mathfrak{B} is an open subset of \mathfrak{M} .

If $\lambda \in \mathfrak{C}_0$, then it is clear that $\mu(\alpha) = \int_0^\infty \exp(\lambda t) d\alpha$ exists by equations (2), (3) and defines a linear multiplicative functional on $\mathfrak{S}(\omega)$. Since $\mu(\alpha)$ does not vanish for all $\alpha \in \mathfrak{V}(\omega)$, it follows that $m_\lambda = [\alpha \mid \mu(\alpha) = 0]$ belongs to \mathfrak{B} . On the other hand if $m \in \mathfrak{B}$, then there exists an $\alpha_0 \in \mathfrak{V}(\omega)$ such that $\mu_m(\alpha_0) \neq 0$. Now $\mu_m(\alpha_0 e_t) = \mu_m(\alpha_0) \mu_m(e_t)$ is continuous in t and hence $\mu_m(e_t) \rightarrow \mu_m(e_0) = 1$ as $t \rightarrow 0$. Further $\mu_m(e_{t+s}) = \mu_m(e_t) \cdot \mu_m(e_s)$. It follows that there exists a complex number λ such that $\mu_m(e_t) = \exp(\lambda t)$. By (2)

$$\mathfrak{R}(\lambda) = t^{-1} \log |\exp(\lambda t)| \leq \inf [t^{-1} \log \|e_t\|] = \omega_0.$$

For any $\beta \in \mathfrak{S}(\omega)$, we have by (6)

$$\begin{aligned} \mu_m(\alpha_0) \cdot \mu_m(\beta) &= \mu_m(\alpha_0 \beta) = \mu_m \left[\int_0^\infty (\alpha_0 e_t) d_t \beta \right] \\ &= \int_0^\infty \mu_m(\alpha_0 e_t) d_t \beta = \int_0^\infty \mu_m(\alpha_0) \exp(\lambda t) d_t \beta \end{aligned}$$

and hence $\mu_m(\beta) = \int_0^\infty \exp(\lambda t) d_t \beta$. Finally since $\exp(\lambda t) = \int_0^\infty \exp(\lambda s) d_s e_t$, it is clear that the correspondence between $m \in \mathfrak{B}$ and $\lambda \in \mathfrak{C}_0$ is one-to-one.

We next show that the correspondence is actually a homeomorphism. Now $\mu_\lambda(\alpha) = \int_0^\infty \exp(\lambda t) d\alpha$ is a continuous function of λ on \mathfrak{C}_0 for each $\alpha \in \mathfrak{S}(\omega)$. Hence an arbitrary neighborhood in \mathfrak{B} , namely,

$$[\lambda \mid |\mu_\lambda(\alpha_i) - \mu_{\lambda_0}(\alpha_i)| < \epsilon; \lambda \in \mathfrak{C}_0; i = 1, \dots, n]$$

corresponds to an open subset of \mathfrak{C}_0 . On the other hand, given a neighborhood $\mathfrak{B} = [\lambda \mid |\lambda - \lambda_0| < \epsilon, \lambda \in \mathfrak{C}_0]$ of λ_0 in \mathfrak{C}_0 we show that it contains a \mathfrak{B} neighborhood. The real part of λ is sufficiently limited by

$$|\Re(\lambda) - \Re(\lambda_0)| = |\log |\mu_\lambda(e_1)| - \log |\mu_{\lambda_0}(e_1)|| < \epsilon/2;$$

in other words by $|\mu_\lambda(e_1) - \mu_{\lambda_0}(e_1)| < |\mu_{\lambda_0}(e_1)| \cdot \epsilon/2$. Further given an $\alpha_0 \in \mathfrak{F}(\omega)$ such that $\mu_{\lambda_0}(\alpha_0) \neq 0$, it follows from the Riemann-Lebesgue Theorem that there exists an N such that $|\mu_\lambda(\alpha_0) - \mu_{\lambda_0}(\alpha_0)| < |\mu_{\lambda_0}(\alpha_0)|/2$ excludes all $\lambda \in \mathfrak{C}_0$ for which $|\Im(\lambda)| > N$. Finally we can exclude all $\lambda \in \mathfrak{C}_0$ for which $\epsilon/2 \leq |\Im(\lambda) - \Im(\lambda_0)| < N$ by the condition $|\mu_\lambda(e_{1/N}) - \mu_{\lambda_0}(e_{1/N})| < |\mu_{\lambda_0}(e_{1/N})| \cdot \sin \epsilon/2N$. It remains to show that \mathfrak{B} is an open subset of \mathfrak{M} and this follows from the fact that \mathfrak{B} is the union of the open sets $[m|\mu_m(\alpha)| > 0]$ where α ranges over $\mathfrak{F}(\omega)$.

THEOREM 2.2. *If μ_0 corresponds to \mathfrak{Z} , then $\mu_0(\alpha) = \alpha(0)$. Further μ_0 is the only multiplicative linear functional such that $\mu(e_t) = 0$ for some $t > 0$.*

Since $\alpha(0) + \beta(0) = (\alpha + \beta)(0)$, $\alpha\beta(0) = \alpha(0) \cdot \beta(0)$, $|\alpha(0)| \leq \|\alpha\|$, and $e_0(0) = 1$, it follows that $\mu_0(\alpha) = \alpha(0)$ is a multiplicative linear functional and hence that \mathfrak{Z} is the corresponding maximal ideal. Suppose μ is a multiplicative linear functional and $\mu(e_{t_0}) = 0$ ($t_0 > 0$). Since $\mu(e_{t+s}) = \mu(e_t e_s) = \mu(e_t) \cdot \mu(e_s)$, it follows that $\mu(e_t) = 0$ for all $t \geq t_0$. Further for any $t > 0$ there is an n such that $nt \geq t_0$, and hence $\mu(e_t) = [\mu(e_{nt})]^{1/n} = 0$. Finally let $\alpha \in \mathfrak{S}(\omega)$ and suppose $\alpha(0) = 0$. We then set $I_\delta = [0, \delta]$ and define $\alpha_\delta(\sigma) = \alpha(\sigma - I_\delta \cap \sigma)$. Then $\lim_{\delta \rightarrow 0} \|\alpha_\delta - \alpha\| = 0$. Further $\alpha_\delta e_{-\delta/2} \in \mathfrak{S}(\omega)$ and $\alpha_\delta = (\alpha_\delta e_{-\delta/2}) e_{\delta/2}$. Hence $\mu(e_{\delta/2}) = 0$ implies that $\mu(\alpha_\delta) = 0$ and by continuity that $\mu(\alpha) = 0$. For arbitrary $\alpha \in \mathfrak{S}(\omega)$, $\mu(\alpha) = \mu(\alpha - \alpha(0)e_0) + \mu(\alpha(0)e_0) = \alpha(0)$; hence $\mu = \mu_0$.

THEOREM 2.3. *The multiplicative linear functional μ corresponds to a maximal ideal in \mathfrak{U} if and only if $\mu[\mathfrak{F}(\omega)] = 0$ and $\mu(e_t) = \exp(\xi t)\chi(t)$ where $\xi \leq \omega_0$ and $\chi(t)$ is a character of the real line.*

It follows from the definition of \mathfrak{U} and the previous two theorems that $\mu \in \mathfrak{U}$ if and only if $\mu[\mathfrak{F}(\omega)] = 0$ and $\mu(e_t)$ is never zero. Now $\mu(e_0) = 1$, $\mu(e_{t+s}) = \mu(e_t) \cdot \mu(e_s)$, and $|\mu(e_t)| \leq \|e_t\| = \exp[\omega(t)]$, which is bounded in every finite interval $[\delta, 1/\delta]$. By a well known theorem on linear functions, $\log |\mu(e_t)| = \xi t$. Further $\xi \leq \inf \omega(t)/t = \omega_0$. Finally $\exp(-\xi t)\mu(e_t)$ is clearly a character of the real line. This character $\chi(t)$ may be either continuous or nonmeasurable.

If $\mu \in \mathfrak{U}$, it is not known how to express $\mu(\alpha)$ in terms of $\mu(e_t)$. However if we limit $\mathfrak{S}(\omega)$ to contain no singular functions, then for $\mu \in \mathfrak{U}$ we have $\mu(\alpha) = \sum_{t \geq 0} \mu(e_t) \cdot \alpha(t)$. Also in this case it can be shown that \mathfrak{B} is dense in \mathfrak{M} . This very important fact leads directly to an inversion theorem of the Wiener-Pitt type. However, since we shall not need this result we do not give the proof and merely remark that it is a consequence of the Riemann-Lebesgue Theorem and the Kronecker Theorem (see [5, pp. 61-62]).

We conclude this section with a characterization of nonmeasurable characters which we shall have occasion to use in section four.

LEMMA 2.1. *Let $\chi(t)$ be a character of the real line and set $\mathfrak{F}_\tau = [\chi(t) | 0 \leq t \leq \tau]$*

and $\mathfrak{F} = \bigcap_{\tau > 0} \overline{\mathfrak{F}}_\tau$. There are two alternatives: either $\mathfrak{F} = 1$ in which case $\chi(t)$ is a continuous character and $\chi(t) = \exp(at)$; or else \mathfrak{F} contains the unit circle and $\chi(t)$ is nonmeasurable.

If \mathfrak{F} consists only of the point 1, then given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\chi(t) - 1| < \epsilon$ for all $t \in [0, \delta]$. Otherwise the closed interval on the unit circle $[|\exp(i\phi) - 1| \geq \epsilon]$ would contain points of $\chi(t)$ for arbitrarily small t and hence would have points in common with \mathfrak{F} . Therefore $\chi(t)$ is continuous at the origin and of the form $\exp(at)$.

