GENERALIZATION OF RUNGE'S THEOREM TO APPROXIMATION BY ANALYTIC FUNCTIONS[1]

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1. **Introduction.** According to Runge's Theorem [4][2], any function $M(z)$ which is analytic in a given region R can be expanded in a sequence of rational functions that converges to $M(z)$ uniformly on any closed subregion interior to R.

Walsh [6, Chap. I; 7; 8] has generalized Runge's Theorem to obtain necessary and sufficient conditions for uniform approximation on an arbitrary closed set C. He supposes points z_k preassigned in a finite number of components of the complement of C and proves the following theorem:

A necessary and sufficient condition that every function $M(z)$ analytic on C can be uniformly approximated on C by a rational function whose poles lie in the points z_k is that at least one point z_k lie in each of the regions into which C separates the plane.

In the present paper it is proved that any function analytic on a "Q-set" S can be approximated on that set by a function which is analytic in the extended plane except possibly at points of a preassigned "$B^*(S)$-set."

Before defining "Q-set" and "$B^*(S)$-set" we define *sequential limit point.* A point which is a limit point of some set of points chosen one from each component of any given set S is called a *sequential limit point of S* (S-s.l. point)[3].

A set consisting of (1) the set B of the s.l. points of a given set S and (2) precisely one point of each component $I_k(S)$ of $C(S)$ such that $I_k(S) \cap B = \emptyset$ will be called a *$B^*(S)$-set.*

A set S whose components are closed and whose s.l. points are in $C(S)$ (the complement of S) will be called a *Q-set.* We note that a Q-set has at most a denumerable number of components.

EXAMPLE 1. Let S be the set whose components $S_1, S_2, \cdots$ are circles (including their interiors) of radii one-fourth with centers at $z=1, 2, \cdots$. This is a Q-set. The point at infinity is the only S-s.l. point, also the only $B^*(S)$-set.

EXAMPLE 2. Let S be a set whose components $S_1, S_2, \cdots$ are closed

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(2) Numbers in brackets refer to the bibliography at the end of the paper.

(3) This definition is based on a definition given by Ketchum [1, pp. 213–214].

circular rings of radii one-fourth with centers at $z=1, 2, \cdots$. This is a Q-set and the set of S-s.l. points consists of just the point at infinity. The set whose points are the point at infinity and the centers of the circles is a $B^*(S)$-set.

Now let us suppose S to be an arbitrary Q-set and B^* a $B^*(S)$-set. It is proved in this paper that, if $M(z)$ is analytic on such a set S and is such that $\log M(z)$ can be chosen as single-valued and continuous on S, the approximating function $L(z)$ can be required to be both analytic and nonvanishing in $C(B^*)$. If $M(z)$ is analytic on S and if no restriction is placed on the vanishing of $L(z)$, then $L(z)$ can be chosen so as to be analytic in $C(B^*)$ and meromorphic except possibly at S-s.l. points.

The approximation obtained is not only uniform on S but is such that the closeness of approximation can be preassigned independently for each component of S.

We consider two simple examples. Let S be a set as described in Example 1. Define $M(z)=i$ on S_i. Then for any sequence of positive constants $\{\delta_i\}$ there exists an integral nonvanishing function $L(z)$ such that—for every i—

$$|L(z) - i| < \delta_i \qquad \text{when } z \in S_i.$$

For our second example we take S to be a set as described in Example 2 previously given and let B^* be the $B^*(S)$-set there defined. We choose as $M(z)$ any function analytic on each ring. ($M(z)$ may be defined by a different analytic function on each S_i.) Then, for any $\{\delta_i\}$, there exists a meromorphic function $L(z)$ such that—for every i—

$$|L(z) - M(z)| < \delta_i \qquad \text{when } z \in S_i.$$

Furthermore, $L(z)$ can be chosen so that its poles lie at the centers of the rings.

2. **Preliminary topology.** Throughout this paper the extended plane is to be understood.

An *infinite subset* $\{S_{n_i}\}$ *of components* of a set S will be called *nested* if it can be arranged in an order $S_{n_1}, S_{n_2}, \cdots$ such that S_{n_i} separates S_{n_j} from S_{n_k} whenever $j<i<k$. An equivalent definition is that, when properly ordered, S_{n_j} separates S_{n_k} from the set of s.l. points of this sequence when $j>k$(4).

An *interior s.l. point of a set* is an s.l. point of a nested subsequence of components.

A *ring* is a closed region bounded by two mutually nonintersecting Jordan curves.

Let us choose a fixed point P as a *base point.* Then, if a given closed set $F \not\ni P$, a point of $C(F)$ will be said to be *inside of* F if it belongs to the component of $C(F)$ to which P belongs, *outside of* F if it belongs to some other component. If no base point is specified, it is to be understood that the origin

(4) The definitions of *nested, interior s.l. point,* and *ring* are essentially those given by Ketchum [1, pp. 212, 214].

is taken as the base point.

The following theorems are given for reference.

(A) *If E is any set and K is a connected set such that $K \cap E \neq \emptyset$ and $K \cap C(E) \neq \emptyset$, then $K \cap F(E) \neq \emptyset$* [3, p. 64] (where $F(E)$ denotes the boundary of E).

(B) *Let B be the set of s.l. points of a Q-set S. Then, if $I(B)$ is any component of $C(B)$, no two points of $I(B)$ are separated by more than a finite number of components of S.*

Proof. If this were not true, some two points p and q of $I(B)$ would be separated by an infinity of components. But then every Jordan arc joining p and q must meet B. However, since $p, q \in I(B)$, a region, they can be joined by an arc in $I(B)$.

(C) *If S is a Q-set and B is its set of s.l. points, any component $I(S)$ of $C(S)$ such that $I(S) \cap B = \emptyset$ is a region of finite connectivity and $F\{I(S)\} \subset S$.*

Proof. $F(S) \subset S \cup B$. Hence, $F\{I(S)\} \subset S \cup B$. Since $I(S) \cap B = \emptyset$ and $S \cap B = \emptyset$, $B \subset \{C(S) - I(S)\}$. Then $F\{I(S)\} \cap B \neq \emptyset$ would imply that a point of some other component of $C(S)$ is a limit point of $I(S)$. This is impossible. Therefore, $F\{I(S)\} \subset S$.

