

A STUDY OF α -VARIATION. I.

BY

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This paper is based on the notion of the higher variation of a function introduced by N. Wiener [9] while studying the Fourier coefficients of a function with bounded variation. L. C. Young applied this idea to derive a new existence theorem for Stieltjes integration and later collaborated with E. R. Love in publishing a number of papers on subjects related to this concept.

PRELIMINARIES

1.1. Suppose that $f(x)$ is a real- or complex-valued function defined over $a \leq x \leq b$. For $0 < \alpha \leq 1$, we define the α -variation of $f(x)$ over this interval as the least upper bound of the sums

$$\left\{ \sum_{n=1}^N |f(x_n) - f(x_{n-1})|^{1/\alpha} \right\}^\alpha$$

taken over all subdivisions $a = x_0 < \dots < x_N = b$, and we denote this upper bound by

$$V_\alpha\{f(x); a \leq x \leq b\} \quad \text{or} \quad V_\alpha\{f(x); x \in I\},$$

where I is the interval $a \leq x \leq b$. Similarly we define the 0-variation (or oscillation) of $f(x)$ over this interval as the least upper bound of the difference $|f(x'') - f(x')|$ for $a \leq x' < x'' \leq b$, and we denote this upper bound by

$$V_0\{f(x); a \leq x \leq b\} \quad \text{or} \quad V_0\{f(x); x \in I\}.$$

It is often convenient to consider the α -variation of a function over an interval, I , which is open or half-open, and we can appropriately define the symbol

$$V_\alpha\{f(x); x \in I\}$$

for $0 \leq \alpha \leq 1$.

Suppose that $\{x_n\}$ is any set of real or complex numbers. For $0 < \alpha \leq 1$, we denote by

$$\left\{ \sum_n |x_n|^{1/\alpha} \right\}^\alpha$$

the least upper bound of all sums

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$$\left\{ \sum_k |x_{n_{k-1}} + \dots + x_{n_k}|^{1/\alpha} \right\}^\alpha,$$

where $\{n_k\}$ is any appropriate finite sequence. When $\alpha=0$, we let

$$\left\{ \sum_n |x_n|^{1/\alpha} \right\}^\alpha \quad \text{and} \quad \left\{ \sum_n \left| \sum x_n \right|^{1/\alpha} \right\}^\alpha$$

denote the least upper bounds for $|x_n|$ and $|x_m + \dots + x_n|$ respectively.

1.2. For $0 \leq \alpha \leq 1$, we say that the function $f(x)$ is in W_α over the interval $0 \leq x \leq 1$ if $f(x)$ has bounded α -variation over this interval; W_0 is simply the class of bounded functions. The sum of two functions in W_α is also in W_α since, by Minkowski's inequality,

$$V_\alpha \{f(x) + g(x); x \in I\} \leq V_\alpha \{f(x); x \in I\} + V_\alpha \{g(x); x \in I\}.$$

Similarly, for $0 \leq \alpha < \beta \leq 1$, it follows from Jensen's inequality that

$$V_\alpha \{f(x); x \in I\} \leq V_\beta \{f(x); x \in I\},$$

and hence a function in W_β is also in W_α .

If, for $0 \leq \alpha \leq 1$, $f(x)$ has period 1 and is in the class $\text{Lip}(\alpha)$, i.e. if for some constant C

$$|f(x'') - f(x')| < C(x'' - x')^\alpha$$

for each $x' < x''$, $f(x)$ is obviously in W_α and the α -variation of $f(x)$ over any interval of length 1 is less than C . The converse is not true since functions in W_α need not be continuous. However we can prove the following result.

THEOREM 1.2.1. (Cf. [11, p. 455].) *Suppose that $0 \leq \alpha \leq 1$, that $f(x)$ is real and continuous with period 1, and that the α -variation of $f(x)$ over any interval of length 1 is less than 1. There exists a continuous increasing function $\phi(t)$ such that $\phi(t+1) = \phi(t) + 1$ and such that*

$$|f\{\phi(t'')\} - f\{\phi(t')\}| < (t'' - t')^\alpha$$

for each $t' < t''$.

We can suppose that $\alpha > 0$ and that $f(x)$ assumes its positive maximum at $x=0$. Define the function $\gamma(x)$ so that $\{\gamma(x)\}^\alpha$ is equal to the α -variation of $f(x)$ over the interval $0 \leq y \leq x$. $\gamma(x)$ is continuous, $\gamma(x+1) \geq \gamma(x) + \gamma(1)$, and $\gamma(1) < 1$. For each $x > 0$ we can select a subdivision $0 = y_0 < \dots < y_N = x + 1$ such that

$$\gamma(x + 1) < \sum_{n=1}^N |f(y_n) - f(y_{n-1})|^{1/\alpha} + \epsilon.$$

If $y_n < 1 \leq y_{n+1}$, we see by periodicity that

$$|f(y_{n+1}) - f(y_n)|^{1/\alpha} \leq |f(y_{n+1}) - f(1)|^{1/\alpha} + |f(1) - f(y_n)|^{1/\alpha},$$

and hence that $\gamma(x+1) < \gamma(x) + \gamma(1) + \epsilon$. For $x > 0$ we conclude that $\gamma(x+1) = \gamma(x) + \gamma(1)$ and we extend $\gamma(x)$ so this holds for all x .

Let $\theta(x) = \gamma(x) + x\{1 - \gamma(1)\}$. $\theta(x)$ is continuous and increasing for all x ; let $\phi(t)$ be the inverse function. Then $\phi(t+1) = \phi(t) + 1$ and

$$|f\{\phi(t'')\} - f\{\phi(t')\}|^{1/\alpha} \leq \gamma\{\phi(t'')\} - \gamma\{\phi(t')\} < \theta\{\phi(t'')\} - \theta\{\phi(t')\}$$

for each $t' < t''$. This completes the proof.

We also have the following

THEOREM 1.2.2. *Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is in W_α . Then*

$$V_\alpha\{f(y); x \leq y \leq x + h\} = O(h^\alpha)$$

almost everywhere, and similarly on the left.

Assume that $\alpha > 0$ and let E_k be the set of points in $0 \leq x < 1$ for which

$$\limsup_{h \rightarrow 0} h^{-\alpha} V_\alpha\{f(y); x \leq y \leq x + h\} > k.$$

For each x in E_k and each $\epsilon > 0$ there exists an interval, $I(x) \equiv x \leq y \leq x + h$, such that

$$V_\alpha\{f(y); x \leq y \leq x + h\} \geq kh^\alpha$$

and such that $h < \epsilon$. By Vitali's covering theorem there exists a sequence, $\{I(x_n)\}$, of nonoverlapping intervals which cover almost all of E_k and

$$\begin{aligned} \{\text{outer meas } E_k\}^\alpha &\leq \left\{ \sum_{n=1}^\infty |I(x_n)| \right\}^\alpha \\ &\leq \frac{1}{k} \cdot \left\{ \sum_{n=1}^\infty V_\alpha\{f(y); y \in I(x_n)\}^{1/\alpha} \right\}^\alpha \\ &\leq \frac{1}{k} \cdot V_\alpha\{f(x); 0 \leq x \leq 1\}. \end{aligned}$$

The set of points in $0 \leq x < 1$ for which

$$V_\alpha\{f(y); x \leq y \leq x + h\} \neq O(h^\alpha)$$

is contained in E_k for each k and must have zero measure.

COROLLARY 1.2.3. *Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is in W_α . Then*

$$|f(x + h) - f(x)| = O(h^\alpha)$$

almost everywhere, and similarly on the left.

Hence a function in W_α satisfies a Lip (α) condition almost everywhere

but of course not uniformly.

1.3. We need the following lemma in order to study the relation between W_α and the Hardy-Littlewood integrated Lipschitz classes [4, p. 612].

