

ON ZERO AND POLE SURFACES OF FUNCTIONS OF TWO COMPLEX VARIABLES⁽¹⁾

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1. **Some problems arising in the study of value distribution of functions of two complex variables.** One of the objectives of modern analysis consists in the generalization of methods of the theory of functions of one complex variable in such a way that the procedures in the revised form can be applied in other fields, in particular, in the theory of functions of several complex variables, in the theory of partial differential equations, in differential geometry, etc. In this way one can hope to obtain in time a unified theory of various chapters of analysis. The method of the kernel function is one of the tools of this kind. In particular, this method permits us to develop some chapters of the theory of analytic and meromorphic functions $f(z_1, \dots, z_n)$ of the class $\mathcal{L}^2(\mathfrak{B}^{2n})$, various chapters in the theory of pseudo-conformal transformations (i.e., of transformations of the domains \mathfrak{B}^{2n} by n analytic functions of n complex variables) etc. On the other hand, it is of considerable interest to generalize other chapters of the theory of functions of one variable, at first to the case of several complex variables. In particular, the study of value distribution of entire and meromorphic functions represents a topic of great interest. Generalizing the classical results about the zeros of a polynomial, Hadamard and Borel established a connection between the value distribution of a function and its growth. A further step of basic importance has been made by Nevanlinna and Ahlfors, who showed not only that the results of Hadamard and Borel in a sharper form can be obtained by using potential-theoretical and topological methods, but found in this way important new relations, and opened a new field in the modern theory of functions.

As one passes from the theory of functions of one variable to the case of two and more variables⁽²⁾, the question of the value distribution becomes more complex and many directions arise which can be considered as a generalization of the above mentioned chapter of the theory of functions of one variable. However, the formulation of the problems which can be answered, and generalization of methods used in one variable, is not immediate.

A function of two complex variables may vanish in a domain on a segment

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⁽²⁾ In the present paper we are considering the case of two variables. We note that a generalization of methods used in the following to the case of n variables, $n > 2$, presents only technical difficulties.

of a (two-dimensional) surface and there exists an infinite variety of independent functionals which we can associate with these segments. Hereby the behavior of different functionals reflects different properties of the segments. Further, a generalization of methods of Nevanlinna and Ahlfors to the theory of functions of two variables is not immediate, and the whole problem of connecting the theory of analytic functions with a conveniently chosen class of real functions for which a generalization of potential-theoretical procedures is possible requires a revision.

Various investigations in the theory of functions of several variables (e.g., arising in the theory of the kernel function) teach us that, in contrast to the situation in one variable, it is useful to classify domains of definition of functions of two complex variables. In particular, so-called domains with a distinguished boundary surface are of great interest; these are domains on whose (three-dimensional) boundary lies a (two-dimensional) *distinguished boundary surface*. In many instances, a generalization of deeper results in the theory of functions of one variable to two variables is possible in the case of these domains, whereby the *distinguished boundary surface* (and not the three-dimensional boundary), *plays the role of the boundary curve*.

In these domains an analytic function of two complex variables assumes the maximum of its absolute value not only on the boundary of the domain, but even on the distinguished boundary surface. There exist a generalized Cauchy formula [B. 1, 2; W. 1. 2; So. 1]^(*) and a development in terms of functions which are orthonormal when integrating along the distinguished boundary surface [B. 11]. In the case of domains with the distinguished boundary surface, these facts permit us to generalize a number of results of the theory of functions of one variable whose generalization is not possible in general domains. If, however, one attempts to generalize results obtained by the use of potential-theoretical methods, one realizes that the class of *B*-harmonic functions (i.e., real parts of functions of two complex variables) is too special, so that not necessarily to every (say, continuous) function defined on the distinguished boundary surface does there exist a function which is *B*-harmonic in the domain and which assumes on the distinguished boundary surface the prescribed values. This situation leads us to extend the class of *B*-harmonic functions in such a way that the extended class possesses all properties needed in various function-theoretical procedures, but is sufficiently large so that the boundary value problem with the values prescribed on the distinguished boundary surface always has a solution. These requirements lead us to introduce certain *functions of the extended class which* (in contrast to the case of one variable) *depend in general upon the domain of definition*.

Once the functions of the extended class are introduced, one can easily define Green's function of the extended class and derive various integral

(*) Numbers in brackets refer to the bibliography. As a rule the name of the author is replaced by the first letter.

formulas of Green's type. It has been shown in [B. 3, 4, 5; B.-S. 2] that, using these formulas, one obtains certain functionals which can be considered as a direct generalization of such notions as a number of points a_ν for which $f(a_\nu) = a$ in a circle $|z| < r$, as $\sum_{|a_\nu| \leq r} |a_\nu|^{-\lambda}$, etc., and that for these functionals one obtains bounds in terms of expressions characteristic for the growth of the functions on certain *three-dimensional* hypersurfaces. There arises, however, a difference in comparison with the two-dimensional case; namely, while a family of circles with increasing radii covers the whole plane, a one parameter family of two-dimensional manifolds forms a *three-dimensional* hypersurface; choosing different families we obtain different hypersurfaces on which the growth of the function is considered.

The continuation of investigations of this kind can proceed in different directions. At first one can use further types of domains with a distinguished boundary surface to obtain analogous formulas connecting functionals of similar type with the growth of the function on hypersurfaces of more complicated structure than those considered in [B. 4]. On the other hand, the variety of the functionals which one can associate with segments of a surface is richer than those which one can associate in the case of one variable with a set of points. Further, in connection with the theory of pseudo-conformal mapping where we consider a mapping of a domain by a pair of functions, it is of interest to associate functionals with a *pair* of functions.