A fixed point p of $I(S)$ belongs to some component of $C(S \cup B)$—say $I(S, B)$. Since $S \cup B$ is a closed set, $F\{I(S, B)\} \subset (S \cup B)$ and $I(S, B)$ is a region. We show that $I(S, B) = I(S)$. Clearly $I(S, B) \subset I(S)$. If $I(S) \not\subset I(S, B)$, (A) would imply that $F\{I(S, B)\} \cap I(S) \neq \emptyset$. But since $I(S) \cap (S \cup B) = \emptyset$ we conclude that $I(S) = I(S, B)$, which is a region.

It remains to be proved that $I(S)$ has finite connectivity. From the fact that $F\{I(S)\} \subset S$, it follows that $F\{I(S)\}$ is contained in only a finite number of S_i's. For otherwise, since $F\{I(S)\}$ is a closed set, $F\{I(S)\} \cap B \neq \emptyset$ and $F\{I(S)\} \not\subset S$. According to a theorem given by Newman [3, p. 118] each component of $C\{I(S)\}$ contains just one component of $F\{I(S)\}$. Hence, no S_i contains more than one component of $F\{I(S)\}$. We conclude that $F\{I(S)\}$ has only a finite number of components and that $I(S)$ is of finite connectivity.

(D) *If two points are separated by a closed set F, they are separated by a component of F* [3, p. 117].

In the proof of Theorem 3 we shall use the following topological theorem. We are given a Q-set S and B^*, a $B^*(S)$-set. The object is to construct a sequence of closed sets $\{F_i\}$, $F_i \subset F_{i+1}$, in such a way that we can apply Runge's Theorem successively to approximate on closed sets obtained by taking the union of F_i and a finite number of certain components of S. In order that we can require that the approximating rational functions have their poles in B^*, we require that condition (e) below be satisfied. Except for this condition, the construction in Theorem 1 would be much simpler.

THEOREM 1. *Suppose S is a Q-set having an infinite number of components S_i. Let B denote the set of S-s.l. points and B^* an arbitrary $B^*(S)$-set. Then* *there exists a sequence of closed sets* $\{F_i\}$ *which have the following properties:*

(a) $F_n \subset C(B)$;

(b) F_n *is interior to* F_{n+1} *and has only a finite number of components;*

(c) *If* $S_t \cap F_n \neq \emptyset$, $S_t \subset F_n$;

(d) *Any closed set* $C \subset C(B)$ *is interior to some* F_i;

(e) *If* $I_r(F_n, S)$ *is any component of* $C(F_n \cup S)$, $I_r(F_n, S) \cap B^* \neq \emptyset$.

In particular, if B does not divide the plane, F_n *can be required to be connected.*

Proof. For each component $I_j(B)$ of $C(B)$ a certain sequence of closed connected sets $\{K_{ij}\}$ will be defined. Then the sets of the sequence $\{F_i\}$ will be taken as the union of a finite number of sets, each chosen from a different sequence $\{K_{ij}\}$.

According to a theorem proved by Ketchum [1, p. 214], there exists in $\{I_j(B) - S\}$ a set R whose components R_i are rings such that: (a) The R-s.l. points, the R-interior s.l. points, and $F\{I_j(B)\}$ are identical sets; (b) If $F\{I_j(B)\}$ has more than one component, then for every ring R_i there are points of $F\{I_j(B)\}$ in both components of $C(R_i)$.

Choose a point P_j of $\{I_j(B) - (S \cup R)\}$ as a base point. Let $O(R_n)$ denote the region outside R_n. Omit every R_n except those with the property that some point b^* of B^* in $O(R_n) \cap \bar{I}_j(B)$ is not separated from R_n by any component of S.

Let R^* be the set whose components are just (1) those rings which have not been omitted and (2) those components of S in $I_j(B)$ which separate the plane. We note that any R^*-s.l. point belongs to $F\{I_j(B)\}$. We let a component of R^* which is separated from P_j by $m-1$ but not by m other components of R^* be denoted by $R_{mj}^{*(k)}$. Every component of R^* is included in the collection $\{R_{ij}^{*(k)}\}$.

We define a sequence of sets $\{E_{ij}\}$—$i=1, 2, \cdots$— by defining E_{ij} as the set of points inside all the $R_{ij}^{*(k)}$'s. Then we define a sequence $\{K_{ij}\}$—$i=1, 2, \cdots$—by letting $K_{ij} = E_{ij} \cup_k (R_{ij}^{*(k)})$. Finally we set

$$
\begin{aligned}
F_1 &= K_{11}, \\
&\cdots\cdots\cdots\cdots, \\
F_n &= K_{n1} \cup K_{n-1,2} \cup \cdots \cup K_{1n}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
$$

(If there are only a finite number N of regions $I_j(B)$, the symbols $K_{n-j+1,j}$ for $j > N$ are to be omitted.)

Before proceeding to prove that the sequence $\{F_i\}$ satisfies the required conditions, we prove the following property:

PROPERTY A. *Any point b of* $F\{I_j(B)\}$ *is separated from* P_j *by an infinite number of components of* R^*.

Proof. According to the definition of R, b is an s.l. point of some nested subsequence of rings $\{R_{n_i}\}$. It is easy to show that—for some $k-R_{n_t}$ separates P_j from b when $t>k$.

For any R_{n_i} which was omitted in the construction of R^* substitute an S_i which separates this R_{n_i} from b. Omit any R_{n_i}'s between the R_{n_i} omitted and this S_i. Thus a subsequence of $R_{ij}^{*(k)}$'s is obtained each of which separates P_j from b.

It is next proved that any K_{mj} is a closed connected set. Then, since F_n is the union of a finite number of these disjunct sets, F_n is closed and has only a finite number of components.

We first show that, for m fixed, there are only a finite number of $R_{mj}^{*(k)}$'s. An infinity of $R_{mj}^{*(k)}$'s would have an s.l. point b. But $b\in F\{I_j(B)\}$, hence is not an s.l. point of components separated from P_j by only m other components. It can now easily be verified that K_{mj} is closed. Clearly K_{mj} is connected.