LEMMA 1.3.1. *Suppose that $0 \leq \alpha \leq 1$, that $f(x)$ is a real-valued function with period 1, and that the α -variation of $f(x)$ over any interval of length 1 does not exceed 1. Then the α -variation of $f(x)$ over any interval of length k does not exceed k^α for each positive integer k .*

We need only show for $0 < \alpha \leq 1$ that

$$1.3.2 \quad V_\alpha\{f(x); 0 \leq x \leq k\} \leq k^\alpha.$$

Pick a subdivision $\sigma, 0 = x_0 < \dots < x_N = k$, so that the left-hand side of 1.3.2 is majorized by

$$\left\{ \sum_\sigma |\Delta f|^{1/\alpha} \right\}^\alpha + \epsilon = S^\alpha + \epsilon.$$

We can assume that σ contains two points, c and d , where $d = c + 1$ and where $f(d) = f(c) = \text{Max}_n f(x_n)$ for adding such points to σ does not decrease S . (Cf. proof for 1.2.1.) We have

$$\begin{aligned} S &= \sum_{\sigma \cap [0, c]} |\Delta f|^{1/\alpha} + \sum_{\sigma \cap [c, d]} |\Delta f|^{1/\alpha} + \sum_{\sigma \cap [d, k]} |\Delta f|^{1/\alpha} \\ &= S_1 + S_2 + S_3. \end{aligned}$$

S_2 is majorized by 1, $\{S_1 + S_3\}^\alpha$ is majorized by the α -variation of $f(x)$ over $0 \leq x \leq k - 1$, and 1.3.2 follows by induction.

THEOREM 1.3.3. (Cf. [10, p. 259].) *Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is a measurable real-valued function with period 1. If the α -variation of $f(x)$ over any interval of length 1 never exceeds 1, then*

$$\left\{ \int_0^1 |f(x+h) - f(x)|^{1/\alpha} dx \right\}^\alpha \leq h^\alpha$$

for every $h > 0$.

Assume that $\alpha > 0$ and let $h = m/n$, where m and n are relatively prime positive integers. Then

$$I(h) = \int_0^1 |f(x+h) - f(x)|^{1/\alpha} dx = \sum_{v=1}^n \int_{(v-1)/n}^{v/n} \left| f\left(x + \frac{m}{n}\right) - f(x) \right|^{1/\alpha} dx$$

and, because $f(x)$ has period 1, this last sum is equal to

$$\int_0^{1/n} \left\{ \sum_{v=1}^n \left| f\left(x + \frac{vm}{n}\right) - f\left(x + \frac{(v-1)m}{n}\right) \right|^{1/\alpha} \right\} dx \leq \int_0^{1/n} m dx = h.$$

Hence the theorem is true for rational h . Since $f(x)$ is bounded, $I(h)$ is continuous and the theorem holds for all h .

COROLLARY 1.3.4. *Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is a measurable real-valued function with period 1. Then*

$$\left\{ \int_0^1 |f(x+h) - f(x)|^{1/\alpha} dx \right\}^\alpha \leq 2^\alpha V_\alpha \{f(x); 0 \leq x \leq 1\} h^\alpha$$

for every $h > 0$.

This follows from 1.3.3 since the α -variation of $f(x)$ over any interval of length 1 is majorized by

$$\{V_\alpha \{f(x); 0 \leq x \leq 1\}^{1/\alpha} + V_0 \{f(x); 0 \leq x \leq 1\}^{1/\alpha}\}^\alpha.$$

If $f(x) = e^{2\pi iz}$ and $\alpha = 1/2$, the α -variation of $f(x)$ over any interval of length 2 exceeds 2^α times the α -variation of $f(x)$ over any interval of length 1. Since 1.2.1 implies 1.3.1 when $f(x)$ is continuous, the restriction that $f(x)$ be real is essential in both of these results. The same is true for 1.3.3.

From 1.3.4 it is obvious that any measurable function with bounded α -variation over some interval, $a \leq x \leq b$, is in the Hardy-Littlewood class $\text{Lip}(\alpha, 1/\alpha)$ over that interval. The converse is not true. Hardy and Littlewood [4, p. 621] point out that if

$$f(x) = \log \frac{1}{|x|}, \quad x \neq 0,$$

then, for $h > 0$,

$$\left\{ \int_{-x}^x |f(x+h) - f(x)|^{1/\alpha} dx \right\}^\alpha = O(h^\alpha), \quad 0 < \alpha < 1,$$

while $f(x)$ is not even bounded in the neighborhood of $x=0$.

1.4. The following lemma generalizes a familiar theorem on uniform continuity.

LEMMA 1.4.1. *Suppose that $0 \leq \alpha \leq 1$, that $f(x)$ has bounded α -variation over $0 \leq x \leq 1$, and that $f(x)$ is continuous in this interval. For $\epsilon > 0$ there exists a $\delta > 0$ such that, for $0 \leq x_0 < x_0 + \delta \leq 1$, we have*

$$V_\alpha \{f(x); x_0 \leq x \leq x_0 + \delta\} < \epsilon.$$

When $0 < \alpha \leq 1$, 1.4.1 is an immediate consequence of the following elementary result.

LEMMA 1.4.2. *Suppose that $0 < \alpha \leq 1$ and that $f(x)$ has bounded α -variation in some right-handed neighborhood of the point $x = x_0$. Then*

$$V_\alpha \{f(x); x_0 < x < x_0 + h\} = o(1)$$

as h approaches 0. A similar result holds on the left.

It follows from 1.4.2 that any function, with bounded α -variation over an open interval, has right- and left-handed limits at each point of the interval.

For $0 \leq \alpha \leq 1$, we say that $f(x)$ is in V_α over the interval $0 \leq x \leq 1$ if, given $\epsilon > 0$, there exists a $\delta > 0$ such that, for any set of disjoint intervals $0 \leq x_1 < y_1 \leq \dots \leq x_N < y_N \leq 1$ for which

$$\left\{ \sum_{n=1}^N |y_n - x_n|^{1/\alpha} \right\}^\alpha < \delta,$$

we have

$$\left\{ \sum_{n=1}^N |f(y_n) - f(x_n)|^{1/\alpha} \right\}^\alpha < \epsilon.$$

V_0 is the class of functions continuous over $0 \leq x \leq 1$ and V_1 is the class of functions absolutely continuous over this interval.

The method of proof used in 1.2.2 gives us the following result.

THEOREM 1.4.3. *Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is in V_α . Then*

$$V_\alpha \{f(y); x \leq y \leq x + h\} = o(h^\alpha)$$

almost everywhere, and similarly on the left.

THEOREM 1.4.4 [6]. *Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is measurable and has period 1. $f(x)$ is in V_α if and only if*

$$V_\alpha \{f(x + h) - f(x); 0 \leq x \leq 1\} = o(1)$$

as h approaches 0.

We see that the class W_α includes V_α and, for $0 < \alpha < 1$, it is not difficult to show that the continuous function

$$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n \pi x$$

is in $W_\alpha - V_\alpha$. Hence a continuous function with bounded α -variation over $0 \leq x \leq 1$ is not necessarily in V_α . However, we can prove, for $0 \leq \alpha \leq 1$, that any continuous function in W_α which possesses a finite derivative everywhere in $0 \leq x \leq 1$, except perhaps on an enumerable set, is in V_α [2, Theorem 2.6].