In the present paper we introduce functionals of a new type, which are considered in special domains \mathfrak{B} . These domains are bounded by two segments of analytic hypersurfaces. One of these segments lies in the analytic hypersurface $[|z_2| = 1]$, so that the family of analytic surfaces $z_2 = \text{const.}$ plays a distinguished role for the structure of the domain under consideration. In domains of this kind (see for details §2), we introduce functionals which are denoted by $\mathcal{M}_k(P)$, $k = 1, 2$, which we associate with a pair (f_1, f_2) of meromorphic functions. These functionals are formed as follows: Let $w(z)$, $w(0) = 0$, $w'(0) > 0$, be the function which maps a simply connected domain \mathfrak{B}^2 of the z plane onto the unit circle. We denote

$$(1) \quad (z, Z; \mathfrak{B}^2) \equiv \left| \frac{w(z) - w(Z)}{1 - \overline{w(Z)}w(z)} \right|,$$

as the $\mathcal{C}(\mathfrak{B}^2)$ -distance between z and $Z^{(*)}$. For every z_2 , $|z_2| \leq 1$, we form the following four products: $P_1(z_2)$ and $P_2(z_2)$ are the products of $\mathcal{C}(\mathfrak{B}^2(z_2))$ -distances between every zero of $f_1 \equiv f_1(z_1, z_2)$ and every zero and pole of f_2 , respectively, and $P_3(z_2)$ and $P_4(z_2)$ are analogous products with zeros of f_1 being replaced by poles of f_1 . $\mathfrak{B}^2(z_2^0)$ means the intersections of the domain \mathfrak{B} under consideration with the plane $z_2 = z_2^0 = \text{const.}$

We introduce further "exceptional points" $(z_1^{(\nu)}, z_2^{(\nu)})$, $\nu = 1, 2, \dots$, of the

(*) (1) is connected in a very simple manner with the non-Euclidean distance, see p. 443).

pair (f_1, f_2) . (These are intersection points of zero and pole surfaces of f_1 with zero and pole surfaces of f_2 , and certain other points. See §2.) We cut out from the projection $|z_2| < 1$ of \mathfrak{B} on the z_2 -plane the projections $|z_2 - z_2^{(v)}| < P$, $P > 0$ sufficiently small, and we denote by \mathfrak{E}_P^2 the remaining part of the unit circle. The function $\mathcal{M}_1(P)$ is the average of the *generalized Blaschke product* $[P_1(z_2)P_4(z_2)/P_2(z_2)P_3(z_2)]$ over \mathfrak{E}_P^2 .

The second functional $\mathcal{M}_2(P)$ is formed in a similar way. We obtain for the sum $\mathcal{M}_1(P) + \mathcal{M}_2(P)$ upper bounds in terms of quantities which are connected with the growth of the functions (f_1, f_2) and certain quantities which we associate with exceptional points of the pair (f_1, f_2) .

In §7 we show that the kernel function of the functions of the extended class normalized in the sense of a conveniently chosen metric is finite.

Thus, in [B. 4] and in the present paper, we derive bounds for functionals characterizing certain properties of segments of zero and pole surfaces of analytic functions of two variables. These results seem to indicate that, in analogy to the case of one variable, in the case of two variables after an extended class has been introduced one can also establish formulas of Green's type which then can be used in deriving bounds for functionals characterizing in this or other respects the behavior of zero and pole surfaces of one or of a pair of functions of two complex variables. In the domains with a distinguished boundary surface of more complicated structure, one obtains further relations using Morse's theory of critical points. [See B. 6].

NOTATION. Manifolds as a rule are denoted by German characters, and the upper index indicates the dimension of the manifold. In the case of four-dimensional manifolds the index 4 is omitted. $\mathfrak{M}^n(z_2^0)$ means the intersection of \mathfrak{M}^n with the plane $z_2 = z_2^0 = \text{const.}$ In operating with the sets we use the usual symbols \mathbf{S} or $+$ (sum set), $-$ (difference set), \cap (intersection), and so on. For instance, the sum set of a family of curves or domains $\mathfrak{B}^n(\zeta_2)$, $n = 1$ or 2 , each of which lies in a different plane $z_2 = \zeta_2$, where ζ_2 varies along a curve or in domain \mathfrak{A}^m , $m \leq 2$, will be denoted by $\mathbf{S}_{z_2 \in \mathfrak{A}^m} \mathfrak{B}^n(z_2)$ or by $[\mathbf{S} \mathfrak{B}^n(z_2), z_2 \in \mathfrak{A}^m]$. Arguments of the functions are abbreviated and omitted as much as possible. z_k , $k = 1, 2$, is used, as a rule, as a complex variable $x_k + iy_k$ and as the point (x_k, y_k) , as well.

A function, say f , is often considered:

1°. in the domain $\mathfrak{B} = \mathbf{S}_{|z_2| < 1} \mathfrak{B}^2(z_2)$, where $\mathfrak{B}^2(z_2)$ is the domain bounded by the curve $\mathfrak{b}^1(z_2) = [z_1 = h(\lambda, z_2), 0 \leq \lambda \leq 2\pi]$, $h(0, z_2) = h(2\pi, z_2)$;

2°. in $\mathfrak{b}^3 = \mathbf{S}_{|z_2| < 1} \mathfrak{b}^1(z_2)$ (\mathfrak{b}^3 represents a part of the boundary of \mathfrak{B}); and

3°. on the distinguished boundary surface $\mathfrak{D}^2 = \mathbf{S}_{0 \leq \phi_2 \leq 2\pi} \mathfrak{b}^1(\exp(i\phi_2))$.

As a rule, a function $f(z_1, z_2)$, $(z_1, z_2) \in \mathfrak{B}$, when considered in \mathfrak{b}^3 , as a function of λ and z_2 , i.e., $f[h(\lambda, z_2), z_2]$, will be denoted by $f^* \equiv f^*(\lambda, z_2)$, and when considered on \mathfrak{D}^2 as a function of λ and ϕ_2 , i.e., $f[h(\lambda, \exp(i\phi_2)), \exp(i\phi_2)]$, will be denoted by \tilde{f} . However, an exception is made in the case of the function $h(\lambda, z_2)$ which is defined originally as a function of (λ, z_2) and only in

very few cases is considered as a function of two complex variables z_1, z_2 . We write then $h_1(z_1, z_2)$ so that $h_1(\exp(i\lambda), z_2) \equiv h(\lambda, z_2)$. We note that we write

$$\tilde{h} \equiv \tilde{h}(\lambda, \phi_2) \text{ for } h(\lambda, \exp(i\phi_2)).$$

Other minor abbreviations or conventions are explained in the body of the paper.