On the other hand suppose $\exp(i\phi_0) \in \mathfrak{F}$ ($0 < \phi_0 < 2\pi$). Then there exists a sequence $t_n \rightarrow 0$ such that $\chi(t_n) \rightarrow \exp(i\phi_0)$ and hence $\chi(kt_n) \rightarrow \exp(ik\phi_0)$ so that $\exp(ik\phi_0)$ likewise belongs to \mathfrak{F} . Further if $\exp(i\phi_1)$ and $\exp(i\phi_2)$ belong to \mathfrak{F} , then similarly $\exp[i(\phi_1 + \phi_2)]$ belongs to \mathfrak{F} . Thus if \mathfrak{F} is not dense on the unit circle it must contain a smallest argument ϕ_0 and \mathfrak{F} consists precisely of the points $[\exp(ik\phi_0) \mid k = 1, 2, \dots, 2\pi/\phi_0]$. Again there exists a sequence $t_n \rightarrow 0$ such that $\chi(t_n) \rightarrow \exp(i\phi_0)$. In this case $\chi(t_n/2)$ will contain a subsequence which converges either to $\exp(i\phi_0/2)$ or to $-\exp(i\phi_0/2)$. Neither of these points is in the set $[\exp(ik\phi_0)]$ and hence \mathfrak{F} must be dense in the unit circle. Since \mathfrak{F} is closed it must consist of all points in the unit circle.

It should be remarked that there is nothing special about $t = 0$ in the above lemma and that a similar result holds for any real t .

Before closing this section it would be well to point out that most of the results in this paper can be obtained without the use of $\mathfrak{S}(\omega)$. However, in proceeding as we do we are able to reduce part of the difficulty to the behavior of the relatively simple Banach algebra $\mathfrak{S}(\omega)$. Furthermore $\mathfrak{S}(\omega)$ will enter directly in the statement of some of our theorems.

3. Banach algebra associated with the semi-group of operators. Let \mathfrak{X} be a complex Banach space, $\mathfrak{C}(\mathfrak{X})$ the bounded linear transformations on \mathfrak{X} to itself, and let $T(t)$ be a semi-group (see [6]) of operators on $[0, \infty)$ to $\mathfrak{C}(\mathfrak{X})$ satisfying the following hypothesis.

Hypothesis H.

$$(a) \quad T(t_1 + t_2) = T(t_1)T(t_2), \quad t_1, t_2 \geq 0,$$

$$T(0) = I,$$

$$(b) \quad T(t)x \text{ is strongly measurable,}$$

$$(c) \quad \int_0^1 \|T(t)\| dt < \infty,$$

$$(d) \quad \lim_{\tau \rightarrow 0} \tau^{-1} \int_0^\tau T(t)x dt = x \text{ for all } x \in \mathfrak{X}.$$

Postulate (H-b) implies that $T(t)$ is strongly continuous for $t > 0$ and hence that $\|T(t)\|$ is lower semi-continuous (see [6] and [10]). It is therefore clear

that the subadditive function,

$$(9) \quad \omega(t) = \log \|T(t)\|,$$

satisfies the conditions (1) and may therefore be used to define a Banach algebra of the type $\mathfrak{S}(\omega)$.

Let A be the infinitesimal generator of a semi-group $T(t)$ of type (H). Then A will be a closed linear operator, in general unbounded, with dense domain $\mathfrak{D}(A)$. Let $R(\lambda; A)$ be the resolvent of A in $\mathfrak{C}(\mathfrak{X})$ defined and holomorphic on the resolvent set for A , namely $\rho(A)$. On this set $R(\lambda; A)$ satisfies the first resolvent equation

$$(10) \quad R(\lambda; A) - R(\zeta; A) = (\zeta - \lambda)R(\lambda; A)R(\zeta; A)$$

as well as the defining equations

$$(11) \quad \begin{aligned} (\lambda I - A)R(\lambda; A)x &= x && \text{for } x \in \mathfrak{X}, \\ R(\lambda; A)(\lambda I - A)x &= x && \text{for } x \in \mathfrak{D}(A). \end{aligned}$$

For $\Re(\lambda) > \omega_0$

$$(12) \quad R(\lambda; A)x = \int_0^\infty \exp(-\lambda t)T(t)x dt.$$

The spectrum of A , denoted by $\Sigma(A)$, is by definition the complementary set to $\rho(A)$ (²). Among other things, we shall be concerned with the relationship between $\Sigma(A)$, the spectrum of $T(t)$, and rate of growth properties of $T(t)$.

Let \mathfrak{A}_1 consist of the operators

$$\begin{aligned} [T(t) \mid t \geq 0], \\ [R(\lambda; T(t)) \mid t \geq 0, \lambda \in \rho[T(t)]], \\ [R(\lambda; A) \mid \lambda \in \rho(A)]. \end{aligned}$$

We shall now show that \mathfrak{A}_1 is an abelian set. In the first place $T(t)$ leaves $\mathfrak{D}(A)$ invariant and for $x \in \mathfrak{D}(A)$ we have $T(t)Ax = AT(t)x$. Hence by (11)

$$\begin{aligned} (\lambda I - A)T(t)R(\lambda; A)x &= T(t)(\lambda I - A)R(\lambda; A)x \\ &= T(t)x = (\lambda I - A)R(\lambda; A)T(t)x \end{aligned}$$

for all $x \in \mathfrak{X}$. Operating on the first and last members of this series of equalities by $R(\lambda; A)$ gives $R(\lambda; A)T(t) = T(t)R(\lambda; A)$. Finally since $R(\lambda; T(t))$ is the inverse of $[\lambda I - T(t)]$, it commutes with everything that commutes with $T(t)$.

We now define \mathfrak{R}_1 to be the strong closure of the set of all polynomials in \mathfrak{A}_1 . Then \mathfrak{R}_1 is an abelian subring of $\mathfrak{C}(\mathfrak{X})$. In fact let \mathfrak{R}' be any abelian subring of $\mathfrak{C}(\mathfrak{X})$ and let \mathfrak{R} be its strong closure. Then if $A, B \in \mathfrak{R}$ it is clear that

(²) The point at infinity is never considered as belonging to $\Sigma(A)$, not even if $R(\lambda; A)$ is singular at infinity.

aA and $A+B$ belong to \mathfrak{R} . Given $(x_1, x_2, \dots, x_n; \epsilon)$, then for any $C' \in \mathfrak{R}'$ there exists an $A' \in \mathfrak{R}'$ such that $\|C'(A-A')x_i\| < \epsilon$ and $\|(A-A')C'x_i\| < \epsilon$ for $i=1, 2, \dots, n$. Hence $AC' = C'A$ belongs to \mathfrak{R} . Likewise there exists a $B' \in \mathfrak{R}'$ such that $\|A(B-B')x_i\| < \epsilon$ and $\|(B-B')Ax_i\| < \epsilon$ for $i=1, \dots, n$. Since as above $AB' = B'A \in \mathfrak{R}$, there exists a $D' \in \mathfrak{R}'$ such that $\|(AB' - D')x_i\| < \epsilon$. Combining these inequalities we get $\|(AB - D')x_i\| < 2\epsilon$ and $\|(BA - D')x_i\| < 2\epsilon$. Hence $AB = BA$ belongs to \mathfrak{R} .

We now adjoin to \mathfrak{R}_1 all inverses in $\mathfrak{G}(\mathfrak{X})$ of elements in \mathfrak{R}_1 . The resulting set \mathfrak{A} is clearly abelian. Finally we define \mathfrak{R} to be the strong closure of all polynomials in \mathfrak{A} . As before \mathfrak{R} will be a commutative Banach algebra with unit⁽³⁾.

Since \mathfrak{R}_1 is the strong closure of polynomials in \mathfrak{A}_1 , it will in particular contain the elements $\Theta(\alpha)$ defined by

$$(13) \quad \Theta(\alpha)x = \int_0^\infty T(t)xd\alpha$$

for $\alpha \in \mathfrak{S}(\omega)$. Further it is clear that

$$\begin{aligned} \Theta(e_0) &= I, \\ \Theta(a\alpha + b\beta) &= a\Theta(\alpha) + b\Theta(\beta), \\ \Theta(\alpha\beta)x &= \int_0^\infty T(t)xd\gamma = \int_0^\infty \int_0^\infty T(u)T(v)xd_\nu\beta d_\mu\alpha \\ (14) \quad &= \int_0^\infty T(u)\Theta(\beta)xd_\mu\alpha = \Theta(\alpha)[\Theta(\beta)x], \\ \|\Theta(\alpha)\| &\leq \int_0^\infty \|T(t)\| |d\alpha| \leq \|\alpha\|. \end{aligned}$$

Hence (13) defines a continuous homeomorphism of $\mathfrak{S}(\omega)$ into \mathfrak{R} which takes the unit e_0 into the identity I . The mapping $\Theta(\alpha)$ can be thought of as defining an operational calculus for the infinitesimal generator A (see [6, chap. 15]). It should also be pointed out that for $x \in \mathfrak{D}(A)$, $A\Theta(\alpha)x = \Theta(\alpha)Ax$. This follows from the fact that A is closed and that $\Theta(\alpha)x = \int_0^\infty T(t)xd\alpha$ is the limit of partial sums each of which lie in $\mathfrak{D}(A)$.