1.5. Suppose that $f(x)$ and $g(x)$ are defined over the interval $0 \leq x \leq 1$ and that $g(x)$ has at most discontinuities of the 1st kind. The Stieltjes integral

1.5.1
$$\int_0^1 f(x) dg(x)$$

exists in the Young sense and is equal to I if, for $\epsilon > 0$, there exists a finite set

E , contained in $0 \leq x \leq 1$, such that, for any subdivision $0 = x_0 < \dots < x_N = 1$ which contains E , we have

$$\left| \sum_{n=1}^N f(\xi_n) \{g(x_n - 0) - g(x_{n-1} + 0)\} + \sum_{n=1}^{N-1} f(x_n) \{g(x_n + 0) - g(x_n - 0)\} \right. \\ \left. + f(0) \{g(0+) - g(0)\} + f(1) \{g(1) - g(1 - 0)\} - I \right| < \epsilon$$

for each set $x_0 < \xi_1 < x_1 < \dots < x_{N-1} < \xi_N < x_N$. L. C. Young [12] has proved the following

THEOREM 1.5.2. *Suppose that $\alpha + \beta > 1$ and that $f(x)$ and $g(x)$ belong to W_α and W_β respectively. If $f(x)$ and $g(x)$ have no common discontinuities, 1.5.1 exists in the Riemann-Stieltjes sense. In any case, 1.5.1 exists in the Young sense.*

This theorem is not true in the limiting case when $\alpha + \beta = 1$. The following result is also an immediate consequence of Young's work.

THEOREM 1.5.3. *Suppose that $\alpha + \beta > 1$ and that $f(x)$ is continuous and in W_α . Suppose also that $\{g_n(x)\}$ is a sequence of uniformly bounded functions with uniformly bounded β -variation over $0 \leq x \leq 1$ which converges to $g(x)$, a function in W_β , on a set which includes the points $x = 0$ and $x = 1$ and which is dense in $0 \leq x \leq 1$. Then*

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) dg_n(x) = \int_0^1 f(x) dg(x), \\ \lim_{n \rightarrow \infty} \int_0^1 g_n(x) df(x) = \int_0^1 g(x) df(x).$$

MOMENT PROBLEMS

2.1. In this chapter we study W_α , the class of functions with bounded α -variation over the interval $0 \leq x \leq 1$, by considering a moment problem. Suppose that $g(x)$ is any function in W_α for $0 < \alpha \leq 1$. We call the numbers

$$\mu_n = \int_0^1 x^n dg(x), \quad n = 0, 1, \dots,$$

the Stieltjes moments of $g(x)$ and we say that $g(x)$ is normalized if

$$g(x) = \frac{1}{2} \{g(x + 0) + g(x - 0)\} \quad \text{for } 0 < x < 1.$$

A uniqueness theorem follows from a known result [8, p. 60] after integration by parts.

THEOREM 2.1.1. *Suppose that $0 < \alpha \leq 1$, that $g(x)$ is normalized and in W_α , and that*

$$\int_0^1 x^n dg(x) = 0$$

for $n=0, 1, \dots$. Then $g(x)$ is identically constant in $0 \leq x \leq 1$.

For an arbitrary sequence of numbers, $\{\mu_n\}$, we define a linear functional over the space of polynomials by letting

$$\mu\{P\} = \mu\left\{\sum_{n=0}^N a_n x^n\right\} = \sum_{n=0}^N a_n \mu_n.$$

Following Hausdorff we define, for $k=0, 1, \dots$ and $n=0, 1, \dots, k$,

$$\lambda_{k,n}(x) = C_n^{k-n} x^n (1-x)^{k-n} \quad \text{and} \quad \lambda_{k,n} = \mu\{\lambda_{k,n}(x)\},$$

where C_r^s is the binomial coefficient

$$\binom{r+s}{r} = \frac{\Gamma(r+s+1)}{\Gamma(r+1)\Gamma(s+1)}.$$

THEOREM 2.1.2. *Suppose that $0 < \alpha \leq 1$. A necessary and sufficient condition that the set of numbers $\{\mu_n\}$ be the Stieltjes moments of a normalized function $g(x)$ in W_α , where $V_\alpha\{g(x); 0 \leq x \leq 1\} \leq 1$, is that*

$$\left\{\sum_{n=0}^k \left|\sum \lambda_{k,n}\right|^{1/\alpha}\right\}^\alpha \leq 1$$

for all k .

Hausdorff [5] has proved this theorem for the case where $\alpha = 1$.

2.2. In order to prove the sufficiency we derive two simple results.

LEMMA 2.2.1. *If $\{\mu_n\}$ is an arbitrary sequence of numbers, then*

$$\sum_{n=0}^k \left(\frac{n}{k}\right)^m \lambda_{k,n} = \mu_m + O\left(\frac{1}{k}\right)$$

for $m=0, 1, \dots$.

Suppose that $f(x)$ is any function defined over the interval $0 \leq x \leq 1$ and consider

$$B_k\{f; x\} = \sum_{n=0}^k f\left(\frac{n}{k}\right) \lambda_{k,n}(x),$$

the Bernstein polynomial of order k for $f(x)$. If $P_m(x)$ is a polynomial of degree m ,

$$B_k\{P_m; x\} = P_m(x) + \sum_{r=1}^{m-1} \frac{Q_{m,r}(x)}{k^r},$$

where the polynomials $Q_{m,r}(x)$ do *not* depend on k and are identically zero when $P_m(x)$ is a constant [7, p. 8]. Setting $P_m(x) = x^m$, we have

$$\sum_{n=0}^k \binom{n}{k}^m \lambda_{k,n} = \mu\{B_k\{x^m; x\}\} = \mu_m + \sum_{r=1}^{m-1} \frac{\mu\{Q_{m,r}\}}{k^r},$$

and 2.2.1 follows.

LEMMA 2.2.2. *Suppose that $0 < \alpha \leq 1$, that $\{g_k(x)\}$ is a sequence of uniformly bounded functions, and that*

$$\liminf_{k \rightarrow \infty} V_\alpha\{g_k(x); 0 \leq x \leq 1\} \leq 1.$$

There exists a function $g(x)$ such that

$$V_\alpha\{g(x); 0 \leq x \leq 1\} \leq 1,$$

and a subsequence $\{k_j\}$ such that $g_{k_j}(x) \rightarrow g(x)$ for every rational x in $0 \leq x \leq 1$, including the end points $x = 0$ and $x = 1$.

We can use the selection principle (or diagonal process) to define $g(x)$ on the rationals. If, for irrational x , we let

$$g(x) = \limsup_{r \rightarrow x} g(r), \quad r \text{ rational,}$$

then $g(x)$ satisfies the conditions of the lemma.

To prove the sufficiency part of 2.1.2, consider the step function $g_k(x)$ where $g_k(0) = 0$ and

$$g_k(x) = \sum_{r=0}^n \lambda_{k,r} \quad \text{for} \quad \frac{n}{k+1} < x \leq \frac{n+1}{k+1}.$$

Since

$$|g_k(x)| \leq V_\alpha\{g_k(x); 0 \leq x \leq 1\} \leq 1,$$

there exists a function $g^*(x)$ such that

$$V_\alpha\{g^*(x); 0 \leq x \leq 1\} \leq 1,$$

and a subsequence $\{k_j\}$ such that $g_{k_j}(x) \rightarrow g^*(x)$ for each rational x in $0 \leq x \leq 1$. Let

$$g(0) = g^*(0), \quad g(1) = g^*(1),$$

2.2.3

$$g(x) = \frac{1}{2} \{g^*(x+0) + g^*(x-0)\} \quad \text{for } 0 < x < 1.$$

The function $g(x)$ is normalized and, from 1.5.3 and 2.2.1, we conclude that

$$\begin{aligned} \mu_m &= \lim_{k \rightarrow \infty} \sum_{n=0}^k \left(\frac{n}{k+1} \right)^m \lambda_{k,n} \\ &= \lim_{k \rightarrow \infty} \int_0^1 x^m dg_k(x) \\ &= \int_0^1 x^m dg^*(x) = \int_0^1 x^m dg(x). \end{aligned}$$

2.3. In order to prove the necessity we require a number of lemmas.

We say that a real function $f(x)$ is unimax in the interval $a \leq x \leq b$ if, for $a \leq c < x < d \leq b$, $f(x) \geq \text{Min} \{f(c), f(d)\}$.