2. Domains with a distinguished boundary surface. Functions of the extended class. Let $h_1(\zeta, z_2)$ be an analytic function of two complex variables ζ, z_2 defined in $[1 - \epsilon \leq |\zeta| \leq 1 + \epsilon, |z_2| \leq 1]$, $\epsilon > 0$. We assume that $\tilde{h} \equiv h[\lambda, \exp(i\phi_2)]$, where $h \equiv h(\lambda, z_2) = h_1(\exp(i\lambda), z)$, and $\tilde{h}_\lambda \equiv \partial \tilde{h} / \partial \lambda$ are continuously differentiable functions of ϕ_2 and λ and that

$$(1) \quad 0 < H_1 \leq |\tilde{h}| \leq H_2 < \infty, \quad |\tilde{h}_\lambda| \leq H_3$$

where $H_k < \infty$ are conveniently chosen constants (Hypothesis 2.I). We assume further that for every fixed z_2 , $|z_2| \leq 1$,

$$(2) \quad \mathfrak{b}^1(z_2) = [z_1 = h, 0 \leq \lambda \leq 2\pi],$$

is a smooth simple curve which bounds a simply-connected, schlicht domain $\mathfrak{B}^2(z_2)$ of the z_1 -plane. It includes the origin $z_1 = 0$ in its interior (Hypothesis 2.II). Using the classical theorems about variable domains [C. 1, 2; R. 1] we conclude that h and h_λ are continuously differentiable functions of x_2 and y_2 and that the function $g(z_1; z_2)$, $g(0; z_2) = 0$, $g'(0; z_2) > 0$, mapping $\mathfrak{B}^2(z_2)$ onto the unit circle is a continuously differentiable function of x_2, y_2 .

In the present paper we consider a special class of domains

$$(3) \quad \mathfrak{B} = \bigcup_{|z_2| < 1} \mathfrak{B}^2(z_2) \equiv [z_1 \in \mathfrak{B}^2(z_2), (|z_2| < 1)].$$

As it can be easily shown (see [B.-S. 2]), the boundary of \mathfrak{B} consists of two segments of analytic hypersurfaces

$$(4) \quad \mathfrak{b}^3 = \bigcup_{|z_2| \leq 1} \mathfrak{b}^1(z_2) \equiv [z_1 \in \mathfrak{b}^1(z_2), (|z_2| \leq 1)],$$

$$(5) \quad \mathfrak{a}^3 = \bigcup_{|z_2| = 1} \mathfrak{B}^2(z_2) \equiv [|z_1| \in \mathfrak{B}^2(z_2), (|z_2| = 1)].$$

The intersection

$$(6) \quad \mathfrak{D}^2 = \mathfrak{a}^3 \cap \mathfrak{b}^3 = \bigcup_{|z_2| = 1} \mathfrak{b}^1(z_2) \equiv [z_1 \in \mathfrak{b}^1(z_2), (|z_2| = 1)]$$

of the two above segments forms the distinguished boundary surface of \mathfrak{B} .

Let $\mathfrak{S}^2(\lambda)$ and $\mathfrak{h}^1(\lambda)$ denote the segment of an analytic surface

$$(7) \quad \mathfrak{S}^2(\lambda) = [z_1 = h, |z_2| \leq 1] \quad (0 \leq \lambda \leq 2\pi)$$

and the curve

$$(8) \quad \mathfrak{h}^1(\lambda) = [z_1 = h, |z_2| = 1] \quad (0 \leq \lambda \leq 2\pi),$$

respectively. We note that we can also write

$$(9) \quad \mathfrak{h}^3 = \sum_{\lambda=0}^{2\pi} \mathfrak{H}^2(\lambda) \equiv [(z_1, z_2) \in \mathfrak{H}^2(\lambda), (0 \leq \lambda \leq 2\pi)],$$

$$(10) \quad \mathfrak{D}^2 = \sum_{\lambda=0}^{2\pi} \mathfrak{h}^1(\lambda) \equiv [(z_1, z_2) \in \mathfrak{h}^1(\lambda), (0 \leq \lambda \leq 2\pi)].$$

Further we assume that $h(\lambda, z_2)$ satisfy the inequalities

$$(11) \quad \left| \frac{h_\lambda}{h_{z_2}} \right| \leq H < \infty, \quad \left| \frac{\partial}{\partial z_2} \left(\frac{h_\lambda}{h_{z_2}} \right) \right| \leq G < \infty$$

(Hypothesis 2.III).

REMARK: If h_{z_2} vanishes (at some points or identically) in $|z_2| \leq 1$, by the transformation

$$(12) \quad z_1^\dagger = z_1 + az_2, \quad z_2^\dagger = z_2,$$

one can make $h^\dagger(\lambda, z_2) = h(\lambda, z_2) + az_2$ satisfy the first of the conditions (11).

As mentioned in §1, for a continuous real function, defined on the distinguished boundary surface \mathfrak{D}^2 , there does not necessarily exist a B -harmonic function defined in \mathfrak{B} , which possesses the prescribed boundary values on \mathfrak{D}^2 . (See [B. 9].) In order to be able to apply potential-theoretical methods in every domain \mathfrak{B} , an "extended class of functions," $\mathcal{E}(\mathfrak{B})$, have been introduced in [B. 4, 6, 9; Bers 1]. However, in the papers mentioned above, only *continuous* boundary values on \mathfrak{D}^2 have been considered. For our purposes, the case is of interest where the boundary value function becomes infinite at finitely many points of \mathfrak{D}^2 . In the following we shall describe the boundary values in which we are interested and define the functions of the extended class for this more general case.

According to (10), \mathfrak{D}^2 is a one-parameter family of curves $\mathfrak{h}^1(\lambda)$, $0 \leq \lambda \leq 2\pi$. Each of these curves bounds a segment $\mathfrak{H}^2(\lambda)$ of the analytic surface $[z_1 = h, \lambda = \text{const}]$. Let a real function \tilde{t} be given on \mathfrak{D}^2 which is continuous, except at finitely many points, say

$$\lambda = \lambda^{\kappa\nu}, \quad \phi_2 = \phi_2^{\kappa\nu}, \quad \kappa = 1, 2, \dots, \kappa_1; \nu = 1, 2, \dots, \nu_1(\kappa_1),$$

where \tilde{t} becomes $+\infty$ or $-\infty$.

A function $e \equiv e(z_1, z_2) \equiv e(z_1, z_2; \tilde{t})$ will be said to be a *function of the extended class* $\mathcal{E}(\mathfrak{B})$ which possesses the boundary values \tilde{t} on \mathfrak{D}^2 if e satisfies the following two conditions.

1°. $e^* = e^*[\lambda, z_2; \tilde{t}]$, $0 \leq \lambda \leq 2\pi$, for every fixed λ in $|z_2| < 1$ is that harmonic function of x_2, y_2 which assumes for $|z_2| = 1$ boundary values \tilde{t} , $\phi_2 = \text{arc } z_2$. If $\{\lambda^{\kappa\nu}, \phi_2^{\kappa\nu}\}$, $\lambda^{\kappa\nu} = \lambda^\nu$, $\kappa = 1, 2, \dots, \kappa_1$, are infinity points of $\tilde{t}(\lambda^\nu, \phi_2)$, then we assume that $[e^*(\lambda^\nu, z_2) - c_{\kappa\nu} \log |z_2 - \exp(i\phi_2^{\kappa\nu})|]$ remain

bounded if we approach the point $z_1 = \tilde{h}(\lambda^\nu, \phi_2^{\kappa\nu})$, $z_2 = \exp(i\phi^{\kappa\nu})$ moving in the surface $z_1 = h(\lambda^\nu, z_2)$ and remaining in the domain whose projection on the z_2 -plane is $-\pi/2 < -\alpha \leq \arg[z_2 \cdot \exp(-i\phi_2^{\kappa\nu}) + 1] \leq \alpha < \pi/2$. Here $c_{\kappa\nu}$ are (positive or negative) integers.

