

HARMONIC FUNCTIONS ON OPEN RIEMANN SURFACES

BY

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In recent years the importance of open (noncompact) Riemann surfaces for function theory has been shown by the work of Nevanlinna, Ahlfors, and Myrberg. In particular there arose the concept of the type of a Riemann surface: Nevanlinna considered Riemann surfaces whose boundary has zero harmonic measure [20], and Myrberg the existence of a Green's function [15]. Elegant results concerning the type of a covering surface were given by Ahlfors [1]. An extension of this idea of type leads to the concept of the classification of Riemann surfaces as suggested by Ahlfors and Sario [21]. The modern theory of open Riemann surfaces has been vigorously developed along these lines not only by these but also by the younger members of the Finnish school: Sario, Virtanen, Laasonen, and Lehto.

It is the purpose of the present paper to consider in some detail the family of harmonic functions on such a Riemann surface and to apply this study to questions concerning the type and classification problems.

In the first two chapters are collected a number of results which are necessary for the later chapters. We have put these results in just that form in which they are needed here, but have omitted proofs in general, since they require only minor modifications from proofs given in the literature.

In the third chapter we consider the structure of the space BD of functions on a Riemann surface which are both bounded and have a finite Dirichlet integral. We introduce a convergence topology into this space and consider some of the linear functionals continuous in this topology. The class of functions which vanish outside a compact set is denoted by K , and its closure in BD denoted by \bar{K} . We then show that for every $f \in BD$ we have

$$f = f_K + u,$$

where $f_K \in \bar{K}$, and $u \in HBD$, i.e., is a harmonic BD function. The decomposition is unique except for those Riemann surfaces for which

$$\bar{K} = BD.$$

We call these surfaces parabolic and show that they have a null boundary in the sense of Nevanlinna and have no Green's function. Conversely, if a surface has a boundary which has zero harmonic measure in the sense of Nevan-

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linna, or if it does not have a Green's function, then it is parabolic.

We also give a criterion in terms of the continuous linear functionals on the space BD that a Riemann surface be hyperbolic (Theorem 4).

In the fourth chapter we extend these results to the case of surfaces with a relative boundary and show that if $f \in BD$, then

$$f = f_0 + u_K + u_N,$$

where $f_0 \in \overline{KO}$, that is, belongs to \overline{K} and vanishes on the relative boundary R of the Riemann surface, $u_K \in H\overline{K}$, that is, belongs to \overline{K} and is harmonic, while $u_N \in HN$, that is, it is an HBD function whose normal derivative vanishes on R . These composition theorems correspond roughly to the classical Dirichlet and Neumann problems. As a consequence it is readily proved that there is in this case a norm preserving isomorphism between the classes HO and HN , where the norm of a function is defined as the supremum of its absolute value.

If an open Riemann surface W is separated by a finite number of compact curves into a finite number of surfaces bounded (relatively) by some of these curves, then we show that the space of HBD functions on W is isomorphic to the linear direct sum of the spaces HO (or equivalently HN) on the bounded surfaces.

In the last chapter we again return to the type problem and obtain some conditions in terms of a triangulation on a Riemann surface that the surface be hyperbolic. The results in this chapter are somewhat similar in form to those of Wittich, but are complementary to his in that we give here sufficient conditions that a Riemann surface be hyperbolic, while he considers necessary conditions.

Much of the background for this paper is to be found in the works of Ahlfors and Nevanlinna in general and in particular in the seminar on open Riemann surfaces given by the former at Harvard in the fall of 1949. I owe much to Professor Lars Ahlfors for his valuable help and kindly encouragement in this investigation.

CHAPTER I. RIEMANN SURFACES: PRELIMINARY CONCEPTS

1. **Definitions.** A Riemann surface W is given by a connected, separable Hausdorff space and a covering $\{U\}$ by open sets together with a collection $\{z\}$ of corresponding homeomorphisms onto open sets in the complex z -plane with the property that whenever two open sets U_1 and U_2 meet, then $z_2 \circ z_1^{-1}$ is a complex analytic function in the region $z_1(U_1 \cap U_2)$. The variable z_1 , which can be used to label points in the neighborhood U_1 , is said to be a local uniformizer valid for the region U_1 or more briefly a uniformizer for U_1 . We shall use the letter W to denote both the Riemann surface and the underlying topological space.

We say that a mapping

$$F: W_1 \rightarrow W_2$$

of one Riemann surface into another is analytic at a point $p \in W_1$ if for a uniformizer z_1 valid near p and a uniformizer z_2 valid near $F(p)$ the function $z_2 \circ F \circ z_1^{-1}$ is a complex analytic function near $z_1^{-1}(p)$. If F is a one-to-one analytic mapping of W_1 into W_2 , we shall call it a conformal mapping into. A conformal mapping is clearly a homeomorphism into. By an analytic arc we mean the conformal image of a closed interval in the complex plane.

By a compact region we shall mean an open connected set whose boundary consists of a finite number of piecewise analytic curves. It is readily shown that any compact set on a Riemann surface can be included in the interior of a compact region.

A sequence $\{\Omega_i\}$ of compact regions with $\bar{\Omega}_i \subset \Omega_{i+1}$ and $\bigcup \Omega_i = W$ will be termed an *exhaustion* of W . The existence of an exhaustion of an open (i.e., noncompact) Riemann surface W is a simple consequence of the separability of W .

A sequence $\{\Omega_i\}$ of disjoint compact regions with $\bigcup \bar{\Omega}_i = W$ is called a subdivision of W provided that a point on the boundary of a region Ω_i which is not a "corner" belongs to the boundary of exactly one other region, and provided also that a given compact set on the Riemann surface meets only a finite number of the Ω_i . We shall usually assume that the compact regions Ω_i are so small that we can find uniformizers z_i such that each z_i is valid in some open set containing $\bar{\Omega}_i$. The existence of such a subdivision is immediate.

Let V be a connected, separable Hausdorff space and W a dense open subset of V . Then we shall say that V is a bounded Riemann surface with interior W and boundary $R = V - W$ provided there is a covering $\{U\}$ of V by open sets and a collection of corresponding homeomorphisms $\{z\}$ into the complex z -plane such that $z(U \cap W)$ is an open set of the complex z -plane, z is real on $U \cap R$, and whenever U_1 and U_2 meet, we have $z_2 \circ z_1^{-1}$ a complex analytic function on $z_1(U_1 \cap U_2)$. We shall use the letter V to denote both the bounded Riemann surface and the underlying topological space. It should be noted that the interior of a bounded Riemann surface is a Riemann surface and that a Riemann surface may be considered to be a bounded Riemann surface with an empty boundary.

The notions of analytic and conformal mappings extend easily to bounded Riemann surfaces.

An important example of a bounded Riemann surface is that obtained by taking $V = \bar{\Omega}$, where Ω is a compact region on a Riemann surface W^* , and setting $W = \Omega$ with a uniformizer for $U \subset W$ defined to be the corresponding uniformizer on W^* . Uniformizers can then be constructed for points belonging to the boundary of Ω . It should be noted that the class of functions analytic on V may differ from the class of functions analytic on $\bar{\Omega}$ in that different requirements are imposed at the corners of Ω .

Remembering that a continuum of the boundary of a bounded Riemann

surface is an analytic arc, we define a compact region on V to be a connected open set whose closure in V is compact and whose boundary is composed of a finite number of piecewise analytic curves. By an exhaustion of V we shall mean a sequence $\{\Omega_i\}$ of compact regions such that $\bar{\Omega}_i \subset \Omega_{i+1} \cup R$, while $W = \bigcup \Omega_i$ and $V = \bigcup \bar{\Omega}_i$. A sequence $\{\Omega_i\}$ of disjoint compact regions is to be called a subdivision of V provided that $V = \bigcup \bar{\Omega}_i$, and that a point other than a corner which belongs to the boundary of some Ω_i belongs either to R or to the boundary of exactly one other compact region. It is to be noted that an exhaustion or subdivision of a bounded Riemann surface is not an exhaustion or subdivision of its interior unless the bounded Riemann surface has an empty boundary.

If V is a bounded Riemann surface, it is well known that V can be extended by a process of symmetrization to a Riemann surface V^\wedge called the double of V . There exists an involutory, indirectly conformal mapping of V^\wedge onto itself such that every point of the boundary R of V remains fixed. The image of $p \in W$ is denoted by \tilde{p} , the image of W by \tilde{W} .

A function f on V^\wedge is called symmetric if $f(\tilde{p}) = f(p)$, while it is called skew-symmetric if $f(\tilde{p}) = -f(p)$. If f is an arbitrary function on V^\wedge , we shall call the function $f_s = [f(p) + f(\tilde{p})]/2$ the symmetrization of f . If f is defined and continuous on V , then there is a unique symmetric function on V^\wedge which is equal to f on V . This function is continuous and is called the symmetric extension of f to V^\wedge . Similarly, if f is defined and continuous on V and has the value zero on the boundary of V , then there is a unique skew-symmetric function defined and continuous on V^\wedge and equal to f on V . This function is called the skew-symmetric extension of f .

2. Harmonic functions and Harnack's theorem. A function u defined on a Riemann surface is said to be harmonic in a region if at each point of the region it is a harmonic function of a uniformizer at that point. The definition is easily seen to be independent of the uniformizers chosen. We shall have considerable use for the following lemma concerning harmonic functions:

LEMMA 1 (HARNACK'S THEOREM). *Let $\{u_i\}$ be a sequence of positive functions on a Riemann surface W with the property that, given any compact region Ω on W , all u_i are harmonic in Ω from some i on. Then either $\{u_i\}$ contains a subsequence which converges to a function u harmonic on W , the convergence being uniform on each compact region of W , or else the sequence $\{u_i\}$ diverges to $+\infty$ uniformly on every compact set.*

The proof is almost identical with the proof of the usual form of Harnack's theorem in the plane and is left to the reader. As an immediate consequence we have the fact that a uniformly bounded sequence of functions which are ultimately harmonic in every compact region must possess a subsequence which converges uniformly on every compact set to a harmonic function.

Let Ω be a compact region on a Riemann surface W . We shall say that the

derivatives of a sequence $\{f_i\}$ converge to the derivatives of f , uniformly in the compact region Ω , if there is a finite set $\{U_k\}$ of neighborhoods covering Ω with the property that for each U_k there is a uniformizer z_k valid in it such that the derivatives of the functions f_i with respect to x_k and y_k ($z_k = x_k + iy_k$) converge to the derivatives of f with respect to x_k and y_k uniformly in the neighborhood U_k . The following lemma is an easy consequence of this definition.

LEMMA 2. *Let $\{u_i\}$ be a sequence of functions on a Riemann surface W with the property that given any compact region Ω on W all u_i are harmonic in Ω from some i on, and suppose that $u_i \rightarrow u$ uniformly on every compact region. Then u is harmonic and the derivatives of $\{u_i\}$ converge to those of u uniformly on every compact region.*

3. Differentials. A first order differential on a Riemann surface W is an expression of the form

$$\alpha = a dx + b dy$$

where a and b are complex-valued functions which depend on the uniformizer $z = x + iy$ in such a way that α is independent of the uniformizer chosen.

A second order differential is an expression

$$\mu = c dx dy,$$

where again c depends on the uniformizer in such a way that the differential is independent of the choice of the uniformizer.

The sum of two differentials of the same order and the product of a function and a differential are defined in the obvious way. The product of two first order differentials $\alpha_1 = a_1 dx + b_1 dy$ and $\alpha_2 = a_2 dx + b_2 dy$ is by definition

$$\alpha_1 \alpha_2 = (a_1 b_2 - a_2 b_1) dx dy,$$

which is easily seen to be independent of the choice of the uniformizer and anti-commutative.

If f is a function whose partial derivatives exist in some sense, we define the differential of f to be

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

and define the differential of a first order differential as

$$d\alpha = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx dy.$$

It is easily verified that these definitions are invariant with respect to changes of uniformizer. We have the important relations

$$d(df) = 0$$

and

$$d(f\alpha) = df\alpha + f d\alpha.$$

In view of the analytic structure of our Riemann surface we may define for each first order differential $\alpha = adx + bdy$ a *dual differential*

$$*\alpha = - bdx + ady.$$

The invariance of this operator is a simple consequence of the Cauchy-Riemann differential equations which hold between different uniformizers.

We have the following identities involving this operation:

$$**\alpha = -\alpha, \quad \alpha_1*\alpha_2 = \alpha_2*\alpha_1, \quad \alpha*\bar{\alpha} = (|a|^2 + |b|^2)dxdy,$$

where we have written $\bar{\alpha} = \bar{a}dx + \bar{b}dy$ for the conjugate of the differential α .

We say that a function f is of class C^k on a region if f together with its first k derivatives are defined and continuous there. We say that a function is of class C^∞ if it has continuous derivatives of all orders. A differential is said to be of class C^k if it has coefficients which are of class C^k .

We say that a differential α vanishes or is zero at a point p_0 if $a(p_0) = b(p_0) = 0$. Let J be an analytic arc, and let z be a uniformizer at a point $p_0 \in J$ chosen so that z is real on J . Then $\alpha = adx + bdy$ is said to vanish *along* J at p_0 if $a(p_0) = 0$. Two first order differentials are said to be equal *along* an analytic arc J if their difference vanishes along J .

The *carrier* (French: support) of a function or differential is the closure of the set of points where the function or differential is not zero.

Let F be an analytic function which maps a Riemann surface W_1 into W_2 . Then the adjoint mapping F^* carries functions on W_2 into functions on W_1 , and we may extend its definition so that it will carry differentials on W_2 into differentials on W_1 by noting that F^* is linear and requiring that

$$dF^*f = F^*df.$$

For further details the reader may consult the excellent treatment given in [11].