To prove that (a)–(c) are satisfied it is sufficient to verify the corresponding properties for the K_{ij}'s when $I_j(B)$ is substituted for $C(B)$.

Proof of (a). This follows from (A), Property (A), and the definition of K_{mj}.

Proof of (b). To prove that K_{mj} is interior to $K_{m+1,j}$ it is sufficient to show that—for any k—$R_{mj}^{*(k)}\subset E_{m+1,j}$, which is a region.

If $R_{mj}^{*(t)}\not\subset E_{m+1,j}$ every Jordan arc joining P_j and some point of $R_{mj}^{*(t)}$ meets $F\{E_{m+1,j}\}\subset \bigcup_k(R_{m+1,j}^{*(k)})$. But then $R_{mj}^{*(t)}$ would be separated from P_j by at least m other components, contrary to the definition of an $R_{mj}^{*(k)}$.

Proof of (c). Clearly, if a point $p\in(S_t\cap E_{mj})$, $S_t\subset E_{mj}\subset K_{mj}$. If—for some q—$p\in(R_{mj}^{*(q)}\cap S_t)$, $R_{mj}^{*(q)}$ is just S_t.

Proof of (d). First we show that for any point p of any $I_j(B)$ there exists m such that $p\in E_{mj}$, which is a region.

If—for every m—p is outside some $R_{mj}^{*(k)}$, p must be separated from P_j by an infinite number of components. But since R^* is a Q-set whose s.l. points are in B, (B) implies this is impossible. We conclude that there exists m such that $p\in E_{mj}$.

To show that (d) is satisfied it is sufficient to prove that any closed set in $C(B)$ is contained in $\bigcup_{t=1}^{i}(E_{i-t+1,t})$ for some i. Assume the contrary. Then there exists a closed set $C\subset C(B)$ and a sequence of points $\{p_i\}$ in C such that $p_i\notin\bigcup_{t=1}^{i}(E_{i-t+1,t})$. A limit point p of this sequence belongs to C, therefore to some $I_j(B)$. It was just proved that—for some m—$p\in E_{mj}$. Hence, p is interior to $\bigcup_t E_{i-t+1,t}$ when $i\geqq m+j-1$. By definition of p_i there exists N such that, for $i\geqq N$, $p_i\notin\bigcup_t(E_{N-t+1,t})$. This leads to a contradiction and completes the proof of (d).

Proof of (e). Let p be an arbitrary point of $I_r(F_n, S)$. Then p belongs to some component—say $I_v(S)$—of $C(S)$.

Case i. $I_v(S)\subset I_r(F_n, S)$.

$I_r(F_n, S)\cap B^*\neq\varnothing$—since, by definition of B^*, $I_v(S)\cap B^*\neq\varnothing$.

Case ii. $I_v(S)\not\subset I_r(F_n, S)$.

First we note that $F_n\cup S$ is a Q-set whose s.l. points are in B. Hence, if $I_r(F_n, S)\cap B=\varnothing$—according to (C)—$F\{I_r(F_n, S)\}\subset(F_n\cup S)$. Then

$$F\{I_r(F_n, S)\}\subset S\cup_k(R_{n1}^{*(k)})\cup_k(R_{n-1,2}^{*(k)})\cup_k\cdots\cup_k(R_{1n}^{*(k)}).$$

Since $I_v(S)\not\subset I_r(F_n, S)$, (A) implies that $F\{I_r(F_n, S)\}\cap I_v(S)\neq\varnothing$. But $S\cap I_v(S)=\varnothing$. Therefore, we have to consider only the case where at least one of these $R_{n-j+1,j}^{*(k)}$'s is a ring—say R_t—which was not omitted.

For some j, $R_t\subset I_j(B)$. We note that $I_r(F_n, S)$ lies outside R_t. From the criteria for the non-omission of a ring we have that some point b^* of B^* in $O(R_t)\cap\bar{I}_j(B)$ is not separated from R_t by any S_i. Clearly there exists a point q of $\{I_r(F_n, S)\cap I_j(B)\}$ not separated from R_t by any S_i and so not separated from b^* by any S_i.

If $b^*\notin I_r(F_n, S)$, every Jordan arc joining q and b^* meets $F\{I_r(F_n, S)\}$. Then (D) would imply that some component of $F\{I_r(F_n, S)\}$ separates b^* and q. For some k, R_t is an $R_{n-j+1,j}^{*(k)}$. Since b^*, $q\in O(R_t)$, no other $R_{n-j+1,j}^{*(k)}$ can separate b^* and q. Since b^*, $q\in\bar{I}_j(B)$, no $R_{lm}^{*(k)}$—$m\neq j$—can separate b^* and q. We conclude that any component of $F\{I_r(F_n, S)\}$ which separates b^* and q must be contained in some S_i. But then q and b^* would be separated by this S_i. This gives a contradiction and completes the proof that $b^*\in I_r(F_n, S)$.

COROLLARY. *The sequence $\{F_i\}$ of the theorem can be chosen so that every F_i contains at least one component of S.*

Proof. According to (d) any preassigned component S_k is interior to some F_{n_k} of the sequence $\{F_i\}$. We now take F_{n_1} as F_1, F_{n_1+1} as F_2, and so on.

The conclusions expressed in the following lemma are included in some topological results obtained by Walsh [6, pp. 7–10].

LEMMA. *Let R be an arbitrary region—not the entire plane. Then there exists a sequence of regions $\{R_k\}$ with the following properties*:

(a) $\bar{R}_n\subset R$;

(b) $\bar{R}_n\subset R_{n+1}$;

(c) *Any closed set $C\subset R$ is interior to some R_k*;

(d) *The connectivity of R_n is not greater than that of R.*

The next theorem is applied in the proof of Theorem 7. We let G denote an open set whose components are $G_1, G_2, \cdots$. By an *F_G-set* we shall mean a set $F\subset G$ such that—for every i—the subset $F_i=F\cap G_i$ is a closed set.