2°. By the requirement 1° the function $e(z_1, z_2; \tilde{l}) \in \mathcal{E}(\mathfrak{B})$ is defined in \mathfrak{b}^3 , see (10). According to (4) the boundary curve $\mathfrak{b}^1(z_2)$ of $\mathfrak{B}^2(z_2)$ lies in \mathfrak{b}^3 . Thus by 1°, e^* is determined on the boundary curve of the domain $\mathfrak{B}^2(z_2)$. For every fixed z_2 the function e is defined to be that harmonic function of x_1, y_1 which on $\mathfrak{b}^1(z_2)$ assumes the boundary value obtained above. As it can be easily shown, these values are for $|z_2| < 1$ continuous. For $|z_2| = 1$ they can have (finitely many) infinity points. If $\tilde{h}(\lambda^{\kappa\nu}, \phi_2^{\kappa\nu})$, $\phi_2^{\kappa\nu} = \phi_2^\kappa$, $\nu = 1, 2, \dots, n_\kappa$, are infinity points which lie on $\mathfrak{b}^1(\exp(i\phi_2^\kappa))$, then we assume that

$$e(z_1, \exp(i\phi_2^\kappa)) - \sum_{\nu=1}^{n_\kappa} c_{\kappa\nu}^* \log |z_1 - \tilde{h}(\lambda^{\kappa\nu}, \phi_2^\kappa)|$$

remains bounded if z_1 converges to $z_1 = \tilde{h}(\lambda_2^{\kappa\nu}, \phi_2^{\kappa\nu})$, along a path lying in the plane $z_2 = \exp(i\phi_2^\kappa)$ and remaining in $\mathfrak{B}^2(i\phi_2^\kappa)$. Here $c_{\kappa\nu}^*$ are conveniently chosen (positive or negative) integers. Using the fact that a function f , belonging to $\mathcal{E}(\mathfrak{B})$ in every lamina $[z_1 = h(\lambda, z_2), |z_2| < 1]$, is a harmonic function of x_2, y_2 , and f in every $[z_1 \in \mathfrak{B}^2(z_2), |z_2| \leq 1]$, is a harmonic function of x_1, y_1 , we can derive for f a generalized Poisson formula (see [B. 11; B.-M. 1; Bers 1]). This formula yields a representation of f when we approach a point $z_1 = \tilde{h}(\lambda^{\kappa\nu}, \phi_2^{\kappa\nu})$, $z_2 = \exp(i\phi_2^{\kappa\nu})$ of \mathfrak{D}^2 along a path in $[\text{Re}[z_1 - z_2 h_{z_2}(\lambda^{\kappa\nu}, \exp(i\phi_2^{\kappa\nu})) - \tilde{h}(\lambda^{\kappa\nu}, \phi_2^{\kappa\nu})] < 0] \times [\text{Re}[z_2 \exp(-i\phi_2^{\kappa\nu}) - 1] < 0]$.

As mentioned in §1, the aim of the present paper is to give bounds for certain functionals $\mathcal{M}_1(\mathbf{P})$ and $\mathcal{M}_2(\mathbf{P})$ (see §6) of segments of zero and pole surfaces of a pair of functions f_1 and f_2 which are meromorphic in $\overline{\mathfrak{B}}$. These functionals are integrals over a domain $\mathfrak{S}_\mathbf{P}^2$. $\mathfrak{S}_\mathbf{P}^2$ is the projection of \mathfrak{B} onto the z_2 -plane (i.e. $[|z_2| < 1]$) from which the projections $[|z_2 - z_2^{(p)}| < \mathbf{P}]$ of the neighborhoods of *exceptional points* $(z_1^{(p)}, z_2^{(p)})$ in \mathfrak{B} of the pair (f_1, f_2) are cut out.

These exceptional points consist of:

(a) Intersection points of the surfaces $[\log |f_1| = \pm \infty]$ and $[\log |f_2| = \pm \infty]$,

(b) On different places, see 4.II.2, 4.III., etc. (see §§4 and 5), we make assumptions about some properties of the functions f_1 and f_2 ; e.g., that f_2, f_1 or functions connected with them have at every point (z_1, z_2) of \mathfrak{B} or \mathfrak{b}^3 series developments whose first coefficients satisfy certain inequalities. In general, these hypotheses are fulfilled. But there can exist some points where these requirements are not fulfilled. These points $(z_1^{(p)}, z_2^{(p)})$ belong also to the set of exceptional points. The projections of these exceptional points do not belong to $\mathfrak{S}_\mathbf{P}^2$ (Hypothesis 2.IV).

f_k , $k=1, 2$, does not vanish or become infinite identically on a surface $z_2=c=\text{const.}$, $|c| \leq 1$ (Hypothesis 2.V).

We assume that there are only finitely many exceptional points (Hypothesis 2.VI).

3. An integral formula for functions of the extended class.

DEFINITIONS. 1. A function $P=P(z_1, z_2)$ will be said to be a function of the class $\mathcal{M}(\mathfrak{B})$, if $(P - \log |f|) \in \mathcal{E}(\mathfrak{B})$, where f is a function of two complex variables which is meromorphic in $\overline{\mathfrak{B}}$.

2. If $z_1=\alpha(z_2)$ is a zero of $f=f(z_1, z_2)$, then $\text{sg}(\alpha(z_2))=1$; if $\alpha(z_2)$ is a pole of f , then $\text{sg}(\alpha(z_2))=-1$.

Formulas of Green's type for harmonic functions, possessing in the domain logarithmic singularities, represent a valuable tool for the study of meromorphic functions of one complex variable. As has been mentioned in §1, in order to be able to apply potential-theoretical methods, and in particular to define some notions analogous to Green's function, we have to introduce functions of the class $\mathcal{M}(\mathfrak{B})$ and to derive theorems of Green's type for these functions.