We next apply the Gelfand theory [3] to the commutative Banach algebra with unit, \mathfrak{R} . Let \mathfrak{M}' be the set of all maximal ideals in \mathfrak{R} , topologized in the usual way to form a compact Hausdorff space. As m' ranges over \mathfrak{M}' , the function $B(m') = \mu_{m'}(B)$ is continuous on \mathfrak{M}' for each $B \in \mathfrak{R}$. The range of $B(m')$ consists precisely of the spectrum of B relative to \mathfrak{R} ⁽⁴⁾. For

(3) Any commutative Banach algebra contained in $\mathfrak{G}(\mathfrak{X})$ and containing \mathfrak{R} could be used instead of \mathfrak{R} .

(4) The symbol $\Sigma(B)$ is used exclusively for the spectrum of B relative to $\mathfrak{G}(\mathfrak{X})$.

each $B \in \mathfrak{R}_1$, \mathfrak{R} includes $R[\lambda; B]$ for $\lambda \in \rho(B)$; hence the range of $B(m')$ is likewise the spectrum of $B \in \mathfrak{R}_1$ relative to $\mathfrak{C}(x)$. One of the important results of the Gelfand theory is that the spectral radius

$$(15) \quad |||B||| = \sup [| B(m') | | m' \in \mathfrak{M}'] = \lim_{n \rightarrow 0} |||B^n|||^{1/n}.$$

Given a maximal ideal $m' \in \mathfrak{M}'$ and its associated multiplicative linear functional μ' , then

$$\mu'[\Theta(\alpha)] = \mu(\alpha)$$

clearly defines a multiplicative linear functional μ on $\mathfrak{S}(\omega)$. Since $\Theta(e_0) = I$, μ is nondegenerate. Thus Θ induces a mapping Φ on \mathfrak{M}' into \mathfrak{M} such that

$$(16) \quad \Theta(\alpha)(m') = \alpha[\Phi(m')].$$

It is easy to see that Φ is a continuous mapping. For, given a neighborhood $\mathfrak{B} = [m | |\alpha_i(m) - \alpha_i(m_0)| < \epsilon; i = 1, \dots, n]$ of $m_0 = \Phi(m'_0)$, then the Φ image of $\mathfrak{B}' = [m' | |\Theta(\alpha_i)(m') - \Theta(\alpha_i)(m'_0)| < \epsilon; i = 1, 2, \dots, n]$ is contained in \mathfrak{B} .

Corresponding to the sets $\mathfrak{B}, \mathfrak{Z}, \mathfrak{U}$ defined by (8), we now define

$$(17) \quad \mathfrak{B}' = \Phi^{-1}(\mathfrak{B}), \quad \mathfrak{Z}' = \Phi^{-1}(\mathfrak{Z}), \quad \mathfrak{U}' = \Phi^{-1}(\mathfrak{U}).$$

It follows from section two that the sets $\mathfrak{B}', \mathfrak{Z}', \mathfrak{U}'$ form a mutually disjoint subdivision of \mathfrak{M}' . Since \mathfrak{B} is an open subset of \mathfrak{M} and since Φ is continuous, \mathfrak{B}' will likewise be an open subset of \mathfrak{M}' . We note that $\Theta(e_t) = T(t)$ and hence that $T(t)(m') = e_t[\Phi(m')]$. Making use of the theory developed in section two it follows that (a) if $m' \in \mathfrak{B}'$, then there exists a complex number $a(m')$ with $\Re[a(m')] \leq \omega_0$ such that $T(t)(m') = \exp [a(m')t]$; (b) if $m' \in \mathfrak{Z}'$, then $T(t)(m') = 1$ for $t=0$ and zero for $t>0$ and hence $0 \in \Sigma(T(t))$ for $t>0$; and finally (c) if $m' \in \mathfrak{U}'$, $T(t)(m') = \chi(t) \cdot \exp (\xi t)$ where $\xi \leq \omega_0$ and $\chi(t)$ is a character of the real line. Moreover if $\alpha \in \mathfrak{X}(\omega)$, then

$$\begin{aligned} \Theta(\alpha)(m') &= \int_0^\infty \exp [a(m')t] d\alpha && (m' \in \mathfrak{B}') \\ &= 0 && \text{elsewhere.} \end{aligned}$$

In particular if $d\alpha([0, t])/dt = \exp (-\lambda t)$ with $\Re(\lambda) > \omega_0$, then by (12)

$$(18) \quad \begin{aligned} R(\lambda; A)(m') &= [\lambda - a(m')]^{-1} && (m' \in \mathfrak{B}') \\ &= 0 && \text{elsewhere.} \end{aligned}$$

THEOREM 3.1. *The relation (18) is valid for all $\lambda \in \rho(A)$ and $\Sigma(A) = [a(m') | m' \in \mathfrak{B}']$. Further $\Re[\Sigma(A)] \leq \omega_0$.*

Suppose now that $\lambda, \zeta \in \rho(A)$, then the first resolvent equation (10) is satisfied. Hence for $\Re(\lambda) > \omega_0$,

$$R(\zeta; A)(m') - [\lambda - a(m')]^{-1} = (\lambda - \zeta)[\lambda - a(m')]^{-1}R(\zeta; A)(m') \quad \text{on } \mathfrak{M}',$$

$$R(\zeta; A)(m') - 0 = 0 \cdot R(\zeta; A)(m') \quad \text{elsewhere,}$$

from which it follows that (18) is valid with λ replaced by ζ . In particular, $a(m')$ must belong to the set $\Sigma(A)$. Conversely suppose that ζ is not in the range of $[a(m') \mid m' \in \mathfrak{M}']$ and that $\Re(\lambda) > \omega_0$, then

$$[I + (\zeta - \lambda)R(\lambda; A)](m') = [\zeta - a(m')][\lambda - a(m')]^{-1} \quad \text{on } \mathfrak{M}'$$

$$= 1 \quad \text{elsewhere.}$$

Since $[I + (\zeta - \lambda)R(\lambda; A)](m') \neq 0$ on \mathfrak{M}' , the inverse of $[I + (\zeta - \lambda)R(\lambda; A)]$ exists in \mathfrak{R} . If we set

$$R(\zeta) = R(\lambda; A)[I + (\zeta - \lambda)R(\lambda; A)]^{-1},$$

then for all $x \in \mathfrak{X}$

$$(\zeta I - A)R(\zeta)x = [(\zeta - \lambda)I + (\lambda I - A)]R(\lambda; A)[I + (\zeta - \lambda)R(\lambda; A)]^{-1}x$$

$$= [(\zeta - \lambda)R(\lambda; A) + I][I + (\zeta - \lambda)R(\lambda; A)]^{-1}x = x,$$

whereas for $x \in \mathfrak{D}(A)$

$$R(\zeta)(\zeta I - A)x = [I + (\zeta - \lambda)R(\lambda; A)]^{-1}R(\lambda; A)[(\zeta - \lambda)I + (\lambda I - A)]x$$

$$= [I + (\zeta - \lambda)R(\lambda; A)]^{-1}[(\zeta - \lambda)R(\lambda; A) + I]x = x.$$

Hence $R(\zeta)$ is actually the resolvent of A at ζ and $\zeta \in \rho(A)$.

This concludes our preliminary remarks. We now go on to a detailed study of various types of semi-groups of operators.

4. $T(t)$ **uniformly continuous at $t=0$** . If $T(t)$ is continuous in the uniform topology at $t=0$, then its infinitesimal generator A is a bounded linear transformation and conversely (see [6, chap. 8]). In fact $A = \lim_{\tau \rightarrow 0} \tau^{-1}[T(\tau) - I]$ exists in the uniform topology and $T(t) = \exp (At)$. Hence A belongs to the ring \mathfrak{R} introduced in section three and has the representation $A(m') = a(m')$ as a continuous function over the maximal ideals \mathfrak{M}' of \mathfrak{R} . Likewise $T(t)(m') = \exp [a(m')t]$.

THEOREM 4.1. *For a semi-group of operators $T(t)$ of type (H), the following statements are equivalent:*

- (a) $T(t) = \exp (At)$ where A is a bounded linear transformation.
- (b) $\mathfrak{M}' = \mathfrak{M}$.
- (c) $T(t)$ has the representation $T(t)(m') = \exp [a(m')t]$ over the maximal ideals \mathfrak{M}' of \mathfrak{R} .

Suppose $T(t) = \exp (At)$ where $A \in \mathfrak{G}(\mathfrak{X})$. Then $T(t)$ is continuous in the uniform topology for $t \geq 0$, and hence $\int_0^t T(t)dt$ exists in the uniform topology. Now the multiplicative linear functionals μ' are continuous in the uniform topology so that

$$\begin{aligned} \mu' \left[\int_0^\tau T(t) dt \right] &= \int_0^\tau \mu' [T(t)] dt = \int_0^\tau \exp [\mu'(A)t] dt \\ &= \tau \text{ or } \{ \exp [\mu'(A)\tau] - 1 \} / \mu'(A) \end{aligned}$$

according as $\mu'(A)$ is or is not zero. Now $\alpha_0([0, t]) = t$ for $0 \leq t \leq \tau$ and $=\tau$ for $t \geq \tau$ belongs to $\mathfrak{R}(\omega)$ and

$$\mu' \left[\int_0^\tau T(t) dt \right] = \mu' [\Theta(\alpha_0)] = \Theta(\alpha_0)(m') = \alpha_0[\Phi(m')] \neq 0$$

for τ sufficiently small. Hence $m' \in \mathfrak{M}'$ for all $m' \in \mathfrak{M}'$. Thus (a) implies (b). By Theorem 2.1, if $m' \in \mathfrak{M}'$ there exists a complex number $a(m')$ such that $T(t)(m') = e_t[\Phi(m')] = \exp [a(m')t]$. Hence (b) implies (c).