Consider a second order differential

$$\mu = cdxdy,$$

and define the new differential

$$|\mu| = |c|dxdy$$

called the absolute value of μ . Its definition is independent of the choice of uniformizer since all changes of uniformizer have a positive Jacobian. We say that μ is locally integrable if every point has a neighborhood U in which

$$\iint_U c dx dy$$

exists in the sense of Lebesgue.

If μ is locally integrable on W , we can define

$$\iint_W |\mu|$$

in the usual manner. We say that μ is integrable over W if

$$\iint_W |\mu| < \infty.$$

In this case it is possible to define

$$\iint_W \mu$$

and we have

$$\left| \iint_W \mu \right| \leq \iint_W |\mu|$$

as well as the usual Lebesgue convergence theorems.

A first order differential α is said to be locally square integrable if $\alpha * \bar{\alpha}$ is locally integrable. We shall call α square integrable (on W) if $\alpha * \bar{\alpha}$ is integrable there. The Schwarz inequality

$$\left\{ \iint |\alpha_1 \alpha_2| \right\}^2 \leq \iint \alpha_1 * \bar{\alpha}_1 \iint \alpha_2 * \bar{\alpha}_2$$

is easily established as well as the fact that $\alpha_1 \alpha_2$ is integrable whenever the right-hand side is finite.

We say that a property holds almost everywhere on a Riemann surface if it holds on a set E which has the property that in any region for which there is a uniformizer z valid z maps E into a set of measure zero in the z -plane. Henceforth, we shall always assume that first order differentials are defined almost everywhere and are locally square integrable. It is also assumed that second order differentials are defined almost everywhere and are locally integrable.

If J is a piecewise analytic arc, and if $\alpha = a dx + b dy$ is a first order differential which is defined and continuous in some region containing J , we define

$$\int_J \alpha = \int_{s_1}^{s_2} \left(a \frac{dx}{ds} + b \frac{dy}{ds} \right) ds,$$

where s is some suitable parameter on J .

4. Continuity and Green's theorem. We shall say that a function f is piecewise continuously differentiable or more briefly piecewise smooth on a Riemann surface W if f is continuous on W and there is a subdivision $\{\Omega_i\}$ of W such that $f \in C^1$ in each Ω_i and

$$\iint_{\Omega} df * d\bar{f} < \infty.$$

It is also useful to consider a type of generalized continuity for differentials which we call g -continuity for brevity: A first order differential $\alpha = adx + bdy$ is said to be g -continuous on a Riemann surface if there is a subdivision $\{\Omega_i\}$ such that

(i) At a corner p of the subdivision there is a uniformizer z such that

$$|a(z)|^2 + |b(z)|^2 = O(|z - z(p)|^{-1}).$$

(ii) If z is a uniformizer which is real on an edge of the subdivision, then $a(z)$ is continuous on the edge, except possibly at the corners.

(iii) In each Ω_i we have $\alpha \in C^1$, and

$$\iint_{\Omega_i} |d\alpha| < \infty.$$

For a bounded Riemann surface V we require that $\{\Omega_i\}$ be a subdivision of V .

It should be noted that the sum as well as the product of two piecewise smooth functions is again piecewise smooth, while the sum of two g -continuous differentials is again g -continuous as well as the product of a g -continuous differential by a piecewise smooth function. Some of the importance of this type of functions and differentials is illustrated by the following easily proved lemma:

LEMMA 3 (GREEN'S THEOREM). *Let Ω be a compact region with the boundary J , and let α be a first order g -continuous differential on Ω . Then*

$$\int_J \alpha = \iint_{\Omega} d\alpha$$

where the direction along J is chosen so that Ω is always to the left.

CHAPTER II. ORTHOGONAL PROJECTION AND THE SPACE Γ

5. Some basic properties. On an open Riemann surface W we define the space $\Gamma = \Gamma(W)$ to consist of those first order differentials α for which

$$\iint \alpha * \bar{\alpha} < \infty.$$

In Γ we define the norm $\|\alpha\| = \|\alpha\|_w$ of α by

$$\|\alpha\|^2 = \iint_w \alpha * \bar{\alpha}.$$

The norm is a non-negative real number which is zero only if α vanishes almost everywhere. It has the following properties:

$$\|\alpha\| = \|\bar{\alpha}\| = \|\ast \alpha\| = \|f\alpha\|$$

provided f is a function with $|f| = 1$ almost everywhere. If we set

$$(\alpha, \beta) = \iint_w \alpha * \bar{\beta},$$

we have

$$\begin{aligned} \|\alpha\|^2 &= (\alpha, \alpha), & (\alpha, \beta) &= (\beta, \alpha)^-, \\ (\ast \alpha, \beta) &= -(\alpha, \ast \beta), & (\lambda_1 \alpha_1 + \lambda_2 \alpha_2, \beta) &= \lambda_1 (\alpha_1, \beta) + \lambda_2 (\alpha_2, \beta) \end{aligned}$$

where λ_1 and λ_2 are complex constants. Thus with (α, β) as a scalar product Γ becomes an incomplete Hilbert space if we identify differentials which are equal almost everywhere. We have at once the Schwarz inequality

$$|(\alpha, \beta)| \leq \|\alpha\| \cdot \|\beta\|$$

and the Minkowski inequality

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|.$$

It follows from the Riesz-Fischer theorem that Γ is complete in this norm and hence is actually a Hilbert space. We say that a sequence $\alpha_i \in \Gamma$ converges, or converges strongly, if

$$\|\alpha_m - \alpha_n\| \rightarrow 0,$$

as $m, n \rightarrow \infty$. The completeness of Γ is then just the assertion that for such a sequence there is an $\alpha \in \Gamma$ such that

$$\|\alpha_m - \alpha\| \rightarrow 0,$$

as $m \rightarrow \infty$. In this case we say that α_i converges to α , and write

$$\alpha_i \rightarrow \alpha.$$

If for every $\beta \in \Gamma$

$$(\alpha_i, \beta) \rightarrow (\alpha, \beta),$$

then we say that α_i converges to β weakly, and write

$$\alpha_i \rightarrow \beta \quad \text{weakly.}$$

Clearly strong convergence implies weak convergence, but not vice versa. A set of elements of Γ is said to be strongly closed if it contains all of its strong limit points, and weakly closed if it contains all of its weak limit points. Thus weak closure implies strong closure.

The following lemmas are well known theorems in the theory of Hilbert space [33].

LEMMA 4. *A linear subspace $A \subset \Gamma$ is weakly closed if and only if it is strongly closed.*

LEMMA 5. *If $\alpha_i \rightarrow \alpha$ weakly, then $\|\alpha\| \leq \liminf \|\alpha_i\|$.*

LEMMA 6. *If $\|\alpha_n\| \leq 1$, then there is a subsequence α_{n_i} which converges weakly; i.e., the unit sphere is compact in the weak topology of Γ .*

LEMMA 7. *If α_i converges, and if $\alpha_i \rightarrow \alpha$ weakly, then $\alpha_i \rightarrow \alpha$ strongly.*

We shall also need the following lemma.

LEMMA 8. *Let α_i be a sequence of differentials in Γ with $\|\alpha_i\|$ uniformly bounded, and suppose that there is a locally square integrable differential α such that for any compact region $\Omega \subset W$ and any $\beta \in \Gamma$ we have*

$$\iint_{\Omega} \alpha_i * \bar{\beta} \rightarrow \iint_{\Omega} \alpha * \bar{\beta}.$$

Then $\alpha \in \Gamma$, and $\alpha_i \rightarrow \alpha$ weakly.

Proof. Let α_{i_k} be an arbitrary subsequence of α_i . By Lemma 6 there is a differential $\alpha_0 \in \Gamma$ and a subsequence $\alpha_{i_{k_j}}$ which converges to α_0 weakly. Let α_{Ω} be equal to $\alpha_0 - \alpha$ on Ω and vanish outside Ω . Thus $\alpha_{\Omega} \in \Gamma$, and

$$\iint \alpha * \bar{\alpha}_{\Omega} = \lim \iint_{\Omega} \alpha_{i_{k_j}} * \bar{\alpha}_{\Omega} = \iint \alpha_0 * \bar{\alpha}_{\Omega}.$$

Therefore

$$\|\alpha - \alpha_0\|_{\Omega}^2 = \iint (\alpha - \alpha_0) * \bar{\alpha}_{\Omega} = 0,$$

and so

$$\alpha = \alpha_0$$

almost everywhere on Ω , and hence on W . Thus $\alpha \in \Gamma$ and $\alpha_{i_{k_j}} \rightarrow \alpha$ weakly. Consequently, $\alpha_i \rightarrow \alpha$ weakly since every subsequence contains a subsequence converging weakly to α .

We introduce the following notation: If A is a linear subspace of Γ , then \bar{A} is the smallest closed linear subspace containing A (strong and weak closures

being the same for linear subspaces by Lemma 4). By A^* is meant the linear subspace consisting of all α such that $*\alpha \in A$. If A and B are two linear subspaces, by $A+B$ we mean the linear subspace consisting of all differentials of the form $\alpha+\beta$, with $\alpha \in A$, and $\beta \in B$.

We say that two differentials α and β of Γ are orthogonal (in symbols $\alpha \perp \beta$) if $(\alpha, \beta) = 0$. We say that a differential β is orthogonal to a linear subspace A (written $\beta \perp A$) if $\beta \perp \alpha$ for every $\alpha \in A$. Two linear subspaces A and B are orthogonal ($A \perp B$) if $\alpha \perp B$ for every $\alpha \in A$. Clearly if $A \perp B$, then $\overline{A} \perp \overline{B}$. We write $A \oplus B$ to mean $A+B$ and $A \perp B$. The orthogonal complement A^\perp of a linear subspace A is the set of all α such that $\alpha \perp A$. Certainly A^\perp is always closed. An important lemma concerning orthogonal complements is the following:

LEMMA 9. *If A is a closed linear subspace of Γ , then $\Gamma = A \oplus A^\perp$. Consequently, $A^{\perp\perp} = A$.*

6. **Weak differentials.** We use K to denote the class of all piecewise smooth functions with a compact carrier. Then to each *locally* square integrable α we associate a linear functional

$$d\alpha[\phi] = - \iint d\phi\alpha$$

on functions ϕ of class K . On a bounded Riemann surface we define $d\alpha$ only on those functions of class K which also vanish on the boundary of the Riemann surface. This linear functional is called the weak differential of α . If there is a real number M such that

$$|d\alpha[\phi]| \leq M$$

for all $\phi \in K$ with $|\phi| \leq 1$, then α is called *regular*. Again an application of Green's theorem gives us the following:

LEMMA 10. *If α is g -continuous,*

$$d\alpha[\phi] = \iint \phi d\alpha,$$

and α is regular if and only if

$$\iint |d\alpha| < \infty.$$

LEMMA 11. *Let ϕ_i be a uniformly bounded sequence of functions of class K which converges to zero uniformly on each compact region $\Omega \subset W$. Then if α is regular,*

$$d\alpha[\phi_i] \rightarrow 0.$$

Proof. Without loss of generality we may assume $|\phi_i| \leq 1$. Let $M = \sup d\alpha[\phi]$ for all $\phi \in K$ with $|\phi| < 1$, and given $\epsilon > 0$, let ϕ_0 be a function of class K with $|\phi_0| < 1$, and $d\alpha[\phi_0] > M - \epsilon$. Let n be chosen so large that $|\phi_i| < \epsilon$ on the carrier of ϕ_0 for all $i \geq n$. Then

$$|\phi_i + (1 - \epsilon)\phi_0| \leq 1,$$

and hence

$$\begin{aligned} M &\geq d\alpha[\phi_i + (1 - \epsilon)\phi_0] \\ &= d\alpha[\phi_i] + (1 - \epsilon)d\alpha[\phi_0] \\ &\geq d\alpha[\phi_i] + M - (M + 1)\epsilon \end{aligned}$$

or

$$d\alpha[\phi_i] \leq (M + 1)\epsilon.$$

Similarly

$$d\alpha[-\phi_i] \leq (M + 1)\epsilon,$$

whence

$$|d\alpha[\phi_i]| \leq (M + 1)\epsilon,$$

and the lemma is proved.

7. Harmonic differentials and the lemma of Weyl. A first order differential ω is called harmonic if every point has a neighborhood in which ω is the differential of a harmonic function. Thus harmonic differentials possess derivatives of all orders. The class of square integrable harmonic differentials will be denoted by Γ_H .

It is readily verified that a necessary and sufficient condition that a first order differential ω be harmonic is that $\omega \in C^1$ and

$$d\omega = d*\omega = 0.$$

As a consequence $*\omega$ is harmonic whenever ω is. Thus in particular $\Gamma_H^* = \Gamma_H$.

If $\phi \in K$, and ω is harmonic, then Green's theorem easily gives

$$\iint d\phi * \bar{\omega} = 0 \quad \text{and} \quad \iint * d\phi * \bar{\omega} = 0.$$

As a converse we have the following lemma:

LEMMA 12 (WEYL). *Let ω be a locally square integrable differential such that*

$$\iint d\phi * \bar{\omega} = 0 \quad \text{and} \quad \iint * d\phi * \bar{\omega} = 0$$

for every $\phi \in K$. Then ω is a harmonic differential.

Since the property of being harmonic is purely a local one, it suffices to prove the lemma for a plane region, and the reader is referred to Weyl [38]. It is worth noting that the lemma says that if the weak differentials of ω and $*\omega$ vanish, then ω is harmonic.