THEOREM 3.1. Let $P_k=P_k(z_1, z_2)$, $k=1, 2$, be two different functions of the class $\mathcal{M}(\mathfrak{B})$, and let $\mathfrak{B}_P=[z_1 \in \mathfrak{B}^2(z_2), z_2 \in \mathfrak{C}_P^2]$, $\mathfrak{C}_P^2 \equiv \mathfrak{C}^2 - \sum_{\sigma=1}^R \mathfrak{b}_\sigma^2$, $\mathfrak{C}^2 = [|z_2| < 1]$. Here $\mathfrak{b}_\sigma^2 = |z_2 - z_2^{(\sigma)}| < P$, $z_2^{(\sigma)}$ being projections of exceptional points⁽⁵⁾ $(z_1^{(\sigma)}, z_2^{(\sigma)})$, $\sigma=1, 2, \dots, \sigma_0 < \infty$, of the pair P_k , $k=1, 2$, in $[|z_2| < 1]$. (P is assumed to be sufficiently small.)

Let further $z_1=\alpha_{k\sigma} \equiv \alpha_{k\sigma}(z_2)$, $\sigma=1, 2, \dots, N_k(z_2)$, $N_k(z_2) \leq N_0 < \infty$, be the singularity surfaces of P_k , $k=1, 2$. Then

$$\begin{aligned} & \mathcal{F}_P(P_1, P_2) + \mathcal{B}_P(P_1, P_2) \\ (1) \quad &= \frac{1}{4} \int_{\mathfrak{B}^2} P_1 P_2 h_{z_2}^{-1} dz_1 dz_2 - \frac{\pi}{2} \int_{\mathfrak{C}_P^2} \sum \text{sg}(\alpha_{1\nu}(z_2)) P_2(\alpha_{1\nu}(z_2), z_2) d\omega_2, \end{aligned}$$

where⁽⁶⁾

$$\begin{aligned} \mathfrak{B}^2 &= \sum_{z_1 \in \mathfrak{B}_P^1} \mathfrak{b}^1(z_2), \quad \mathcal{F}_P(P_1, P_2) \equiv \int_{\mathfrak{B}_P} P_{1,z_1} P_{2,\bar{z}_1} d\omega, \quad \mathcal{B}_P = \sum_{z_1 \in \mathfrak{P}^2} \mathfrak{B}^2(z_2), \\ (2) \quad \mathcal{B}_P(P_1, P_2) &\equiv \frac{1}{2i} \int_{\mathfrak{C}_P^2} \int_{\lambda=0}^{2\pi} \left[P_2^* p_1 h_{z_2}^{-1} + P_1^* \frac{\partial(P_2^* h_{\lambda} h_{z_2}^{-1})}{\partial z_2} \right] d\lambda d\omega_2, \\ P_{k,z_n} &\equiv \frac{\partial P_k}{\partial z_n}, \quad k=1, 2; \quad n=1, 2, \quad p_k = \left(\frac{\partial P_k}{\partial z_2} \right)_{z_1=h(\lambda, z_2)}, \\ h_{\lambda} &= \frac{\partial h}{\partial \lambda}, \quad h_{z_2} = \frac{\partial h}{\partial z_2}, \quad d\omega_k = dx_k dy_k, \quad d\omega = d\omega_1 d\omega_2. \end{aligned}$$

⁽⁵⁾ See the end of § 2.

⁽⁶⁾ $\mathfrak{B}^2(z_2^{(0)}) = \mathfrak{B} \cap [z_2 = z_2^{(0)}]$ have been defined on p. 415.

Here \mathfrak{B}_P^1 is the boundary of \mathfrak{S}_P^2 , $\mathfrak{b}^1(z_2)$ is the boundary of $\mathfrak{B}^2(z_2)$.

Proof. Let $z_2 \in \mathfrak{S}_P^2$ and let $\alpha_{k\nu}$ be the infinity points of P_k , $k=1, 2$, in $\mathfrak{B}^2(z_2)$, while $\alpha_{k\mu}$ are infinity points on the boundary curve $\mathfrak{b}^1(z_2)$ of $\mathfrak{B}^2(z_2)$. By $\mathfrak{b}^{*1}(z_2)$ we denote the boundary of the domain

$$(3) \quad \mathfrak{B}^{*2} = \mathfrak{B}^2(z_2) - \sum_{k=1}^2 \sum_{\mu} [|z_1 - \alpha_{k\mu}(z_2)| \leq \epsilon_2],$$

and by $\mathfrak{b}^{**1}(z_2)$ the boundary of

$$(4) \quad \mathfrak{B}^{**2}(z_2) = \mathfrak{B}^2(z_2) - \sum_{k=1}^2 \left\{ \sum_{\nu} [|z_1 - \alpha_{k\nu}(z_2)| \leq \epsilon_1] \right. \\ \left. + \sum_{\mu} [|z_1 - \alpha_{k\mu}(z_2)| \leq \epsilon_2] \right\}.$$

Here $\epsilon_k = \epsilon_k(z_2)$, $k=1, 2$, are chosen so small that all circles

$$|z_1 - \alpha_{k\nu}(z_2)| = \epsilon_1 \quad \text{and} \quad |z_1 - \alpha_{k\mu}(z_2)| = \epsilon_2$$

have no intersection points with each other and that $|z_1 - \alpha_{k\nu}(z_2)| \leq \epsilon_1$ lies completely inside of $\mathfrak{B}^2(z_2)$. Applying (37b), p. 57 of [B. 8], we obtain

$$(5) \quad \int_{\mathfrak{B}^{**2}(z_2)} P_{1,z_1} P_{2,\bar{z}_1} d\omega_1 = \frac{1}{2i} \int_{\mathfrak{b}^{*1}} P_{1,z_1} P_2 dz_1 \\ - \frac{1}{2i} \sum_{k=1}^2 \sum_{\nu} \int_{|z_1 - \alpha_{k\nu}(z_2)| = \epsilon_1(z_2)} P_{1,z_1} P_2 dz_1 \\ - \int P_{1,z_1 \bar{z}_1} P_2 d\omega_1.$$

Here the last (double) integral in the right-hand side of (5) is taken over $\mathfrak{B}^{**2}(z_2)$. This last integral vanishes since P_1 is there a regular harmonic function, and therefore $P_{1,z_1 \bar{z}_1} \equiv \Delta P_{1/4} = 0$ in $\mathfrak{B}^{**2}(z_2)$.

