SCHLICHT SOLUTIONS OF $W''+pW=0$

BY

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1. Introduction. Let

$$z^2p(z) = p_0 + p_1z + \cdots + p_nz^n + \cdots \tag{1.1}$$

be regular for $|z| < 1$. For the differential equation

$$\frac{d^2W}{dz^2} + p(z)W = 0, \tag{1.2}$$

the origin is a regular singular point (or an ordinary point when p_0 and p_1 are both zero), and the indicial equation is

$$\lambda^2 - \lambda + p_0 = 0, \tag{1.3}$$

with roots α and β, $\alpha+\beta=1$, $\Re\alpha \geqq 1/2 \geqq \Re\beta$. Corresponding to the root α with the larger real part (or to either root if the real parts are equal), there exists a unique solution of the form

$$W = W(z) = z^\alpha \sum_{n=0}^{\infty} a_nz^n, \qquad a_0 = 1, \tag{1.4}$$

valid in the unit circle $|z| < 1$. Let $F(z)$ be defined as

$$F(z) = \{W(z)\}^{1/\alpha} = z + \cdots , \tag{1.5}$$

where that branch of the function is chosen for which $F'(0) = 1$. In this paper we shall obtain sufficient conditions on $p(z)$, of a fairly general nature, so that $F(z)$ is schlicht in $|z| < 1$ ($F(z)$ takes on no value more than once in the unit circle).

It will be observed immediately that every analytic function $f(z)$, schlicht in $|z| < 1$,

$$f(z) = z + \mu_2z^2 + \cdots + \mu_nz^n + \cdots , \tag{1.6}$$

satisfies an equation of the form (1.2) where

$$zp(z) = \frac{-zf''(z)}{f(z)} \tag{1.7}$$

is regular for $|z| < 1$. In this case, $p_0 = 0$ and $\alpha = 1$. It follows then, for the special instance $\alpha = 1$, that our sufficiency conditions on $p(z)$ can be rephrased in terms of $f(z)$ and $f''(z)$ to give many new tests for an arbitrary analytic function $f(z)$ for which $f(0) = 0$, $f'(0) = 1$, to be schlicht in $|z| < 1$.

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A related problem was recently solved by Nehari [5], who showed that in the special case $p_0=p_1=0$, $\alpha=1$, no solution of (1.2) can take on the value zero more than once in $|z|<1$, provided either

(i) $$|p(z)| \leqq (1-|z|^2)^{-2}, \qquad |z|<1,$$

or

(ii) $$|p(z)| \leqq \pi^2/4, \qquad |z|<1.$$

In either of these two cases the ratio of two independent solutions of (1.2) is schlicht in $|z|<1$. The conditions (i) and (ii) may be replaced by

$$|p(z)| \leqq \begin{cases} 2^{3\lambda-2}\pi^{2(1-\lambda)}(1-|z|^2)^{-\lambda}, & \text{for } 0 \leqq \lambda \leqq 1, \\ 2^{2-\lambda}(1-|z|^2)^{-\lambda}, & \text{for } 1 \leqq \lambda \leqq 2, \end{cases}$$

a refinement of Nehari's result due to Pokornyi [6]. A condition analogous to (ii) applied to analytic functions in a convex domain was obtained recently by Ryll-Nardzewski [8]. The problem at hand now, however, is to find sufficient conditions on $p(z)$ so that certain individual solutions can take on no value more than once in $|z|<1$ (even when p_1 is different from zero). The "Green's Transform" of (1.2), used so successfully by Nehari [5] and of fundamental importance in the earlier papers of Hille [1; 2; 4] on the existence of zero-free regions for solutions of (1.2), also plays an important role in this investigation.

Our aim is to derive a fairly general "parent" theorem, involving a somewhat arbitrary function $p^*(z)$, whose "offspring" will be theorems corresponding to each selected function $p^*(z)$. Each such $p^*(z)$ will have associated with it a universal constant $A=A(p^*)$ (often times a root of a transcendental equation) which will give a sharp character to the corresponding theorem. By varying $p^*(z)$ innumerable tests for the univalency of $F(z)$ of (1.5) may be obtained. A few of these examples will be explored in §7 of this paper. Because of the length of a satisfactory and complete statement of the main theorem, we postpone this until the proof is at hand in §6.

2. **Preliminary definitions.** Let

$$f(z) = z + \mu_2 z^2 + \cdots + \mu_n z^n + \cdots \tag{2.1}$$

be regular for $|z|<1$. We denote by S the class of functions $f(z)$, $f(0)=0$, $f'(0)=1$, given by (2.1), which are schlicht, or univalent, in $|z|<1$. Let $S(\gamma)$ be the subclass of S whose members $f(z)$ satisfy, for some real constant γ ($|\gamma| \leqq \pi/2$) and $|z|<1$, the inequality

$$\Re\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} \geqq 0. \tag{2.2}$$

It was shown by Špaček [9] that the inequality (2.2) is a sufficient condition (when $f'(0)\neq 0$) for $f(z)$ of (2.1) to be schlicht in $|z|<1$. In general, a member of $S(\gamma)$ maps $|z|<1$ onto a spiral-like domain. We shall call $f(z)$ "spiral-like" if it is a member of $S(\gamma)$.

The subclass $S(0)$ of $S(\gamma)$, corresponding to $\gamma=0$, contains only those members of S which are star-like with respect to the origin. Each member $f(z)$ of $S(0)$ maps $|z|<1$ onto a star-domain, one which has the property that every ray from the origin contains an open segment, from the origin to a boundary point, which lies entirely within the domain. We shall call $f(z)$ "star-like" if it is a member of $S(0)$. Every star-like function is also spiral-like since $S(0)\subset S(\gamma)$.

We shall denote by K the subclass of $S(0)$ whose members $f(z)$ map $|z|<1$ onto a convex domain, one which has the property that, if W_1 and W_2 are points within the domain, the line segment joining W_1 and W_2 lies entirely within the domain. It is well known that $f(z)$, with $f(0)=0$ and $f'(0)=1$, is convex in $|z|<1$ if, and only if, $zf'(z)$ is star-like in $|z|<1$. Since (2.2), with $\gamma=0$, is both necessary and sufficient for $f(z)$ to be star-like in $|z|<1$, it follows (as is well known) that the necessary and sufficient condition that $f(z)$ (with again $f(0)=0, f'(0)=1$) be convex in $|z|<1$ is that

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \qquad \text{for } |z|<1. \tag{2.3}$$

Another important subclass of S, to be denoted by K^*, is the class whose members $f(z)$ are real on the real axis and each $f(z)$ maps $|z|<1$ onto a domain which is convex in the direction of the imaginary axis. This means that if W_1 is any point within the domain, then the line segment joining W_1 and its conjugate point $\overline{W}_1$ lies entirely within the domain. It is known that the necessary and sufficient condition for $f(z)$ to belong to K^* is that $zf'(z)=g(z)$ be typically-real for $|z|<1$. Following Rogosinski [7] we say $g(z)$ ($g(0)=0$, $g'(0)=1$) is typically-real for $|z|<1$ if $g(z)$ is regular in $|z|<1$, is real on the real axis, and if the imaginary part of $g(z)$ vanishes for $|z|<1$ only when z is real. We shall speak of $f(z)$ as being "convex in the direction of the imaginary axis" if it is a member of K^*.