8. Closed differentials and periods. A first order differential α is called *closed* if its weak differential vanishes.

LEMMA 13. *Let α be a closed g.-continuous differential. Then each point has a neighborhood U in which there is a function f defined so that $\alpha = df$ in U .*

Proof. Let U be a circle $|z - z_0| < r$. If J_1 and J_2 are two piecewise analytic arcs in U with the same beginning and end points, then J_1 and $-J_2$ bound a region $\Omega \subset U$ and

$$\int_{J_1} \alpha - \int_{J_2} \alpha = \iint_{\Omega} d\alpha = 0$$

by Green's theorem. Thus if we define

$$f(p) = \int_J \alpha$$

where J is a piecewise analytic arc beginning at p_0 and ending at p , we have a definition independent of the arc J chosen, and it is easily verified that f is piecewise smooth and that

$$\alpha = df.$$

We say that a piecewise analytic closed curve J is homologous to zero, or that it bounds, if there is a compact region whose boundary is J . We call two closed piecewise analytic curves J_1 and J_2 homologous if they form the boundary of a compact region Ω such that Ω lies to the right as we traverse J_1 and to the left as we traverse J_2 or vice versa. From Green's theorem we have the following lemma:

LEMMA 14. *If J_1 and J_2 are homologous piecewise analytic closed curves and α a closed g.-continuous differential, then*

$$\int_{J_1} \alpha = \int_{J_2} \alpha.$$

Let h be a homology class (class of homologous, piecewise analytic closed curves) and α a closed g.-continuous differential. Then we define the period of α with respect to h as

$$P_h(\alpha) = \int_J \alpha$$

where $J \in h$. By Lemma 14 the definition is independent of the choice of J .

If we wish to extend this concept to closed differentials which are not g -continuous, we must try to make a definition of $P_h(\alpha)$ which does not depend on line integration but rather on area integration. To do this we resort to the following artifice: Suppose J is a piecewise analytic closed curve belonging to the homology class h . Then we define a differential η_h which has a compact carrier and derivatives of all orders as follows: Let U be a compact region containing J which J separates into two parts U_1 and U_2 . The existence of such a region follows easily from the definition of a piecewise analytic closed curve. In U_1 we define a function $\phi \in C^\infty$ which is identically one in a neighborhood of J and which vanishes identically in a neighborhood of the remainder of the boundary of U_1 . Then $\eta_h = - * d\phi$ is a differential, whose dual is closed, of class C^∞ whose carrier is contained in U_1 , and hence is compact. If α is a closed g -continuous differential, we have

$$\begin{aligned} \iint \alpha * \eta_h &= \iint_{U_1} \alpha * \eta_h \\ &= \iint_{U_1} \alpha d\phi \\ &= \int_J \alpha \\ &= P_h(\alpha) \end{aligned}$$

by Green's theorem, since $\phi = 1$ on J and $\phi = 0$ on the rest of the boundary of U_1 . Thus we may define

$$P_h(\alpha) = \iint \alpha * \eta_h$$

for all closed differentials α which are locally square integrable, and this corresponds to our earlier definition if α is g -continuous. It is not apparent whether or not this definition depends on the choice of the functions η_h if α is an arbitrary locally square integrable closed differential. However, it will be shown in the next section that if α is a closed differential belonging to the space Γ , then $P_h(\alpha)$ is actually independent of the choice of η_h and depends only on the homology class h .

The following lemma is readily proved:

LEMMA 15. *A closed g -continuous differential α is the differential of a func-*

tion if and only if

$$P_h(\alpha) = \int \int \alpha * \eta_h \doteq 0$$

for each homology class h .

9. Orthogonal projection. As in §7 we denote by K the class of piecewise smooth functions whose carriers are compact, and we use Γ_K to denote the subspace of Γ consisting of the differentials of functions of class K . If $f \in K$, and $\omega \in \Gamma_H$, then by Green's theorem we have, since $d * \omega = 0$,

$$(df, \omega) = \int \int_{\Omega} df * \bar{\omega} = 0$$

where Ω is a compact region containing the carrier of f . Since $*\omega \in \Gamma_H$, we also have

$$(*df, \omega) = - (df, *\omega) = 0.$$

Thus Γ_H is orthogonal to both Γ_K and Γ_K^* . Now if f_1 and f_2 belong to K , then

$$\begin{aligned} (df_1, *df_2) &= \int \int_{\Omega} df_1 * * \bar{df}_2 \\ &= - \int \int_{\Omega} df_1 \bar{df}_2 = 0 \end{aligned}$$

by Green's theorem, where Ω is a compact region containing the carriers of f_1 and f_2 . Thus Γ_K and Γ_K^* are orthogonal, and consequently Γ_H , Γ_K , and Γ_K^* are mutually orthogonal linear subspaces. However, Lemma 12 asserts that if $\omega \perp \Gamma_K$ and $\omega \perp \Gamma_K^*$, then $\omega \in \Gamma_H$. Thus

$$(\Gamma_K + \Gamma_K^*)^{\perp} \subseteq \Gamma_H \subseteq (\Gamma_K + \Gamma_K^*)^{\perp},$$

whence $\Gamma_H = (\Gamma_K + \Gamma_K^*)^{\perp}$. Thus in view of Lemma 9 we have established the following proposition:

PROPOSITION 1. *The linear subspace Γ_H is closed, and Γ has the orthogonal decomposition*

$$\Gamma = \Gamma_H \oplus \bar{\Gamma}_K \oplus \Gamma_K^*.$$

When we note that the weak differential of a differential in $\bar{\Gamma}_K$ vanishes, this proposition has the following form in terms of differentials.

PROPOSITION 2. *If $\alpha \in \Gamma$, then*

$$\alpha = \omega + \alpha_1 + *\alpha_2$$

where $\omega \in \Gamma_H$, $\alpha_1, \alpha_2 \in \overline{\Gamma}_K$. Moreover, the weak differential of α is that of $*\alpha_2$ and the weak differential of $*\alpha$ that of $*\alpha_1$.

PROPOSITION 3. *The space Γ_{cl} of closed square integrable differentials has the orthogonal decomposition*

$$\Gamma_{cl} = \Gamma_H \oplus \overline{\Gamma}_K.$$

Hence Γ_{cl} is a closed linear subspace and

$$\Gamma_{cl} = (\Gamma_K^*)^\perp = (\overline{\Gamma}_K^*)^\perp.$$

Proof. If $\alpha \in \Gamma_H \oplus \overline{\Gamma}_K$, α is certainly closed. On the other hand if α is closed, we decompose α by Proposition 2 into

$$\alpha = \omega + \alpha_1 + *\alpha_2$$

whence $*\alpha_2 = \alpha - \omega - \alpha_1$ is closed. But $\alpha_2 \in \overline{\Gamma}_K$, and hence must also be closed. Thus by Lemma 12, $\alpha_2 \in \Gamma_H$. Since Γ_H and $\overline{\Gamma}_K$ are orthogonal, $\alpha_2 = 0$, proving the proposition.

PROPOSITION 4. *If $\alpha \in \Gamma$ is closed, then*

$$\alpha = \omega + \alpha_1$$

where $\omega \in \Gamma_H$ and $\alpha_1 \in \overline{\Gamma}_K$. The weak differential of $*\alpha$ is that of $*\alpha_1$, while the periods of ω are those of α . Consequently, the periods $P_h(\alpha)$ depend only on the homology class h and not on the function η_h .

Proof. It is necessary only to show that α_1 has no periods. But if α_1 is in Γ_K , then $\alpha_1 = df$ with f piecewise smooth, and

$$\begin{aligned} P_h(\alpha_1) &= (\alpha_1, \eta_h) \\ &= \iint \alpha_1 * \bar{\eta}_h \\ &= \int_J \alpha_1 \\ &= \int_J df = 0, \end{aligned}$$

where J is that piecewise analytic closed curve of class h used in defining η_h . If α_1 belongs to $\overline{\Gamma}_K$ but not to Γ_K , then there is a sequence $\alpha^r \rightarrow \alpha_1$, $\alpha^r \in \Gamma_K$. But

$$(\alpha^r, \eta_h) = 0,$$

and hence

$$(\alpha_1, \eta_h) = 0$$

by the continuity of the scalar product.

We define the space D to be the space of all piecewise smooth functions for which $df \in \Gamma$, and write

$$D(f, g) = (df, dg)$$

and

$$D(f) = (df, df).$$

We have the following decomposition theorem for the space D :

PROPOSITION 5. *If $f \in D$, then*

$$f = u + f_0$$

where u is a harmonic function and $df_0 \in \bar{\Gamma}_K$. Moreover, du and df_0 are orthogonal.

Proof. Since $df \in \Gamma_{cl}$, we have by Proposition 3 that $df = \omega + \alpha_1$ where $\omega \in \Gamma_H$, and $\alpha_1 \in \bar{\Gamma}_K$. Furthermore, ω has no period since df has none. Thus by Lemma 15 and the fact that $\omega \in C^\infty$, $\omega = du$, where u is a harmonic function. If we set $f_0 = f - u$, then $df_0 = \alpha_1 \in \bar{\Gamma}_K$ and the theorem is proved.

10. **The Dirichlet principle.** One of the more powerful tools at the disposal of the analyst is the Dirichlet principle. Formulated first by Lord Kelvin in 1847, it was used extensively by both Dirichlet and Riemann. We shall use it in the following form:

THEOREM 1. *Let Ω be a compact region with boundary R on a Riemann surface, and let g be a piecewise smooth function on $\bar{\Omega}$. Then there is a unique piecewise smooth function u on $\bar{\Omega}$ which is harmonic in Ω and such that $g - u$ vanishes identically on R . Moreover, $D_\Omega(u) \leq D_\Omega(g)$.*

Proof. Considering Ω as a Riemann surface in its own right, we apply Proposition 5 to the function g to get

$$g = v + g_0$$

where v is harmonic and $dg_0 \in \Gamma_K(\Omega)$, i.e., there is a sequence ϕ_i of piecewise smooth functions whose carriers are contained in Ω such that $D(g_0 - \phi_i) \rightarrow 0$. Since dv and dg_0 are orthogonal in $\Gamma(\Omega)$, $D(g) = D(v) + D(g_0)$. Hence to prove the theorem it suffices to show that $g - v$ approaches a constant as we approach R . The proof of this fact is somewhat laborious, and we refer the reader to [10].

This theorem shows the utility of piecewise smooth functions by the fact that if we have a piecewise smooth function f defined on the Riemann surface W and a compact region $\Omega \subset W$, then we can define a new function f_1 which is again piecewise smooth and which is harmonic in Ω and identically equal to f outside Ω . Moreover $D(f_1) \leq D(f)$.

By doubling Ω with respect to R_1 , we establish the following generalization of Theorem 1:

THEOREM 1. *Let Ω be a compact region whose boundary consists of the two parts R_1 and R_2 , and let g be a piecewise smooth function on $\bar{\Omega}$. Then there is a piecewise smooth function u harmonic in Ω which has a vanishing normal derivative on R_1 , while $g - u$ vanishes on R_2 . Moreover $D(u) \leq D(g)$.*

If Ω is a compact region with boundary R , we call $g(p, q)$ its Green's function if it is harmonic (as a function of p) in $\Omega - q$, approaches zero as $p \rightarrow R$, and has a logarithmic pole at q , i.e., for a uniformizer z at q we have $g(p, q) + \log |z(p) - z(q)|$ a harmonic function near q . We have the following well known proposition:

PROPOSITION 6. *Given a compact region Ω and a point $q \in \Omega$, there exists a Green's function $g(p, q)$. Moreover, we have*

$$g(p, q) = g(q, p).$$

11. The capacity of a ring domain. If Ω is a compact region whose boundary consists of two disjoint nonempty compact sets R_1 and R_2 , we call the triad consisting of Ω , R_1 , and R_2 a ring domain. For example, a region in the plane bounded by a finite number of analytic curves becomes a ring domain when the boundary curves are divided into two nonempty classes. The reader should have little difficulty in constructing a piecewise smooth function g on $\bar{\Omega}$ which has the value one on R_1 and vanishes on R_2 . As a consequence of the Dirichlet principle (Theorem 1) there is a harmonic function u with a finite Dirichlet integral which is one on R_1 and zero on R_2 . This function is called the harmonic measure of R_1 with respect to Ω and written $u = u(p, R_1, \Omega)$.

We call

$$C = D(u) = \iint_{\Omega} du * du$$

the capacity of the ring domain. Among the homology classes of piecewise analytic closed curves in Ω , let h_0 denote that one to which R_1 belongs when traversed in such a direction that Ω lies to the left. Then it is easily verified that $P_{h_0}(u) = C$.

PROPOSITION 7. *If $\alpha \in \Gamma(\Omega)$, and if⁽²⁾*

$$\int_J \alpha \geq 1$$

for every piecewise analytic curve J beginning on R_2 and ending on R_1 , then

$$\iint_{\Omega} \alpha * \alpha \geq \iint_{\Omega} du * du = C$$

⁽²⁾ We thus implicitly impose upon α the condition that for all piecewise analytic arcs beginning on R_2 and ending on R_1 the integral must either exist or diverge to $+\infty$.

with equality only if $\alpha = du$ almost everywhere.

PROPOSITION 7'. If $\beta \in \Gamma(\Omega)$, and if

$$\int_J \beta \geq 1$$

for all $J \in h_0$, then

$$\iint \beta * \beta \geq \iint \frac{du}{C} * \frac{du}{C} = \frac{1}{C}$$

with equality only if $\beta = *du/C$ almost everywhere.

These propositions give upper and lower bounds for the capacity of a ring domain and are known in electricity (in a somewhat less general form) as Dirichlet's and Thomson's principles respectively, although both are due to W. Thomson, Lord Kelvin. Since the proofs of the two propositions are nearly identical, we prove the former and leave the latter to the reader. The proof given below is modeled after a proof given by Ahlfors and Beurling in the theory of extremal distances. The central idea of the proof, however, stems from the length-area principle of Grötzsch.