LEMMA 2.3.1. *Suppose that $0 \leq \alpha \leq 1$ and that $X_n = \sum_{m=1}^M a_{n,m} x_m$ for $n = 1, \dots, N$, where $a_{n,m}$ is a non-negative unimax function of m for each n , and where $\sum_{n=1}^N a_{n,m} \leq 1$ for each m . Then*

$$\left\{ \sum_{n=1}^N |X_n|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{m=1}^M | \sum x_m |^{1/\alpha} \right\}^\alpha.$$

We consider 2.3.1 in a slightly different form.

LEMMA 2.3.2. *Suppose that $0 \leq \alpha \leq 1$ and that $X_n = \sum_{m=1}^M a_{n,m} x_m$ for $n = 1, \dots, N$, where $a_{n,m}$ is a non-negative integer and is unimax as a function of m for each n , and where $\sum_{n=1}^N a_{n,m} = R$ for each m . Then*

$$\left\{ \sum_{n=1}^N |X_n|^{1/\alpha} \right\}^\alpha \leq R \left\{ \sum_{m=1}^M | \sum x_m |^{1/\alpha} \right\}^\alpha.$$

LEMMA 2.3.3. *With the hypotheses of 2.3.2 we can write*

$$X_n = \sum_{r=1}^R y_{n,r} = \sum_{r=1}^R \left\{ \sum_{m=1}^M a_{n,m,r} x_m \right\},$$

where $a_{n,m,r}$ is either 0 or 1 and is unimax in m for each n and r , and where $\sum_{n=1}^N a_{n,m,r} = 1$ for each m and r .

Now 2.3.3 is true when $R=1$. Suppose it true for $R=k-1$ and consider the case where $R=k$.

A. There exists n_1 such that $a_{n_1,1} \geq 1$. Let m_1 be the largest m for which $a_{n_1,m} \geq 1$ and define

$$a_{n_1,m,1} = \begin{cases} 1 & \text{for } 1 \leq m \leq m_1, \\ 0 & \text{everywhere else.} \end{cases}$$

It is not difficult to see that $a_{n_1,m} - a_{n_1,m,1}$ is a non-negative integer and is unimax as a function of m .

B. If $m_1 < M$, there exists $n_2 \neq n_1$ such that $a_{n_2, m_1} < a_{n_2, m_1+1}$. Let m_2 be the largest m for which $a_{n_2, m} \geq 1$ and define

$$a_{n_2, m, 1} = \begin{cases} 1 & \text{for } m_1 < m \leq m_2, \\ 0 & \text{everywhere else.} \end{cases}$$

Again $a_{n_2, m} - a_{n_2, m, 1}$ is a non-negative integer and is unimax in m .

C. If $m_2 < M$, we can find $n_3 \neq n_2, n_1$ such that $a_{n_3, m_2} < a_{n_3, m_2+1}$, and we can define m_3 and the set $\{a_{n_3, m, 1}\}$ for $m = 1, \dots, M$. After a finite number of steps we arrive at the place where $m_j = M$. We shall have defined a sequence of distinct integers, n_1, n_2, \dots, n_j , and a set of coefficients, $\{a_{n, m, 1}\}$, for $n = n_1, \dots, n_j$ and $m = 1, \dots, M$. Let $a_{n, m, 1} = 0$ for $n \neq n_1, \dots, n_j$ and $m = 1, \dots, M$.

The set $\{a_{n, m, 1}\}$ satisfies the conditions in 2.3.3, the set $\{a_{n, m} - a_{n, m, 1}\}$ satisfies the hypotheses of 2.3.2 with $R = k - 1$, and 2.3.3 follows by applying the induction hypothesis to

$$X'_n = \sum_{m=1}^M \{a_{n, m} - a_{n, m, 1}\} x_m.$$

For a fixed r , consider the set $\{y_{n, r}\}$. Since each element here is simply the sum of consecutive x_m , we see from 2.3.3 and Minkowski's inequality that

$$\begin{aligned} \left\{ \sum_{n=1}^N |X_n|^{1/\alpha} \right\}^\alpha &= \left\{ \sum_{n=1}^N \left| \sum_{r=1}^R y_{n, r} \right|^{1/\alpha} \right\}^\alpha \\ &\leq \sum_{r=1}^R \left\{ \sum_{n=1}^N |y_{n, r}|^{1/\alpha} \right\}^\alpha \\ &\leq R \left\{ \sum_{m=1}^M | \sum x_m |^{1/\alpha} \right\}^\alpha. \end{aligned}$$

Now 2.3.2 implies 2.3.1 when all the $a_{n, m}$ are rational and $\sum_{n=1}^N a_{n, m} = c_m = 1$. A simple limiting process removes the restriction that the $a_{n, m}$ be rational. When $0 \leq c_m \leq 1$, we can apply our results to the set of linear forms

$$\begin{aligned} \sum_{m=1}^M a_{n, m} x_m, & \quad n = 1, \dots, N, \\ (1 - c_m) x_m, & \quad m = 1, \dots, M, \end{aligned}$$

and show that

$$\left\{ \sum_{n=1}^N |X_n|^{1/\alpha} + \sum_{m=1}^M |(1 - c_m) x_m|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{m=1}^M | \sum x_m |^{1/\alpha} \right\}^\alpha.$$

This completes the proof for 2.3.1. We can assume that no x_m is zero. Hence when $\alpha > 0$, we get strict inequality if, for any m , $c_m < 1$.

LEMMA 2.3.4. *Suppose that $k=0, 1, \dots$, and that $0 \leq m \leq n \leq k$. The function $f(x) = \sum_{\nu=m}^n \lambda_{k,\nu}(x)$ is unimax in $0 \leq x \leq 1$.*

We can assume that $k \geq 1$ and 2.3.4 follows immediately from the identity

$$2.3.5 \quad \frac{d}{dx} \lambda_{k,n}(x) = \begin{cases} -k\lambda_{k-1,n}(x), & n = 0, \\ k\{\lambda_{k-1,n-1}(x) - \lambda_{k-1,n}(x)\}, & 0 < n < k, \\ k\lambda_{k-1,n-1}(x), & n = k. \end{cases}$$

To complete the proof for 2.1.2, fix k and consider any finite sequence $0 = \nu_0 < \dots < \nu_N = k + 1$. If

$$\theta_n = \sum_{\nu_{n-1} \leq \nu < \nu_n} \lambda_{k,\nu} \quad \text{and} \quad \theta_n(x) = \sum_{\nu_{n-1} \leq \nu < \nu_n} \lambda_{k,\nu}(x),$$

then

$$\theta_n = \int_0^1 \theta_n(x) dg(x)$$

and this integral exists in the Riemann-Stieltjes sense for each n . For each subdivision $0 = y_0 < \dots < y_M = 1$, let

$$X_n = \sum_{m=1}^M \theta_n(y_m) \{g(y_m) - g(y_{m-1})\}.$$

$\theta_n(y_m)$ is a non-negative unimax function of m and

$$\sum_{n=1}^N \theta_n(y_m) = \sum_{\nu=0}^k \lambda_{k,\nu}(y_m) = 1.$$

Applying 2.3.1 we get

$$\left\{ \sum_{n=1}^N |X_n|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{m=1}^M \left| \sum g(y_m) - g(y_{m-1}) \right|^{1/\alpha} \right\}^\alpha$$

which completes the argument.