Let $f(z)$ be a member of S, and let $Cf(z)$ (C any constant not zero) map $|z|<1$ onto a simply-connected domain D. Let $h(z)$ be regular in $|z|<1$, and $h(0)=0$. We shall say that $h(z)$ is "subordinate" to $Cf(z)$ in $|z|<1$, and write

$$h(z)\prec\prec Cf(z)$$

whenever $h(z)$ lies within D for all z, $|z|<1$.

3. **Green's transform.** Following Hille [2], we adjoin to the differential equation

$$W''+p(z)W=0, \qquad \left(W''=\frac{d^2W}{dz^2}\right), \tag{3.1}$$

the "Green's transform" of (3.1) obtained as follows. We multiply (3.1) by $\overline{W}dz$ and integrate from 0 to z, $|z|<1$. This gives

$$\int_0^z \overline{W}(z)W''(z)dz + \int_0^z p(z)\,|\,W(z)\,|^2 dz = 0, \tag{3.2}$$

where $p(z)$ is given as in (1.1) and $W=W(z)$ is the solution (1.4), $\Re\alpha \geqq 1/2$. For the present we shall assume $\Re\alpha > 1/2$ (the case $\Re\alpha = 1/2$ will be considered separately). Since $\Re\alpha > 1/2$ the integrals appearing in (3.2) exist. Integrating by parts, we obtain

$$[\overline{W}(z)W'(z)]_0^z - \int_0^z |\,W'(z)\,|^2 dz + \int_0^z p(z)\,|\,W(z)\,|^2 dz = 0. \tag{3.3}$$

Let the path of integration in (3.3) be the straight line segment, $\theta=$constant, joining the origin to the point $z=re^{i\theta}$, $r<1$. Now multiply (3.3) by $ze^{i\gamma}$ and equate the real part of the resulting equation to zero. We then have

$$\begin{aligned} &|\,W(z)\,|^2 \Re\left\{e^{i\gamma}\frac{zW'(z)}{W(z)}\right\} \\ &\qquad = r\cos\gamma\int_0^r |\,W'\,|^2 d\rho - r\int_0^r \Re\{e^{i\gamma}z^2p(z)\}_{|z|=\rho}\frac{|\,W\,|^2 d\rho}{\rho^2}, \end{aligned} \tag{3.4}$$

where $\Re F(z)$ denotes the real part of $F(z)$. The Green's transform, written in the form (3.4), will be of fundamental importance in the paragraphs to follow.

4. A fundamental inequality of integrals. Let

$$z^2p^*(z) = p_0^* + p_1^*z + \cdots + p_n^*z^n + \cdots \tag{4.1}$$

be regular for $|z|<1$, real on the real axis, and $p_0^* \leqq 1/4$. If C is any non-negative constant, the differential equation

$$\frac{d^2W}{dz^2} + \left\{C\left(p^*(z) - \frac{p_0^*}{z^2}\right) + \frac{p_0^*}{z^2}\right\}W = 0 \tag{4.2}$$

has its indicial equation

$$\lambda^2 - \lambda + p_0^* = 0 \tag{4.3}$$

independent of C. Let α^*, β^* be the roots of (4.3). Since $p_0^* \leqq 1/4$, α^* and β^* are real and so we have $\alpha^* \geqq 1/2 \geqq \beta^*$, $\alpha^*+\beta^*=1$, where

$$2\alpha^* = 1 + (1-4p_0^*)^{1/2}, \qquad 2\beta^* = 1 - (1-4p_0^*)^{1/2}. \tag{4.4}$$

Let

$$W_C = W_C(z) = z^{\alpha^*}\sum_{n=0}^{\infty} a_n^*(C)z^n, \qquad a_0^*(C) = 1, \tag{4.5}$$

be the unique solution of (4.2), corresponding to the real root α^* of (4.3), and depending upon the non-negative parameter C. The coefficients $a_n^* = a_n^*(C)$, all real, are determined from the recurrence relation

$$(4.6) \qquad a_n^* = -\frac{C[p_n^* + p_{n-1}^* a_1^* + \cdots + p_1^* a_{n-1}^*]}{n(n + 2\alpha^* - 1)}, \qquad n \geqq 1.$$

From (4.6) it is readily seen that $a_n^*(C)$ is a polynomial in C of degree not exceeding n, and, in particular, a continuous function of C for $\alpha^* > 0$.

We shall show now that $W_C(z)$ is continuous in each of the three variables C, α^*, and z for $C \geqq 0$, $\alpha^* > 0$, $|z| < 1$. If all the coefficients p_k^*, $k = 1, 2, \cdots$, in the expansion (4.1) are replaced by their absolute values, and if p_0^* is replaced by $\epsilon - \epsilon^2$, ϵ arbitrarily small and positive, we have a new function which we shall call $\tilde{p}(z)$ and

$$(4.7) \qquad z^2 \tilde{p}(z) = \epsilon - \epsilon^2 + \sum_{k=1}^{\infty} | p_k^* | z^k.$$

Let $C_0 > 0$ be chosen arbitrarily large. Then the differential equation

$$(4.8) \qquad \frac{d^2 W}{dz^2} + \left\{ - C_0 \left(\tilde{p}(z) - \frac{\epsilon - \epsilon^2}{z^2} \right) + \frac{\epsilon - \epsilon^2}{z^2} \right\} W = 0$$

has the unique solution

$$(4.9) \qquad \tilde{W}_{C_0}(z) = z^\epsilon \sum_{n=0}^{\infty} A_n(C_0) z^n, \qquad A_0(C_0) = 1,$$

corresponding to the smaller root $\epsilon > 0$ of the indicial equation. Since $z^2 \tilde{p}(z)$ is regular in $|z| < 1$ whenever $z^2 p^*(z)$ is, the series for $\tilde{W}_{C_0}(z)$ converges for $|z| < 1$ for all C_0. Moreover, the coefficients $A_n(C_0)$ are determined by the recurrence relation

$$(4.10) \qquad n(n + 2\epsilon - 1) A_n(C_0) = C_0[| p_n^* | + | p_{n-1}^* | A_1 + \cdots + | p_1^* | A_{n-1}].$$

For $0 \leqq C \leqq C_0$, $\alpha^* > \epsilon > 0$, it is readily seen that the coefficients $a_n^*(C)$ determined by (4.6) satisfy

$$(4.11) \qquad n(n + 2\epsilon - 1) | a_n^*(C) | < C_0[| p_n^* | + | p_{n-1}^* a_1^* | + \cdots + | p_1^* a_{n-1}^* |],$$

$$(4.12) \qquad | a_1^*(C) | < \frac{C_0 | p_1^* |}{2\epsilon} = A_1(C_0).$$

Thus

$$(4.13) \qquad | a_n^*(C) | < A_n(C_0) \qquad \text{for } n \geqq 1,\ 0 \leqq C \leqq C_0,\ \alpha^* > \epsilon.$$

Since the series $\sum_{n=0}^{\infty} A_n(C_0) z^n$ converges uniformly in z for $|z| < R < 1$, it follows that the series $\sum_{n=0}^{\infty} a_n^*(C) z^n$ converges uniformly in the three variables C, α^*, and z for $0 \leqq C \leqq C_0$, $\alpha^* > \epsilon > 0$, and $|z| < R < 1$. A similar statement holds for the derived series $\sum_{n=0}^{\infty} (n+\alpha^*)a_n^*(c)z^n$ if α^* is bounded above. Since the coefficients $a_n^*(C)$ are continuous functions of C and α^* and the series $\sum_{n=0}^{\infty} a_n^*(C)z^n$ converges uniformly in C, α^*, and z, it follows that $W_C(z)$ and its derivatives with respect to z are continuous functions of C, α^*, and z for $C \geqq 0$, $\alpha^* > 0$, $|z| < 1$. Similar arguments apply to $W(z)$ if $\Re\alpha > 0$ in (1.4).