First I maintain that u can have only a finite number of critical points, i.e., points where $du = 0$. For if there were an infinite number, they must have a limit point, $\bar{\Omega}$ being compact. If this limit point were an interior point of Ω , we would have a neighborhood of it in which u is a harmonic function of the uniformizer variable which has a limit point of critical points as a point of harmonicity, when u would be identically constant, which is impossible. On the boundary of Ω , we note that u is a harmonic function of the boundary uniformizer by the Schwarz reflection principle, and the same argument applies.

Let v be the conjugate function of u , i.e., the function such that $dv = *du$. Then v is possibly a many-valued function, but the curves where $v = \text{constant}$ are well determined. Let Ω' be a subregion of Ω bounded by curves J_0 on which v is constant and such that Ω' contains no critical points of u . Clearly

$$\iint_{\Omega'} du * du \rightarrow C,$$

and

$$\iint_{\Omega'} \alpha * \alpha \rightarrow \iint_{\Omega} \alpha * \alpha$$

as $\Omega' \rightarrow \Omega$. But in Ω' we may, in each sufficiently small neighborhood, use $u + iv$ as a uniformizer (u and v are seen to be real from their definition). Thus if $\alpha = adu + bdv$, we have

$$1 \leq \int_0^1 a du,$$

since the curves J where $v = \text{constant}$ are analytic curves beginning on R_2 and ending on R_1 . By the Schwarz inequality

$$1 \leq \int_0^1 |a|^2 du,$$

and if J' denotes that portion of a curve $u = \text{constant}$ which lies in Ω' ,

$$\begin{aligned} \iint d u d v &= \int_{J'} d v \\ &\leq \iint_{\Omega'} |a|^2 d u d v \\ &\leq \iint_{\Omega'} (|a|^2 + |b|^2) d u d v \\ &= \iint_{\Omega'} \alpha * \alpha \end{aligned}$$

which, if we let Ω' tend to Ω , proves the theorem.

CHAPTER III. THE SPACE BD ON AN OPEN RIEMANN SURFACE

12. **The space B .** In §9 we obtained some results concerning the class D of all piecewise smooth functions whose differentials belong to Γ . However the space D with the topology defined by

$$f_i \rightarrow f \quad \text{in } D$$

whenever

$$d f_i \rightarrow d f \quad \text{in } \Gamma$$

has two disadvantages: A sequence in which $f_i = f$ converges to $f + c$ as well as to f , and this topology gives no information concerning the upper bound of the absolute value of a function, a quantity most naturally associated with functions on an arbitrary topological space. This leads us to consider the space $B = B(W)$ consisting of all piecewise smooth functions on the Riemann surface W which are bounded there. When it comes to defining a topology in B , the first idea to come to mind is probably to make B a metric space by defining a norm

$$\|f\| = \sup |f|,$$

but this has the disadvantage that we cannot approach all functions in B

by means of sequences of functions which vanish or are constant outside a compact region, these being the functions on W about which we have the most complete information. In order to overcome these disadvantages we shall make B into a convergence space by defining

$$f_i \rightarrow f \quad \text{in } B$$

whenever $|f_i|$ is uniformly bounded and $f_i \rightarrow f$ uniformly on every compact region.

The reader should be warned that if we define the closure of a set of elements in B as the the adjunction of the limit elements, then the closure of a closed set need not be closed, as simple examples will show. Actually it is customary to define the closure of a set as the smallest closed set containing the given set, a set being closed if it contains all of its limit elements. In this case, however, it must be remembered that not all frontier elements of a set are accessible by sequences of elements belonging to the set. That is, equivalently, neither countability axiom is satisfied.

In §6 we introduced the weak differential of a locally square integrable differential as a linear functional on the space K defined by

$$d\alpha[\phi] = - \iint d\phi\alpha$$

and saw (Lemma 11) that if $\phi_i \in K$ and

$$\phi_i \rightarrow 0 \quad \text{in } B,$$

then

$$d\alpha[\phi_i] \rightarrow 0$$

for each regular α .

LEMMA 16. *The weak differential of a locally square integrable differential α admits of a unique extension to the space B which is continuous in the sense that*

$$d\alpha[f_i] \rightarrow d\alpha[f]$$

whenever

$$f_i \rightarrow f \quad \text{in } B.$$

Proof. Let $f \in B$. Then there is easily constructed a sequence $\phi_i \in K$ such that

$$\phi_i \rightarrow f \quad \text{in } B.$$

Now

$$\phi_i - \phi_j \rightarrow 0 \quad \text{in } B$$

as $i, j \rightarrow \infty$. By Lemma 11

$$d\alpha[\phi_i - \phi_j] \rightarrow 0,$$

and hence $d\alpha[\phi_i]$ converges to a number which we take as the definition of $d\alpha[f]$. This definition is independent of the sequence chosen since the difference of any two such sequences must tend to zero in B and hence lead to the same definition of $d\alpha[f]$.

Now it is clear from the definition that

$$\max_{|\phi| \leq 1} |d\alpha[f]| = \max_{|\phi| \leq 1} |d\alpha[\phi]| \quad \phi \in K.$$

Thus the same proof as was used for Lemma 11 shows that if

$$f_i \rightarrow 0 \quad \text{in } B,$$

then

$$d\alpha[f_i] \rightarrow 0,$$

from which the required continuity of $d\alpha$ follows, proving the lemma.

We shall say that

$$f_i \rightarrow f \quad \text{weakly in } B$$

if for every regular α

$$d\alpha[f_i] \rightarrow d\alpha[f].$$

Clearly convergence in B implies weak convergence in B . It should be noted, however, that our definition of weak convergence differs from the standard definition of weak convergence in a linear vector space which requires

$$L[f_i] \rightarrow L[f]$$

for all continuous linear functionals L . However, the definition which we have given is the one that proves to be most useful for our purposes.

We note in passing that both weak topologies are actually the same. Indeed, it can be shown, although with considerable effort, that the convergence of f_i to f in either weak topology is equivalent to having $|f_i|$ uniformly bounded and f_i converging to f everywhere. However, the use of the definition of weak convergence which we have given absolves us of the need of proving this fact here.

13. **The space BD .** We shall obtain more fruitful results, however, when we consider the space BD consisting of all bounded, piecewise continuously differentiable functions f for which df belongs to Γ , i.e., consisting of these functions which belong to both B and D . We introduce a topology in the space BD by defining

$$f_i \rightarrow \quad \text{in } BD$$

whenever

$$f_i \rightarrow f \quad \text{in } B$$

and

$$df_i \rightarrow df \quad \text{in } \Gamma.$$

We shall also introduce a weak topology by saying that

$$f_i \rightarrow f \quad \text{weakly in } BD$$

whenever

$$f_i \rightarrow f \quad \text{weakly in } B$$

and

$$df_i \rightarrow df \quad \text{weakly in } \Gamma.$$

As in §6, we denote by K the subclass of BD consisting of those functions with a compact carrier. We say that a function f of class BD is of class \bar{K} if there is a sequence of functions ϕ_i in K such that

$$\phi_i \rightarrow f \quad \text{in } BD.$$

As was mentioned before, there is a priori no reason to suppose that \bar{K} is closed. In the next section, however, it will be shown that \bar{K} is actually closed in the weak topology of BD and hence also in the strong topology. Clearly $\Gamma_{\bar{K}} \subset \Gamma_K$. The following proposition shows that functions of the class \bar{K} behave in some respects as though they "had zero boundary values":

PROPOSITION 8. *If $\alpha \in \Gamma$ is a regular differential, and if $f \in \bar{K}$, then*

$$d\alpha[f] = - \int \int df\alpha.$$

Proof. There is a sequence $\phi_i \in K$ such that $\phi_i \rightarrow f$ in BD . Now for $\phi_i \in K$, we have by definition

$$d\alpha[\phi_i] = - \int \int d\phi_i\alpha.$$

But

$$d\alpha[\phi_i] \rightarrow d\alpha[f]$$

since $\phi_i \rightarrow f$ in B , and

$$\int \int d\phi_i\alpha \rightarrow \int \int df\alpha$$

since $d\phi_i \rightarrow df$ in Γ .

As a direct consequence of Lemmas 5, 7, and 8, we have the following condition for convergence:

LEMMA 17. Let $\{f_i\}$ be a uniformly bounded sequence of functions of the space BD which converge uniformly together with their derivatives on every compact region to a function f . If $D(f_i)$ is uniformly bounded, then $f \in BD$, and

$$f_i \rightarrow f \quad \text{weakly in } BD.$$

Moreover,

$$D(f) \leq \liminf D(f_i).$$

If, furthermore,

$$D(f_i - f_j) \rightarrow 0$$

as $i, j \rightarrow \infty$, then

$$f_i \rightarrow f \quad \text{strongly in } BD.$$

If we use HBD to denote the space of harmonic functions which belong to the space BD , we have the following decomposition theorem:

THEOREM 2. If $f \in BD$, then

$$f = f_K + u,$$

where $f_K \in \overline{K}$ and $u \in HBD$. Moreover

$$\|u\| \leq \|f\|.$$

Proof. Choose an exhaustion $\{\Omega_i\}$ of W , and set $f = \phi_i + u_i$, where u_i is that piecewise smooth function which is identically equal to f outside Ω_i and harmonic inside Ω_i . The existence of u_i is guaranteed by Theorem 1. Since on every compact region u_i is eventually a uniformly bounded (by $\|f\|$) sequence of harmonic functions, there is by Lemmas 1 and 2 a subsequence (which we shall again call u_i) which converges in B to a harmonic function u , and for which du_i converges to du uniformly on every compact set.

By the Dirichlet principle $D(u_j) \leq D(u_i)$ for $i < j$, whence $D(u_i)$ converges. Since $u_i - u_j$ vanishes outside Ω_j while u_j is harmonic inside Ω_j , we have by Green's theorem that

$$0 = D(u_j, u_i - u_j) = D(u_i, u_j) - D(u_j).$$

Hence

$$\begin{aligned} D(u_i - u_j) &= D(u_i) - 2D(u_i, u_j) + D(u_j) \\ &= D(u_i) - D(u_j). \end{aligned}$$

Thus du_i converges in Γ , and by Lemma 17

$$\phi_i \rightarrow f_K = f - u$$

in BD .

Consequently, $f_K \in \overline{K}$, since $\phi_i \in K$.

COROLLARY 1. *If $f \in BD$ and $df \in \overline{\Gamma}_K$, then there is a constant c such that $f - c \in \overline{K}$.*

Proof. In the decomposition $f = f_K + u$ we must have $du = 0$, since Γ_H and Γ_K are both orthogonal while df and df_K both belong to $\overline{\Gamma}_K$.

COROLLARY 2. *We have*

$$\Gamma_{\overline{K}} = \overline{\Gamma}_K.$$

14. Harmonic measures and the type of a Riemann surface. We shall say that an open Riemann surface W is of parabolic type if $1 \in \overline{K}$. Otherwise we say that W is of hyperbolic type. In the next section it will be shown that this definition is equivalent to the usual one in terms of the existence of Green's functions. It should be noted that if $1 \in \overline{K}$, then by Proposition 8

$$d\alpha[1] = - \iint d1\alpha = 0$$

for every regular differential $\alpha \in \Gamma$. In order to establish the converse of this we first establish some theorems concerning harmonic measure.

Let C be a set consisting of a finite number of piecewise analytic closed curves, and let W_1 be an open subset of W which is closed in $W - C$, i.e., which is bounded by a part of C . Let $\{\Omega_i\}$ be a sequence of compact regions such that $\overline{\Omega}_i \subset \Omega_{i+1}$, $C \cap \overline{W}_1 \subset \overline{\Omega}_1$, and $\bigcup \Omega_i = W_1$. The existence of such a sequence is a consequence of the compactness of C . In each Ω_i we construct that function u_i which is piecewise smooth on W_1 , harmonic in Ω_i , has the boundary value one on C , and vanishes identically in $W_1 - \Omega_i$. The existence of u_i is again a simple consequence of the compactness of C . By the maximum principle

$$0 \leq u_i \leq u_j \leq 1$$

in Ω_i provided that $i < j$. By Harnack's theorem (Lemmas 1 and 2) the functions u_i converge in B to a harmonic function u , while du_i converges to du uniformly on every compact set of W_1 . By the Dirichlet principle (Theorem 1) we have

$$D(u_i) \geq D(u_j)$$

for $i < j$. Hence $D(u_i)$ converges. But if $i < j$, we have by Green's theorem

$$\begin{aligned} D(u_i - u_j) &= D(u_i) - D(u_j) + 2D(u_j, u_i - u_j) \\ &= D(u_i) - D(u_j) \end{aligned}$$

whence du_i converges in $\Gamma(W_1)$.

The function u which we have constructed is called the harmonic measure of C with respect to W_1 and is denoted by the expression $u(p, C, W_1)$. That this harmonic measure does not depend on the particular exhaustion of W_1 which is chosen is a consequence of the fact that the harmonic measure is characterized by being the smallest non-negative piecewise smooth function on $\overline{W_1}$ which is harmonic in W_1 and identically equal to one on C .

An important special case is that in which C is the boundary of a compact region Ω , and $W_1 = W - \overline{\Omega}$. In this case u is simply called the harmonic measure of Ω , and written $u(p, \Omega)$. In the remainder of this chapter we restrict ourselves to harmonic measures of compact regions. If we define u and its approximating functions u_i to be identically one on the compact region Ω , then u and u_i are piecewise smooth on W and

$$u_i \rightarrow u \qquad \text{in } BD.$$

But the u_i are functions of the class K . Hence we have proved the following proposition:

PROPOSITION 9. *The harmonic measure $u = u(p, \Omega)$ of a compact region belongs to the class \overline{K} .*

THEOREM 3. *If some compact region $\Omega \subset W$ has a constant harmonic measure, then W is parabolic. If W is of parabolic type, then the harmonic measure of each compact region $\Omega \subset W$ is identically equal to one.*

Proof. If $u(p, \Omega)$ is constant, it must be identically one since $u = 1$ on C . Thus $1 = u(p, \Omega)$ belongs to \overline{K} by Proposition 9.