2.4. We have discussed the Stieltjes moments of a function $g(x)$, i.e. the sequence $\{\mu_n\}$ where

$$\mu_n = \int_0^1 x^n dg(x), \quad n = 0, 1, \dots$$

We can also consider the moment sequence

$$\mu_n = \int_0^1 x^n g(x) dx, \quad n = 0, 1, \dots,$$

where the integral is interpreted in the Lebesgue sense. We call such a se-

quence of numbers the Lebesgue moments of $g(x)$ and we have the following

THEOREM 2.4.1. *Suppose that $0 < \alpha \leq 1$. A necessary and sufficient condition that the set of numbers $\{\mu_n\}$ be the Lebesgue moments of a normalized function $g(x)$ in W_α , where*

$$V_\alpha\{g(x); 0 \leq x \leq 1\} \leq 1,$$

is that

$$(k + 1) \left\{ \sum_{n=1}^k \left| \sum \lambda_{k,n} - \lambda_{k,n-1} \right|^{1/\alpha} \right\}^\alpha \leq 1$$

for all k .

The necessity follows immediately from 2.1.2 since, with the help of 2.3.5, we can write

$$\begin{aligned} (k + 1) \{ \lambda_{k,n} - \lambda_{k,n-1} \} &= \int_0^1 (k + 1) \{ \lambda_{k,n}(x) - \lambda_{k,n-1}(x) \} g(x) dx \\ &= \int_0^1 \lambda_{k+1,n}(x) dg(x) \end{aligned}$$

for $0 < n \leq k$.

For the sufficiency, observe that

$$2.4.2 \quad \left| \mu_0 - (k + 1)\lambda_{k,m} \right| \leq \sum_{n=0}^k \left| \lambda_{k,n} - \lambda_{k,m} \right| \leq 1$$

for $0 \leq m \leq k$. Consider the step function $g_k(x)$ where $g_k(0) = (k + 1)\lambda_{k,0}$ and

$$g_k(x) = (k + 1)\lambda_{k,n} \quad \text{for } \frac{n}{k + 1} < x \leq \frac{n + 1}{k + 1}.$$

From 2.4.2 and 2.2.2 it follows that there exists a function $g^*(x)$ such that

$$V_\alpha\{g^*(x); 0 \leq x \leq 1\} \leq 1,$$

and a subsequence $\{k_j\}$ such that $g_{k_j}(x) \rightarrow g^*(x)$ for each rational x in $0 \leq x \leq 1$. Define $g(x)$ as in 2.2.3. From 1.5.3, 2.2.1, and 2.4.2 we can conclude that

$$\begin{aligned} \mu_m &= \lim_{k \rightarrow \infty} \sum_{n=0}^k \left(\frac{n}{k + 1} \right)^m \lambda_{k,n} \\ &= \lim_{k \rightarrow \infty} \int_0^1 x^m g_k(x) dx \\ &= \int_0^1 x^m g^*(x) dx = \int_0^1 x^m g(x) dx. \end{aligned}$$

2.5. Turning back to 2.3.1, we deduce two alternative forms for this inequality which are useful in later work.

LEMMA 2.5.1. *Suppose that $0 \leq \alpha \leq 1$ and that $Y_n = \sum_{m=1}^M b_{n,m}(y_m - y_0)$ for $n = 0, \dots, N$, where $b_{n,m}$ is non-negative for all m and n , where $\sum_{m=1}^M b_{n,m}$ is nondecreasing in n and bounded by 1, and where, for each $0 \leq n' < n'' \leq N$, $b_{n'',m} - b_{n',m}$ is at first nonpositive and then non-negative as m increases from 1 to M . Then*

$$\left\{ \sum_{n=1}^N \left| \sum Y_n - Y_{n-1} \right|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{m=1}^M \left| \sum y_m - y_{m-1} \right|^{1/\alpha} \right\}^\alpha.$$

LEMMA 2.5.2. *Suppose that $0 \leq \alpha \leq 1$ and that $Y_n = \sum_{m=0}^M b_{n,m}y_m$ for $n = 0, \dots, N$, where $b_{n,m}$ is non-negative for all m and n , where $\sum_{m=0}^M b_{n,m} = 1$, and where, for each $0 \leq n' < n'' \leq N$, $b_{n'',m} - b_{n',m}$ is at first nonpositive and then non-negative as m increases from 0 to M . Then*

$$\left\{ \sum_{n=1}^N \left| \sum Y_n - Y_{n-1} \right|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{m=1}^M \left| \sum y_m - y_{m-1} \right|^{1/\alpha} \right\}^\alpha.$$

For 2.5.1, let $x_m = y_m - y_{m-1}$ and pick any sequence of integers, $0 = k_0 < \dots < k_{N'} = N$. If

$$X_n = Y_{k_n} - Y_{k_{n-1}} = \sum_{m=1}^M a_{n,m}x_m,$$

then

$$a_{n,m} = \sum_{\mu=m}^M \{ b_{k_n,\mu} - b_{k_{n-1},\mu} \}.$$

The difference $a_{n,m+1} - a_{n,m} = -\{ b_{k_n,m} - b_{k_{n-1},m} \}$ is at first non-negative and then nonpositive as m increases; hence $a_{n,m}$ is unimax in m . We see that

$$a_{n,1} = \sum_{m=1}^M b_{k_n,m} - \sum_{m=1}^M b_{k_{n-1},m} \geq 0,$$

$$a_{n,M} = b_{k_n,M} - b_{k_{n-1},M} \geq 0,$$

and the unimax property ensures that $a_{n,m}$ is non-negative for all m and n . Finally

$$\sum_{n=1}^{N'} a_{n,m} = \sum_{\mu=m}^M \{ b_{N',\mu} - b_{0,\mu} \} \leq 1,$$

and we can apply 2.3.1.

For 2.5.2 observe that the set $\{ b_{n,m} \}$, for $n = 0, \dots, N$ and $m = 0, \dots, M$, satisfies the hypotheses of 2.5.1. Since the sums

$$\sum_{m=0}^M b_{n,m}(y_m - y_0) \quad \text{and} \quad \sum_{m=0}^M b_{n,m}y_m$$

differ by a constant which is independent of n , the conclusion follows immediately.

From 2.5.2 we can deduce the following result concerning Bernstein polynomials.

THEOREM 2.5.3. *If $0 \leq \alpha \leq 1$,*

$$V_\alpha \{B_k\{f; x\}; 0 \leq x \leq 1\} \leq \left\{ \sum_{m=1}^k \left| \sum f\left(\frac{m}{k}\right) - f\left(\frac{m-1}{k}\right) \right|^{1/\alpha} \right\}^\alpha.$$

Let $0 < x_0 < \dots < x_N < 1$ be any subdivision of the interval $0 < x < 1$. We see that

$$B_k\{f; x_n\} = \sum_{m=0}^k f\left(\frac{m}{k}\right) \lambda_{k,m}(x_n)$$

where $\lambda_{k,m}(x_n)$ is non-negative for all m and n , and where $\sum_{m=0}^k \lambda_{k,m}(x_n) = 1$. For $0 \leq n' < n'' \leq N$,

$$\lambda_{k,m}(x_{n''}) - \lambda_{k,m}(x_{n'}) = \lambda_{k,m}(x_{n''}) \left\{ 1 - \left(\frac{x_{n'}}{x_{n''}}\right)^m \left(\frac{1-x_{n'}}{1-x_{n''}}\right)^{k-m} \right\}.$$

Since $0 < x_{n'} < x_{n''} < 1$, the bracketed quantity is negative for $m=0$, positive for $m=k$, and strictly increasing in m . Applying 2.5.2 completes the proof.

In conclusion we add that 2.3.1, 2.5.1, and 2.5.2 are valid when M and/or $N = \infty$.

A FALTUNG THEOREM

3.1. In one of his papers [11], L. C. Young considered a Stieltjes Faltung of the form

$$3.1.1 \quad s(x) = \int_0^1 f(x, y) dg(y).$$

We present here a theorem suggested by Young's results.