Now that we are dealing with a function $W_C(z)$ continuous in C, and because, for $n \geqq 1$, $a_n^*(0) = 0$, $a_0^*(0) = 1$, it is easily seen that

$$\lim_{C\to 0} W_C(z) = W_0(z) = z^{\alpha^*}, \tag{4.14}$$

uniformly for $|z| \leqq R$ for any positive $R < 1$, and $W_0(z)$ is the solution of (4.2) when $C = 0$. By z^α we shall mean $\exp(\alpha \log z)$, the principal branch of $\log z$ being chosen.

Although $W_C(z)$ is in general not single-valued in the neighborhood of the origin, the logarithmic derivative $W_C'(z)/W_C(z)$ is single-valued and has a simple pole at the origin. Since the coefficients are all real and $a_0^*(C) = 1$, $\alpha^* > 0$ in (4.5), it is seen that for each $C \geqq 0$ we have

$$W_C'(\rho) > 0 \qquad \text{for a range } 0 < \rho < r(C), \tag{4.15}$$

and for the same range at least we also have

$$W_C(\rho) = \int_0^\rho W_C'(\rho)d\rho > 0. \tag{4.16}$$

Thus

$$\frac{\rho W_C'(\rho)}{W_C(\rho)} > 0 \qquad \text{for } 0 \leqq \rho < r \leqq 1, \tag{4.17}$$

where for C fixed, r is the smallest positive zero of $W_C'(\rho)$ (as a function of ρ) or one, whichever is smaller. By taking C sufficiently small we may obviously have r as near to one as we like. We shall see later that, under certain restrictions (not very severe) on $p^*(z)$, by taking $C = C(R)$ sufficiently large we can make $W_C'(R)$ vanish for any given $R < 1$.

We are now ready to prove the following inequality of integrals which is of fundamental importance in the proof of our main theorem. We state the inequality as a lemma.

LEMMA. *Let $y(\rho)$, $dy(\rho)/d\rho = y'(\rho)$ be real functions, continuous in the real variable ρ for $0 < \rho < 1$. For small values of ρ let*

$$y(\rho) = O(\rho^\delta), \qquad y'(\rho) = O(\rho^{\delta-1}), \qquad \delta > 1/2.$$

Then

$$\int_0^r \{C(\rho^2 p^*(\rho) - p_0^*) + p_0^*\} y^2(\rho) \frac{d\rho}{\rho^2} \tag{4.18}$$
$$\leqq \int_0^r \{y'(\rho)\}^2 d\rho - \frac{W_C'(r)}{W_C(r)} \cdot y^2(r), \qquad 0 < r < 1,$$

where $W_C(z)$ is the solution (4.5) *of* (4.2), *and where $C(\geqq 0)$ is chosen small enough so that* (4.17) *holds. Equality in* (4.18) *holds if, and only if, $y(\rho) = kW_C(\rho)$, $\alpha^* > 1/2$, where k is an arbitrary real constant.*

The conditions $y(\rho) = O(\rho^\delta)$, $y'(\rho) = O(\rho^{\delta-1})$, $\delta > 1/2$ guarantee the existence of the integrals involved. The lemma is proved with the use of the following identity and partial integration.

$$\begin{aligned}
&\int_0^r \left[y'(\rho) - \frac{W_C'(\rho)}{W_C(\rho)} y(\rho)\right]^2 d\rho \\
&= \int_0^r \{y'(\rho)\}^2 d\rho - \int_0^r 2y'(\rho)y(\rho) \frac{W_C'(\rho)}{W_C(\rho)} d\rho \\
&\quad + \int_0^r \left\{\frac{W_C'(\rho)}{W_C(\rho)}\right\}^2 y^2(\rho) d\rho \\
&= \int_0^r \{y'(\rho)\}^2 d\rho - \left[y^2(\rho) \frac{W_C'(\rho)}{W_C(\rho)}\right]_0^r \\
&\quad + \int_0^r y^2(\rho) \left[\frac{d}{d\rho}\left(\frac{W_C'}{W_C}\right) + \left(\frac{W_C'}{W_C}\right)^2\right] d\rho \\
&= \int_0^r \{y'(\rho)\}^2 d\rho - y^2(r) \frac{W_C'(r)}{W_C(r)} + \int_0^r \frac{W_C''(\rho)}{W_C(\rho)} y^2(\rho) d\rho \\
&= \int_0^r \{y'(\rho)\}^2 d\rho - y^2(r) \frac{W_C'(r)}{W_C(r)} \\
&\quad - \int_0^r \{C(\rho^2 p^*(\rho) - p_0^*) + p_0^*\} y^2(\rho) \frac{d\rho}{\rho^2}.
\end{aligned} \tag{4.19}$$

Since the left-hand side of the identity (4.19) is non-negative, and zero only if $y(\rho) = kW_C(\rho)$, the inequality (4.18) follows. When equality exists in (4.18) it is necessary that $\alpha^* > 1/2$, $p_0^* < 1/4$. However, the inequality holds for $\alpha^* \geqq 1/2$ if $\delta > 1/2$. This completes the proof of the lemma.

5. **Some new universal constants.** Let $z^2 p(z)$ be regular in $|z| < 1$ and be given as in (1.1). Let

$$W = W(z) = z^\alpha \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1, \ |z| < 1, \tag{5.1}$$

be the solution (1.4) of the differential equation (1.2) associated with a given $p(z)$. We have a (1-1) correspondence between the function $p(z)$ of (1.1) and the solution $W(z)$ of (1.4). Similarly, we have a (1-1) correspondence between the function $p^*(z)$ of (4.1) and the associated solution $W_C(z)$ in (4.5) of the differential equation (4.2).

We shall now restrict $p(z)$ by making it satisfy an inequality involving $p^*(z)$ ($p^*(z)$ regarded as a given fixed function). Then we shall deduce an inequality involving the associated functions $W(z)$ and $W_C(z)$.

Let $C\geqq 0$, γ ($|\gamma|\leqq\pi/2$) be assigned constants. Let $p(z)$ be restricted so that

$$\Re\{e^{i\gamma}z^2p(z)\} \leqq \cos\gamma[C\{|z|^2p^*(|z|) - p_0^*\} + p_0^*] \tag{5.2}$$

for $|z|<1$, and let $\Re\alpha>1/2$. Let C be chosen small enough so that (4.17) holds. Taking $z=0$, we note that (5.2) implies in particular that

$$\Re(e^{i\gamma}p_0) \leqq p_0^* \cos\gamma \leqq (1/4)\cos\gamma. \tag{5.3}$$

We prove now the following preliminary theorem, comparing the solutions $W(z)$ and $W_C(z)$.