Suppose on the other hand that $1 \in \overline{K}$. Then there is a sequence $\phi_i \in K$ such that

$$\phi_i \rightarrow 1 \qquad \text{in } BD.$$

Let Ω_i be a compact region containing a given compact region Ω and the carriers of all ϕ_j with $j \leq i$. Since $\phi_i \rightarrow 1$ uniformly on Ω , we may insure that $\phi_i > 1 - \epsilon$ on Ω simply by omitting a finite number of terms from the sequence $\{\phi_i\}$. Now $\Omega_i - \Omega$ becomes a ring domain if we take R_1 to be the boundary of Ω and R_2 to be the boundary of Ω_i . On R_1 we have $\phi_i \geq 1 - \epsilon$, while ϕ_i vanishes on R_2 . Hence by Proposition 7

$$D(\phi_i) \geq (1 - \epsilon)^2 D(u_i),$$

where we have set u_i equal to the harmonic measure of R_1 with respect to $\Omega_i - \Omega$. Hence $D(u_i) \rightarrow 0$, since $D(\phi_i) \rightarrow 0$. But

$$u_i \rightarrow u(p, \Gamma) \qquad \text{in } BD,$$

whence $D(u) = 0$. Therefore u is identically constant, and thus $u = 1$ iden-

tically, proving the theorem.

THEOREM 4. *A necessary and sufficient condition that a Riemann surface W be hyperbolic is the existence of a regular differential $\alpha \in \Gamma$ such that*

$$d\alpha[1] \neq 0.$$

Proof. As we noted before $da[1]=0$ for every regular differential $\alpha \in \Gamma$ on a parabolic Riemann surface. Thus it remains only to prove that on a hyperbolic Riemann surface there is a regular differential $\alpha \in \Gamma$ such that $d\alpha[1] \neq 0$.

Let U be a neighborhood $|z| < r$ in which the uniformizer z is valid, and let Ω be the compact region $|z| < r' < r$. If we set $u = u(p, \Omega)$, we know by Theorem 3 that u is not constant. Moreover, u is a harmonic function in the annulus $r' < |z| < r$ and is identically one on $|z| = r'$. Thus by the Schwarz reflection principle u has a continuous normal derivative on $|z| = r'$, i.e., $*du$ is g.-continuous in $W - \Omega$. Therefore, we have by Green's theorem

$$\begin{aligned} d*du[\phi] &= - \int \int_W d\phi * du = - \int \int_{W-\Omega} d\phi * du \\ &= - \int_{|z|=r'} \phi * du \end{aligned}$$

for a function $\phi \in K$, and so $*du$ is regular. But

$$d*du[1] = - \int_{|z|=1} *du = -D(u) < 0.$$

COROLLARY. *If W is parabolic, then there are no nonconstant harmonic functions in the class BD .*

Proof. If u were such a function, then $u * du$ is a regular differential belonging to Γ , and

$$d(u * du)[1] = \int \int du * du \neq 0.$$

We are now in a position to prove the following proposition.

PROPOSITION 10. *The space \overline{K} is a weakly closed linear subspace of BD ; i.e., if $f \in BD$ and there is a sequence $f_i \in \overline{K}$ such that*

$$f_i \rightarrow f \qquad \text{weakly in } BD,$$

then $f \in \overline{K}$.

Proof. Since $\overline{\Gamma_K}$ is a strongly closed linear subspace of Γ , it is also weakly closed by Lemma 4. Thus $df \in \overline{\Gamma_K}$. By the first corollary of Theorem 2 we may

write $f=f_K+c$, whence $f_K \in \bar{K}$. If W is parabolic, then $c \in \bar{K}$, and so $f \in \bar{K}$.

If W is hyperbolic, let α be a regular differential in Γ with $d\alpha[1] \neq 0$. Now

$$\begin{aligned} d\alpha[f] &= \lim d\alpha[f_i] \\ &= -\lim \iint df_i\alpha \\ &= -\iint df\alpha \end{aligned}$$

by weak convergence in BD and Proposition 8. Also by Proposition 8

$$d\alpha[f_K] = -\iint df_K\alpha.$$

Hence, since $c=f-f_K$,

$$d\alpha[c] = -\iint dc\alpha = 0.$$

Consequently, $c=0$ since $d\alpha[c]=cd\alpha[1]$. Thus $f=f_K \in \bar{K}$.

15. **Green's functions on an open Riemann surface.** Let $\{\Omega_i\}$ be an exhaustion of a Riemann surface W such that Ω_1 contains the point q . Let $G_i(p, q)$ be the Green's function of Ω_i with a pole at q . By the maximum principle $G_i(p, q)$ is a monotone increasing sequence of harmonic functions, and so by Harnack's theorem either $G_i(p, q)$ converges together with its derivatives to a harmonic function $G(p, q)$, the convergence being uniform on each compact set not containing q , or else $G_i(p, q)$ diverges to $+\infty$ uniformly on every compact set. In the first case we call $G(p, q)$ the Green's function for W with a pole at q . The Green's function clearly has the property of being the smallest non-negative function on W , which is harmonic except at q and which has a logarithmic singularity there. Thus the function $G(p, q)$ is independent of the exhaustion used in its definition.

PROPOSITION 11. *If $G(p, q)$ exists for some q , then W is of hyperbolic type, while if W is of hyperbolic type, then $G(p, q)$ exists for every q .*

Proof. If $G(p, q)$ does not exist for some q , let

$$\gamma_i = \min [1, G_i(p, q)].$$

Since $G_i(p, q)$ diverges to $+\infty$ uniformly on every compact region, γ_i must be identically one on each compact region provided that i is large enough. By Green's theorem

$$D(\gamma_i) = \int_{G_i=1} *dG_i = 2\pi.$$

Thus by Lemma 8, $d\gamma_i \rightarrow 0$ weakly in Γ , and by Proposition 10 it follows that $1 \in \bar{K}$, since 1 is the limit of γ_i in B . Thus W is parabolic.

On the other hand, if $G(p, q)$ exists for some q , then $G_i(p, q)$ converges together with its derivatives to $G(p, q)$ uniformly on each compact set not containing q . From Lemma 8 it follows that $G(p, q)$ has a finite Dirichlet integral over any region not containing q . Thus if we let g be a twice continuously differentiable function which is identically equal to $G(p, q)$ outside a compact region Ω , then $*dg$ is a regular differential belonging to Γ . By Green's theorem

$$d * dg [1] = \int_R * dg = 2\pi,$$

where R is the boundary of Γ . Thus by Theorem 4 we have W hyperbolic.

This proposition shows that our definition of the type of a Riemann surface is equivalent to the usual one, but has the advantage of being independent from the start of such questions as the dependence of the Green's function on the parameter.

PROPOSITION 12. *If c is so large that $G(p, q) > c$ only in a compact region containing q , and if $\gamma = \min [c, G]$, then $\gamma \in \bar{K}$.*

Proof. If $\gamma_i = \min [c, G_i]$, it is easily verified that

$$\gamma_i \rightarrow \gamma \qquad \text{weakly in } BD.$$

Since $\gamma_i \in K$, we have from Proposition 10 that $\gamma \in \bar{K}$.

PROPOSITION 13. *Let $\alpha \in \Gamma$ have the orthogonal decomposition*

$$\alpha = \omega + \alpha_1 + * \alpha_2$$

of Proposition 1, and suppose that $\alpha_1 = df$ where $f \in \bar{K}$. Then

$$\iint \alpha * dG(p, q)$$

exists and is equal to $2\pi f(q)$.

Proof. Clearly

$$\iint * \alpha_2 * dG = 0$$

since any approximation to G has a closed differential and $*\alpha_2$ is orthogonal to all closed differentials. If we take a small circular neighborhood of q and take c to be larger than the maximum of G on the boundary of this neighborhood, it is readily verified that the region Ω where $G > c$ is simply connected. By the Riemann mapping theorem we may choose a uniformizer z

so that Ω is the set $|z| < 1$, and $z(q) = 0$. Thus

$$\begin{aligned} \iint \alpha * dG &= \iint (\omega + df) * dG \\ &= \iint (\omega + df) * d\gamma + \iint (\omega + df) * d(G - \gamma). \end{aligned}$$

Now

$$\iint \omega * d\gamma = 0,$$

since $\gamma \in K$, and therefore $d\gamma$ is orthogonal to ω . By Proposition 8 and Green's theorem

$$(1) \quad \iint df * d\gamma = -d * d\gamma[f] = \int_R f * dG$$

where R is the boundary of Ω . Now $G - \gamma$ vanishes on R and is harmonic in Ω except for a logarithmic pole at $z = 0$. Hence $G - \gamma = -\log |z|$, and we have

$$\begin{aligned} - \iint_{\Omega} (\omega + df) * d \log |z| &= - \iint_{\Omega} d(u + f) * d \log |z| \\ &= 2\pi[u(0) + f(0)] - \int_R (u + f) d\phi \\ (2) \quad &= 2\pi f(q) - \int_R f d\phi \\ &= 2\pi f(q) - \int_R f * dG, \end{aligned}$$

where u is a harmonic function such that $\omega = du$ in Ω , and the proposition follows by adding (1) and (2).

PROPOSITION 14. *We have*

$$G(p, q) = G(q, p).$$

Proof. Consider the symmetric function

$$I(p, q) = \iint d_r G(r, p) * d_r G(r, q) = \iint d_r G(r, q) * dG_r(r, p).$$

Both integrals exist for $p \neq q$ since the differential of a Green's function is dominated by $|dz|/|z|$ at a pole. Hence if we set $\gamma(c) = \min [c, G(r, p)]$ and $\gamma'(c') = \min [c', G(r, q)]$,

$$I(p, q) = \lim_{c, c' \rightarrow \infty} \iint d\gamma * d\gamma'.$$

Letting $c' \rightarrow \infty$, we have from Proposition 13 that

$$I(p, q) = \lim_{c \rightarrow \infty} \gamma(c) = G(p, q),$$

and the symmetry of G follows from that of I .

16. The classes *HBD* and *HD*.

PROPOSITION 15. *A necessary and sufficient condition for the existence of a nonconstant function of class HBD is the existence of a regular differential $\alpha \in \Gamma$ and a function $f \in BD$ such that*

$$d\alpha[1] = 0$$

and

$$d\alpha[f] \neq - \iint df\alpha.$$

Proof. By Theorem 2 we have $f = f_K + u$, with $u \in HBD$.

$$\begin{aligned} d\alpha[u] &= d\alpha[f - f_K] \\ &= d\alpha[f] - d\alpha[f_K] \\ &= d\alpha[f] + \iint df_K\alpha \end{aligned}$$

by Proposition 8. Hence

$$d\alpha[u] + \iint du\alpha = d\alpha[f] + \iint df\alpha \neq 0.$$

Therefore u cannot be constant for

$$d\alpha[c] = 0 = - \iint dc\alpha.$$

On the other hand, if u is a nonconstant *HBD* function, then $\alpha = *du$ and $f = u$ satisfy the requirements of the theorem.

We denote by *HD* those functions f for which $df \in \Gamma_H$. The following proposition shows that the space *HD* is essentially the same as the space *HBD*:

PROPOSITION 16. *If $u \in HD$, then there exists a sequence $u_i \in HBD$ such that $du_i \rightarrow du$ in Γ .*

Proof. Let A be the space of all differentials of functions of class *HBD*

and A' the space of all differentials of functions of class HD . If \bar{A}' were different from \bar{A} , then there would be a harmonic function u such that $du \in A'$ and $du \perp A$. But if we set, for $c > c_1$,

$$f_1 = \max [u, c_1] \quad \text{where } c_1 < \sup u$$

and

$$f = \min [f_1, c] \quad \text{where } c > \inf f_1,$$

then $f \in BD$, and f is not constant. Hence $f = f_K + u_1$, where $u_1 \in HBD$ and $f_K \in \bar{K}$. Now

$$\iint du_1 * d\bar{u} = 0$$

by orthogonality, and

$$\iint df_K * d\bar{u} = d * d\bar{u}[f_K] = 0$$

by Proposition 8. Therefore

$$\iint df * d\bar{f} = \iint df * d\bar{u} = 0,$$

a contradiction, since f was not constant. Hence the proposition.

17. Summary. Since Proposition 11 shows that a Riemann surface is hyperbolic if and only if it possesses a Green's function, we shall use N_G to denote the class of parabolic Riemann surfaces. Similarly, we use N_{HD} and N_{HBD} to denote the classes of Riemann surfaces whose only HD and HBD functions are constants. By Proposition 16 and the corollary to Theorem 4 we have

$$N_G \subset N_{HD} = N_{HBD}.$$

We shall return to the classification of Riemann surfaces again in §23.

Theorem 2 tells us that

$$BD = \bar{K} \oplus HBD,$$

while on parabolic surfaces $BD = \bar{K}$, by the corollary to Theorem 4. Hence we may say that the surfaces of class N_G are characterized by having HBD empty, while the surfaces of class N_{HBD} may be characterized by having HBD consist of at most constants.

There is a rather striking analogy between the behavior of BD functions on an arbitrary Riemann surface and the classical case of BD functions on a plane region bounded by n continua. In this case Theorem 2 asserts nothing more than the solvability of the Dirichlet problem, while in passing to an

arbitrary Riemann surface we define a function to have "zero boundary values" if it belongs to the space \overline{K} , and see (Proposition 8) that such a function has the same properties with respect to Green's theorem as have the functions which vanish on the boundary in the classical case. The statement that a Riemann surface is parabolic means that its boundary is so weak that all functions of class BD have zero boundary values, which in the classical case must mean that the boundary continua have all degenerated to points. Proposition 13 shows that even on an arbitrary (hyperbolic) Riemann surface we can still find the component of a function which has "zero boundary values" simply by forming the integral

$$f_K(q) = \frac{1}{2\pi} \iint df * dG.$$

CHAPTER IV. THE SPACE BD ON A BOUNDED RIEMANN SURFACE

18. **Some decomposition theorems.** On a bounded Riemann surface V we consider the class BD of functions which are piecewise smooth on V , bounded, and have differentials belonging to $\Gamma = \Gamma(W)$, where W is the interior of V . In addition to the class K of functions having compact carriers it is useful to consider the class O of BD functions which vanish on the boundary R of V .