It remains to prove that (c) implies (a). Here we assume merely that the representation of $T(t)$ is of the form $\exp [a(m')t]$ and that this function is continuous in m' for each $t \geq 0$. We first show that $a(m')$ is itself continuous on \mathfrak{M}' and later that there exists a bounded linear transformation A such that $T(t) = \exp (At)$.

LEMMA 4.1. *If $g_t(m') = \exp [a(m')t]$ is continuous for each $t \geq 0$ in the compact Hausdorff space \mathfrak{M}' , then $a(m')$ is itself continuous on \mathfrak{M}' .*

Since $g_1(m') \neq 0$ and is continuous on \mathfrak{M}' , $0 < c \leq |g_1(m')| \leq k$. Hence $\Re[a(m')]$ is bounded. Suppose that $|\Im[a(m')]|$ were not bounded. Then we can find a sequence $\{m'_n\}$ such that $\Re[a(m'_n)] \rightarrow b$ and $\Im[a(m'_n)] \rightarrow \infty$ (or $-\infty$). Since $|g_t(m')|$ is bounded on \mathfrak{M}' , we can then find a subsequence (which we renumber) such that $g_t(m'_n)$ converges to a limit as $n \rightarrow \infty$ for each t of a denumerable dense subset \mathfrak{P} of $[0, \infty)$. Since \mathfrak{M}' is compact there exists at least one point of condensation m'_0 of the set $\{m'_n\}$. Now $a(m'_0)$ will be of the form $(b + id)$. Further since $g_t(m')$ is continuous in each variable, it follows that

$$h_t(m'_n) \equiv \exp [\{a(m'_n) - (b + id)\}t] \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

for $t \in \mathfrak{P}$. We now define

$$\Omega_n = [t \mid 0 \leq t \leq 2\pi; \pi/2 \leq \arg h_t(m'_n) \leq 3\pi/2].$$

Because of the fact that $\Im[a(m'_n) - (b + id)] \rightarrow \infty$ with n , we have, for sufficiently large n , $\text{meas}(\Omega_n) \geq 1$. Hence $\text{meas}(\limsup \Omega_n) \geq 1$. For any $t_0 \in \limsup \Omega_n$, we choose a subsequence of the n 's (again we renumber) such that $h_{t_0}(m'_n) \rightarrow \exp(i\phi_0)$ with $\pi/2 \leq \phi_0 \leq 3\pi/2$. We now have

$$(19) \quad \begin{aligned} g_t(m'_n) &\rightarrow \exp [(b + id)t], & t \in \mathfrak{P}, \\ h_{t_0}(m'_n) &\rightarrow \exp(i\phi_0) \exp [(b + id)t_0]. \end{aligned}$$

For no condensation point m' of this final subsequence can $g_t(m')$ be of the

form $\exp [a(m')t]$ and hence continuous in t since it must be equal to the right sides of (19) both on the dense set \mathfrak{B} and on t_0 . It therefore follows that $|\Im[a(m')]|$ is bounded on \mathfrak{M}' . This implies that $(g_r(m') - 1)/\tau$ converges uniformly on \mathfrak{M}' as $\tau \rightarrow 0$. Thus $a(m')$ is the uniform limit of continuous functions and is therefore itself continuous on \mathfrak{M}' .

We now return to the proof of Theorem 4.1. By our lemma $|a(m')|$ is bounded, say by k . Hence for $t < \pi/2k$, $\exp [a(m')t]$ lies in the interior of the right half-plane for all $m' \in \mathfrak{M}'$. Hence $t^{-1} \log [T(t)] \equiv A_t$ is a well defined element of \mathfrak{K} (see Hille [6, chap. 5]), and $A_t(m') = a(m')$. If \mathfrak{K} had no radical, then we could assert that A_t is independent of t and hence that $T(t) = \exp (At)$. Allowing for the possibility of a radical we argue as follows. For notational convenience suppose that $2k < \pi$ and set $A_1 = A$. It is clear that $T(n) = \exp (An)$ and therefore that the semi-group

$$S(t) = \exp (-At)T(t)$$

is periodic of period one. Since $T(t)$ satisfies the hypothesis (H) and since $\exp (-At)$ is continuous in the uniform topology, it follows that $S(t)$ also satisfies the hypothesis (H). Now $S(t)$, being of period one, must be strongly continuous everywhere and uniformly bounded in norm and this remains true even if we extend $S(t)$ periodically over $(-\infty, \infty)$. Finally the representation of $S(t)$ is given by $S(t)(m') = \exp [-a(m')t] \exp [a(m')t] = 1$ and hence $S(t)$ differs from the identity at most by a generalized nilpotent element of \mathfrak{K} . It now follows from a rather deep theorem due to Gelfand [4] (see also [6, p. 493] and [15]) that $S(t) = I$ and hence that $T(t) = \exp (At)$. For the sake of completeness we shall give a simpler proof of this fact which applies to our special situation. We define

$$J_n(x) = \int_0^1 S(t)x \exp (-2\pi int) dt.$$

It can be shown by direct computation that $J_n J_m = \delta_{mn} J_n$; that is, the J_n 's form an orthogonal system of projection operators. Further $J_n S(t)x = J_n(x) \exp (2\pi int)$. By the analogue of the classical Fejér Theorem on Fourier series, $S(t) = (C-1) \sum_{-\infty}^{\infty} J_n(x) \exp (2\pi int)$ and in particular $S(0)x = x = (C-1) \sum_{-\infty}^{\infty} J_n(x)$. Since J_n is idempotent, $J_n(m') = 0$ or 1 and since $\|J_n\| = \lim_{k \rightarrow \infty} \|J_n^k\|^{1/k} = \lim_{k \rightarrow \infty} \|J_n\|^{1/k}$, it follows that $J_n(m') \equiv 0$ if and only if $J_n = 0$. Hence

$$J_n(m') = J_n(m') \cdot S(t)(m') = [J_n S(t)](m') = J_n(m') \exp (2\pi int) \quad \text{for all } t$$

implies that $J_n(m') \equiv 0$ and therefore that $J_n = 0$ for $n \neq 0$. Thus $J_0 = S(0) = I$ and $S(t) \equiv I$.

COROLLARY 4.1. *If $T(t)$ is a semi-group of operators of type (H) and if there exists an unbounded open connected set \mathfrak{G} containing the origin such that for*

some interval $0 \leq a < t < b$, $\Sigma[T(t)] \cap \mathfrak{G} = \phi$, then the infinitesimal generator A is bounded and $T(t) = \exp(At)$.

Since zero does not belong to $\Sigma[T(t)]$ for some $t_0 > 0$, $T(t_0)(m') \neq 0$ and therefore \mathfrak{Z}' is vacuous. If for some m'_0 , $T(t)(m'_0) = \exp(\xi t)\chi(t)$ where $\chi(t)$ is a non-measurable character of the real line, then the values of $\arg [T(t)(m_0)]$ would, by Lemma 2.1, be dense on the unit circle in every interval $0 \leq a < t < b$ and this is easily seen to contradict our hypothesis on \mathfrak{G} . Hence we conclude that $T(t)(m')$ must be of the form $\exp [a(m')t]$ and the result now follows directly from Theorem 4.1. This corollary generalizes a theorem due to Hille [6, Theorem 13.7.1] by relaxing conditions on \mathfrak{G} , on the interval (a, b) , and on the semi-group of operators $T(t)$.

COROLLARY 4.2. *If $T(t)$ is a semi-group of operators of type (H), then the following statements are equivalent:*

- (a) *The infinitesimal generator A of $T(t)$ is unbounded.*
- (b) *\mathfrak{B}' is a proper subset of \mathfrak{M}' .*
- (c) *$0 \in \Sigma[R(\lambda; A)]$ for all λ of $\rho(A)$ ⁽⁵⁾.*

If we now define for $\alpha \in \mathfrak{S}(\omega)$ the function

$$(20) \quad f_\alpha(\lambda) = \int_0^\infty \exp(\lambda t) d\alpha$$

for $\Re(\lambda) \leq \omega_0$, then it is clear by Theorem 3.1 that $f_\alpha[\Sigma(A)]$ is precisely the range of values of $\Theta(\alpha)(m')$ on \mathfrak{B}' . Thus we obtain the following theorem.

THEOREM 4.2. *If the semi-group of operators $T(t)$ is uniformly continuous at $t=0$, then $\Sigma[\Theta(\alpha)] = f_\alpha[\Sigma(A)]$ for all $\alpha \in \mathfrak{S}(\omega)$. If the semi-group of operators $T(t)$ of type (H) has an unbounded infinitesimal generator A , then $\Sigma[\Theta(\alpha)] \supset f_\alpha[\Sigma(A)]$ for all $\alpha \in \mathfrak{S}(\omega)$ whereas for $\alpha \in \mathfrak{L}(\omega)$, $\Sigma[\Theta(\alpha)] = f_\alpha[\Sigma(A)] \cup 0$.*

It is clear that for $\alpha \in \mathfrak{L}(\omega)$, $\Theta(\alpha)(m') = 0$ on $\mathfrak{M}' - \mathfrak{B}'$ and hence that $0 \in \Sigma[\Theta(\alpha)]$ when A is unbounded⁽⁵⁾. Theorem 4.2 in essentially⁽⁶⁾ this form was first proved by Hille [6, Theorem 15.5.1].