THEOREM A. *Let $z^2p(z)$ be regular in $|z|<1$ and satisfy* (5.2). *Let the root α of* (1.3) *be the one for which $\Re\alpha\geqq 1/2$. Let*

$$W(z) = z^\alpha\sum_{n=0}^{\infty} a_nz^n, \qquad a_0 = 1,\ |z|<1,$$

be the unique solution of (1.2) *corresponding to α. Let $z^2p^*(z)$ be regular in $|z|<1$ and real on the real axis with $\lim_{z\to 0} z^2p^*(z) = p_0^*\leqq 1/4$. Let*

$$W_C(z) = z^{\alpha^*}\sum_{n=0}^{\infty} a_n^*(C)z^n, \qquad a_0^*(C) = 1,$$

be the solution of (4.2) *where α^* is given by* (4.4), $\alpha^*\geqq 1/2$. *Then*

$$\Re\left\{e^{i\gamma}\frac{zW'(z)}{W(z)}\right\} \geqq \frac{|z|\,W_C'(|z|)}{W_C(|z|)}\cos\gamma \geqq 0, \qquad |z|\leqq R<1, \tag{5.4}$$

for those values C, $0\leqq C\leqq C(R)$, for which

$$W_C'(r) > 0 \qquad in\ 0<r<R. \tag{5.5}$$

If we assume further that

$$\max_{|z|=r<1}\Re\{z^2p^*(z)\} = r^2p^*(r), \tag{5.6}$$

then

$$\Re\left\{\frac{zW_C'(z)}{W_C(z)}\right\} \geqq \frac{|z|\,W_C'(|z|)}{W_C(|z|)} > 0, \qquad |z|<R<1, \tag{5.7}$$

$0 \leqq C \leqq C(R)$, and $rW_C'(r)/W_C(r)$ is a nonincreasing positive function of r for $0 \leqq r \leqq R$ when $0 \leqq C \leqq C(R)$, and for $C = C(R)$ decreases from α^* to zero as r increases from 0 to $R<1$.

To prove Theorem A we use the Green's transform as given in the form (3.4), and assume $\Re\alpha > 1/2$ to begin with. From (3.4), (5.2), and (4.18) of the fundamental lemma we have for $z = re^{i\theta}$, $r<1$, θ constant,

$$
\begin{aligned}
& | W(z) |^2 \Re \left\{ e^{i\gamma} \frac{zW'(z)}{W(z)} \right\} \\
(5.8) \qquad & \geqq r \cos\gamma \int_0^r | W' |^2 d\rho - r \cos\gamma \int_0^r \frac{C\{\rho^2 p^*(\rho) - p_0^*\} + p_0^*}{\rho^2} | W |^2 d\rho \\
& \geqq \frac{rW_C'(r)}{W_C(r)} \cos\gamma \, | W(z) |^2,
\end{aligned}
$$

which gives (5.4) when $\Re\alpha > 1/2$. But (5.4) holds also when $\Re\alpha = 1/2$ since, as we have seen previously, $zW'(z)/W(z)$ is a continuous function of α when $\Re\alpha > 0$.

In particular, if $p(z)$ is chosen so that

$$
(5.9) \qquad z^2 p(z) \equiv C\{z^2 p^*(z) - p_0^*\} + p_0^*,
$$

and if $p^*(z)$ is chosen so that (5.6) holds, then it follows that the condition (5.2) is satisfied when $\gamma = 0$. Furthermore, the solution $W(z)$ of (1.2) given in (1.4) becomes identical with the solution $W_C(z)$ of (4.2) given in (4.5). Thus we may replace $W(z)$ by $W_C(z)$ in (5.4) when $\gamma = 0$ and obtain (5.7). Obviously, equality occurs in (5.7) when z is positive. Thus (5.7) shows that

$$
\min_{|z|=r} \Re \frac{zW_C'(z)}{W_C(z)} = \frac{rW_C'(r)}{W_C(r)} \cdot
$$

Because the minimum of a harmonic function does not occur at an interior point of a domain, it follows that $rW_C'(r)/W_C(r)$ is a nonincreasing positive function of r for $0 \leqq r \leqq R < 1$. We shall see a little later that this function for $C = C(R)$ decreases from α^* to zero as r increases from 0 to $R<1$.

We have seen that for a given $R<1$ there exists a range for C, $0 \leqq C \leqq C(R)$, for which

$$
(5.10) \qquad W_C'(r) > 0 \qquad \text{in } 0 < r < R.
$$

We shall now show that (5.10) cannot hold for sufficiently large values of C whenever (5.6) holds, and when $z^2 p^*(z)$ is not identically a constant. Because of (5.7)

$$
(5.11) \qquad \phi(z) \equiv \frac{zW_C'(z)}{W_C(z)} = \alpha^* + b_1 z + \cdots + b_n z^n + \cdots
$$

has a positive real part for $|z|<R<1$. Hence the coefficients b_n satisfy the inequalities

$$|b_n| \leqq \frac{2\alpha^*}{R^n}, \qquad n = 1, 2, \cdots . \tag{5.12}$$

We conclude from (5.12) that $|b_n|$ is a bounded function of C. On the other hand b_n is a polynomial in C with coefficients which are functions of the coefficients p_k^*, $k\geqq 1$, and $\alpha^*\geqq 1/2$. For example,

$$b_1 = a_1^*(C) = -\frac{Cp_1^*}{2\alpha^*}, \qquad b_2 = 2a_2^*(C) - (a_1^*(C))^2 = -\frac{[C^2p_1^{*2} + 4\alpha^2Cp_2^*]}{4\alpha^2(2\alpha+1)}.$$

It follows from this point of view that $|b_n|$ cannot be a bounded function of C unless $p_k^*=0$ for $k\geqq 1$. The apparent contradiction is eliminated only if either

(5.13) (i) $W_C'(r)>0$ for all r in the interval $0<r<R<1$ whenever $0\leqq C \leqq C(R)<\infty$, while at the same time $W_C'(r)<0$ for some value of $r<R$ and for every C such that $C(R)<C<C(R)+\delta$, $\delta>0$ arbitrarily small;

or

(5.14) (ii) $z^2p^*(z)$ is identically a constant.

In the second case ($z^2p^*(z)=$constant p_0^*) the solution $W_C(z)$ of (4.2) is the same as $W_0(z)$ $(C=0)$ which we have seen in (4.14) to be z^{α^*}. For this function $W_C'(r)>0$ for arbitrary $C\geqq 0$ and all positive r. Thus $C(R)=\infty$. In all other cases $C(R)$ is finite. In what follows we shall suppose that this trivial case is ruled out.

We shall show now that for any fixed R in the range $0<R<1$

$$W'_{C(R)}(R) = 0. \tag{5.15}$$

Thus it is possible to determine the value of $C(R)$ by finding, for fixed R, the smallest positive root $C=C(R)$ of the equation $W_C'(R)=0$.

To prove (5.15) we note by (5.13) that for each $\delta>0$, and for some $r=r(\delta)$, $0<r(\delta)<R$, we have $W'_{C(R)+\delta}\{r(\delta)\}\leqq 0$. Let $\{\delta_n\}$ be a sequence of values of δ for which $\delta_n>0$, $\lim_{n\to\infty}\delta_n=0$, $\lim_{n\to\infty} r(\delta_n)=r_0$ exists. Then, obviously, $0\leqq r_0 \leqq R$. We have already seen that $W_C'(r)$ is continuous in C and r. Consequently, since $W'_{C(R)+\delta_n}\{r(\delta_n)\}\leqq 0$, we have in the limit as $\delta_n\to 0$ the inequality $W'_{C(R)}(r_0)\leqq 0$. But $W'_{C(R)}(r)>0$ for $0<r<R$, so that in particular $W'_{C(R)}(r_0)\geqq 0$. We must conclude, therefore, that not only does $W'_{C(R)}(r_0)=0$, but $r_0=0$ or R. However, $\lim_{r\to 0} W'_{C(R)}(r)$ is never zero if $\alpha^*>0$. This implies then that $r_0\neq 0$. Thus $r_0=R$ and $W'_{C(R)}(R)=0$.