In the neighborhood of $\alpha_{1\nu}(z_2)$, $P_1 = \text{sg}(\alpha_{1\nu}(z_2)) \log |z_1 - \alpha_{1\nu}(z_2)| + \text{regular harmonic function}$; and

$$\frac{\partial \log |z_1 - \alpha_{1\nu}(z_2)|}{\partial z_1} = \frac{1}{2} \left[\frac{\partial \log (z_1 - \alpha_{1\nu}(z_2))}{\partial z_1} - \frac{\partial \log (\bar{z}_1 - \overline{\alpha_{1\nu}(z_2)})}{\partial z_1} \right] \\ = \frac{1}{2} (z_1 - \alpha_{1\nu}(z_2))^{-1} = \frac{1}{2} r^{-1} e^{-i\psi}$$

where $re^{i\psi} \equiv z_1 - \alpha_{1\nu}(z_2)$. Since for $z_2 = \text{const.}$ along the circles $|z_1 - \alpha_{1\nu}(z_2)| = r$, $dz_1 = d(z_1 - \alpha_{1\nu}(z_2)) = ire^{i\psi} d\psi$ we conclude in the usual way that

$$(6) \quad \lim_{\epsilon_1 \rightarrow 0} \int_{|z_1 - \alpha_{1\nu}(z_2)| = \epsilon_1} P_{1,z_1} P_2 dz_1 = i\pi \cdot \text{sg}(\alpha_{1\nu}(z_2)) \cdot P_2(\alpha_{1\nu}(z_2), z_2).$$

REMARK. We note that $P_2(\alpha_{1r}(z_2), z_2)$, $z_2 \in \mathfrak{E}_P^2$, is always finite since, according to the hypothesis 2.IV, the projections on the z_2 -plane of the intersection points of $[P_1 = \pm \infty]$ and $[P_2 = \pm \infty]$ do not belong to \mathfrak{E}_P^2 .

Since P_{1,z_1} is regular in $|z_1 - \alpha_{2r}(z_2)| \leq \epsilon_1$ and P_2 is there logarithmically infinite, we conclude in an analogous way

$$(7) \quad \lim_{\epsilon_1 \rightarrow 0} \int_{|z_1 - \alpha_{2r}(z_2)| = \epsilon_1} P_{1,z_1} P_2 dz_1 = 0.$$

Thus we finally obtain

$$(8) \quad \int_{\mathfrak{B}^{*2}(z_2)} P_{1,z_1} P_{2,z_1} d\omega_1 = \frac{1}{2i} \int_{\mathfrak{B}^{*1}(z_2)} P_{1,z_1} P_2 dz_1 - \frac{\pi}{2} \sum_r \operatorname{sg}(\alpha_{1r}(z_2)) P_2(\alpha_{1r}(z_2), z_2).$$

We proceed now to the study of the four-dimensional integrals and to the proof that the improper integrals $\epsilon_2 \rightarrow 0$ can be replaced by ordinary ones. Obviously

$$(9) \quad \int_{\mathfrak{E}_P^2} d\omega_2 \int_{\mathfrak{B}^{*2}(z_2)} P_{1,z_1} P_{2,z_1} d\omega_1 = \int_{\mathfrak{B}^*} P_{1,z_1} P_{2,z_1} d\omega, \quad \mathfrak{B}^* = \bigcup_{z_2 \in \mathfrak{E}_P^2} \mathfrak{B}^{*2}(z_2).$$

According to our assumptions the integrand is infinite, not higher than of first order on a two-dimensional manifold and therefore

$$(10) \quad \lim_{\epsilon_2 \rightarrow 0} \int_{\mathfrak{B}^*} P_{1,z_1} P_{2,z_1} d\omega = \int_{\mathfrak{B}} P_{1,z_1} P_{2,z_1} d\omega.$$

Since $P_{1,z_1} P_2$ in $\mathfrak{b}_2^3 = \mathbf{S}_{z_2 \in \mathfrak{E}_P^2} \mathfrak{b}^1(z_2)$ is infinite of the first order in a one-dimensional manifold,

$$(11) \quad \frac{1}{2i} \lim_{\epsilon_2 \rightarrow 0} \int_{\mathfrak{b}^{*3}} P_{1,z_1} P_2 dz_1 d\omega_2 = \frac{1}{2i} \int_{\mathfrak{b}_2^3} P_{1,z_1} P_2 dz_1 d\omega_2 \equiv \mathfrak{I}_1, \\ \mathfrak{b}^{*3} \equiv \bigcup_{z_2 \in \mathfrak{E}_P^2} \mathfrak{b}^{*1}(z_2).$$

According to (2), see p. 416,

$$P_k^* \equiv P_k[h(\lambda, z_2), z_2], \quad p_1 \equiv (\partial P_1 / \partial z_2)_{z_1 = h(\lambda, z_2)}.$$

Thus $\partial P_1^* / \partial z_2 \equiv (P_{1,z_1})^* h_{z_2} + p_1$ and therefore

$$(12) \quad (P_{1,z_1})^* = h_{z_2}^{-1} \left[\frac{\partial P_1^*}{\partial z_2} - p_1 \right] \quad \left(\frac{\partial P_1^*}{\partial z_2} \equiv \frac{\partial P_1^*(\lambda, z_2)}{\partial z_2} \right),$$

and

$$\begin{aligned}
 \mathfrak{I}_1 &= \mathfrak{I}_3 - \mathfrak{I}_4, \\
 (13) \quad \mathfrak{I}_3 &\equiv \frac{1}{2i} \int_{\{\mathfrak{E}_P^2 \times t^1\}} P_2^* \frac{\partial P_1^*}{\partial z_2} h_\lambda h_{z_2}^{-1} d\lambda d\omega_2, \quad \mathfrak{I}_4 \equiv \frac{1}{2i} \int_{\{\mathfrak{E}_P^2 \times t^1\}} P_2^* p_1 h_\lambda h_{z_2}^{-1} d\lambda d\omega_2 \\
 t^1 &= [0 \leq \lambda \leq 2\pi].
 \end{aligned}$$