It is worth remarking that if for some $\alpha \in \mathfrak{L}(\omega)$, $[\Theta(\alpha)]^{-1}$ exists, then $\mathfrak{B}' = \mathfrak{M}'$ and hence $T(t) = \exp(At)$ where A is a bounded linear operator. On the other hand if for some $\alpha \in \mathfrak{S}(\omega)$ with $\alpha(0) = 0$, $[\Theta(\alpha)]^{-1}$ exists, then \mathfrak{Z}' must be vacuous and therefore $T(t)^{-1}$ exists. In this case $T(t)$ can be embedded in a strongly continuous group of operators defined on $(-\infty, \infty)$.

5. $T(t)$ uniformly continuous for $t \geq c > 0$. In this section we shall deal with semi-groups of operators $T(t)$ of type (H) which are continuous in the

⁽⁵⁾ Both $\Theta(\alpha)$ and $R(\lambda; A)$ belong to \mathfrak{R}_1 and hence their spectra relative to $\mathfrak{C}(\mathfrak{X})$ are precisely the ranges of $\Theta(\alpha)(m')$ and $R(\lambda; A)(m')$ respectively.

⁽⁶⁾ There is an unresolved difficulty about 0 belonging to $\Sigma[\Theta(\alpha)]$ in Hille's proof for the case $\alpha \in \mathfrak{L}(\omega)$ and A unbounded.

uniform topology for $t \geq c > 0$ ⁽⁷⁾. We shall explicitly assume that $T(t)$ is not continuous in the uniform topology at $t=0$ and hence that the infinitesimal generator A is unbounded. For any linear multiplicative functional μ' on \mathfrak{X} , $\mu'(T(t)) = \mu'[\Theta(e_t)] = \mu(e_t)$ will be continuous for $t \geq c > 0$. This is enough to exclude nonmeasurable characters. However more can be asserted; for clearly

$$\mu' \left[\tau^{-1} \int_c^{c+\tau} T(t) dt \right] = \tau^{-1} \int_c^{c+\tau} \mu' [T(t)] dt \rightarrow \mu' [T(c)]$$

as $\tau \rightarrow 0$. But $\tau^{-1} \int_c^{c+\tau} T(t) dt = \Theta(\alpha)$ where $\alpha([0, t]) = 0$ for $t < c$, $= (t-c)/\tau$ for $c \leq t \leq c+\tau$, $= 1$ for $t > c+\tau$, and α belongs to $\mathfrak{X}(\omega)$. Hence if $\mu' [T(c)] = 0$ the corresponding $m' \in \mathfrak{Z}'$ and otherwise $m' \in \mathfrak{W}'$. Since we have assumed the infinitesimal generator A to be unbounded, $\mathfrak{Z}' = \mathfrak{W}' - \mathfrak{X}'$ will not be empty (Corollary 4.2). It now follows from Theorems 2.1 and 2.2 that

$$(21) \quad \Sigma[\Theta(\alpha)] = f_\alpha[\Sigma(A)] \cup \alpha(0)$$

for all $\alpha \in \mathfrak{C}(\omega)$. We shall even be able to obtain a correspondence between the detailed spectrum of $\Theta(\alpha)$ and A . To this end we prove the following lemma.

LEMMA 5.1. *If the semi-group of operators $T(t)$ is of type (H) and continuous in the uniform topology for $t \geq c > 0$, then $\Sigma(A)$ lies to the left of a bounding curve $\xi = \psi(|\eta|)$ ($\lambda = \xi + i\eta$) where $\lim_{\eta \rightarrow \infty} \psi(\eta) = -\infty$.*

By Theorem 3.1, $\Sigma(A) = [a(m') | m' \in \mathfrak{W}']$. Hence if the lemma were false, there would exist a subsequence $\{m'_n\} \subset \mathfrak{W}'$ such that $\mathfrak{R}[a(m'_n)] \rightarrow \xi_0 \leq \omega_0$ and $\mathfrak{I}[a(m'_n)] \rightarrow \infty$ (or $-\infty$). By the Riemann-Lebesgue Theorem, we have

$$(22) \quad \lim_{n \rightarrow \infty} \int_0^\infty \exp [a(m'_n)t] d\alpha = 0$$

for all $\alpha \in \mathfrak{X}(\omega)$. However, since \mathfrak{W}' is compact, $\{m'_n\}$ has at least one point of condensation, say m'_0 . It follows from (22) that $\Theta(\alpha)(m'_0) = 0$ for all $\alpha \in \mathfrak{X}(\omega)$ and hence that $m'_0 \notin \mathfrak{W}'$. On the other hand, $|\Theta(e_t)(m'_n)| \rightarrow \exp(\xi_0 t)$ and hence $m'_0 \in \mathfrak{Z}'$. This is contrary to the fact that $\mathfrak{W}' = \mathfrak{X}' \cup \mathfrak{Z}'$.

If we impose further restrictions on the semi-group of operators, the function $\psi(\eta)$ can be further delimited. Hille [7] has considered semi-group operators (strongly continuous for $t > 0$) which map \mathfrak{X} into $\mathfrak{D}(A)$. In this case $d^n T(t)x / (dt)^n = A^n T(t)x$ defines a bounded operator on X for $t > 0$ [7, Theorem 1].

THEOREM 5.1. *Let $T(t)$ be a semi-group of operators of type (H) such that $T(t)$ maps \mathfrak{X} into $\mathfrak{D}(A)$ if $t > 0$. Then $\Sigma(A)$ lies to the left of a bounding curve $\xi = K - \log |\eta|$ ($\lambda = \xi + i\eta$).*

⁽⁷⁾ The assumption that $T(t)$ is continuous in the uniform topology for $t \geq c > 0$ is equivalent to $T(t)$ being weakly measurable and $[T(t) | t \geq c > 0]$ being a separable subset of $\mathfrak{C}(\mathfrak{X})$. This follows from Theorems 3.3.1 and 8.3.1 of Hille [6].

It is readily seen that $T(t)$ will be continuous in the uniform topology for $t > 0$. For

$$T(b) - T(a) = \int_a^b AT(t)dt, \quad 0 < a < b,$$

and since $AT(a)$ is a bounded transformation,

$$\|T(b) - T(a)\| \leq \|AT(a)\| \int_0^{b-a} \|T(t)\|dt.$$

The right side converges to zero as $b \rightarrow a$ uniformly for $1/\delta < a < \delta < \infty$. More is true; namely, the derivative exists in the uniform topology. This is seen from the equation

$$[T(a + \Delta) - T(a)]/\Delta - AT(a) = \Delta^{-1} \int_a^{a+\Delta} [AT(t) - AT(a)]dt \quad (a, \Delta > 0)$$

from which it follows that

$$\|[T(a + \Delta) - T(a)]/\Delta - AT(a)\| \leq \|AT(a/2)\| \int_0^\Delta \|T(t - a/2) - T(a/2)\|dt.$$

Since $T(t)$ is continuous in the uniform topology for $t > 0$, the right-hand side of this inequality goes to zero with Δ . A similar argument holds for $\Delta < 0$. Making use of the above representation argument, we see that $T(t)(m') = \exp [a(m')t]$ on \mathfrak{B}' and 0 on \mathfrak{Z}' for $t > 0$. Since the derivative exists in the uniform topology, we see that $(AT)(m') = a(m') \exp [a(m')t]$ in \mathfrak{B}' and 0 on \mathfrak{Z}' for $t > 0$. Hence

$$\text{LUB } [|a(m')| \exp [a(m')t] | m' \in \mathfrak{B}'] \leq K(t) \equiv \|AT(t)\|.$$

In other words, $\Sigma(A)$ lies to the left of the curve $(\xi^2 + \eta^2)^{1/2} \exp [\xi t] = K(t)$ and a fortiori to the left of $|\eta| \exp [\xi t] = K(t)$, which can be written as $\xi = K - \log |\eta|$ when $t = 1$.

DEFINITION. The operator T has the property (see Hille [6, p. 313]):

P₁ if there is an $x_0 \neq 0$ such that $Tx_0 = 0$;

P₂ if the range of T is not dense in X .

P₃ if the T -image of the intersection of the domain of T with the shell of the unit sphere is not bounded away from the zero element.

THEOREM 5.2. Let $T(t)$ be a semi-group of operators of type (H) continuous in the uniform topology for $t \geq c > 0$ and let $\alpha \in \mathfrak{S}(\omega')$ where $\omega'(t) = \max [\omega(t), \omega_1 t]$ for some $\omega_1 > \omega_0$ ⁽⁸⁾. If $\nu \in \Sigma(A)$ and if $[\nu I - A]$ has property P_j, then $[f_\alpha(\nu)I$

(8) The maximum of two subadditive functions is again a subadditive function.

$-\Theta(\alpha)]$ has the same property P_j . Conversely $[\mu I - \Theta(\alpha)]$, $\mu \neq \alpha(0)$, has property P_j only if there is a $\nu \in \Sigma(A)$ with $f_\alpha(\nu) = \mu$ such that $[\nu I - A]$ has the property P_j . There is one exception: if $[\mu I - \Theta(\alpha)]$ has P_3 together with P_1 or P_2 , then P_3 may possibly not hold for any $[\nu I - A]$, where $f_\alpha(\nu) = \mu$ and $\nu \in \Sigma(A)$.