We note then by Theorem A and equality (5.15) that the function $[rW'_{C(R)}(r)/W_{C(R)}(r)]$ decreases from α^* to 0 as r increases from 0 to $R<1$.

Since it is possible to determine $C(R)$, and since $C(R)$ is obviously a nonincreasing function of R bounded below by zero, it is natural to seek the one-sided limit of $C(R)$ as $R\to 1-0$. Thus, to each given function $z^2p^*(z)$ there corresponds a universal constant $A=A(p^*)$ defined as

$$A = A(p^*) = \lim_{R\to 1-0} C(R) \tag{5.16}$$

which is finite, except when $z^2p^*(z)\equiv$constant p_0^* in which case $A=\infty$. In other words, A is the largest value of C for which $W_C'(r)>0$ for all values of r in the interval $0<r<1$. To see this we note from (5.13) that $W_C'(r)>0$ for all r in $0<r<R<1$ when $0\leqq C\leqq C(R)$, and in particular $W_C'(r)>0$ for all r in $0<r<R<1$ when $0\leqq C\leqq A$. Since A is independent of R and R may be taken as near to 1 as we like, we have $W_C'(r)>0$ for all r in $0<r<1$ for $0\leqq C\leqq A$. Thus

$$W_A'(r) > 0 \qquad \text{for all } r \text{ in } 0 < r < 1. \tag{5.17}$$

On the other hand, we have, from (5.13), $W_C'(r)\leqq 0$ for $C(R)<C<C(R)+\delta$ for all small $\delta>0$ for at least one value of r in $0<r<1$. It follows then that for small δ and R near enough to 1 we have, for all $\epsilon>0$ arbitrarily small,

$$A \leqq C(R) < A + \epsilon < C(R) + \delta$$

in which case

$$W_C'(r) \leqq 0 \qquad \text{for } C = A + \epsilon \tag{5.18}$$

for all small $\epsilon>0$ for some r in $0<r<1$. Because of (5.17) and (5.18) we have shown that for each $\epsilon>0$ there exists some $r=r(\epsilon)$ in $0<r<1$ for which

$$W_A'(r) > 0, \qquad W_{A+\epsilon}'(r) \leqq 0. \tag{5.19}$$

(5.17) and (5.19) show that A is the largest value of C for which $W_C'(r)>0$ for all r in $0<r<1$.

We remark also that for every $\delta_1>0$, there exists a $\delta\leqq\delta_1$ for which $C(R)=A+\delta$ for some R in $0<R<1$, in which case $W_{A+\delta}'(R)=0$. If this were not so, since $C(R)$ is nonincreasing and $A=\lim_{r\to 1-0} C(R)$ we would have $C(R)=A$ for an interval $1-\epsilon<R<1$. In that case $W_A'(R)=0$ for an $R<1$. This contradicts the fact that $W_A'(r)>0$ for all r in $0<r<1$ as we have shown above by (5.17). We conclude then that A is the largest value of C for which $|z|\, W_C'(|z|)/W_C(|z|)>0$, when $z^2p^*(z)$ is not a constant, and $|z|<1$.

We shall presently give examples of functions $p^*(z)$ for which positive constants $A(p^*)$ are determined.

It is clear also that Theorem A may be restated with $A(p^*)$ replacing C, and inequalities (5.4), (5.5), and (5.6) then hold for $|z|<1$ with $C=A(p^*)$.

6. **The main theorem.** Let us now define $F(z)$ as in (1.5)

$$F(z) = \{W(z)\}^{1/\alpha} = z + \cdots . \tag{6.1}$$

Similarly, we write

$$F_A(z) = \{W_A(z)\}^{1/\alpha^*} = z + \cdots . \tag{6.2}$$

Then both $F(z)$ and $F_A(z)$ are regular and single-valued in $|z|<1$. Furthermore,

$$\Re\left\{\alpha e^{i\gamma}\frac{zF'(z)}{F(z)}\right\} = \Re\left\{e^{i\gamma}\frac{zW'(z)}{W(z)}\right\}, \tag{6.3}$$

$$\alpha^*\Re\left\{\frac{zF_A'(z)}{F_A(z)}\right\} = \Re\left\{\frac{zW_A'(z)}{W_A(z)}\right\}. \tag{6.4}$$

By Theorem A we have

$$\Re\left\{\alpha e^{i\gamma}\frac{zF'(z)}{F(z)}\right\} \geqq \frac{|z|\,W_A'(|z|)}{W_A(|z|)}\cos\gamma \geqq 0, \qquad |z|<1, \tag{6.5}$$

$$\Re\left\{\frac{zF_A'(z)}{F_A(z)}\right\} \geqq \frac{1}{\alpha^*}\,\frac{|z|\,W_A'(|z|)}{W_A(|z|)} > 0, \qquad |z|<1. \tag{6.6}$$

Thus, $F(z)$ is schlicht and spiral-like in $|z|<1$ for $R\alpha \geqq 1/2$. Furthermore, $F_A(z)$ is schlicht and star-like in $|z|<1$ for $\alpha^* \geqq 1/2$. Since equality signs hold in (6.6) when z is positive, and since we have seen that $W'_{A+\epsilon}(R)=0$ for some R in $0<R<1$ and arbitrarily small but positive ϵ, we conclude that $F_{A+\epsilon}(z)$ is not schlicht no matter how small $\epsilon>0$ is taken.

We shall show now that, if $A(p^*)>0$ and $z^2p^*(z)$ is not identically a constant, then the radius of univalency (defined to be the largest circle with center at the origin within which the function is both regular and schlicht) of $F_A(z)$ is precisely one. To begin with, let us suppose that $p^*(z)$ has a singularity on $|z|=1$ and $A(p^*)>0$. From the differential equation (6.10) below it follows that $W_A''(z)/W_A(z)$ also has a singularity on $|z|=1$. Thus $W_A(z)$ either has a zero or a singularity on $|z|=1$. In either case $F_A(z)$ cannot be both regular and schlicht in any circle containing the unit circle $|z|=1$. In the second place, if $p^*(z)$ is regular on $|z|=1$, then so is the function $W_A(z)$. In this case, assuming $A(p^*)>0$ and $z^2p^*(z)$ not a constant, we may take $R=1$ in Theorem A, $A(p^*)=C(1)$, and $W_A'(1)=W'_{C(1)}(1)=0$. Thus, in this second case, the derivative of $F_A(z)$ vanishes on the unit circle. In either of the two cases we conclude that the radius of univalency for $F_A(z)$ is one.

It seems desirable at this point to summarize our conclusions in the following theorem, the principal object of this paper.