LEMMA 18. *The linear space \overline{KO} of functions which belong to both \overline{K} and O is closed in the BD topology.*

Proof. Since \overline{K} is closed, it is necessary only to show that O is closed. But this is trivial since convergence in B implies uniform convergence on every compact set, whence the limit in B of a function with zero boundary values must again have zero boundary values.

PROPOSITION 17. *If $\alpha \in \Gamma$ is a regular differential, and $f \in \overline{KO}$, then*

$$d\alpha[f] = - \iint df\alpha.$$

Proof. Since there exists a sequence ϕ_i of functions in KO whose limit is f , we have

$$\begin{aligned} d\alpha[f] &= \lim d\alpha[\phi_i] \\ &= - \lim \iint d\phi_i\alpha \\ &= - \iint df\alpha. \end{aligned}$$

THEOREM 5. *On a bounded Riemann surface V the space BD has the decompositions:*

$$BD = \overline{KO} \oplus HBD = \overline{KO} \oplus HK \oplus HN,$$

where HK denotes the class of functions belonging to both HBD and \overline{K} , and HN the class of those HBD functions whose normal derivative vanishes on R . That is to say every $f \in BD$ has the decomposition

$$(1) \quad f = f_0 + u_K + u_N,$$

where $f_0 \in \overline{KO}$, $u_K \in HK$, $u_N \in HN$, and the decomposition is unique to within a constant. If for $g \in BD$ we set

$$\|g\| = \sup_V |g|,$$

we have $\|u_K\| \leq \|f\|$ and $\|u_N\| \leq \|f\|$. Moreover, the decomposition is orthogonal in the sense that

$$df = df_0 + du_K + du_N$$

is an orthogonal decomposition.

Proof. Let $\{\Omega_i\}$ be an exhaustion of V , and let u_i be that piecewise smooth function, guaranteed by Theorem 1, which is harmonic in Ω_i and identically equal to f outside Ω_i . Then as in the proof of Theorem 2 there is a subsequence (which we again call u_i) converging in BD to a harmonic function u . Since $f - u_i \in KO$, $f - u$ must belong to \overline{KO} by Lemma 18. The orthogonality of du and $df_0 = d(f - u)$ follows from Proposition 17.

Hence there remains only to show that u may be split into $u_K + u_N$. To do this we consider the double W^\wedge of V and define u on W^\wedge by its symmetric extension. Then by Theorem 2

$$u = u_1 + u_2$$

where $u_1 \in \overline{K}(W^\wedge)$ and $u_2 \in HBD(W^\wedge)$. Since symmetrization preserves the classes $\overline{K}(W^\wedge)$ and $HBD(W^\wedge)$, we may take u_1 and u_2 to be symmetric. Hence u_2 must have a vanishing normal derivative on R and thus (when restricted to V) belongs to HN . But u_1 is the BD limit on W^\wedge of functions of the class $K(W^\wedge)$, whence u_1 (restricted to V) is the BD limit of functions of class $K(V)$. Thus $u_1 \in \overline{K}$, and since $u_1 = u - u_2 \in H(V)$, $u_1 \in HK$. Since u_1 and u_2 are symmetric on W^\wedge , and du_1 and du_2 are orthogonal there, du_1 and du_2 must also be orthogonal over V . This completes the proof.

19. Relative harmonic measures and the type of bounded Riemann surface. We shall say that a bounded Riemann surface V is of (relative) parabolic type if $1 \in \overline{K}(V)$. Otherwise we say that V is of (relative) hyperbolic type. It should be noted that the interior W of V considered as a Riemann surface in its own right is always hyperbolic as soon as the boundary of V is not empty.

Let Ω be a compact region on V and let $\{\Omega_i\}$ be an exhaustion of V such that $\Omega \subset \Omega_i$. In $\Omega_i - \Omega$ we construct that piecewise smooth harmonic function

u_i which is identically one on the boundary of Ω , identically zero on the boundary of Ω_i , and has a vanishing normal derivative on $R \cap (\Omega_i - \Omega)^-$. If we extend the definition of u_i so that $u_i \equiv 1$ on Ω and $u_i \equiv 0$ outside Ω_i , then u_i is of class K . By the Dirichlet principle $D(u_j) \leq D(u_i) \leq D(u_1)$, and by the maximum principle $0 \leq u_i \leq u_j \leq 1$, for $i < j$. Hence by Lemmas 1 and 2 there is a subsequence which converges together with its derivatives uniformly on every compact set to a harmonic function u , which we call the relative harmonic measure of Ω . Clearly, u is independent of the exhaustion chosen, since u may be characterized as the smallest non-negative harmonic function in $V - \Omega$ which is one on Ω and has zero normal derivative on R .

By Green's theorem

$$D(u_i - u_j, u_j) = 0, \quad i < j,$$

whence

$$\begin{aligned} D(u_i - u_j) &= D(u_i) - D(u_j) - 2D(u_i - u_j, u_j) \\ &= D(u_i) - D(u_j). \end{aligned}$$

Since $D(u_i)$ is a decreasing sequence of positive numbers, it must converge, and so $D(u_i - u_j)$ must tend to zero. Hence by Lemma 17

$$u_i \rightarrow u \quad \text{in } BD.$$

Thus we have proved the following proposition:

PROPOSITION 18. *The relative harmonic measure of a compact region Ω on a bounded Riemann surface V belongs to the class \overline{K} .*

THEOREM 6. *If V is of hyperbolic type, then every compact region has a nonconstant relative harmonic measure. If V is parabolic, then the relative harmonic measure of each compact region is identically one.*

Proof. The first part is a direct consequence of Proposition 18, for if any relative harmonic measure were constant, the constants would belong to \overline{K} , making V parabolic. The proof of the second half is identical with the proof of the second part of Theorem 3.

THEOREM 7. *A necessary and sufficient condition that a bounded Riemann surface be of hyperbolic type is the existence of a regular differential $\alpha \in \Gamma$ such that*

$$d\alpha[\phi] = - \iint d\phi\alpha$$

for all $\phi \in K$, and

$$d\alpha[1] \neq 0.$$

Proof. For such a differential

$$d\alpha[f] = - \iint df\alpha$$

for every $f \in \bar{K}$, by the continuity of $d\alpha$ in B and of $\iint df\alpha$ in D . Hence if $1 \in \bar{K}$, no such differential can exist.

If on the other hand V is hyperbolic, a compact region must by Theorem 6 have a nonconstant relative harmonic measure u . If we set $\alpha = *du$, then

$$d\alpha[\phi] = - \iint d\phi\alpha = \int_{\gamma} \phi * du$$

by Green's theorem, where γ is the boundary of the compact region. Thus α is a regular differential in Γ , and

$$d\alpha[1] = \int_{\gamma} * du = \int_{\gamma} u * du = D(u) \neq 0,$$

proving the theorem.

PROPOSITION 19. *The space \bar{K} is weakly closed in BD .*

Proof. By Theorem 5, $f = f_K + u_N$. But f_K is the weak limit of functions of K , and hence u_N must be a constant c , since du is orthogonal to Γ_K and hence to $\bar{\Gamma}_K$. If the surface V is parabolic, $c \in \bar{K}$, and therefore $f \in \bar{K}$. If the surface is hyperbolic, we choose α by Theorem 7. Now

$$d\alpha[f_K] = - \iint df_K\alpha$$

and by weak convergence

$$d\alpha[f] = - \iint df\alpha.$$

Hence

$$0 = - \iint dc\alpha = d\alpha[c] = cd\alpha[1],$$

and $c=0$, whence $f \in \bar{K}$.

PROPOSITION 20. *The space $\bar{K}O$ is weakly closed in BD .*

Proof. If

$$f_i \rightarrow f \quad \text{weakly in } BD$$

while $f_i \in \bar{K}O$, then $f = f_0 + c$ as before. Let α be a differential which is g.-con-

tinuous and has a compact carrier and the property that

$$\int_R \alpha = 1.$$

Then by Proposition 17

$$d\alpha[f_i] = - \int \int df_i \alpha.$$

Whence

$$d\alpha[f] = - \int \int df \alpha$$

by weak convergence. Also

$$d\alpha[f_0] = - \int \int df_0 \alpha.$$

Therefore

$$d\alpha[c] = 0.$$

But

$$d\alpha[c] = c \int_R \alpha = c$$

whence $c=0$, and the proposition is proved.

PROPOSITION 21. *If V is parabolic, then HN contains only constants. Hence $BD = \bar{K}$, and we adopt the convention that HN is empty (i.e., does not even contain constants). Under this convention the decomposition of Theorem 5 is unique.*

Proof. Suppose there were a nonconstant $u \in HN$. Then the hypothesis of Theorem 7 is valid with $\alpha = u * du$, and the surface is hyperbolic.

20. Surfaces with a compact boundary. In this section we assume that V is a bounded Riemann surface whose boundary is compact. If we use the symbol \dagger to denote a direct sum, in contrast to \oplus which denotes an orthogonal direct sum, we can then prove the following theorem:

THEOREM 8. *Let V be a bounded Riemann surface with a compact boundary. Then*

$$BD = \bar{K}O \dagger HK \dagger HO = \bar{K} \dagger HO.$$

That is to say if $f \in BD$, then

$$(2) \quad f = f_K + u_0$$

where $f_K \in \bar{K}$, $u_0 \in HO$ and the decomposition is unique. Moreover,

$$\|u_0\| \leq \|f\|.$$

Proof. In view of Theorem 5 we have

$$\bar{K} = \bar{K}O + HK,$$

and so it suffices to prove that

$$BD = \bar{K} + HO.$$

Let $\{\Omega_i\}$ be an exhaustion of V such that $R \subset \Omega_1$, and R separated from the remainder of the boundary of Ω_1 . Let f_1 be a piecewise smooth function which is identically equal to f outside Ω_1 , and vanishes on R . Set u_i equal to that piecewise smooth function given by Theorem 1 which vanishes on R , is harmonic in Ω_i , and is identically equal to $f (=f_1)$ outside Ω_i . Then

$$0 \leq D(u_i) \leq D(u_i) \leq D(f_1)$$

for $i < j$. Also

$$u_i \leq \sup_{p \in \Omega_j} |f(p)| = \sup_{p \in V} |f(p)|.$$

Hence by Harnack's principle there is a subsequence (which we again call u_i) of functions which converge together with their derivatives to a harmonic function u_0 which vanishes on R , the convergence being uniform on every compact set. Thus by Lemma 17

$$u_i \rightarrow u_0 \quad \text{in } BD.$$

Since $f - u_i \in K$, $f - u_0 \in \bar{K}$, and it remains only to prove that the decomposition (2) is unique. But this is certainly true, for if $u \in HO$ and $u \in \bar{K}$, then $u \in HBD$ and $u \in \bar{K}O$, whence, by Theorem 5, u must be a constant, and the constant must be zero since u vanishes on R .

THEOREM 9. *On a bounded Riemann surface with a compact boundary there is a natural algebraic isomorphism π_0 between the spaces HN and HO (provided we adopt the convention of Proposition 21 concerning constants). Moreover $\|u\| = \|\pi_0 u\|$.*

Proof. The algebraic isomorphism follows from Theorems 5 and 8 since HN and HO are both isomorphic to the quotient space BD/\bar{K} . In order to show that the norms are preserved, however, we shall actually construct π_0 .

If $u \in HN$, then the decomposition of Theorem 8 gives

$$u = u_0 + u_K,$$

where $u_0 \in HO$ and $u_K \in \bar{K}$, with $\|u_0\| \leq \|u\|$. But by the uniqueness of the

decompositions in Theorems 5 and 9, we must have for u_0 the decomposition

$$u_0 = u - u_K$$

with $\|u\| \leq \|u_0\|$ by Theorem 5.

Thus the mapping $\pi_0: u \rightarrow u_0$ is one-to-one isometric onto, and is easily seen to be linear, thus proving the theorem.

At this point I consider it a reasonable conjecture that π_0 is actually bicontinuous in the BD topology. However, this seems to be a difficult thing to prove because of our meager knowledge of the BD topology.

It should be noted that $\pi_0 1 = 1 - u$ where u is the harmonic measure of R with respect to V , for it is easily proved that $u \in \overline{K}$. In view of Theorem 9 we have the following characterization of the type of a bounded Riemann surface V with a compact boundary:

PROPOSITION 22. *A bounded Riemann surface V with a compact boundary R is of (relative) hyperbolic type if and only if the harmonic measure of R with respect to V is not constant.*

In [25] there are given counterexamples which show that the results of this section are no longer valid if we do not require V to have a compact boundary.

21. Separations by a compact set of curves.

THEOREM 10. *Let W be an open Riemann surface, and let R consist of a finite number of closed analytic curves which separate W into the regions $W_i, i=1, \dots, N$. Take V_i to be the bounded Riemann surface whose interior is W_i and whose boundary is $R_i = R \cap W_i$. Then under the convention of Proposition 21 there are algebraic isomorphisms π_1 and π_2 which map $HBD(W)$ onto*

$$HO(V_1) \dot{+} \dots \dot{+} HO(V_N)$$

and onto

$$HN(V_1) \dot{+} \dots \dot{+} HN(V_N)$$

respectively. Moreover

$$\|u\| = \|\pi_1 u\| = \|\pi_2 u\|.$$

Proof. In view of Theorem 9 it suffices to prove the statements for π_1 . We first note that a function ϕ which is piecewise smooth on W has a compact carrier if and only if the restrictions of ϕ to each of the V_i each have compact carriers. Consequently, a function f of class BD on W belongs to $\overline{K}(W)$ if and only if the restrictions of f to V_i each belong to $\overline{K}(V_i)$.