Let $z_2 = \beta_{k\mu}(\lambda)$ denote the singularities of P_k^* in \mathfrak{E}_P^2 , and $z_2 = \beta_{k\nu}(\lambda)$ the singularities of P_k^* on the boundary \mathfrak{E}_P^1 of \mathfrak{E}_P^2 . According to 2.IVb, for fixed λ the integrand of \mathfrak{I}_3 becomes (at the most at finitely many points) infinite at the most of the first order, and we can write (see (37a), p. 57 of [B. 8])

$$\begin{aligned}
 (14) \quad & \int_{\lambda=0}^{2\pi} \int_{\mathfrak{E}_P^{**2}} \frac{\partial P_1^*}{\partial z_2} \cdot P_2^* h_\lambda \cdot h_{z_2}^{-1} d\omega_2 d\lambda = - \frac{1}{2i} \int_{\lambda=0}^{2\pi} \int_{\mathfrak{E}_P^1} P_1^* P_2^* h_\lambda h_{z_2}^{-1} dz_2 d\lambda \\
 & + \frac{1}{2i} \sum_{k=1}^2 \int_{\lambda=0}^{2\pi} \sum_{\nu} \int_{|z_2 - \beta_{k\nu}(\lambda)| = \epsilon_3} P_1^* P_2^* h_\lambda h_{z_2}^{-1} dz_2 d\lambda \\
 & - \int_{\lambda=0}^{2\pi} \int_{\mathfrak{E}_P^{**2}} P_1^* \frac{\partial(P_2^* h_\lambda h_{z_2}^{-1})}{\partial z_2} d\omega_2 d\lambda,
 \end{aligned}$$

where

$$\begin{aligned}
 \mathfrak{E}_P^{**2} &\equiv \mathfrak{E}_P^{**2}(\lambda) \equiv \mathfrak{E}_P^2 - \sum_{k=1}^2 \left\{ \sum_{\nu} [|z_2 - \beta_{k\nu}(\lambda)| \leq \epsilon_3] \right. \\
 & \quad \left. + \sum_{\mu} [|z_2 - \beta_{k\mu}(\lambda)| \leq \epsilon_4] \right\},
 \end{aligned}$$

\mathfrak{E}_P^{*1} denotes the boundary of $\mathfrak{E}_P^{*2} \equiv \mathfrak{E}_P^{*2}(\lambda) \equiv \mathfrak{E}_P^2 - \sum_{k=1}^2 \sum_{\mu} [|z_2 - \beta_{k\mu}(\lambda)| \leq \epsilon_4]$. Here $\epsilon_3 \equiv \epsilon_3(\lambda)$ and $\epsilon_4 \equiv \epsilon_4(\lambda)$ are chosen so small that all circles $|z_2 - \beta_{k\nu}(\lambda)| = \epsilon_3$ and $|z_2 - \beta_{k\mu}(\lambda)| = \epsilon_4$ have no intersection points, and so that $|z_2 - \beta_{k\nu}(\lambda)| \leq \epsilon_3$ lie completely inside \mathfrak{E}_P^2 .

Since $P_1^* P_2^*$ becomes at the most at finitely many points at the most logarithmically infinite, the second integral in the right-hand side of (14) vanishes if we go to limit $\epsilon_3 \rightarrow 0$.

In the last integral on the right-hand side of (14) the integrand becomes infinite at the most of the first order along a one-dimensional manifold. We can therefore also pass to the limit and write

$$\begin{aligned}
 (15) \quad \mathfrak{I}_3 &= \frac{1}{2i} \int_{\lambda=0}^{2\pi} \int_{\mathfrak{E}_P^2} \frac{\partial P_1^*}{\partial z_2} P_2^* h_\lambda h_{z_2}^{-1} d\omega_2 d\lambda = \frac{1}{4} \int_{\lambda=0}^{2\pi} \int_{\mathfrak{E}_P^1} P_1^* P_2^* h_\lambda h_{z_2}^{-1} dz_2 d\lambda \\
 & - \frac{1}{2i} \int_{\lambda=0}^{2\pi} \int_{\mathfrak{E}_P^2} P_1^* \frac{\partial(P_2^* h_\lambda h_{z_2}^{-1})}{\partial z_2} d\omega_2 d\lambda.
 \end{aligned}$$

Combining (8), (10), (11), (13), and (15), we obtain (1).

4. **A bound for** $|\mathcal{F}_P(-\log |f_1|, -\log |f_2| + e_2)|$. Formula (3.1) establishes a relation between the sum of the integral $\mathcal{F}_P(P_1, P_2)$ (over the domain \mathfrak{B}_P) and the integral $\mathcal{B}_P(P_1, P_2)$ (over \mathfrak{b}_2^3 a part of \mathfrak{b}^3 , see (2.9), p. 422) on one side, and the sum of certain integrals over the distinguished boundary surface, certain two-dimensional surfaces lying in \mathfrak{b}^3 , and segments of zero and pole surfaces, on the other side. If we choose

$$(1) \quad P_1 = -\log |f_1|, \quad P_2 = -\log |f_2| + e_2, \quad e_2 = e(z_1, z_2; \log |f_2|),$$

see p. 418, then, as we shall show in §6, the terms appearing on the right-hand side of (3.1) can be interpreted as certain functionals associated with the segments of zero and pole surfaces of f_1 and f_2 . (See for details §6.)

In the present section bounds for

$$(2) \quad |\mathcal{F}_P(-\log |f_1|, -\log |f_2| + e_2)|,$$

and in the next section bounds for

$$(3) \quad |\mathcal{B}_P(-\log |f_1|, -\log |f_2| + e_2)|,$$

will be derived. Using (3.1), we obtain in this way bounds for the above mentioned functionals.

On the other hand, in order to be able to obtain bounds of this kind for $|\mathcal{F}_P(-\log |f_1|, -\log |f_2| + e_2)|$, we have to make a number of hypotheses about the structure of the pole and zero surfaces of functions f_1, f_2 . We proceed now to the formulation of these hypotheses.

Around every point

$$(4) \quad [z_1 = \alpha_{k\sigma}(z_2)^0] = [-\operatorname{sg}(\alpha_{k\sigma}(z_2)^0 \log |f_k(z_1, z_2)^0| = \infty]$$

we draw the circle

$$(5) \quad \mathfrak{G}_{k\sigma}^2(z_2, \rho) = [|z_1 - \alpha_{k\sigma}(z_2)^0| < \rho]$$

and form around every branch of singularity surface

$$(6) \quad \mathfrak{a}_{k\sigma}^2 = [z_1 = \alpha_{k\sigma}(z_2)], z_2 \in \mathfrak{C}_P^2$$

a (four-dimensional) tube

$$(7) \quad \mathfrak{t}_{k\sigma}(\rho) = \bigcup_{z_2 \in \mathfrak{C}_P^2} \mathfrak{G}_{k\sigma}^2(z_2, \rho).$$

We assume that ρ is chosen so small that the tubes $\mathfrak{t}_{k\sigma}(\rho)$ have no intersection points⁽⁷⁾.