Suppose that $f(x, y)$ and $g(y)$ have period 1 in y and define $F(x, y)$ as the integral $\int_0^1 f(x, y+t)g(t)dt$ which we assume exists in the Lebesgue sense for all $0 \leq x, y \leq 1$. Let

$$3.1.2 \quad \begin{aligned} s_n(x) &= 2^n \{F(x, 0) - F(x, 2^{-n})\} \\ &= 2^n \int_0^1 f(x, t) \{g(t) - g(t - 2^{-n})\} dt \end{aligned}$$

for $n=0, 1, \dots$. Then $s_0(x) \equiv 0$ and the following is easily verified.

LEMMA 3.1.3. *Suppose that $g(y)$ is continuous and that the integral 3.1.1 exists in the Riemann-Stieltjes sense for $x = x_0$. Then $s(x_0) = \lim_{n \rightarrow \infty} s_n(x_0)$ and, for $n \geq 1$, we have*

$$s_n(x) - s_{n-1}(x) = 2^{n-1} \int_0^1 \{f(x, t) - f(x, t + 2^{-n})\} \{g(t) - g(t - 2^{-n})\} dt.$$

Our principal result is as follows.

THEOREM 3.2. (Cf. [11, Theorem 6.1].) *Suppose that $0 < \alpha, \beta, \gamma \leq 1$, $0 < \lambda = \beta + \gamma - 1$, and $\mu = \alpha\lambda/\beta$. Suppose also that $f(x, y)$ and $g(y)$ have period 1 in y , that $g(y)$ is continuous, and that*

$$\begin{aligned} 3.2.1 \quad & V_\alpha \{f(x, y); 0 \leq x \leq 1\} \leq A && \text{for each } y, \\ 3.2.2 \quad & V_\beta \{f(x, y); 0 \leq y \leq 1\} \leq B && \text{for each } x, \\ 3.2.3 \quad & V_\gamma \{g(y); 0 \leq y \leq 1\} \leq C. \end{aligned}$$

If $s(x)$ is the Stieltjes Faltung 3.1.1, then

$$3.2.4 \quad V_\mu \{s(x); 0 \leq x \leq 1\} \leq k(\lambda) A^{\lambda/\beta} B^{1-\lambda/\beta} C,$$

where $k(\lambda)$ is a finite constant.

If $B = 0$ and/or $C = 0$, $s(x) \equiv 0$ and 3.2.4 follows immediately. Hence we can assume that $B = C = 1$.

Obviously we can suppose that $f(x, y)$ and $g(y)$ are real. By an argument similar to that used in 1.2.1, we can find a strictly increasing continuous function $\phi(t)$ such that

$$\phi(0) = 0 \text{ and } \phi(t+1) = \phi(t) + 1$$

for all t , and such that

$$|g\{\phi(t'')\} - g\{\phi(t')\}| < 2(t'' - t')^\alpha$$

for each $t' < t''$. Furthermore, for each $0 \leq x \leq 1$,

$$V_\beta \{f(x, \phi(t)); 0 \leq t \leq 1\} = V_\beta \{f(x, y); 0 \leq y \leq 1\},$$

$$\int_0^1 f\{x, \phi(t)\} dg\{\phi(t)\} = \int_0^1 f(x, y) dg(y),$$

and, by performing a change of variable, we replace condition 3.2.3 by the condition

$$3.2.5 \quad |g(y'') - g(y')| < 2(y'' - y')^\alpha$$

for each $y' < y''$.

Since $g(y)$ is continuous and $\beta + \gamma > 1$, the Faltung $s(x)$ exists in the Riemann-Stieltjes sense for each x and is equal to $\lim_{n \rightarrow \infty} s_n(x)$. Using 3.1.3, 3.2.5, Jensen's inequality, 3.2.2, and 1.3.4 we have

$$\begin{aligned}
 |s_n(x) - s_{n-1}(x)| &\leq 2^{n-1} \int_0^1 |f(x, t + 2^{-n}) - f(x, t)| |g(t) - g(t - 2^{-n})| dt \\
 &\leq 2^{n(1-\gamma)} \left\{ \int_0^1 |f(x, t + 2^{-n}) - f(x, t)|^{1/\beta} dt \right\}^\beta \\
 &\leq 2^\beta \cdot 2^{-n\lambda},
 \end{aligned}$$

and summing on n we get

$$3.2.6 \quad |s(x) - s_n(x)| \leq 2^\beta \sum_{\nu=n+1}^{\infty} 2^{-\nu\lambda} = c(\lambda)2^{-n\lambda}$$

for $n=0, 1, \dots$. Fix n and consider $0 \leq x' < x'' \leq 1$. From 3.1.2, 3.2.5, and Jensen's inequality we have

$$\begin{aligned}
 3.2.7 \quad |s_n(x'') - s_n(x')| &\leq 2^n \int_0^1 |f(x'', t) - f(x', t)| |g(t) - g(t - 2^{-n})| dt \\
 &\leq 2^{n(1-\gamma)} \cdot 2\Delta
 \end{aligned}$$

where

$$\Delta = \left\{ \int_0^1 |f(x'', t) - f(x', t)|^{1/\alpha} dt \right\}^\alpha.$$

By considering three different cases we prove that

$$3.2.8 \quad |s(x'') - s(x')| \leq k(\lambda)\Delta^{\mu/\alpha},$$

where $k(\lambda)$ is a finite constant.

A. Suppose that $1 < \Delta < \infty$. Setting $n=0$ in 3.2.6 gives us 3.2.8 if we choose $k(\lambda) \geq 2c(\lambda)$.

B. Suppose that $0 < \Delta \leq 1$. Choose $n \geq 1$ so that $2^{-n\beta} < \Delta \leq 2^{-(n-1)\beta}$, and with 3.2.6 we have

$$|s(x) - s_n(x)| < c(\lambda)\Delta^{\mu/\alpha}$$

for $0 \leq x \leq 1$. From 3.2.7 we get

$$|s_n(x'') - s_n(x')| \leq 4\Delta^{\mu/\alpha},$$

and 3.2.8 follows if we choose $k(\lambda) \geq 2(\lambda) + 4$.

C. Suppose that $\Delta = 0$. We see from 3.1.3 and 3.2.7 that

$$|s(x'') - s(x')| = \lim_{n \rightarrow \infty} |s_n(x'') - s_n(x')| = 0,$$

and 3.2.8 follows if we choose $k(\lambda) \geq 0$.

To complete the proof for 3.2, take any subdivision $0 = x_0 < \dots < x_N = 1$. From 3.2.8 we have

$$\left\{ \sum_{n=1}^N |s(x_n) - s(x_{n-1})|^{1/\mu} \right\}^\mu \leq k(\lambda) \left\{ \int_0^1 \sum_{n=1}^N |f(x_n, t) - f(x_{n-1}, t)|^{1/\alpha} dt \right\}^\mu$$

$$\leq k(\lambda) A^{\lambda/\beta}.$$

The proof introduces unnecessary restrictions. For example, Young's argument [11, p. 459] allows us to consider the case where $g(y)$ is not continuous. We conclude this chapter by simply stating the following generalization of 3.2.

THEOREM 3.3. *Suppose that $0 < \alpha, \beta, \gamma \leq 1$, $0 < \lambda = \beta + \gamma - 1$, and $u = \alpha\lambda/\beta$. Suppose also that 3.2.1, 3.2.2, and 3.2.3 hold. If $s(x)$ is the Stieltjes Faltung 3.1.1, then*

$$V_\mu \{s(x) - \delta f(x, 0); 0 \leq x \leq 1\} \leq k(\lambda) A^{\lambda/\beta} B^{1-\lambda/\beta} C,$$

where $\delta = g(1) - g(0)$ and $k(\lambda)$ is a finite constant.