If we use $HO(W)$ to denote those BD functions on W which are harmonic in $W - R$ and vanish on R , then clearly $HO(W)$ is isomorphic to $HO(V_1) \dot{+} \dots \dot{+} HO(V_N)$ in a natural way which preserves norms. But in Theorem 8 we see that $f \in BD$ has the unique decomposition

$$(3) \quad f = f_K + u_0$$

where $f_K \in \overline{K}(W)$ and $u_0 \in HO(W)$, with $\|u_0\| \leq \|f\|$. Also a function $f \in BD$ has the unique decomposition

$$(4) \quad f = f'_K + u$$

with $f'_K \in \overline{K}(W)$ and $u \in HBD(W)$ with $\|u\| \leq \|f\|$. Thus for $u \in HBD(W)$ we have by (3) the unique decomposition

$$(5) \quad u = u_K + u_0$$

with $\|u_0\| \leq \|u\|$. But by (4) we have

$$(6) \quad u_0 = -u_K + u$$

with $\|u\| \leq \|u_0\|$, where u_K and u must be the same as in (5) by the uniqueness of the decompositions (3) and (4). Thus $\|u_0\| = \|u\|$, and $\pi_1: u \rightarrow u_0$ is an isometric isomorphism onto.

22. The unrestricted maximum principle and the character of a bounded Riemann surface. We say that the unrestricted maximum principle does not hold on a bounded Riemann surface V , or that V is of hyperbolic character, if there is a real bounded harmonic function u which is nonpositive on the boundary R of V and which is positive somewhere in V . We shall say that the unrestricted maximum principle does not hold on an open Riemann surface W if there is at least one subregion of W on which the unrestricted maximum principle does not hold.

PROPOSITION 23. *A necessary and sufficient condition that a Riemann surface W be hyperbolic is that the unrestricted maximum principle does not hold on W .*

Proof. By Theorem 3 there exists a compact region Ω which has a non-constant harmonic measure u . The harmonic function $1 - u$ violates the unrestricted maximum principle in $W - \Omega$.

Suppose on the other hand that there is a subregion $W_1 \subset W$ in which the unrestricted maximum principle does not hold. We may assume a compact region $\Omega \subset W - W_1$ since the validity of the unrestricted maximum principle is not affected by the addition or subtraction of a compact region. Let v be a real bounded harmonic function in W_1 which is nonpositive on the boundary R of W_1 and positive somewhere in W_1 . We normalize v so that it is bounded by unity. Hence by the maximum principle for compact regions we have $u_i \leq 1 - v$, where u_i is the sequence of harmonic measures used to define the harmonic measure u of the region Ω . Hence $u \leq 1 - v$, and u is not constant since $1 - v < 1$ at some points of W_1 . Thus W is hyperbolic by Theorem 3.

PROPOSITION 24. *A necessary and sufficient condition that a Riemann surface W have a nonconstant bounded harmonic function on it is the existence on it of two disjoint subregions of hyperbolic character.*

The proof is as in [24].

PROPOSITION 25. *A bounded Riemann surface V whose boundary R is compact is of hyperbolic character if and only if it is of hyperbolic type.*

Proof. If it is of hyperbolic character, let v with $v < 1$ be nonpositive on R and positive somewhere in V . Then if we define the harmonic measure u of R with respect to V as the limit of the harmonic measures u_i in an exhaustion of V , we have $u_i \leq 1 - v$, whence $u \leq 1 - v$. Thus u is not identically one, since $1 - v$ is somewhere less than one. Consequently, V is of hyperbolic type by Proposition 22.

If on the other hand V is of hyperbolic type, then by Proposition 22 the boundary R must have a nonconstant harmonic measure u with respect to V , and $1 - u$ violates the unrestricted maximum principle.

PROPOSITION 26. *If a bounded Riemann surface V has a compact boundary R , then the existence of a nonconstant bounded harmonic function with a vanishing normal derivative on R implies that V is of hyperbolic character.*

Proof. A harmonic function with a vanishing normal derivative on R can take neither its maximum nor its minimum on R if it is not constant. Hence such a function violates the unrestricted maximum principle if it is bounded.

Counterexamples are given in [25] which show that the restrictions to compact boundaries are essential.

23. The classification of Riemann surfaces. If we denote the class of parabolic surfaces by N_G and the class of Riemann surfaces on which there are no nonconstant positive harmonic functions by N_{HP} , then I maintain

$$N_G \subseteq N_{HP}.$$

For let v be a nonconstant positive harmonic function on a Riemann surface W , and assume that the greatest lower bound of v is zero. Then a component W_1 of the set where $v < 1$ is a subregion in which $1 - v$ violates the unrestricted maximum principle, and hence W is hyperbolic by Proposition 23.

Clearly

$$N_{HP} \subseteq N_{HB},$$

since the real part of a bounded harmonic function becomes a positive harmonic function by the addition of a suitable constant.

By virtue of Proposition 16 we have

$$N_{HD} = N_{HBD}.$$

Hence we may summarize these relations in the following proposition:

PROPOSITION 27. *We have*

$$N_G \subseteq N_{HP} \subseteq N_{HB} \subset N_{HBD} = N_{HD}.$$

In [25] an example is given which shows that the inclusion of N_{HB} in N_{HD} is strict. In [24] it was shown that for analytic functions

$$N_{AB} \subset N_{AD}.$$

It is known [9] that N_{AB} is not contained in N_{HD} , but it is not known whether or not the inclusion holds in the opposite direction. In the "schlichtartig" case

$$N_G = N_{HD}$$

and these surfaces are characterized by being mapped onto plane regions whose complements have logarithmic capacity zero.

The characterization of N_G by means of the maximum principle is due to Myrberg [15]. The inclusion of N_{HB} in N_{HD} was shown by Virtanen [35]. For the "schlichtartig" case see [9] and [15].

CHAPTER V. SOME APPLICATIONS TO THE TYPE PROBLEM

24. Triangulation of a Riemann surface. By a polygon s^2 on a Riemann surface W we mean a pair (S^2, G) consisting of a polygon S^2 in the plane and a mapping G which is a conformal mapping of the interior of S^2 into W which satisfies the following conditions:

- (i) On (the interior of) each edge of S^2 the mapping G is analytic and one-to-one.
- (ii) At a vertex S^0 of S^2 either G is analytic or else for any compact region $\Omega_0 \subset W$ there is a neighborhood U of S^0 such that $G(U)$ does not meet Ω_0 .

Thus, roughly speaking, a polygon on a Riemann surface is the conformal image of a Euclidean polygon. However, some of the vertices of the Euclidean polygon may not be mapped into interior points of W , but rather so to say onto the boundary of W . The images under G of the interior, edges, and vertices of S^2 will be called the interior, edges, and vertices, respectively, of s^2 . Thus the edges of s^2 are analytic arcs, which may, however, overlap one another. The points in the interior, edges, and vertices of s^2 are the points of s^2 .

A triangulation of W is a collection of polygons $s_i^2 = (S_i^2, G_i)$ such that:

- (i) Every point $p \in W$ is a point of some polygon s_i^2 .
- (ii) If a point $p \in W$ is an interior point of some s_i^2 , then it belongs to no other polygon.
- (iii) If a point $p \in W$ belongs to an edge of some s_i^2 , then it belongs to exactly one other edge (which may belong to s_i^2 or to some other polygon).

That we may take the indices to run over the integers is a consequence of the fact that a Riemann surface is separable.

The fact that we have not required the vertices of a triangulation to be points of the Riemann surface has the advantage that if we have a triangulation of a Riemann surface W , it is also a triangulation of the Riemann surface

W' formed by removing one or more of the points of W which are vertices of the triangulation. Also we have allowed two edges of a single polygon to meet, which has the advantage that any compact Riemann surface W (as well as the surface W' formed by removing a finite number of points from W) can be triangulated with a single polygon, e.g., by making a finite number of analytic cuts which reduce the surface to a simply-connected one and mapping this onto a suitable plane polygon.

For our purposes a two-dimensional cochain A^2 on a triangulation may be defined as a mapping of the polygons of the triangulation into the real numbers. We shall write

$$(1) \quad A^2 = \sum_{i=1}^{\infty} a_i^2 s_i^2,$$

where a_i^2 is the value of the mapping on s_i^2 . If a polygon s_j^2 does not appear in the formal sum (1), then we mean that $a_j^2 = 0$.

A one-dimensional cochain A^1 on a triangulation is a mapping of the edges of the Euclidean polygons S_i^2 into the real numbers having the property that two edges which have a common image on the Riemann surface are mapped into numbers which are the negatives of one another. We define an oriented edge s_i^1 of the triangulation to be a pair (S_j^1, S_k^1) of edges of Euclidean polygons S_j^2 , and S_k^2 , which have the same image on the Riemann surface, and write

$$(S_j^1, S_k^1) = - (S_k^1, S_j^1).$$

Then we may write

$$(2) \quad A^1 = \sum_{i=1}^{\infty} a_i^1 s_i^1,$$

where only one of the pairs $s_i^1, -s_i^1$ appears in the sum, and a_i^1 is the value of the cochain on the first element of s_i^1 .

The p -dimensional cochains ($p=1$ or 2) form an Abelian group under the definition

$$A^p + B^p = \sum_{i=1}^{\infty} (a_i^p + b_i^p) s_i^p.$$

We may define a homomorphism δ of the group of one-dimensional cochains as follows: If $s^1 = (S_j^1, S_k^1)$,

$$(3) \quad \delta s_i^1 = s_{j'}^2 - s_{k'}^2,$$

where $s_{j'}^2$ is the polygon such that S_j^1 is an edge of $S_{j'}^2$, and $s_{k'}^2$ the polygon such that S_k^1 is an edge of $S_{k'}^2$. On A^1 we define δ by

$$\delta A^1 = \sum_{i=1}^{\infty} a_i^1 \delta s_i^1.$$

25. **The β -measure of a polygon.** Let S be an n -sided polygon in the plane, and let γ be a differential which is defined and continuous on S , except possibly at the vertices of S . Suppose that $\{m_k\}$ is a set of n real numbers whose sum is zero.

If γ is (absolutely) integrable along each edge C_k of S (note that γ need be neither defined nor continuous at the end points of C_k), and if

$$I_k = \int_{C_k} \gamma \neq 0, \quad k = 1, \dots, n,$$

we can construct a function $v = v(m_1, \dots, m_n; \gamma)$ which is harmonic in S and which satisfies

$$(5) \quad * dv = m_k \gamma / I_k \quad \text{along } C_k,$$

and consequently

$$(6) \quad \int_{C_k} * dv = m_k.$$

Indeed, to prescribe $* dv$ along C_k is merely to prescribe $\partial v / \partial n$ on C_k . In order to infer the existence of v by the classical solution of the Neumann problem one has only to observe that

$$\int_{\Sigma C_k} \frac{\partial v}{\partial n} = \sum m_k = 0.$$

It is now possible to define a measure $\lambda_\gamma(S)$ of the polygon with respect to the differential γ . We set $\lambda_\gamma(S)$ equal to the maximum for all sets $\{m_k\}$ satisfying

$$(7) \quad \sum m_k = 0, \quad \sum |m_k|^2 = 1$$

of the Dirichlet integral of $v(m_1, \dots, m_n; \gamma)$, provided $v(m_1, \dots, m_n; \gamma)$ can be defined for all m_k satisfying (7). If not, e.g., if the integral of γ over some C_k vanishes or if γ is not absolutely integrable over some C_k , we set $\lambda_\gamma(S) = \infty$.

As a consequence one has the inequality

$$(8) \quad D_S v(m_1, \dots, m_n; \gamma) \leq \lambda_\gamma(S) \sum_{k=1}^n |m_k|^2,$$

with the function $v(m_1, \dots, m_n; \gamma)$ existing whenever the right-hand side of (8) is finite. For

$$D_S(v) = c^2 D_S\left(\frac{1}{c} v\right),$$

and we need only take $c^2 = \sum m_k^2$. In the right-hand side of (8) we may adopt the convention that $0 \cdot \infty = 0$, since

$$v(0, \dots, 0; \gamma) = 0.$$

This definition of measure can be extended to a polygon

$$s_i^2 = (S_i^2, G)$$

and a differential β on W by setting

$$(9) \quad \lambda_\beta(s_i^2) = \lambda_{G^*\beta}(S_i^2).$$

Thus the β -measure of a polygon on a Riemann surface is unchanged by a conformal mapping provided the differential β undergoes the adjoint mapping.

The function $v = v(m_1, \dots, m_n; G^*\beta)$ on S_i^2 can be extended to a harmonic function $u = u(m_1, \dots, m_n; \beta)$ defined on the image of S_i^2 by setting

$$(10) \quad u = G_i^{-1*} v.$$

It is then clear that we have

$$(11) \quad \int_{GC_k} * du = m_k$$

and

$$(12) \quad D(u) \leq \lambda_\beta(s_i^2) \sum_{k=1}^n |m_k|^2,$$

the function u being defined whenever the right-hand side is finite (with the convention still that $0 \cdot \infty = 0$).

We define the β -measure of an edge of a triangulation to be the sum of the β -measures of the two polygons of which it is an edge.

With this in mind we define the β -norm of a one-dimensional cochain

$$A^1 = \sum_{i=1}^{\infty} A_i^1 s_i^1$$

as

$$(13) \quad N_\beta(A^1) = \sum_{i=1}^{\infty} |a_i^1|^2 \lambda_\beta(s_i^1)$$

with the convention that $0 \cdot \infty = 0$.