The proof of this theorem makes use of a method due to Hille [6, Theorem 15.5.2]. Suppose first that $\nu \in \Sigma(A)$ and that $\alpha \in \mathfrak{C}(\omega')$. Then

$$f_\beta(\lambda) = [f_\alpha(\nu) - f_\alpha(\lambda)]/(\nu - \lambda) = \int_0^\infty \exp(\lambda t) d_t \beta$$

where

$$\beta'(t) = \int_0^t \exp[-\nu(t-s)] d_s [f_\alpha(\nu)e_0 - \alpha] = \int_t^\infty \exp[\nu(s-t)] d_s \alpha.$$

It is easily seen that $\beta \in \mathfrak{C}(\omega)$. For by Theorem 3.1, $\Re(\nu) \leq \omega_0$ and hence

$$\begin{aligned} \|\beta\| &\leq \int_0^\infty \exp[\omega(t)] \int_t^\infty \exp[\omega_1(s-t)] |d_s \alpha| dt \\ &\leq \int_0^\infty \exp[\omega(t) - \omega_1 t] dt \int_0^\infty \exp(\omega_1 s) |d_s \alpha| < \infty. \end{aligned}$$

Thus $\Theta(\beta)$ exists. We should like to show that

$$(23) \quad \begin{aligned} f_\alpha(\nu)I - \Theta(\alpha) &= [\nu I - A]\Theta(\beta) && \text{on } \mathfrak{X} \\ &= \Theta(\beta)[\nu I - A] && \text{on } \mathfrak{D}(A). \end{aligned}$$

For $\gamma = e_0 + (\nu - \omega_1) \int_0^\infty \exp(-\omega_1 s) ds \in \mathfrak{C}(\omega)$, we have

$$f_\gamma(\lambda) = \int_0^\infty \exp(\lambda t) d\gamma = 1 + (\nu - \omega_1)(\omega_1 - \lambda)^{-1} = (\nu - \lambda)(\omega_1 - \lambda)^{-1}.$$

Thus $\Theta(\gamma) = I + (\nu - \omega_1)R(\omega_1; A) = (\nu I - A)R(\omega_1; A)$. Further

$$f_{\beta\gamma}(\lambda) = f_\beta(\lambda)f_\gamma(\lambda) = [f_\alpha(\nu) - f_\alpha(\lambda)](\omega_1 - \lambda)^{-1}$$

from which it follows that $\beta\gamma = [f_\alpha(\nu)e_0 - \alpha] * \int_0^\infty \exp(-\omega_1 s) ds^{(*)}$. Hence

$$\begin{aligned} (\nu I - A)\Theta(\beta)R(\omega_1; A) &= \Theta(\beta)(\nu I - A)R(\omega_1; A) = \Theta(\beta\gamma) \\ &= [f_\alpha(\nu)I - \Theta(\alpha)]R(\omega_1; A). \end{aligned}$$

Since the range of $R(\omega_1; A)$ is precisely $\mathfrak{D}(A)$, the second equation of (23) follows directly. On the other hand, $\mathfrak{D}(A)$ is dense in \mathfrak{X} , so that given $x_0 \in \mathfrak{X}$ there exist $x_n \in \mathfrak{D}(A)$ such that $x_n \rightarrow x_0$. Now both $\Theta(\beta)x_n$ and $[f_\alpha(\nu)I - \Theta(\alpha)]x_n$ converge to limits and, since A is closed, it follows that $(\nu I - A)\Theta(\beta)x_0 = [f_\alpha(\nu)I - \Theta(\alpha)]x_0$. The first part of our theorem then follows immediately from (23).

(*) We use * to indicate the convolution.

To prove the converse, suppose that $\mu \in \Sigma[\Theta(\alpha)]$ ($\mu \neq \alpha(0)$). Now $\lim_{|\lambda| \rightarrow \infty} f_\alpha(\lambda) = \alpha(0)$ uniformly with respect to λ in every sector $\pi/2 + \epsilon \leq \arg \lambda \leq 3\pi/2 - \epsilon$. Hence by Lemma 5.1, $f_\alpha(\lambda) = \mu$ can have at most a finite number of roots in $\Sigma(A)$, say $\nu_1, \nu_2, \dots, \nu_n$. By equation (21), $\mu \in f_\alpha[\Sigma(A)]$. Hence there is at least one root. We now set $Q(\lambda) = \prod_{k=1}^n (\nu_k - \lambda)$ and define

$$(24) \quad f_\beta(\lambda) = [\mu - f_\alpha(\lambda)](\omega_1 - \lambda)^n / Q(\lambda).$$

If we split $(\omega_1 - \lambda)^n / Q(\lambda)$ into partial fractions, $f_\beta(\lambda)$ becomes a sum of terms of the type $f_{\beta_j}(\lambda) = [\mu - f_\alpha(\lambda)] [\nu - \lambda]^{-j}$. For $j=1$ we showed above that $\beta_1 \in \mathfrak{C}(\omega'')$ where $\omega''(t) = \max [\omega(t), \omega_2 t]$ and $\omega_0 < \omega_2 < \omega_1$. For $j=2$, $f_{\beta_1}(\nu) = 0$ and hence $f_{\beta_2}(\lambda) = [f_{\beta_1}(\nu) - f_{\beta_1}(\lambda)](\nu - \lambda)^{-1}$ so that the same argument proves that $\beta_2 \in \mathfrak{C}(\omega''')$ where $\omega'''(t) = \max [\omega(t), \omega_3 t]$ and $\omega_0 < \omega_3 < \omega_2$. It follows by induction that $\beta \in \mathfrak{C}(\omega)$. We see from (24) that $f_\beta(\lambda)$ is different from zero in $\Sigma(A)$ and that $\beta(0) = \mu - \alpha(0) \neq 0$. Hence $\Theta(\beta)(m') \neq 0$ on $\mathfrak{M}' = \mathfrak{X}' \cup \mathfrak{Z}'$. By the Gelfand theory, $B = \Theta(\beta)^{-1}$ exists in \mathfrak{K} . Again as before for $f_\gamma(\lambda) = Q(\lambda) / (\omega_1 - \lambda)^n$, $\gamma \in \mathfrak{C}(\omega)$ and $\Theta(\gamma) = Q(A)R(\omega_1; A)^n$. Finally $f_{\beta\gamma}(\lambda) = f_\beta(\lambda)f_\gamma(\lambda) = \mu - f_\alpha(\lambda)$ from which it follows that

$$\Theta(\beta)Q(A)R(\omega_1; A)^n = \mu I - \Theta(\alpha),$$

and hence

$$(25) \quad Q(A)R(\omega_1; A)^n = B[\mu I - \Theta(\alpha)] = [\mu I - \Theta(\alpha)]B.$$

If we set $O_k(A) = (\nu_k I - A)R(\omega_1; A)$, then (25) becomes

$$(26) \quad \prod_{k=1}^n O_k(A) = B[\mu I - \Theta(\alpha)] = [\mu I - \Theta(\alpha)]B.$$

It should be noted that

$$O_k(A) = [(\nu_k - \omega_1)I + (\omega_1 I - A)]R(\omega_1; A) = (\nu_k - \omega_1)R(\omega_1; A) + I$$

is a bounded operator. Suppose now that $[\mu I - \Theta(\alpha)]$ has P_1 . Then by (26) one of the $O_k(A)$ has P_1 and since $R(\omega_1; A)$ does not, the corresponding $(\nu_k I - A)$ must have P_1 . If $[\mu I - \Theta(\alpha)]$ has P_2 , then the range of $\prod_{k=1}^n O_k(A)$ is not dense in \mathfrak{X} . If the range of $\prod_{k=2}^n O_k(A)$ is dense in \mathfrak{X} , then the range of $O_1(A)$ could not be dense in \mathfrak{X} since otherwise $\prod_{k=2}^n O_k(A)$, being bounded, would take a dense set again into a dense set. If, on the other hand, $\prod_{k=2}^n O_k(A)$ does not have a dense range, we proceed by induction and find some $O_k(A)$ with a nondense range. Since the range of $R(\omega_1; A)$ is precisely $\mathfrak{D}(A)$, $(\nu_k I - A)$ must then have P_2 . Finally if $[\mu I - \Theta(\alpha)]$ has P_3 but neither P_1 nor P_2 , then since the nonvacuous set $\{\nu_k\} \subset \Sigma(A)$ and since none of the $(\nu_k I - A)$ can have P_1 or P_2 , it follows by the first part of the theorem that at least one ν_k must have P_3 .

Before concluding this section, two remarks are in order. In the first place we cannot hope for a classification of $T(t)$ continuous in the uniform topology

for $t \geq c > 0$ on the basis of the decomposition of \mathfrak{M}' alone as was possible for $T(t)$ uniformly continuous at $t=0$. This follows from the existence of semi-groups of operators strongly continuous but nowhere uniformly continuous for which $T(t)$ is quasi-nilpotent for all $t > 0$ (see for example Hille [6, section 16.4]). In this case of course $\mathfrak{M}' \equiv \mathfrak{Z}'$.

For $T(t)$ continuous in the uniform topology ($t \geq c > 0$) we have

$$\lim_{t \rightarrow \infty} t^{-1} \log \|T(t)\| = \log \| \|T(1)\| \| = \sup \mathfrak{R}[\Sigma(A)].$$

Hence if we set $c(x)$ equal to the abscissa of convergence for $R(\lambda; A)x = \int_0^\infty \exp(-\lambda t) T(t)x dt$, then $\sup c(x) = \lim_{t \rightarrow \infty} t^{-1} \log \|T(t)\|$. It is not known whether this result holds in general for $T(t)$ strongly continuous on $(0, \infty)$.