THE MAIN THEOREM. *Let the nonconstant function*

$$z^2p^*(z) = p_0^* + p_1^*z + \cdots + p_n^*z^n + \cdots \tag{6.7}$$

be regular for $|z|<1$, *real on the real axis and* $p_0^* \leqq 1/4$. *Let*

$$\Re\{z^2p^*(z)\} \leqq |z|^2p^*(|z|) \qquad \textit{for } |z|<1. \tag{6.8}$$

Let $A = A(p^*)$ be the universal constant associated with $p^*(z)$ as determined by (5.15) *and* (5.16). *Let*

$$(6.9)\qquad W_A(z) = z^{\alpha^*}\sum_{n=0}^{\infty} a_n^* z^n, \qquad a_0^* = 1,\ |z| < 1,$$

be the unique solution of

$$(6.10)\qquad W'' + \left\{A\left(p^*(z) - \frac{p_0^*}{z^2}\right) + \frac{p_0^*}{z^2}\right\} W = 0$$

corresponding to the larger root α^* of the indicial equation. Then the function

$$(6.11)\qquad F_A(z) = \{W_A(z)\}^{1/\alpha^*} = z + \cdots$$

is regular, single-valued, schlicht and star-like with respect to the origin in $|z| < 1$, and is not both regular and schlicht in any larger circle whenever $A(p^*) > 0$, and $z^2p^*(z)$ is not a constant. For arbitrarily small and positive ϵ the function $F_{A+\epsilon}(z)$ is not schlicht in $|z| < 1$.

Let $z^2p(z)$ be regular for $|z| < 1$, and γ a real constant ($|\gamma| \leqq \pi/2$) for which in $|z| < 1$

$$(6.12)\qquad \Re\{e^{i\gamma}z^2p(z)\} \leqq \cos\gamma\{A(|z|^2p^*(|z|) - p_0^*) + p_0^*\}$$

where $A = A(p^*)$. Let

$$(6.13)\qquad W(z) = z^{\alpha}\sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1,\ |z| < 1,$$

be the unique solution of

$$(6.14)\qquad W'' + p(z)W = 0$$

corresponding to the root α, with the larger real part, of the indicial equation. Then the function

$$(6.15)\qquad F(z) = \{W(z)\}^{1/\alpha} = z + \cdots$$

is regular, single-valued, schlicht and spiral-like in $|z| < 1$. The constant $A = A(p^*)$ is the largest possible one.

We remark that if $z^2p^*(z)$ is a constant p_0^*, then $A(p^*) = \infty$ and, for all C, $W_C(z) = z^{\alpha^*}$, $2\alpha^* = 1 + (1 - 4p_0^*)^{1/2}$, in which case the function (6.11) is the trivial function z. However, in this case the right-hand side of (6.12) is indeterminate, as indeed is (6.10). If it should happen that for some functions $p(z)$ the real part of $\{e^{i\gamma}z^2p(z)\}$ is bounded above by some constant K, then we may deduce from the theory of functions with bounded real part that

$$(6.16)\qquad \Re\{e^{i\gamma}z^2p(z)\} \leqq \frac{2K|z|}{1+|z|} + \frac{1-|z|}{1+|z|}\cdot\Re(p_0e^{i\gamma}), \qquad |z| < 1.$$

In this case, rather than take $z^2p^*(z)$ a constant in our theorem above we may take $z^2p^*(z)=z(1+z)^{-1}+p_0^*$, and the value of $A(p^*)$ which goes with this choice. In this way an appropriate value for K is determined.

COROLLARY 1. *The main theorem holds in particular if* (6.8) *is replaced by the condition that* $\{z^2p^*(z)-p_0^*\}$ *be convex in the direction of the imaginary axis for* $|z|<1$, *and if* (6.12) *is replaced by the conditions that the function* $\{z^2p(z)-p_0\}$ *be subordinate to the function* $A(p^*)\{z^2p^*(z)-p_0^*\}$ *in* $|z|<1$ *and that* $\Re p_0\leqq p_0^*$.

Since $\{z^2p^*(z)-p_0^*\}$ is to be convex in the direction of the imaginary axis, and real on the real axis, it follows that the $\max_{|z|=r}\Re\{z^2p^*(z)\}$ occurs for $z=r$ and (6.8) then holds. If also

$$\{z^2p(z) - p_0\} \prec\prec A(p^*)\{z^2p^*(z) - p_0^*\}, \tag{6.17}$$

and if $\Re p_0\leqq p_0^*$, we have

$$\Re\{z^2p(z) - p_0\} \leqq A(p^*)\{r^2p^*(r) - p_0^*\} \tag{6.18}$$

and

$$\Re\{z^2p(z)\} \leqq A(p^*)\{r^2p^*(r) - p_0^*\} + p_0^*. \tag{6.19}$$

Thus (6.12) holds for $\gamma=0$. This completes the proof of Corollary 1.

7. **Illustrative examples.** Since $z^2p^*(z)$ was chosen to be not a constant in the main theorem, we shall take the next simplest case for our first illustration.

EXAMPLE 1. Let $z^2p^*(z)=p_0^*+z$, $p_0^*\leqq 1/4$. It will be convenient to write

$$p_0^* = \alpha^* - \alpha^{*2}, \qquad \alpha^* \geqq 1/2.$$

Equation (4.2) becomes

$$\frac{d^2W}{dz^2} + \left(\frac{C}{z} + \frac{\alpha^* - \alpha^{*2}}{z^2}\right)W = 0. \tag{7.1}$$

The solution (4.5) of (7.1) is

$$W_C(z) = \Gamma(2\alpha^*)z^{\alpha^*}\sum_{n=0}^{\infty}\frac{(-Cz)^n}{n!\Gamma(n+2\alpha^*)}, \tag{7.2}$$

$$W_C(z) = \frac{\Gamma(2\alpha^*)}{C^{\alpha^*}}(Cz)^{1/2}J_{2\alpha^*-1}(2(Cz)^{1/2}). \tag{7.3}$$

The equation $W_C'(R)=0$ leads to

$$J_{2\alpha^*-1}(2(CR)^{1/2}) + 2(CR)^{1/2}J'_{2\alpha^*-1}(2(CR)^{1/2}) = 0. \tag{7.4}$$

Thus

$$C(R) = \frac{X_1^2(\alpha^*)}{4R}, \tag{7.5}$$

where $X_1 = X_1(\alpha^*)$ is the smallest positive root of the equation

$$J_{2\alpha^*-1}(X) + XJ'_{2\alpha^*-1}(X) = 0. \tag{7.6}$$

$$A = A(p^*) = \lim_{R\to 1-0} C(R) = \frac{X_1^2(\alpha^*)}{4}\cdot \tag{7.7}$$

$$W_A(z) = \left(\frac{2}{X_1}\right)^{2\alpha^*-1} \Gamma(2\alpha^*) z^{1/2} J_{2\alpha^*-1}(X_1 z^{1/2}). \tag{7.8}$$

$$F_A(z) = \left[\left(\frac{2}{X_1}\right)^{2\alpha^*-1} \Gamma(2\alpha^*) z^{1/2} J_{2\alpha^*-1}(X_1 z^{1/2})\right]^{1/\alpha^*} = z + \cdots. \tag{7.9}$$

The function $F_A(z)$ of (7.9) is schlicht and star-like in $|z| \leqq 1$, and its derivative vanishes at $z=1$. Thus the radius of univalency of $F_A(z)$ has the value 1.