APPLICATIONS TO INFINITE SERIES

4.1. In this chapter we apply the notion of α -variation to the study of infinite series. We say that the series

4.1.1
$$\sum_{n=0}^\infty a_n$$

is α -convergent if, given $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$\left\{ \sum_{\nu=m}^n \left| \sum a_\nu \right|^{1/\alpha} \right\}^\alpha < \epsilon$$

for $N(\epsilon) \leq m < n$. 0-convergence is ordinary convergence and 1-convergence is absolute convergence. If $0 \leq \alpha < \beta \leq 1$, we have by Jensen's inequality

$$\left\{ \sum_{\nu=m}^n \left| \sum a_\nu \right|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{\nu=m}^n \left| \sum a_\nu \right|^{1/\beta} \right\}^\beta,$$

and thus a β -convergent series is always α -convergent.

We can extend the notion of α -convergence to sequences. We call $\{S_n\}$ an α -convergent sequence if S_n is the n th partial sum of an α -convergent series. From Minkowski's inequality we see that any finite linear combination of α -convergent series (sequences) is itself an α -convergent series (sequence). We also have the following result.

LEMMA 4.1.2. (Cf. Lemma 1.4.2.) *Suppose that $0 < \alpha \leq 1$. The series 4.1.1 is α -convergent if and only if*

$$\left\{ \sum_{n=0}^\infty \left| \sum a_n \right|^{1/\alpha} \right\}^\alpha < \infty.$$

The same type of result is true for sequences.

Any series derived from a 1-convergent series by a rearrangement of terms is convergent to the sum of the original series. However, when $\alpha < 1$, an α -convergent series is "conditionally convergent" and little can be said about rearrangement.

A second important property of 1-convergent series is found in multiplication theorems. For example it is well known that the Cauchy product of a 1-convergent series by a 0-convergent series is 0-convergent to the product of the sums of the series. We have the following extension of this result.

THEOREM 4.1.3. *Suppose that $0 < \alpha, \beta \leq 1$, and that $0 < \gamma = \alpha + \beta - 1$. Then the Cauchy product of an α -convergent series by a β -convergent series is γ -convergent to the product of the sums.*

This theorem follows easily from the following specialization of 3.3.

THEOREM 4.1.4. *Suppose that $0 < \alpha, \beta \leq 1$, and that $0 < \gamma = \alpha + \beta - 1$. If $f(x)$ has bounded α -variation over $0 \leq x < \infty$ and if $g(x)$ has bounded β -variation over $0 \leq x < \infty$, then the Stieltjes Faltung*

$$s(x) = \int_0^x f(x-y)dg(y)$$

exists in the Young sense for each x and has bounded γ -variation over $0 \leq x < \infty$.

Theorem 4.1.3 holds in the limiting case where $\alpha + \beta = 1$ if and only if $\alpha = 0$ or 1 ; 4.1.3 is also true when one considers the more general Dirichlet product [3, p. 239] instead of the Cauchy product.

4.2. We can apply our scale to the study of Cesaro and Abel summability. Suppose that S_n^k is the n th Cesaro mean of order k for the series 4.1.1. We say that 4.1.1 is summable $(C, k; \alpha)$ to S if the sequence $\{S_n^k\}$ is α -convergent to S . Thus $(C, k; 0)$ summability is ordinary Cesaro summability and $(C, k; 1)$ summability is absolute Cesaro summability. Consider the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and assume that this series converges for $0 \leq x < 1$. We say that 4.1.1 is summable $(A; \alpha)$ to S if $f(x)$ has bounded α -variation over $0 \leq x < 1$ and if $\lim_{x \rightarrow 1^-} f(x) = S$. $(A; 0)$ summability is ordinary Abel summability and $(A; 1)$ summability is absolute Abel summability. (See [1, p. 11] for references on absolute summability.) A linear combination of series, summable $(C, k; \alpha)$ for some k and $0 \leq \alpha \leq 1$, is itself a series summable $(C, k; \alpha)$ and similarly for the $(A; \alpha)$ method.

When $0 \leq \alpha < \beta \leq 1$, a series summable $(C, k; \beta)$ to S is summable $(C, k; \alpha)$ to S and a series summable $(A; \beta)$ to some limit is summable $(A; \alpha)$ to the

same limit. We can also establish the following consistency result.

THEOREM 4.2.1. *Suppose that $0 \leq \alpha \leq 1$ and that $k > j > -1$. A series summable $(C, j; \alpha)$ to S is summable $(C, k; \alpha)$ and $(A; \alpha)$ to S . If S_n^j and S_n^k are the n th Cesaro means of order j and k respectively for 4.1.1, we have*

$$4.2.2 \quad \left\{ \sum_{n=1}^{\infty} \left| \sum S_n^k - S_{n-1}^k \right|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{n=1}^{\infty} \left| \sum S_n^j - S_{n-1}^j \right|^{1/\alpha} \right\}^\alpha,$$

$$4.2.3 \quad V_\alpha \left\{ \sum_{n=0}^{\infty} a_n x^n; 0 \leq x \leq 1 \right\} \leq \left\{ \sum_{n=1}^{\infty} \left| \sum S_n^k - S_{n-1}^k \right|^{1/\alpha} \right\}^\alpha.$$

For the first of these inequalities let $k - j = i > 0$ and write

$$S_n^k = \sum_{m=0}^n \frac{C_{n-m}^{i-1} C_m^j}{C_n^k} S_m^j = \sum_{m=0}^n b_{n,m} S_m^j,$$

where C_r^s is the binomial coefficient

$$\binom{r+s}{r}.$$

$b_{n,m}$ is non-negative for all m and n and

$$\sum_{m=0}^n b_{n,m} = \frac{1}{C_n^k} \sum_{m=0}^n C_{n-m}^{i-1} C_m^j = 1.$$

When $0 \leq n' < n'' < \infty$,

$$b_{n'',m} - b_{n',m} = b_{n',m} \left\{ \frac{(n'' - m + i - 1) \cdots (n' - m + i)}{(n'' - m) \cdots (n' - m + 1)} \cdot \frac{n'' \cdots (n' + 1)}{(n'' + k) \cdots (n' + k + 1)} - 1 \right\}$$

for $0 \leq m \leq n'$. If $0 < i < 1$, the bracketed quantity is negative for $0 \leq m \leq n'$ and, since $b_{n',m} = 0$ for $m > n'$, we can apply 2.5.2.

For any subdivision $0 < x_0 < \cdots < x_N < 1$ let

$$\sum_{m=0}^{\infty} a_m (x_n)^m = \sum_{m=0}^{\infty} C_m^k (1 - x_n)^{k+1} (x_n)^m S_m^k = \sum_{m=0}^{\infty} b_{n,m} S_m^k.$$

Again $b_{n,m}$ is non-negative for all m and n ,

$$\sum_{m=0}^{\infty} b_{n,m} = (1 - x_n)^{k+1} \sum_{m=0}^{\infty} C_m^k (x_n)^m = 1,$$

and, for $0 \leq n' < n'' \leq N$,

$$b_{n'',m} - b_{n',m} = b_{n',m} \left\{ \left(\frac{x_{n''}}{x_{n'}} \right)^m \left(\frac{1 - x_{n''}}{1 - x_{n'}} \right)^{k+1} - 1 \right\}.$$

The bracketed factor here is an increasing function of m and 4.2.3 follows from 2.5.2.

The rest of the theorem follows from these two inequalities and a classical result.

4.3. The direct converse of 4.2.1 is not true. The coefficients in the power series expansion for

$$f(x) = e^{1/(1+x)}$$

constitute a series which is summable $(A; 1)$ but which is not summable $(C, k; 0)$ for any finite k . However, we can prove some "corrected converses" and the following theorems extend two results due to A. Tauber.

LEMMA 4.3.1. 1. $\alpha_n = \sum_{m=1}^{n-1} (1 - (1 - 1/n)^m)/m$ is a positive, increasing sequence bounded by 1 for $n \geq 2$.

2. $\beta_n = (1 - (1 - 1/n)^n)/n$ is a positive, decreasing sequence bounded by 1 for $n \geq 1$.