6. Analytical semi-groups. The basic material on analytical semi-groups is to be found in Hille [6, Chap. 13]. We shall limit ourselves to some generalizations of Hille's work on the rate of growth of $T(\zeta)$. These results follow quite naturally from our approach, whereas the Laplace transform methods used by Hille do not seem to yield them.

DEFINITION. An essentially angular semi-module Γ is by definition an open simply-connected semi-module ($\zeta_1, \zeta_2 \in \Gamma \supset \zeta_1 + \zeta_2 \in \Gamma$) having two asymptotic angles ϕ_1, ϕ_2 ($-\pi/2 \leq \phi_1 < 0 < \phi_2 \leq \pi/2$) such that for each $\epsilon > 0$ there exists an r_ϵ for which the infinite sector

$$\phi_1 + \epsilon \leq \arg \zeta \leq \phi_2 - \epsilon, \quad |\zeta| \geq r_\epsilon,$$

lies entirely within Γ . Γ does not contain the origin.

THEOREM 6.1. *Let the semi-group of operators $T(\zeta)$ be defined and holomorphic in the essentially angular semi-module Γ . Define the indicator*

$$(27) \quad \sigma(\phi) = \lim_{r \rightarrow \infty} r^{-1} \log \|T[r \exp(i\phi)]\| \quad \text{for } \phi_1 < \phi < \phi_2.$$

Then either (a) $\sigma(\phi) \equiv -\infty$ in which case $T(\zeta)$ is a quasi-nilpotent element of $\mathfrak{E}(\mathfrak{X})$ for all $\zeta \in \Gamma$; or else (b) $\sigma(\phi)$ is always finite-valued in which case $\sigma(\phi)$ is the function of support for a closed convex set \mathfrak{D} which will be called the indicator diagram for the semi-group $T(\zeta)$.

We consider a commutative ring \mathfrak{R}' containing the operators $[T(\zeta) | \zeta \in \Gamma]$ and the identity I . Let \mathfrak{M}' be the set of all maximal ideals m' in \mathfrak{R}' . By the Gelfand theory [3], $T(\zeta)$ has a representation as a continuous function $T(\zeta)(m')$ in \mathfrak{M}' . Since $T(\zeta)$ is holomorphic in Γ , so likewise is $T(\zeta)(m')$ for each m' . By the semi-group property

$$T(\zeta_1 + \zeta_2)(m') = T(\zeta_1)(m') \cdot T(\zeta_2)(m') \quad \text{for } \zeta_1, \zeta_2 \in \Gamma.$$

The only such complex-valued function holomorphic in the simply-connected semi-module Γ is either identically zero or else of the form $\exp [a(m')\zeta]$. In conformity with our previous notation we shall say that $m' \in \mathfrak{Z}'$ if $T(\zeta)(m')$

$\equiv 0$ and $m' \in \mathfrak{B}'$ otherwise.

Now the limit in (27) is shown to exist by the familiar argument (see [6, p. 135]). Hence

$$\sigma(\phi) = \lim_{n \rightarrow \infty} (rn)^{-1} \log \|T[r \exp(i\phi)]^n\| = r^{-1} \log \| \|T[r \exp(i\phi)]\| \|.$$

In other words

$$\sigma(\phi) = \sup [r^{-1} \log |T[r \exp(i\phi)](m')| \mid m' \in \mathfrak{M}', r \exp(i\phi) \in \Gamma].$$

If $\mathfrak{M}' \equiv \mathfrak{Z}'$, then $T(\zeta)(m') \equiv 0$ and $\sigma(\phi) \equiv -\infty$; in this case each $T(\zeta)$ is quasinilpotent. On the other hand if \mathfrak{B}' is nonvacuous, then $T(\zeta)(m') = \exp[a(m')\zeta]$ in \mathfrak{B}' and

$$\sigma(\phi) = \sup [\mathcal{R}[a(m') \exp(i\phi)] \mid m' \in \mathfrak{B}'].$$

Thus $\sigma(\phi)$ is the supremum of the projections of $a(m')^*$ on the unit vector $\exp(i\phi)^{(10)}$. Thus if we set

$$(28) \quad \mathfrak{D} = \text{closed convex extension of } [a(m')^* \mid m' \in \mathfrak{B}'],$$

then $\sigma(\phi)$ is the function of support for \mathfrak{D} .

It is customary to define an extremal point of a closed convex point set \mathfrak{D} as a point on the boundary of \mathfrak{D} which is not an interior point of any line segment belonging to the boundary of \mathfrak{D} . An extremal point λ_0 is called exceptional if it is the end point of a line segment on the boundary of \mathfrak{D} and the extension of this line segment is the only line of support of \mathfrak{D} passing through λ_0 . It is clear that every exceptional extremal is the limit of ordinary extremal points.

We now seek to relate the indicator diagram \mathfrak{D} to the spectrum of the infinitesimal generator A . It is clear that even to define A we must impose further conditions on $T(\zeta)$ near the origin.

THEOREM 6.2. *Let $T(\zeta)$ be a semi-group of linear transformations of type (H) on $[0, \infty)$ and holomorphic on the essentially angular semi-module Γ . Then the indicator diagram $\mathfrak{D} = \text{closed convex extension } \Sigma(A)^*$. $R(\lambda; A)$ exists and is holomorphic outside of \mathfrak{D}^* and every extremal point of \mathfrak{D}^* is a singular point of $R(\lambda; A)$. Finally if $\sigma(\phi) \equiv -\infty$ in (ϕ_1, ϕ_2) , then $R(\lambda; A)$ is an entire function of λ .*

We have merely to replace \mathfrak{M}' by the ring \mathcal{R} defined in section three. Since $T(\zeta)$ is continuous in the uniform topology for $\xi \geq 0$ ($\zeta = \xi + i\eta$) and sufficiently large, it follows that the subdivision of \mathfrak{M}' into \mathfrak{B}' and \mathfrak{Z}' which we made in the proof of Theorem 6.1 agrees with that which we previously made. By Theorem 3.1, $\Sigma(A) = [a(m') \mid m' \in \mathfrak{B}']$ and therefore $\mathfrak{D} = \text{closed convex extension of } \Sigma(A)^*$. Now $R(\lambda; A)$ is holomorphic outside of $\Sigma(A)$ and a

⁽¹⁰⁾ We denote the complex conjugate of a by a^* .

fortiori outside of \mathfrak{D}^* . To show that every extremal point of \mathfrak{D}^* is a singular point of $R(\lambda; A)$ it is clearly enough to do this for ordinary extremal points. However if an ordinary extremal point λ_0 were a regular point for $R(\lambda; A)$, then we could separate a small neighborhood of λ_0 from the rest of \mathfrak{D}^* by a suitable parallel to one of the lines of support of \mathfrak{D}^* through λ_0 in such a manner that the part of \mathfrak{D}^* cut off by this parallel lies in this neighborhood. In this case, however, λ_0 would not be a boundary point of the closed convex extension of $\Sigma(A)$, contrary to our choice of λ_0 .

7. Semi-groups in Hilbert space. If the semi-group of operators can be embedded in a commutative self-adjoint algebra \mathfrak{R} of operators on a Hilbert space, then it is possible to obtain exceedingly precise results out of the representation theorem for \mathfrak{R} . One can, in fact, derive directly the spectral resolution theorems for semi-groups of normal operators due originally to Stone [11], Sz. Nagy [8], and Hille [6]. For purposes of illustration we shall obtain the spectral resolution theorem for a group of unitary operators where the group involved is any locally compact abelian group \mathfrak{G} . Stone first discovered this result for the additive group of real numbers and this was later generalized by Ambrose [1] and Neumark [9] to the locally compact abelian groups.

We assume that we are given a strongly continuous unitary representation $U(g)$ of the group \mathfrak{G} . That is

- (a) $U(g)$ is a unitary operator in the Hilbert space \mathfrak{X} ,
- (b) $U(g)x$ is continuous on \mathfrak{G} in the Hilbert space metric,
- (c) $U(g_1 + g_2) = U(g_1)U(g_2)$.

We now set \mathfrak{R} equal to the strong closure of the set of all polynomials in $[U(g) | g \in \mathfrak{G}]$. Then \mathfrak{R} is isomorphic and isometric with the ring $\mathfrak{C}^*(\mathfrak{M})$ of all continuous complex-valued functions defined on a totally disconnected compact Hausdorff space \mathfrak{M} of maximal ideals m in \mathfrak{R} (see [12], [13], [14], and [16]). The clopen (closed and open) subsets of \mathfrak{M} are in one-to-one correspondence with the projection operators in \mathfrak{R} , and this correspondence is such that if $\sigma \leftrightarrow E(\sigma)$, then $E(\sigma)(m)$ is the characteristic function of the clopen set σ . Further, the closure of any open set is clopen. Thus each open set differs from a unique clopen set by a set of the first category. From this it follows that to each Borel measurable subset σ of \mathfrak{M} there corresponds a unique clopen set $\Psi(\sigma)$ which differs from σ only on a set of the first category. The projection operators $E[\Psi(\sigma)]$ over the Borel measurable subsets of \mathfrak{M} have the following properties:

- (a) If $\sigma_1 \cap \sigma_2 = \phi$, then $E[\Psi(\sigma_1)]E[\Psi(\sigma_2)] = 0$.
- (b) $E[\Psi(\mathfrak{M})] = I$ and $E[\Psi(\phi)] = 0$.
- (c) If $\sigma_i \cap \sigma_j = \phi$ for $i \neq j$, then $E[\Psi(\cup \sigma_i)] = \sum_{i=1}^{\infty} E[\Psi(\sigma_i)]$ where the partial sums converge in the strong operator topology.