THEOREM 2. *Let G be an open set—not the entire plane—and let B^* be any $B^*(G)$-set. Suppose F is a given F_G-set. Then there exists a sequence of Q-sets $S^{(1)}, S^{(2)}, \cdots, S^{(n)}, \cdots$ such that—for every j—*

(a) $F\subset S^{(j)}\subset G$;

(b) *$S^{(j)}$ is interior to $S^{(j+1)}$*;

(c) *For any closed set* $C \subset G$ *there exists* N *such that* C *is interior to* $S^{(j)}$ *for* $j > N$;

(d) B^* *contains a* $B^*(S^{(j)})$*-set, and every* $S^{(j)}$*-s.l. point is a* G*-s.l. point.*

Proof. The lemma implies that for each component G_i of G there exists a sequence of regions $R_1^{(i)}, R_2^{(i)}, \cdots$ having the properties listed in the lemma. Since—by definition of F—F_i is a closed set, it follows from (c) of the lemma that we may suppose these $R_k^{(i)}$'s chosen so that $F_i \subset R_k^{(i)}$, for every k.

Next let us define sets $S^{(1)}, S^{(2)}, \cdots$ thus:

$$S^{(1)} = \bigcup_i (\overline{R}_1^{(i)}),$$

$$\cdots\cdots\cdots,$$

$$S^{(n)} = \bigcup_i (\overline{R}_n^{(i)}),$$

$$\cdots\cdots\cdots.$$

Now for every j, $F \subset S^{(j)}$, which is a Q-set, and $S^{(j)} \subset G$. Clearly $S^{(j)}$ is interior to $S^{(j+1)}$. Thus conditions (a) and (b) are satisfied.

If (c) were not true there must exist a closed set $C \subset G$ and a sequence of points $\{p_k\}$ such that $p_k \in C$ but $p_k \notin S^{(k)}$. A limit point p of this sequence belongs to C, therefore to some G_i. It follows from (c) of the lemma that—for some j—p is interior to $S^{(j)}$. But this leads us to a contradiction and we conclude that (c) is satisfied.

We have yet to verify (d). Clearly any $S^{(j)}$-s.l. point is a G-s.l. point and so belongs to B^*. Hence, to prove that B^*, which was defined as a $B^*(G)$-set, contains a $B^*(S^{(j)})$-set, it is sufficient to show that $I_k(S^{(j)}) \cap B^* \neq \emptyset$ for an arbitrary component $I_k(S^{(j)})$ of $C(S^{(j)})$.

It is evident from (d) of the lemma and the definition of $S^{(j)}$ that there is a component—say $I_t(G)$—of $C(G)$ such that $I_t(G) \cap I_k(S^{(j)}) \neq \emptyset$. Since—if $I_k(S^{(j)})$ contains an $S^{(j)}$-s.l. point—$I_k(S^{(j)}) \cap B^* \neq \emptyset$, the only case that remains to be considered is that where $I_k(S^{(j)})$ contains no $S^{(j)}$-s.l. point. In this case (C) implies that $F\{I_k(S^{(j)})\} \subset S^{(j)}$, and hence $F\{I_k(S^{(j)})\} \subset G$. Now, if $I_t(G) \not\subset I_k(S^{(j)})$, (A) implies that $F\{I_k(S^{(j)})\} \cap I_t(G) \neq \emptyset$. But this would give a contradiction to the conclusion that $F\{I_k(S^{(j)})\} \subset G$. We conclude that $I_t(G) \subset I_k(S^{(j)})$.

Then, since $I_t(G) \cap B^* \neq \emptyset$, $I_k(S^{(j)}) \cap B^* \neq \emptyset$. This completes the proof that in no case is $I_k(S^{(j)}) \cap B^* = \emptyset$.

3. **Approximation theorems.** Throughout this paper it is to be understood that the functions are single-valued. A function is said to be *analytic on a set* if it is analytic in a neighborhood of every point of the set. We note that a function analytic on a Q-set S may be defined by a different analytic function on each component of S.

THEOREM 3. *Let S be a Q-set, B the set of S-s.l. points, and B^* any $B^*(S)$-set. Suppose $M(z)$ is any function analytic on S. Then, for any sequence $\{\delta_i\}$ of positive constants, there exists a function $L(z)$ analytic in $C(B^*)$ and meromorphic in $C(B)$ such that—for every i—*

$$\left| M(z) - L(z) \right| < \delta_i$$

when z is any point of the component S_i.

Proof. If S is the null set or the whole extended plane, the conclusion is trivial. If S has only a finite number of components, S is a closed set and the desired result follows directly from Runge's Theorem.

Otherwise, there exists a sequence of closed sets $\{F_i\}$ as described in Theorem 1. Let us choose such a sequence as specified in the corollary. We note that "$F_n \subset C(B)$" implies F_n intersects only a finite number of S_i's.

Denote the components of S contained in F_1 by $S_1, S_2, \cdots, S_{m_1}$ and those in $F_n \cap C(F_{n-1})$ by $S_{m_{n-1}+1}, \cdots, S_{m_n}$. Change the subscripts for δ_i's of the given $\{\delta_i\}$ so that each δ_i corresponds to its original S_i. Then let $\delta^{(n)}$ denote the minimum of $\delta_1, \delta_2, \cdots, \delta_{m_n}$.