3. $\gamma_n = \sum_{m=n+1}^{\infty} ((1 - 1/n)^m)/m$ is a positive, increasing sequence bounded by 1 for $n \geq 1$.

Consider the first sequence. If $r > 1$, $x^r - y^r > ry^{r-1}(x - y)$ for any two positive and unequal x and y . Hence for $n \geq 2$ we have

$$\begin{aligned} \alpha_{n+1} - \alpha_n &= \sum_{m=1}^{n-1} \frac{(1 - 1/n)^m - (1 - 1/(n+1))^m}{m} + \frac{1 - (1 - 1/(n+1))^n}{n} \\ &\geq -\frac{1}{n(n+1)} \sum_{m=0}^{n-2} \left(1 - \frac{1}{n+1}\right)^m + \frac{1 - (1 - 1/(n+1))^n}{n} \\ &\geq \frac{1}{n} \left\{ \left(1 - \frac{1}{n+1}\right)^{n-1} - \left(1 - \frac{1}{n+1}\right)^n \right\} > 0. \end{aligned}$$

For the boundedness we see that

$$\alpha_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} \leq \frac{n - 1}{n} < 1.$$

The proofs of 4.3.1.2 and 4.3.1.3 follow along similar lines.

We can now generalize Tauber's first theorem.

THEOREM 4.3.2. Suppose that $0 \leq \alpha \leq 1$ and that 4.1.1 is summable $(A; \alpha)$ to S . If the sequence $\{na_n\}$ is α -convergent to 0, 4.1.1 is α -convergent to S .

We can assume that $0 < \alpha \leq 1$. For $0 \leq x < 1$ let

$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

and let S_n be the n th partial sum for 4.1.1, i.e.

$$S_n = \sum_{\nu=0}^n a_{\nu}.$$

For $n \geq 2$ we can write

$$S_n - f\left(1 - \frac{1}{n}\right) = A_n + B_n - C_n,$$

where

$$A_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \alpha_{n,m} (ma_m),$$

$$B_n = \frac{1 - (1 - 1/n)^n}{n} (na_n) = \beta_n (na_n),$$

$$C_n = \sum_{m=n+1}^{\infty} \frac{(1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \gamma_{n,m} (ma_m).$$

$\alpha_{n,m}$ and $\gamma_{n,m}$ are non-negative for all m and n and, by 4.3.1, both $\sum_{m=1}^{\infty} \alpha_{n,m}$ and $\sum_{m=1}^{\infty} \gamma_{n,m}$ are increasing in n and bounded by 1. These two sets of coefficients satisfy the hypotheses of 2.5.1. Hence

$$\left\{ \sum_{n=3}^{\infty} \left| \sum A_n - A_{n-1} \right|^{1/\alpha} \right\}^{\alpha} \quad \text{and} \quad \left\{ \sum_{n=3}^{\infty} \left| \sum C_n - C_{n-1} \right|^{1/\alpha} \right\}^{\alpha}$$

are majorized by

$$\left\{ \sum_{m=1}^{\infty} \left| \sum ma_m - (m-1)a_{m-1} \right|^{1/\alpha} \right\}^{\alpha},$$

and $\{A_n\}$ and $\{C_n\}$ are α -convergent sequences. The sequence $\{B_n\}$ is also α -convergent since $\{\beta_n\}$ is 1-convergent to 0. Because 4.1.1 is $(A; \alpha)$ summable, the sequence $\{f(1 - 1/n)\}$ is α -convergent, and we conclude that $\{S_n\}$ is α -convergent to S .

THEOREM 4.3.3. *Suppose that $0 \leq \alpha \leq 1$ and that the series 4.1.1 is summable $(A; \alpha)$ to S . 4.1.1 is α -convergent to S if and only if the sequence*

$$\left\{ \frac{a_1 + \cdots + na_n}{n} \right\}$$

is α -convergent to 0.

For the necessity let S_n^1 be the n th Cesaro mean of order 1 for 4.1.1 and we see from 4.2.1 that the sequence

$$S_n - S_{n-1}^1 = \frac{a_1 + \cdots + na_n}{n}$$

is α -convergent to 0.

For the sufficiency set $b_0 = 0$ and let

$$\sum_{m=0}^n b_m = \frac{a_1 + \cdots + na_n}{n}$$

for $n \geq 1$. If $a_n = b_n + c_n$ for all n , then

$$c_n = \frac{a_1 + \cdots + (n-1)a_{n-1}}{n(n-1)}$$

for $n \geq 2$. $\sum_{n=0}^\infty a_n$ is $(A; \alpha)$ summable to S , $\sum_{n=0}^\infty b_n$ is α -convergent to 0, and hence the series $\sum_{n=0}^\infty c_n$ is also $(A; \alpha)$ summable to S . By 4.3.2 we see that this series is then α -convergent to S and this completes the proof.

We saw in 4.3.2 how the Tauberian condition

$$\{na_n\} \text{ is } \alpha\text{-convergent to } 0$$

allowed us to pass from summability $(A; \alpha)$ to summability $(C, 0; \alpha)$ or α -convergence. Actually more is true and we have the following result.

THEOREM 4.3.4. *Suppose that $0 \leq \alpha \leq 1$ and that 4.1.1 is α -convergent. If the sequence $\{na_n\}$ is α -convergent to 0, 4.1.1 is summable $(C, k; \alpha)$ to its sum for every $k > -1$.*

Pick $\delta > 0$. Let S_n^δ and $S_n^{\delta-1}$ be the n th Cesaro means of order δ and $\delta-1$ respectively for 4.1.1, and let T_n^δ be the n th Cesaro mean of order δ for the series whose n th partial sum is na_n . From the identity $(n+\delta)C_{n-\nu}^{\delta-1} - \delta C_{n-\nu}^\delta = \nu C_{n-\nu}^{\delta-1}$, we see that

$$S_n^{\delta-1} - S_n^\delta = \frac{1}{\delta} T_n^\delta$$

By 4.2.1, $\{S_n^\delta\}$ and $\{T_n^\delta\}$ are α -convergent to S and 0 respectively and thus $\{S_n^{\delta-1}\}$ is α -convergent to S .

In conclusion we construct, for $0 \leq \alpha < 1$, a series which is summable $(C, k; \alpha)$ for every $k > -1$ and which is not summable $(A; \beta)$ for any $\beta > \alpha$.

Let $\{b_k\}$ be any positive decreasing sequence of numbers which approach zero such that

$$\left\{ \sum_{k=1}^\infty b_k^{1/\alpha} \right\}^\alpha < \infty \quad \text{and} \quad \left\{ \sum_{k=1}^\infty b_k^{1/\beta} \right\}^\beta = \infty$$

for each $\beta > \alpha$. Define a sequence of integers, $1 = n_0 < n_1 < \dots$, and a set of positive numbers, $\{c_k\}$, such that

$$\sum_{n_{k-1} \leq n < n_k} \frac{1}{n} \geq 2^k b_k = c_k \quad \sum_{n_{k-1} \leq n < n_k} \frac{1}{n}$$

for $k = 1, 2, \dots$. Set $a_0 = 0$ and $a_n = ((-1)^k/n)2^{-k}c_k$ for $n_{k-1} \leq n < n_k$. Then

$$\left\{ \sum_{n=n_k}^{\infty} \left| \sum a_n \right|^{1/\rho} \right\} = \left\{ \sum_{j=k+1}^{\infty} b_j^{1/\rho} \right\}^{\rho}$$

for each $0 \leq \rho \leq 1$, and the series is α -convergent but not β -convergent for any $\beta > \alpha$. Since

$$\sum_{n=1}^{\infty} \left| na_n - (n-1)a_n \right| = 2 \sum_{k=1}^{\infty} 2^{-k} c_k < 2,$$

the sequence $\{na_n\}$ is 1-convergent to 0, and we see from 4.3.2 and 4.3.4 that this series has the desired properties.

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