If $\alpha^* = 1$ $(p_0^* = 0)$, we have as a special case the result that the function

$$\phi(z) = \frac{2}{X_1} z^{1/2} J_1(X_1 z^{1/2}) = z + \cdots, \tag{7.10}$$

where X_1 is the smallest positive zero of $J_0(X)$, $X_1 = 2.405 \cdots$, is schlicht and star-like in $|z| \leqq 1$, but is not schlicht in any larger circle. As a consequence we have the theorem

THEOREM 1. *Let $zp(z)$ be regular for $|z| < 1$, and*

$$\Re\{z^2 p(z)\} \leqq \frac{X_1^2}{4} |z| \qquad \text{for } |z| < 1,$$

where X_1 is the smallest positive zero of $J_0(X)$ $(X_1^2/4 = 1.4460 \cdots)$. Let

$$W = W(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \qquad |z| < 1,$$

be the unique solution $W = W(z)$, $W(0) = 0$, $W'(0) = 1$, of the equation

$$W'' + p(z) W = 0.$$

Then $W = W(z)$ is schlicht and star-like in $|z| < 1$. The constant $X_1^2/4$ is a best possible one.

EXAMPLE 2. *Let $z^2 p^*(z) = z^2$, $\alpha^* = 1$.* Equation (4.2) becomes

$$\frac{d^2 W}{dz^2} + CW = 0. \tag{7.11}$$

The solution (4.5) of (7.11) is

$$W_C(z) = C^{-1/2} \sin (C^{1/2}z). \tag{7.12}$$

We find $W'_C(R)=0$ for $C^{1/2}R=\pi/2$. Thus

$$C(R) = \frac{\pi^2}{4R^2}, \qquad A(p^*) = \frac{\pi^2}{4} \cdot \tag{7.13}$$

$$W_A(z) = \frac{2}{\pi} \sin \left(\frac{\pi}{2} z\right) = z + \cdots . \tag{7.14}$$

$W_A(z)$ of (7.14) has a radius of univalency equal to 1 and its derivative vanishes at $z=1$. We then have the theorem

THEOREM 2. *Let $zp(z)$ be regular in $|z|<1$ and $\Re\{z^2p(z)\} \leqq (\pi^2/4)|z|^2$ in $|z|<1$. Then the unique solution $W=W(z)$, $W(0)=0$, $W'(0)=1$ of*

$$W'' + p(z)W = 0$$

is schlicht and star-like for $|z|<1$. The constant $\pi^2/4$ is a best possible one.

EXAMPLE 3. Let $z^2p^*(z)=z/(1+z)$, $\alpha^*=1$. In this case $z^2p^*(z)$ is a convex function, real on the real axis. Thus for $|z|=r<1$

$$\max_{|z|=r} \Re\{z^2p^*(z)\} = \frac{r}{1+r} = |z|^2p^*(|z|). \tag{7.15}$$

The solution $W_C(z)$, $W_C(0)=0$, $W'_C(0)=1$ of

$$W'' + \frac{C}{z(1+z)} W = 0 \tag{7.16}$$

is

$$W_C(z) = \frac{-1}{C} \sum_{n=1}^{\infty} \prod_{k=1}^{n} \left\{\frac{(2k-3)^2-1}{4} + C\right\} \frac{(-z)^n}{(n-1)!n!}, \tag{7.17}$$

$$W_C(z) = zF\left(\frac{1+(1-4C)^{1/2}}{2}, \frac{1-(1-4C)^{1/2}}{2}; 2; -z\right) \tag{7.18}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)n!} z^n, \qquad |z| < 1. \tag{7.19}$$

The equation $W'_C(R)=0$ leads to

$$F(\alpha, \beta; 2; -R) - RF'(\alpha, \beta; 2; -R) = 0, \tag{7.20}$$

where

$$2\alpha = 1 + (1-4C)^{1/2}, \qquad 2\beta = 1 - (1-4C)^{1/2}, \qquad \alpha + \beta = 1. \tag{7.21}$$

The equation (7.20) may be written as

$$(7.22) \qquad (1-\beta)F(\alpha, \beta; 2; -R) + \beta F(\alpha, \beta+1; 2; -R) = 0.$$

The series for $F(\alpha, \beta; 2; -1)$ converges absolutely while the series for $F(\alpha, \beta+1; 2; -1)$ converges conditionally for the values of α and β given in (7.21).

Using the integral representation

$$(7.23) \qquad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,$$

valid when $|z|<1$, $\Re\gamma>\Re\beta>0$, in (7.22) we find that the left side of equation (7.22) is positive for $0<C\leqq 1/4$. Letting

$$(7.24) \qquad \begin{aligned} s &= \frac{1}{2}(4C-1)^{1/2} > 0, \\ C &= s^2 + \frac{1}{4}, \end{aligned}$$

and equating to zero the real part of the integral representation of the left side of (7.22) we obtain, after considerable simplification, the equation

$$(7.25) \qquad \int_0^1 \frac{\cos\left[s \log\left((1-t)/(t+Rt^2)\right)\right]}{\left[t(1-t)(1+Rt)\right]^{1/2}} dt = 0.$$

$$(7.26) \qquad A(p^*) = \lim_{R\to 1-0} C(R) = \lim_{R\to 1} s^2(R) + \frac{1}{4} = s_1^2 + \frac{1}{4}$$

where s_1 is the smallest positive zero of the equation (7.25) when $R=1$. Putting $e^x=(1-t)(t+t^2)^{-1}$ in (7.25) we find that s_1 is the smallest positive zero of the function $\phi(s)$ defined as

$$(7.27) \qquad \phi(s) = \int_0^\infty (3+\cosh x)^{-1/2} \cos sx dx.$$

By considering the contour integral

$$(7.28) \qquad \oint e^{(si+1/2)z}(e^{2z}+6e^z+1)^{-1/2}dz = 0,$$

where the contour is the rectangle with corners at $z=0$, R, $R+2\pi i$, $2\pi i$ and a slit, parallel to the real axis and joining the points πi and $2\ln(2^{1/2}+1)+\pi i$, it is possible to show, after letting $R\to\infty$ and taking real parts, that

$$(7.29) \qquad \phi(s) \equiv \operatorname{sech} s\pi \int_0^{2\ln(2^{1/2}+1)} (3-\cosh x)^{-1/2} \cos sx dx.$$

From (7.29) it can be shown that

$$(7.30)\qquad \begin{aligned} &\phi(s) > 0, \quad 0 \leqq s \leqq \frac{\pi}{4}\,[\ln(2^{1/2}+1)]^{-1}, \\ &\phi(s) < 0, \quad \frac{\pi}{2}\,[\ln(2^{1/2}+1)]^{-1} < s < \frac{3\pi}{4}\,[\ln(2^{1/2}+1)]^{-1}. \end{aligned}$$

The existence of the zero s_1 of $\phi(s)$ follows from (7.30). It lies between 1.3 and 1.4. We omit the details of the proof of this statement.