$M(z)$ is analytic on the closed set $\bigcup_{i=1}^{m_1} S_i$ and every component of $C(\bigcup_{i=1}^{m_1} S_i)$ contains a point of B^*. Then Runge's Theorem as generalized by Walsh implies there exists a rational function $r_1(z)$ whose poles lie in B^* such that

$$\left| M(z) - r_1(z) \right| < \frac{\delta^{(1)}}{2} \qquad \text{when } z \in \bigcup_{i=1}^{m_1} S_i.$$

For any integer $k > 1$ there exists an open set G_{k-1} containing F_{k-1} and disjunct from some open set $O_k \supset \bigcup_{q=1}^{m_k - m_{k-1}} S_{m_{k-1}+q}$ in which $\{M(z) - \sum_{j=1}^{k-1} r_j(z)\}$ is analytic. Let us define

$$\begin{aligned} M_k(z) &= 0 \text{ in } G_{k-1} \\ &= \left\{ M(z) - \sum_{j=1}^{k-1} r_j(z) \right\} \text{ in } O_k. \end{aligned}$$

Now $M_k(z)$ is analytic on the closed set $F_{k-1} \bigcup_{q=1}^{m_k - m_{k-1}} (S_{m_{k-1}+q})$ and a point of B^* lies in each region into which this set separates the plane—by (e) of Theorem 1. Hence, according to Walsh's generalization of Runge's Theorem, there exists a rational function $r_k(z)$ whose poles lie in $B^* \cap C\{F_{k-1} \bigcup_{q=1}^{m_k - m_{k-1}} S_{m_{k-1}+q}\}$ such that

$$\left| M_k(z) - r_k(z) \right| < \delta^{(k)}/2^k$$

for z any point of F_{k-1} or of $S_{m_{k-1}+q}$—$q = 1, 2, \cdots, (m_k - m_{k-1})$.

Let D be any region in $C(B)$ such that $\overline{D} \subset C(B)$. According to (d) of Theorem 1—for some integer t—$\overline{D}$ is interior to F_{t-1}. Then, for all $j \geqq t$, $r_j(z)$ is analytic on $\overline{D}$ and

$$|r_j(z)| < \delta^{(t)}/2^j \qquad \text{when } z \in \overline{D}.$$

Since $\sum_{j=t}^{\infty} \delta^{(t)}/2^j$ converges, $\sum_t^{\infty} r_j(z)$ converges uniformly in $\overline{D}$—hence is analytic in D. Except possibly for a finite number of poles in (B^*-B), $L(z)=\sum_1^{\infty} r_j(z)$ is analytic in D. Since $\overline{D}$ was chosen as any closed region in $C(B)$, $L(z)$ is analytic in $C(B)$ except possibly for poles in (B^*-B).

We wish to prove that—for any n—

$$|M(z) - L(z)| < \delta_n \qquad \text{when } z \in S_n.$$

For some k, $m_{k-1} < n \leqq m_k$. We have that

$$\left| M(z) - \sum_{j=1}^{k} r_j(z) \right| < \frac{\delta^{(k)}}{2^k} \leqq \frac{\delta_n}{2^k} \qquad \text{when } z \in S_n.$$

Now

$$|M(z) - L(z)| \leqq \left| M(z) - \sum_{j=1}^{k} r_j(z) \right| + \sum_{j=k+1}^{\infty} |r_j(z)|$$

$$< \frac{\delta_n}{2^k} + \sum_{j=k+1}^{\infty} \frac{\delta_n}{2^j} \leqq \delta_n \qquad \text{when } z \in S_n.$$

COROLLARY 1. *If $M(z)$ is any function analytic on a Q-set S which has no s.l. point in the finite plane, then—for any $\{\delta_i\}$—there exists a meromorphic function $L(z)$ such that, for every i,*

$$|M(z) - L(z)| < \delta_i \qquad \textit{when } z \in S_i.$$

If points are preassigned, one in each of the bounded components of $C(S)$, $L(z)$ can be chosen so that its poles lie in these points.

In particular, if S does not divide the plane and does not contain the point at infinity, $L(z)$ can be required to be an integral function.

COROLLARY 2. *Let $G(z)>0$ be any function bounded away from zero on each component of a Q-set S. Let B be the set of S-s.l. points and B^* any $B^*(S)$-set. Then, if $M(z)$ is any function which is analytic on S, there exists a function $L(z)$ meromorphic in $C(B)$ such that*

$$|M(z) - L(z)| < G(z) \qquad \textit{when } z \in S.$$

Furthermore, $L(z)$ can be chosen so that its poles lie in (B^*-B).

Proof. This follows directly from the theorem when δ_i is chosen as G.L.B. $G(z)$ on the component S_i.

The next theorem shows that in order that every function analytic on a given Q-set can be even uniformly approximated on that set it is actually necessary to allow the approximating function a possible singularity in each component of $C(S)$.

LEMMA. *Let R be a region of finite connectivity such that $\overline{R}$ is not the entire extended plane. Let $M(z)$ be a function analytic on $F(R)$. Then, if there exists a sequence of functions $\{f_n(z)\}$ analytic on $\overline{R}$ which converges uniformly to $M(z)$ on $F(R)$, $M(z)$ can be continued analytically from $F(R)$ to all of R.*

Proof(5). Let R' denote the region obtained by suppressing all components of $F(R)$ which are isolated points. Now $R\subset R'$, $\overline{R}'=\overline{R}$, and $F(R')\subset F(R)$. Since R has finite connectivity so does R'.

By hypothesis, there exists a sequence of functions $\{f_n(z)\}$ analytic on $\overline{R}'$ which converges uniformly to $M(z)$ on $F(R')$. By a well known theorem, such a sequence converges uniformly on $\overline{R}'$ to some function $M^*(z)$ which is analytic in R' and continuous in $\overline{R}'$. On $F(R')$, $M^*(z)\equiv M(z)$. The function $f(z)=M^*(z)-M(z)$ is single-valued and analytic interior to R' in the neighborhood of $F(R')$ and $\lim_{z\to F(R')} f(z)=0$ for z interior to R'. It follows from a theorem given by Walsh [6, p. 23] that $f(z)$ vanishes identically interior to R' in the neighborhood of $F(R')$.

Thus $M^*(z)$ gives an analytic extension of $M(z)$ from $F(R')$ to all of R', hence from $F(R)$ to all of R.

THEOREM 4. *Let S be a Q-set (which consists of more than one point), B the set of S-s.l. points, and E a set in $C(S)$ such that $B\subset E$. Then a necessary and sufficient condition that every function which is analytic on S can be uniformly approximated on S by some function analytic in $C(E)$ is that every component of $C(S)$ contain a point of E.*

Proof. The sufficiency of the condition follows directly from Theorem 3 when the sequence $\epsilon, \epsilon, \cdots$ is taken for $\{\delta_i\}$ of the theorem. We proceed to the proof of the necessity.