In other words $E[\Psi(\sigma)]$ is a resolution of the identity relative to the Borel measurable subsets of \mathfrak{M}

THEOREM 7.1. *Let $U(g)$ be a strongly continuous unitary representation of the locally compact abelian group \mathfrak{G} . Then there exists a resolution of the identity $F(\sigma')$ relative to the Borel measurable subsets of the character group $\mathfrak{G}^* \equiv [\chi(g)]$ such that*

$$U(g) = \int_{\mathfrak{G}^*} \chi(g) dF(\sigma')$$

where the integral converges in the uniform operator topology.

It is clear for a fixed m that $U(g_1 + g_2)(m) = U(g_1)(m)U(g_2)(m)$ and $|U(g)(m)| \leq \|U(g)\| = 1$ and hence that $U(g)(m)$ is a character of the group \mathfrak{G} . However, $U(g)(m)$ need not be a continuous character. We shall show that, except for a set of the first category in \mathfrak{M} , $U(g)(m)$ will be a continuous character. To this end we consider the class \mathfrak{X} of Lebesgue summable functions $w(g)$ on \mathfrak{G} (relative to the Haar measure dg). Since $U(g)$ is strongly continuous, the Bochner integral $\int w(g)U(g)xdg$ exists for each $x \in \mathfrak{X}$ and defines a bounded linear operator A_w belonging to \mathfrak{R} . We now split \mathfrak{M} into two sets:

- (a) \mathfrak{B} : m_0 belongs to \mathfrak{B} if there exists a $w_0 \in \mathfrak{X}$ for which $A_{w_0}(m_0) \neq 0$.
- (b) \mathfrak{U} : m_0 belongs to \mathfrak{U} if $A_w(m_0) = 0$ for all $w \in \mathfrak{X}$.

We see that $\mathfrak{B} \cup \mathfrak{U} = \mathfrak{M}$ and that $\mathfrak{B} \cap \mathfrak{U} = \emptyset$. Further \mathfrak{B} is an open subset of \mathfrak{M} since it is the union of the open sets $[m | A_w(m) \neq 0]$ over all $w \in \mathfrak{X}$.

If $m_0 \in \mathfrak{B}$, then $U(g)(m_0)$ is a continuous character. For let $w_0 \in \mathfrak{X}$ be such that $A_{w_0}(m_0) \neq 0$. Then

$$U(h)A_{w_0} = \int U(h + g)w_0(g)dg = \int U(g)w_0(g - h)dg$$

and hence

$$\|U(h_1)A_{w_0} - U(h_2)A_{w_0}\| \leq \int \|U(g)\| |w_0(g - h_1) - w_0(g - h_2)| dg.$$

It follows that $U(h)A_{w_0}$ is continuous in the uniform operator topology and a fortiori that $U(h)(m_0)A_{w_0}(m_0) = [U(h)A_{w_0}](m_0)$ is continuous. Since $A_{w_0}(m_0) \neq 0$, we have that $U(h)(m_0)$ is itself continuous. If $m_0 \in \mathfrak{B}$ we can obtain the further result that $A_w(m_0) = \int U(g)(m_0)w(g)dg$ for all $w \in \mathfrak{X}$. For since $U(h)A_{w_0}$ is continuous in the uniform operator topology, the integral $\int w(h)U(h)A_{w_0}dh$ will converge in this topology. Hence the multiplicative linear functionals commute with this integration. On the other hand

$$\int w(h)U(h)A_{w_0}(x) dh = A_w[A_{w_0}(x)].$$

Combining these two facts, we get

$$\begin{aligned} A_w(m_0)A_{w_0}(m_0) &= [A_w A_{w_0}](m_0) = \int w(h)[U(h)A_{w_0}](m_0)dh \\ &= \int w(h)U(h)(m_0)A_{w_0}(m_0)dh = A_{w_0}(m_0) \int w(h)U(h)(m_0)dh. \end{aligned}$$

The result now follows from the fact that $A_{w_0}(m_0) \neq 0$.

We shall now show that $\overline{\mathfrak{B}} = \mathfrak{M}$. Suppose the contrary were so. Then \mathfrak{B} being open implies that $\overline{\mathfrak{B}}$ is clopen and hence that $\sigma = \mathfrak{M} - \overline{\mathfrak{B}}$ is clopen. Clearly $E(\sigma)(m) = 0$ on \mathfrak{B} . On the other hand let \mathfrak{B} be a neighborhood of the identity in \mathfrak{G} with finite measure c_b . We define $w_b = c_b^{-1}$ on \mathfrak{B} and zero elsewhere. The neighborhoods \mathfrak{B} form a directed set ($\mathfrak{B}_2 \supseteq \mathfrak{B}_1$ means that $\mathfrak{B}_2 \subset \mathfrak{B}_1$) and since $U(g)$ is strongly continuous, $\lim_b A_{w_b}(x) = x$ for each $x \in \mathfrak{X}$. Finally since $w_b \in \mathfrak{L}$, $A_{w_b}(m) = 0$ on $\cup \sigma$. Hence $E(\sigma)(m)A_{w_b}(m) \equiv 0$ on \mathfrak{M} so that $E(\sigma)A_{w_b} = 0$. But then

$$0 = (E(\sigma)A_{w_b}x, x) = (A_{w_b}x, E(\sigma)x) \rightarrow (x, E(\sigma)x)$$

for all $x \in \mathfrak{X}$ and hence $E(\sigma) = 0$ contrary to assumption. This incidently shows that $\mathfrak{M} - \mathfrak{B}$ is of the first category and also that $\Psi(\mathfrak{B}) = \mathfrak{M}$.

We next define a mapping Φ on \mathfrak{B} into \mathfrak{G}^* : each $m \in \mathfrak{B}$ maps into the continuous character $\chi(g) \equiv U(g)(m)$. In order to show that Φ is a continuous mapping it is convenient to introduce the equivalent neighborhood system for \mathfrak{G}^* defined by $\epsilon > 0$ and finite subsets w_1, w_2, \dots, w_n of \mathfrak{L} as

$$\mathfrak{N}'(\chi_0) \equiv \left[\chi \left| \left| \int \chi(g)w_i(g)dg - \int \chi_0(g)w_i(g)dg \right| < \epsilon; i = 1, 2, \dots, n \right].$$

It is clear for $\chi_0 = \Phi(m_0)$ for $m_0 \in \mathfrak{B}$ that

$$\mathfrak{N}(m_0) \equiv [m \mid |A_{w_i}(m) - A_{w_i}(m_0)| < \epsilon; i = 1, 2, \dots, n; m \in \mathfrak{B}]$$

maps into $\mathfrak{N}'(\chi_0)$. Since Φ is continuous on \mathfrak{B} into \mathfrak{G}^* , and since \mathfrak{B} is an open subset of \mathfrak{M} , it follows that Φ^{-1} maps open sets in \mathfrak{G}^* into open subsets of \mathfrak{M} , closed sets in \mathfrak{G}^* into the complements of open subsets of \mathfrak{M} relative to \mathfrak{B} ; and more generally Borel subsets of \mathfrak{G}^* into Borel subsets of \mathfrak{M} .

We are now essentially through. For given a Borel measurable subset σ' of \mathfrak{G}^* , $\Phi^{-1}(\sigma')$ is a Borel measurable subset of \mathfrak{M} ; and we can define

$$F(\sigma') = E[\Psi(\Phi^{-1}(\sigma'))].$$

From what we have said earlier it is clear that $F(\sigma')$ is a completely additive (in the strong topology) resolution of the identity. Finally for a given $g \in \mathfrak{G}$, let $\{\sigma'_1, \sigma'_2, \dots, \sigma'_n\}$ be a subdivision of \mathfrak{G}^* into disjoint Borel sets such that for some numbers $\{a_1, a_2, \dots, a_n\}$ we have $|\chi(g) - a_i| \leq \epsilon$ for all $\chi \in \sigma'_i$. By the definition of Φ it follows that $|U(g)(m) - a_i| \leq \epsilon$ for all $m \in \Phi^{-1}(\sigma'_i)$, and hence for all $m \in \Psi[\Phi^{-1}(\sigma'_i)]$ except perhaps for a set of the first category. Thus, except for a set of the first category,

$$\left| U(g)(m) - \sum_{i=1}^n a_i F(\sigma'_i)(m) \right| \leq \epsilon^{(11)}.$$

On the other hand, both $U(g)(m)$ and $\sum_{i=1}^n a_i F(\sigma'_i)(m)$ are continuous functions on \mathfrak{M} and since they can differ by more than ϵ only on a set of the first category, it follows that they can nowhere differ by more than ϵ . In other words $\|U(g) - \sum_{i=1}^n a_i F(\sigma'_i)\| \leq \epsilon$. Finally since $\chi(g)$ is Borel measurable on \mathfrak{G}^* , we have $U(g) = \int_{\mathfrak{G}^*} \chi(g) dF(\sigma')$, where the integral converges in the uniform operator topology.

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⁽¹¹⁾ Note that for each $m \in \mathfrak{M}$, only one of the $F(\sigma'_i)(m)$ has the value one, the remainder are zero-valued.