26. A condition for a Riemann surface to be hyperbolic.

THEOREM 11. *A sufficient condition that a Riemann surface be hyperbolic is the existence of a triangulation for which there is a cochain A^1 and a differential β such that*

$$N_\beta(A^1) < \infty$$

and

$$\delta A^1 = s_0^2,$$

where s_0^2 is a polygon of the triangulation.

Proof. Let $A^1 = \sum a_i^1 s_i^1$. We shall define a differential as follows: Let s_{ik}^1 be the edges of an n -sided polygon s_j^2 , $j \neq 0$. Then we set $m_k = \pm a_{ik}^1$ according to whether the first or second element of s_{ik}^1 is an edge of S_j^2 . Since s_j^2 does not appear in the expression for A^1 , we must have

$$\sum_{k=1}^n m_k = 0.$$

Since also

$$\lambda_\beta(s_j^2) \sum_{k=1}^n |m_k|^2 \leq \sum_{k=1}^n \lambda_\beta(s_{jk}^1) |a_{jk}^1|^2 \leq N(A^1),$$

we have $u = u(m_1, \dots, m_n; \beta)$ defined and we set

$$\alpha = * du$$

in the interior of s_j^2 .

On S_0^2 we define α by setting it equal to an arbitrary continuously differentiable differential which is a constant multiple of β along each of the s_{ik}^1 which is an edge of s_0^2 and requiring

$$\int_{s_{ik}^1} \alpha = m_k = \pm a_{ik}^1$$

as before. Clearly

$$\int_{s_0^2} \alpha * \bar{\alpha} = \sum_{k=1}^n m_k = 1,$$

since s_0^2 is the coboundary of A^1 .

Now

$$\|\alpha\|^2 = \int \int_{s_0^2} \alpha * \bar{\alpha} + \sum_{j=1}^{\infty} D_j^2 u(m_1, \dots, m_n; \beta).$$

By inequality (12)

$$\|\alpha\|^2 \leq \int \int_{s_0^2} \alpha * \bar{\alpha} + \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} |a_{i_k}^1|^2 \lambda_{\beta}(s_j^2),$$

where $s_{i_k}^1$ are the edges of s_j^2 . Since $\lambda_{\beta}(s_j^2) \leq \lambda_{\beta}(s_{i_k}^1)$, we have

$$\|\alpha\|^2 = \int \int_{s_0^2} \alpha * \bar{\alpha} + \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} |a_{i_k}^1|^2 \lambda_{\beta}(s_{i_k}^1).$$

Since each $s_{i_k}^1$ occurs in this sum at most twice,

$$\|\alpha\|^2 \leq \int \int_{s_0^2} \alpha * \bar{\alpha} + 2 \sum_{i=1}^{\infty} |a_i^1|^2 \lambda_{\beta}(s_i^1) \leq \int \int_{s_0^2} \alpha * \bar{\alpha} + 2N(A^1).$$

Thus $\alpha \in \Gamma$.

Now α is a piecewise continuous and piecewise continuously differentiable differential which has the edges s_i^1 as its only lines of discontinuity. But along s_i^1 we have

$$\alpha = \frac{a_i^1 \beta}{\int_{s_i^1} \beta}$$

no matter which side we approach s_i^1 from. Thus α is g -continuous and by Lemma 10 we have

$$d\alpha[f] = \int \int_{s_0^2} f d\alpha$$

and so α is regular. But

$$d\alpha[1] = \int \int_{s_0^2} d\alpha = 1 \neq 0.$$

Hence W is hyperbolic by Theorem 4. This completes the proof of our theorem.

If a triangulation $\{s_i^2\}$ has the property that there exists a differential β such that

$$(14) \quad \lambda_{\beta}(s_i^2) < M$$

for all i , then we call the triangulation *uniform*. We shall define the norm of a one-dimensional cochain to be

$$(15) \quad N(A^1) = \sum_{i=1}^{\infty} |a_i^1|^2.$$

$$D(v) \leq nD \sum_{i=1}^n |m_i|^2.$$

Thus

$$\lambda_\beta(s^2) \leq nD < \infty.$$

If we set $\tilde{\beta} = F^*\beta$, then

$$\lambda_{\tilde{\beta}}(s_i^2) = \lambda_\beta(s^2)$$

by the invariance of the β -measure under conformal mapping. Thus the triangulation $\{s_i^2\}$ is uniform and the corollary to Theorem 11 becomes Theorem 12:

THEOREM 12. *Let (\tilde{W}, F) be a covering surface of a Riemann surface W' obtained by removing a finite number of points from a compact Riemann surface. Let $\{s_i^2\}$ be a triangulation of W arising from a triangulation of W' by a single polygon. Then a sufficient condition that W be of hyperbolic type is the existence of a one-dimensional cochain A^1 on $\{s_i^2\}$ such that*

$$N(A^1) < \infty$$

and

$$\delta A^1 = s_0^2.$$

The reader should have no difficulty in generalizing this theorem to triangulations $\{s_i^2\}$ of W which arises from any finite triangulation of W' .

28. The Speiser linear graph. Let $\{s_i^2\}$ be a triangulation of a covering surface (\tilde{W}, F) arising from the triangulation of the base surface W' by a single polygon. We may then construct a linear graph dual to the triangulation in the following way: Let t_0^i be a point in the interior of s_i^2 for each i . If two polygons s_i^2 and s_j^2 have an edge s_k^1 in common, we join t_0^i and t_0^j by an analytic arc t_1^k lying entirely in the interiors of s_i^2, s_j^2 , and s_k^1 . If, say,

$$\delta s_k^1 = s_i^2 - s_j^2,$$

we orient the arc t_1^k so that its boundary satisfies the relation

$$\partial t_1^k = t_0^i - t_0^j.$$

This linear graph is known as the Speiser linear graph (Streckenkomplex) of W , and has been considered in some detail by R. Nevanlinna [19]. By duality Theorem 12 becomes Theorem 12':

THEOREM 12'. *A sufficient condition that a covering surface (\tilde{W}, F) of a Riemann surface W' obtained by removing a finite number of points from a compact Riemann surface be hyperbolic is that there be a one-dimensional chain*

$$A_1 = \sum_{i=1}^{\infty} a_i^i t_i^i$$

on the linear graph of \bar{W} with the properties that

$$\partial A_1 = t_0^0$$

and

$$N(A_1) = \sum_{i=1}^{\infty} |a_i^i|^2 < \infty.$$

29. Covering surfaces of type S. By a circular domain is meant a plane region bounded by a finite number of disjoint circles $C_k, k=1, \dots, n$. If a circular domain is considered as a sort of generalized polygon with the sides C_k and without vertices, it is possible in some cases to define a generalized triangulation of a Riemann surface W by the conformal images of circular domains. In particular if W is a compact Riemann surface, it is possible to construct a generalized triangulation of W by means of a single generalized polygon, e.g., by making W into a surface of genus zero ("schlichtartig") by a finite number of disjoint cuts and mapping this surface onto the exterior of a finite number of circles. Since the generalized polygon is no longer simply-connected, we can no longer infer that such a triangulation induces a triangulation of an arbitrary covering surface of W .

This leads us to consider covering surfaces of W with the property that there is a generalized triangulation on them which arises from a triangulation of the base surface (which must necessarily be compact) by means of a single generalized polygon. Such covering surfaces we shall call covering surfaces of type S. Clearly we may extend the notion of cochains to generalized triangulations of such surfaces.

PROPOSITION 28. *In order that a covering surface of type S be hyperbolic it is necessary and sufficient that the generalized triangulation arising from the triangulation of the base surface by a single circular domain has on it a one-dimensional cochain A^1 so that*

$$\delta A^1 = s_0^2$$

and

$$N(A^1) < \infty.$$

Proof. The sufficiency of the condition is proved in exactly the same manner as Theorem 12. On the other hand suppose that there is a Green's function on the covering surface W with a pole in the interior of the polygon s_0^2 . To each edge s_i^1 of the triangulation we attach the value

$$a_k^1 = \frac{1}{2\pi} \int_{s_k^1} *dG$$

where the direction of integration on s_k^1 is such that the interior of s_k^2 is to the right where $s_k^1 = s_j^2 - s_i^2$. Now

$$A^1 = \sum_{i=1}^{\infty} a_i^1 s_i^1$$

is clearly a cochain on the triangulation. Since the integral of $*dG$ over the boundary of any polygon not containing the pole of the Green's function vanishes, we have

$$\delta A^1 = s_0^2.$$

Hence to complete the proof of the theorem it need only be shown that

$$\sum_{i=1}^{\infty} |a_i^1|^2 < \infty.$$

Let R be the circular domain in the plane whose images by the functions G_i are the polygons s_i^2 . Then the function G_i^*G is a harmonic function in the interior of R , and the numbers a_{jk}^1 representing the integrals of $*dG_i^*G$ over the boundary circles C_k of R are nothing more than the periods of $*dG_i^*G$ over the different homology classes of R . Let m be the maximum of the numbers $|a_{jk}^1|$. Then the norm in R of a harmonic differential which has at least one period larger than m cannot become arbitrarily small by Proposition 7', and we denote the smallest possible norm by $M(m)$. From the quadratic nature of the norm we have

$$M(m) = m^2 M(1)$$

with $M(1) > 0$. Let R have n boundaries. Then we have

$$\begin{aligned} \|dG\|_{s_i^2} &= \|*dG_i^*G\|_R \\ &\geq M(m) = m^2 M(1) \\ &\geq \frac{1}{n} M(1) \sum_{k=1}^n |a_{ik}^1|^2. \end{aligned}$$

Thus

$$\sum_{i=1}^{\infty} \|dG\|_{s_i^2} \geq \frac{M(1)}{n} \sum_{j=1}^{\infty} |a_j^1|^2.$$

But the left-hand side is just the Dirichlet integral of G over the exterior of the polygon s_0^2 and is thus finite. Hence

$$\sum_{i=1}^{\infty} |a_i^1|^2 < \infty$$

and we have proved the proposition.

The method of proof also yields the following interesting corollary:

COROLLARY. *If we define*

$$\pi u = A_u^1 = \sum_{j=1}^{\infty} a_j^1 s_j$$

where

$$a_j^1 = \int_{s_j^1} * du,$$

then π is an isomorphism of the class HD onto the group of cocycles of finite norm. Moreover, there exist two nonzero constants N_1 and N_2 such that

$$N_1 D(u) \leq N(A_u^1) \leq N_2 D(u)$$

for all u belonging to the class HD .

30. The Schottky covering surface. A covering surface of type S which is of genus zero is called a Schottky covering surface. For a covering surface of type S to have genus zero it is necessary and sufficient that there be no finite cycles in the Speiser linear graph of W , for such a cycle in the graph corresponds to a compact cycle on W which intersects just once some of the edges of a generalized triangulation of W .

Starting from a point t_0^0 of the graph, we call the segments emanating from it first generation segments and their remote (i.e., other than t_0^0) end points first generation points. In general we call segments which are not of the $(n-1)$ st generation, but which emanate from points of the n th generation, n th generation segments and their remote end points $(n+1)$ st generation points. Since there are no finite cycles in the graph, it follows that each point and each segment of the graph belong to a unique generation. If the base surface W of \tilde{W} has genus $p \geq 1$, then p cuts are required to reduce W to a surface of genus zero, and hence a circular domain which effects a generalized triangulation of W has $2p$ disjoint circles for boundaries. Thus $2p$ segments emanate from each point of the graph, and there are $2p(2p-1)^{n-1}$ segments in the n th generation.

If we assume the n th generation segments to be oriented so that the positive direction is from the $(n-1)$ st generation end point to the n th generation end point, we define a chain A_1 on the linear graph which has the value

$$-1/2p(2p-1)^{n-1}$$

on each oriented segment of the n th generation. Now

$$\partial A_1 = t_0^0,$$

for at t_0^0 we have $2p$ outgoing segments where the value of A_1 is $-1/(2p)$, while at the other points of the graph we have one incoming and $2p-1$ outgoing segments with the value of the chain on the incoming segment being $-(2p-1)$ times its value on each of the outgoing segments. Since there are $2p(2p-1)^{n-1}$ segments in the n th generation, we have

$$\begin{aligned} N(A_1) &= \sum_{n=1}^{\infty} 2p(2p-1)^{n-1} [2p(2p-1)^{n-1}]^{-2} \\ &= \sum_{n=1}^{\infty} [2p(2p-1)^{n-1}]^{-1} \\ &= \frac{2p-1}{4p(p-1)} \end{aligned}$$

if $p > 1$. Thus in view of Proposition 28 we have the following proposition which has as a corollary an important result due to P. J. Myrberg [16].

PROPOSITION 29. *If \tilde{W} is a Schottky covering surface of a compact surface whose genus is greater than one, then it is hyperbolic.*

COROLLARY. *The singular set of a Schottky group in the plane has positive capacity provided that the group has more than one generator.*

31. A condition for capacity in the plane. We conclude this chapter by proving the following condition for capacity:

PROPOSITION 30. *Let Ω be a region in the plane containing the point $z = \infty$, and let $\{s_i^2\}$ be a triangulation of Ω with the property that there are only a finite number of the s_i^2 which are dissimilar in the sense of Euclidean geometry. In order that the complement of Ω have positive capacity it is sufficient that there be a cochain A^1 on s_i^2 with*

$$N(A^1) < \infty$$

and

$$\partial A^1 = s_0^2.$$

Proof. By virtue of the corollary to Theorem 11 it suffices to show that the triangulation is uniform. But the dz -measure of two similar polygons is the same, as one can readily verify. Thus $\lambda_{dz}(s_i^2)$ has only a finite number of values and hence a finite upper bound. Consequently, the triangulation is uniform and the proposition is proved.

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