For the sake of argument, let us suppose that there is a component—say $I_k(S)$—of $C(S)$ such that $I_k(S)\cap E=\varnothing$. Since $B\subset E$, $I_k(S)\cap B=\varnothing$. Then (C) implies that $F\{I_k(S)\}\subset S$ and that $I_k(S)$ is a region of finite connectivity.

Case I. $\overline{I}_k(S)$ is the entire plane.

The hypothesis implies that, for any preassigned function $M(z)$ as described, there exists a sequence of functions analytic in the entire plane, that is, a sequence of constants, which converges uniformly to $M(z)$ on S. This is impossible for an arbitrary function $M(z)$.

Case II. $\overline{I}_k(S)$ is not the entire plane.

Let us choose $M(z)=1/(z-p)$, where $p\in I_k(S)$. It follows from the hypothesis that—for every positive integer n and for any $\epsilon>0$—there exists a function $f_n(z)$ analytic in $C(E)$ such that

$$|M(z)-f_n(z)|<\epsilon/n$$

on S, hence on $F\{I_k(S)\}$.

(5) This proof is essentially the same as that of a similar statement by Walsh [6, p. 25].

Now for the function $M(z)$, which is analytic on $F\{I_k(S)\}$—where $I_k(S)$ is a region of finite connectivity—we have the sequence of functions $\{f_n(z)\}$ analytic on $\bar{I}_k(S)$ which converges uniformly to $M(z)$ on $F\{I_k(S)\}$. Then the lemma implies that $M(z)$ can be extended analytically from $F\{I_k(S)\}$ to all of $I_k(S)$. But this is impossible since $M(z)=1/(z-p)$ has a pole at p, a point of $I_k(S)$. This completes the proof of the theorem.

The approximating function of the next theorem is required to be not only analytic but also nonvanishing in the complement of a preassigned $B^*(S)$-set.

THEOREM 5. *Let S be a Q-set and B^* a $B^*(S)$-set. Suppose $M(z)$ is any function which is analytic and nonvanishing on S and such that log $M(z)$ can be chosen as single-valued and continuous on S. Then, for any $\{\epsilon_i\}$, there exists a function $f(z)$ nonvanishing and analytic in $C(B^*)$ such that—for every i—*

$$|M(z) - f(z)| < \epsilon_i \qquad \textit{when } z \in S_i.$$

Proof. For each i choose δ_i so that when $z \in S_i$ and $|w - \log M(z)| < \delta_i$, then $|e^w - M(z)| < \epsilon_i$.

According to Theorem 3—when log $M(z)$ is chosen as the function to be approximated—there exists a function $L(z)$ analytic in $C(B^*)$ such that, for every i,

$$|\log M(z) - L(z)| < \delta_i \qquad \text{when } z \in S_i.$$

Then, for every i,

$$|M(z) - e^{L(z)}| < \epsilon_i \qquad \text{when } z \in S_i.$$

Clearly $e^{L(z)}$ is analytic and nonvanishing in $C(B^*)$. Hence, $e^{L(z)}$ may be taken as the required function $f(z)$.

COROLLARY. *Suppose $M(z)$ is analytic and nonvanishing on a Q-set S which does not separate the plane. Then—for any $\{\epsilon_i\}$—there exists a function $f(z)$ which is nonvanishing and analytic except possibly at S-s.l. points (provided S has an infinite number of components) such that, for every i,*

$$|M(z) - f(z)| < \epsilon_i \qquad \textit{when } z \in S_i.$$

In particular, if S has no finite s.l. point and does not contain the point at infinity, $f(z)$ can be required to be a nonvanishing integral function. If S has only a finite number of components, we allow $f(z)$ a singularity at an arbitrary preassigned point of $C(S)$.

It is easy to show that the singularities of an approximating function satisfying the conditions required in Theorem 5 cannot in general be restricted to at most poles at points of B^* which are not S-s.l. points.

THEOREM 6. *Let S be a closed set and let B^* be any set of points chosen one* *from each component of $C(S)$. Suppose $M(z)$ is any function such that log $M(z)$ can be chosen as single-valued and analytic on S. Then $M(z)$ can be uniformly approximated on S by a function $f(z)$ which is analytic and nonvanishing in the entire plane except at a finite number of points of B^*.*

Proof. Given $\epsilon>0$, choose $\delta>0$ so that when $z\in S$ and $|w-\log M(z)|<\delta$, then $|e^w-M(z)|<\epsilon$.

According to Walsh's theorem on approximation on a closed set [6, p. 15] —when log $M(z)$ is chosen as the function to be approximated—there exists a rational function $r(z)$ whose poles lie in B^* such that

$$|\log M(z) - r(z)| < \delta, \qquad z \in S.$$

Then

$$|M(z) - e^{r(z)}| < \epsilon \qquad \text{when } z \in S$$

and $e^{r(z)}$ satisfies the conditions required of $f(z)$.

In the next theorem our approximation results are expressed in terms of expansion of a function analytic in an open set G in a sequence of functions analytic (or analytic and nonvanishing) except possibly at preassigned points of $C(G)$. Our definitions of an F_G-set and of F_i were previously given for Theorem 2.

THEOREM 7. *Let G be an open set, B the set of G-s.l. points, and B^* any $B^*(G)$-set. Suppose $M(z)$ is any function analytic in G. Then there exists a sequence of functions $\{f_j(z)\}$ meromorphic in $C(B)$ and analytic in $C(B^*)$ which converges to $M(z)$ in G, uniformly on any closed set interior to G(6). If F is a preassigned F_G-set, $\{f_j(z)\}$ can be chosen so that—when $z\in F_i$—*

$$|M(z) - f_j(z)| < \epsilon_i^{(j)}$$

for any preassigned $\{\epsilon_i^{(j)}\}$.

If log $M(z)$ can be chosen as single-valued and continuous in G the functions $f_j(z)$ can be required to be analytic and nonvanishing in $C(B^*)$(7).

Proof. If G is the null set or the whole extended plane, the conclusions are trivial. These cases are henceforth disregarded.