If $zp(z)$ is regular in $|z|<1$ and

$$(7.31)\qquad \Re\{z^2p(z)\} \leqq A(p^*)/2 \qquad \text{in } |z| < 1,$$

where $A(p^*)=s_1^2+1/4$, and s_1 is the smallest positive zero of $\phi(s)$, determined by (7.27) or (7.29), then for $|z|<1$

$$(7.32)\qquad \Re\{z^2p(z)\} \leqq A(p^*)\,\frac{|z|}{1+|z|} = A(p^*)\,|z|^2p^*(|z|),$$

and (6.12) is satisfied. From this we have the theorem:

THEOREM 3. *If $zp(z)$ is regular in $|z|<1$, and if*

$$\Re\{z^2p(z)\} \leqq \frac{1}{2}s_1^2 + \frac{1}{8}, \qquad |z| < 1,$$

where s_1 is the smallest positive zero of the function

$$\phi(s) = \int_0^\infty (3+\cosh x)^{-1/2}\cos sx\,dx,$$

then the unique solution $W=W(z)$ of the form

$$W = z + \sum_{n=2}^{\infty} a_nz^n, \qquad |z| < 1,$$

of the equation

$$\frac{d^2W}{dz^2} + p(z)W = 0,$$

is schlicht and star-like in $|z|<1$. The constant $s_1^2/2+1/8$ cannot be replaced by a larger one.

We remark that for $C=A(p^*)=s_1^2+1/4$ the hypergeometric function (multiplied by z) in (7.18) is schlicht in $|z|\leqq 1$ and its derivative vanishes at $z=1$. This solution corresponds to a choice of

$$(7.33)\qquad z^2p(z) = A(p^*)z^2p^*(z) = A(p^*)z(1+z)^{-1}$$

in which case $p_0=0$, $\alpha=1$.

If $p_0\neq 0$ and if $z^2p(z)$ is regular in $|z|<1$, Theorem 3 could have been stated in a somewhat more general form provided we assume $\Re p_0\leqq 0$ and use (6.16) with $\gamma=0$. In this case $\alpha\neq 1$.

EXAMPLE 4. Let

$$z^2p^*(z) = \sum_{n=0}^{\infty} p_n^* z^n, \qquad p_0^* \leqq \frac{1}{4}, \tag{7.34}$$

be regular in $|z|<1$ with $p_n^*\geqq 0$, $n=1, 2, \cdots$. Suppose $|z|=1$ is a natural boundary for $z^2p^*(z)$. This is the situation if, for instance, the series has sufficiently large gaps. Since none of the coefficients is negative, the condition

$$\max_{|z|=r} \Re\{z^2p^*(z)\} = |z|^2p^*(|z|) \tag{7.35}$$

of the main theorem is fulfilled. We then determine the constant $A=A(p^*)$ by (5.15) and (5.16). If $A>0$ we see from (6.9) and (6.10) that the solution $F_A(z)$ of (6.11) corresponding to our choice of $z^2p^*(z)$ in this example is schlicht and star-like in $|z|<1$, and, moreover, has the unit circle as a natural boundary. Thus we have a device for constructing schlicht functions with natural boundaries whenever $A(p^*)$ can be determined in a constructive way, and provided it is not zero.

EXAMPLE 5. That $A=A(p^*)$ can sometimes be zero is shown by the following illustration. Let

$$z^2p^*(z) = z^2(1-z^2)^{-2}. \tag{7.36}$$

Here $p_0^*=0$. The solution [3] corresponding to $\alpha^*=1$ of

$$W'' + \frac{C}{(1-z^2)^2} W = 0, \qquad C \geqq 0, \tag{7.37}$$

is

$$W_C(z) = (1-z^2)^{1/2}\cdot\frac{((1+z)/(1-z))^{\delta/2} - ((1-z)/(1+z))^{\delta/2}}{2\delta}, \tag{7.38}$$

$$\delta = (1-4C)^{1/2} \neq 0.$$

$$W_C'(r) = [(\delta-r)(1+r)^{\delta} + (\delta+r)(1-r)^{\delta}] \div 2\delta(1-r^2)^{(1+\delta)/2}. \tag{7.39}$$

Let $y=y(\delta)$ be the numerator of (7.39). For values of r sufficiently close to, but less than, one and for $\epsilon>0$ arbitrarily small we have

$$y = 2r(1-r)^r > 0 \qquad \text{when } \delta = r < 1,$$

$$y = -\epsilon r(1+r)^{(1-\epsilon)r} + (2-\epsilon)r(1-r)^{(1-\epsilon)r} < 0 \qquad \text{when } \delta = (1-\epsilon)r.$$

Thus $y=0$ for at least one root $\delta=\delta_0$, $(1-\epsilon)r<\delta_0<r$. As $r\to 1$, $\delta_0\to 1$, since ϵ

may be taken arbitrarily small. In this case $C=C(r)\to 0$. Thus $A(p^*)=0$, $W_A(z)\equiv z$.

Although $W_C(z)$ in (7.38) is indeterminate when $\delta=0$ ($C=1$), a limiting process gives

$$W_1(z) = \frac{1}{2}(1-z^2)^{1/2}\log\left(\frac{1+z}{1-z}\right). \tag{7.40}$$

The derivative of $W_1(z)$ vanishes within the unit circle. Since $W_1(z)$ is therefore not schlicht in $|z|<1$, it is sufficient to consider as we did only the range $0\leqq C<1$ for $W_C(z)$ in (7.38) (δ real and positive).

8. **Concluding remarks.** Throughout this paper we have confined our investigation to the solution of

$$W'' + p(z)W = 0, \qquad z^2p(z) \text{ regular in } |z| < 1, \tag{8.1}$$

which corresponds to that root α of the indicial equation for solutions about the origin for which the real part of α is the larger (or, if the real parts are both equal, to a solution about the origin which does involve $\log z$). The reason for this is fairly obvious: the integrals in the Green's Transform (3.4) do not exist for $\Re\alpha<1/2$.

This, however, poses the question as to whether our main theorem may not still have a counterpart for the other root β, if we assume $\Re\beta>0$ and employ a modified method of proof. I am leaving this question open for further investigation, but point out here that the Green's transform may be rewritten so that the integrals exist for $\Re\beta>0$. We multiply (8.1) by $z\overline{W}dz$ and integrate from 0 to z, $|z|<1$. This gives

$$\int_0^z z\overline{W}(z)W''(z)dz + \int_0^z zp(z)\,|\,W(z)\,|^2dz = 0. \tag{8.2}$$

Integrating by parts, we obtain

$$\begin{aligned} |\,W(z)\,|^2\frac{zW'(z)}{W(z)} &= \int_0^z |\,W'(z)\,|^2 z\overline{dz} + \int_0^z W'(z)\overline{W}(z)dz \\ &\quad - \int_0^z zp(z)\,|\,W(z)\,|^2dz. \end{aligned} \tag{8.3}$$

If the path of integration from 0 to $z=re^{i\theta}$ is a straight line segment, $\theta=$constant, we have

$$\begin{aligned} |\,W(z)\,|^2\Re\left\{\frac{zW'(z)}{W(z)}\right\} &= \int_0^r |\,W'\,|^2\rho d\rho + \int_0^r \Re\left\{\frac{zW'}{W}\right\}_{|z|=\rho} |\,W\,|^2\frac{d\rho}{\rho} \\ &\quad - \int_0^r \Re\{z^2p(z)\}_{|z|=\rho}|\,W\,|^2\frac{d\rho}{\rho}. \end{aligned} \tag{8.4}$$

It is seen at once that in this modified form Green's transform involves integrals which exist for $\Re\beta > 0$. However, an additional term has been added to the formula which means that some further modifications of attack on the problem are necessary to obtain results for the case $\Re\beta > 0$ analogous to those found in this paper.

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