In order to avoid the consideration of special cases, if an F_G-set is not preassigned we choose one arbitrarily—which may be the null set.

For fixed i, we may suppose each $\epsilon_i^{(j)}\leqq 1/j$.

Let us choose a sequence of Q-sets $S^{(1)}, S^{(2)}, \cdots$ satisfying the conditions

(6) When no further restriction is made, the $f_j(z)$ can be required to be rational functions with their poles in B^* [6, p. 16].

(7) In this case the $f_j(z)$ are not required to be meromorphic in $C(B)$.

The proof of this statement is similar to that of the first part of the theorem. Necessary modifications in the proof are indicated in parentheses.

listed in Theorem 2.

We note that $M(z)$ is analytic on each $S^{(j)}$. Then according to Theorem 3 (Theorem 5)—when applied successively—there exist functions $f_j(z)$ analytic in $C(B^*)$ and meromorphic in $C(B)$ (nonvanishing and analytic in $C(B^*)$) such that, for every i,

$$
\begin{aligned}
&| M(z) - f_1(z) | < \epsilon_i^{(1)} \quad \text{when} \quad z \in S_i^{(1)}; \quad \text{hence when} \quad z \in F_i, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&| M(z) - f_n(z) | < \epsilon_i^{(n)} \quad \text{when} \quad z \in S_i^{(n)}; \quad \text{hence when} \quad z \in F_i, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
$$

By applying (c) of Theorem 2 we see that by taking $\{\epsilon_i^{(1)}\} = 1, 1, \cdots$ and, in general, $\{\epsilon_i^{(j)}\} = 1/j, 1/j, \cdots$, a sequence $\{f_j(z)\}$ can be obtained which converges uniformly to $M(z)$ on any closed set interior to G. Since we supposed that each $\epsilon_i^{(j)} \leqq 1/j$, the original $\{f_j(z)\}$ obtained converges uniformly to $M(z)$ on any closed set $C \subset G$.

Corollary. *If the components of the open set G are simply connected and bounded, if G has no finite s.l. point, and if $M(z)$ has no zero in G, the functions of the sequence $\{f_j(z)\}$ in the theorem can be chosen as integral nonvanishing functions.*

We next obtain what might be called a Weierstrass factor-approximation-theorem. We let J denote an isolated set and suppose a positive integer assigned each point of J. According to the Weierstrass-factor-theorem [2], there exists a function $g(z)$ analytic except at limit points of J which has zeros of the prescribed orders at precisely the prescribed points.

Let us suppose that $M(z)$ is any function which is analytic in a neighborhood of each point of J and which has zeros of the prescribed orders at points of J. (Of course, $M(z)$ may be defined by as many different analytic functions as there are points in the set J.) Then, according to Theorem 9, there exists a function $h(z)$ analytic except at limit points of J which not only satisfies the conditions required in the Weierstrass theorem but which also approximates $M(z)$ arbitrarily closely in a neighborhood of each point of J, where the closeness of approximation may be preassigned independently for each point of J.

Theorem 8. *Let S be a Q-set which does not separate the plane. If S has an infinite number of components, we let B denote the set of S-s.l. points; if S has only a finite number of components, we let B consist of an arbitrary point of $C(S)$. Suppose $M(z)$ is analytic on S and not identically zero on any component S_i. Then, for any $\{\epsilon_i\}$, there exists a function $h(z)$ such that*

(a) *$h(z)$ is nonvanishing in $C(S \cup B)$ and is analytic in $C(B)$;*

(b) *The zeros of $h(z)$ coincide with the zeros of $M(z)$ on S and are of the same orders;*

(c) *For every* i,

$$|M(z) - h(z)| < \epsilon_i \qquad \textit{when } z \in S_i.$$

Proof. The set $J = \{z | M(z) = 0; z \in S\}$ is an isolated set. Hence, Weierstrass's factor-theorem [2] implies the existence of a function $g(z)$ analytic in $C(B)$ which has zeros at precisely the points of J of the same orders as the zeros of $M(z)$.

We let $M_i = \text{L.U.B. } g(z)$ on S_i.

The function $F(z) = M(z)/g(z)$ is analytic and nonvanishing on S. Then by the corollary of Theorem 5 there exists a function $f(z)$ nonvanishing and analytic in $C(B)$ such that, for every i,

$$|F(z) - f(z)| < \epsilon_i/M_i \qquad \text{when } z \in S_i.$$

Now

$$\begin{aligned} |M(z) - f(z)\cdot g(z)| &= \left| \left\{ \frac{M(z)}{g(z)} - f(z) \right\} \cdot g(z) \right| \\ &= \left| \frac{M(z)}{g(z)} - f(z) \right| \cdot |g(z)| < \frac{\epsilon_i}{M_i} |g(z)| \leqq \epsilon_i \end{aligned}$$

when $z \in S_i$.

The function $f(z)\cdot g(z)$ satisfies the conditions required of the function $h(z)$.

The following lemma is easily verified. Modifications in the statement for a similar lemma are indicated in parentheses.

LEMMA. *Let* $M(z)$ *be any function which is meromorphic (analytic and not identically zero) in a neighborhood of each point of a given isolated set* J. *Then there exists a neighborhood* N_j *for each point* j *of* J *such that*

(a) $\overline{N}_i \cap \overline{N}_j = \emptyset$ *for* $i \neq j$;

(b) $N = \bigcup \overline{N}_j$ *is a Q-set which does not separate the plane and whose set of s.l. points is just the set of limit points of* J.

(c) $M(z)$ *is meromorphic (analytic) on* N *and is analytic (nonvanishing) on* N *except possibly at points of* J.

Such a set of neighborhoods for the points of an isolated set J and a given function $M(z)$ will be called an *M.J.-collection of neighborhoods.*

The following theorem, which is an extension of the Weierstrass-factor-theorem, follows directly from Theorem 8 when the Q-set S of the theorem is chosen as $\bigcup \overline{N}_i$.

THEOREM 9. *Let* J *be any isolated set. Assign to each point of* J *a positive integer as order. Let* $M(z)$ *be any function which is analytic in a neighborhood of each point of* J *and which has zeros of the prescribed orders at these points. Then, for any M.J.-collection of neighborhoods* $\{N_i\}$ *and for any* $\{\epsilon_i\}$ *there exists a function* $h(z)$ *analytic except at limit points of* J *(or—in case* J *has only* *a finite number of points—analytic except at a preassigned point of* $C(\bigcup \overline{N}_i)$) *such that*

(a) $h(z)$ *has zeros to the prescribed orders at the points of* J *and is nonvanishing elsewhere*;

(b) *For every* i,

$$|M(z) - h(z)| < \epsilon_i \qquad \text{when } z \in \overline{N}_i.$$

We note that such a function $h(z)$ necessarily has an essential singularity at each limit point of J.

Next we obtain an extension of the Mittag-Leffler partial-fractions theorem [2] analogous to our extension of the Weierstrass factor-theorem. We prove the existence of a function which not only has poles with assigned principal parts at prescribed points but which also satisfies a preassigned approximation condition arbitrarily closely in a neighborhood of each prescribed point.

THEOREM 10. *We define sets* S *and* B *as in Theorem* 8. *Let* $M(z)$ *be any function which is meromorphic on* S. *Then, for any* $\{\epsilon_i\}$, *there exists a function* $N(z)$ *such that*

(a) $N(z)$ *is analytic in* $C(B)$ *except at poles of* $M(z)$ *on* S;

(b) *The poles of* $N(z)$ *coincide with those of* $M(z)$ *on* S *and have the same principal parts*;

(c) *For every* i

$$|M(z) - N(z)| < \epsilon_i \qquad \text{when } z \in S_i,$$

except at poles of $M(z)$ *on* S_i.

Proof. The set of points at which $M(z)$ has poles on S is an isolated set. Hence, Mittag-Leffler's partial-fractions theorem [2] implies the existence of a function $g(z)$ analytic in $C(B)$ which has poles at precisely the same points as $M(z)$ on S with the same principal parts.

The function $F(z) = M(z) - g(z)$ is analytic on S. Hence, Theorem 3 implies the existence of a function $L(z)$ analytic in $C(B)$ such that—when $z \in S_i$—

$$|F(z) - L(z)| < \epsilon_i$$

or

$$|M(z) - [g(z) + L(z)]| < \epsilon_i.$$

We take $g(z) + L(z)$ for the required function $N(z)$.

The following theorem, which is an extension of the Mittag-Leffler partial-fractions theorem, follows directly from Theorem 10 when the Q-set S of the theorem is chosen as $\bigcup \overline{N}_i$.

THEOREM 11. *Let* J *be any isolated set. Assign to each point of* J *a principal* *part. Let $M(z)$ be any function which is meromorphic in a neighborhood of each point of J and which has a pole at each point of J with the assigned principal part. Then, for any M.J.-collection of neighborhoods $\{N_i\}$ and for any $\{\epsilon_i\}$, there exists a function $N(z)$ meromorphic except at limit points of J (or—in case J has only a finite number of points—meromorphic except at a preassigned point of $C(\bigcup \overline{N}_i)$ such that*

(a) *$N(z)$ has poles at precisely the points of J with the prescribed principal parts;*

(b) *For every i,*

$$|M(z) - N(z)| < \epsilon_i \qquad \textit{when } z \in \overline{N}_i,$$

except at points of J.

We note that such a function $N(z)$ necessarily has an essential singularity at each limit point of J.

4. **On the order of the approximating function.** One might expect that, when the "rate of growth" of the function $M(z)$ to be approximated on S in the corollary of Theorem 5 is properly restricted, the approximating function could be required to be of finite order. We next determine a necessary restriction on $M(z)$ when S is unbounded.

THEOREM 12. *Let $M(z)$ be analytic and nonvanishing on an unbounded Q-set S which does not separate the plane. Suppose that, for any given $\{\epsilon_i\}$, there exists an integral nonvanishing function $f(z)$ of finite order such that—for every i—*

$$|M(z) - f(z)| < \epsilon_i \qquad \textit{when } z \in S_i.$$

Then $M(z)$ is necessarily an integral nonvanishing function of finite order.

Proof. For any $\delta > 0$ and for any i we can choose ϵ_i so that when $|w - M(z)| < \epsilon_i$ and $z \in S_i$, then $|\log w - \log M(z)| < \delta$. By hypothesis there exists an integral nonvanishing function $f(z)$ of finite order such that, when $z \in S_i$,

$$|f(z) - M(z)| < \epsilon_i$$

holds for every i. We may, according to Hadamard's Theorem [5, p. 250], write $f(z) = e^{L(z)}$ where $L(z)$ is some polynomial. Since—for every i—

$$|e^{L(z)} - M(z)| < \epsilon_i \qquad \text{when } z \in S_i,$$

it follows from the choice of ϵ_i that

$$|L(z) - \log M(z)| < \delta \qquad \text{when } z \in S_i.$$

Now the function $\log M(z)$ has been uniformly approximated on the unbounded set S by the polynomial $L(z)$. Walsh [6, p. 25] has observed that a function which can be uniformly approximated on an unbounded point set by a polynomial is itself a polynomial. Hence, we conclude that $\log M(z)$ is a polynomial. Then $M(z)$ $(=e^{\log M(z)})$ is an integral nonvanishing function of finite order.

REFERENCES

1. P. W. Ketchum, *On the possible rate of growth of an analytic function*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 211–228.
2. G. Mittag-Leffler, *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*, Acta Math. vol. 4 (1884) pp. 1–79.
3. M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1939.
4. C. Runge, *Zur Theorie der eindeutigen analytischen Funktionen*, Acta Math. vol. 6 (1885) pp. 229–244
5. E. C. Titchmarsh, *The theory of functions*, 2d ed., Oxford University Press, 1939.
6. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1935.
7. ———, *On approximation by rational functions to an arbitrary function of a complex variable*, Trans. Amer. Math. Soc. vol. 31 (1929) pp. 477–502.
8. ———, *On approximation to an arbitrary function of a complex variable by polynomials*, Trans. Amer. Math. Soc. vol. 30 (1928) pp. 472–482.

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