HYPOTHESIS 4.I.

$$(8) \quad |(z_1 - \alpha_{k\sigma}(z_2))^{\tau} f_k|^{-1} \leq B_k \equiv B_k(\rho), \quad \tau = -\operatorname{sg}(\alpha_{k\sigma}(z_2)),$$

⁽⁷⁾ This is possible since the singularity surfaces of $\log |f_1|$ and $\log |f_2|$ have no intersection points in \mathfrak{B}_P .

$$(9) \quad \left| \frac{\partial [(z_1 - \alpha_{k\sigma}(z_2))^\tau f_k]}{\partial z_1} \right| \leq D_k \equiv D_k(\rho)$$

hold for $(z_1, z_2) \in t_{k\sigma}(\rho)$, and

$$(10) \quad \frac{1}{A_k} \leq |f_k| \leq A_k = A_k(\rho) < \infty, \quad A_k > 1,$$

$$(11) \quad \left| \frac{\partial f_k}{\partial z_1} \right| \leq C_k = C_k(\rho) < \infty$$

for $(z_1, z_2) \in \mathfrak{B}_P - \sum_{\sigma} t_{k\sigma}(\rho)$. Here ρ , B_k , D_k , A_k , C_k are conveniently chosen constants.

HYPOTHESIS 4.II. We assume that the zero and pole surfaces of f_k , $k=1, 2$, intersect the distinguished boundary surface \mathfrak{D}^2 at the most at finitely many points:

$$\begin{aligned} f_2 \text{ at } J_\nu &= \{z_1^{(\nu)}, z_2^{(\nu)}\}, & \nu &= 1, 2, \dots, n, \\ f_1 \text{ at } J_\nu &= \{z_1^{(\nu)}, z_2^{(\nu)}\}, & \nu &= n+1, \dots, n_1, \end{aligned}$$

and we make the following hypotheses about the behavior of the functions f_k , $k=1, 2$ in the neighborhood of the points J_ν .

1. Through every $\{z_1^{(\nu)}, z_2^{(\nu)}\}$, $\nu=1, 2, \dots, n_1$, goes only one singularity surface of $\log |f_k|$, where either $k=1$ or 2 .

2. Let $z_1 = \alpha_{2\sigma}(z_2)$ be the equation of the singularity surface of $\log |f_2|$ in the neighborhood of $J_\nu = \{z_1^{(\nu)}, z_2^{(\nu)}\}$, $\nu=1, 2, \dots, n$. Then we assume that in a sufficiently small neighborhood of J_ν the inequality⁽⁸⁾

$$(12) \quad |h(\lambda, z_2) - \alpha_{2\sigma}(z_2)| \geq a[|\lambda - \lambda^{(\nu)}| + |z_2 - z_2^{(\nu)}|]$$

holds where $0 < a < 1$.

HYPOTHESIS 4.III. We assume that the intersections $[f_1=0] \cap \mathfrak{b}^3$ and $[f_1, z_1=\infty] \cap \mathfrak{b}^3$ represent curves possessing the property that they can be divided into finitely many intervals i_ν^1 , such that in the neighborhood of each $i_\nu^1 = [z_1 = h(\lambda, Z_2(\lambda)), z_2 = Z_2(\lambda)]$, $Z_2(\lambda) \equiv Z_2^{(\nu)}(\lambda)$,

$$(13) \quad f_1(h(\lambda, z_2), z_2) \equiv f_1^* = K_1(\lambda)(z_2 - Z_2(\lambda)) + \dots \text{ and } |K_1(\lambda)| \geq K > 0.$$

HYPOTHESIS 4.IV. In our further considerations we need auxiliary functions $s_\nu(z_1, z_2)$ which we associate with every intersection point J_ν , $\nu=1, 2, \dots, n$, of singularities of $\log |f_2|$ with the distinguished boundary surface. So as to be able to form these functions s_ν we need some further additional hypotheses.

(8) The hypothesis 4. II. 2 refers to the intersection of \mathfrak{D}^2 with singularities of $\log |f_2|$ only.

Considering intersections of singularities of $\log |f_2|$ with the distinguished boundary surface, we distinguish two cases⁽⁹⁾.

Case 1.

$$(14) \quad \alpha_{2\nu}(\exp(i\phi_2)) \in \mathfrak{B}^2(\exp(i\phi_2)) \text{ for } \phi_2 > \phi_2^{(\nu)},$$

$$(15) \quad \alpha_{2\nu}(\exp(i\phi_2)) \text{ outside of } \mathfrak{B}^2(\exp(i\phi_2)) \text{ for } \phi_2 < \phi_2^{(\nu)}.$$

Case 2. Where the opposite occurs, i.e., (14) holds for $\phi_2 < \phi_2^{(\nu)}$, and (15) for $\phi_2 > \phi_2^{(\nu)}$. Let

$$z_1^{(\nu)} = h(\lambda_\nu, \exp(i\phi_2^{(\nu)})), \quad z_2^{(\nu)} = \exp(i\phi_2^{(\nu)}).$$

In case 1, the boundary values $\bar{s}_\nu \equiv \bar{s}_\nu(\lambda, \phi_2)$ on \mathfrak{D}^2 are defined as follows:

$$(16) \quad \bar{s}_\nu = \log |h(\lambda, \exp(i\phi_2)) - \alpha_{2\nu}(\exp(i\phi_2))|$$

$$\text{for } \lambda_0/2 > |\lambda - \lambda_\nu|, \quad \Phi/2 \geq \phi_2 - \phi_2^{(\nu)} \geq 0.$$

For $\lambda_0 \geq |\lambda - \lambda_\nu| \geq \lambda_0/2$, $(\Phi/2) \geq \phi_2 - \phi_2^{(\nu)} \geq 0$ and for $\lambda_0 \geq |\lambda - \lambda_\nu|$, $\Phi \geq \phi_2 - \phi_2^{(\nu)} > \Phi/2$, \bar{s}_ν goes smoothly to zero, so that second derivatives with respect to λ and to ϕ_2 are continuous. These values are chosen in such a way that \bar{s}_ν satisfies the inequality $s_\nu^* \geq -\log a + \log |\lambda - \lambda_\nu|$, see (12) and p. 416. Φ is a conveniently chosen positive constant, and $\lambda_0 < 1$.

Everywhere else, i.e. for $\lambda_0 > |\lambda - \lambda_\nu| \geq 0$, $0 > \phi_2 - \phi_2^{(\nu)}$ or $\phi_2 - \phi_2^{(\nu)} > \Phi$, $\bar{s}_\nu = 0$. At $\phi_2 = \phi_2^{(\nu)}$, $\lambda_0 > |\lambda - \lambda_\nu|$, \bar{s}_ν has a jump.

In case 2, \bar{s}_ν is defined in an analogous manner with the only difference that in the definitions everywhere $\phi_2 - \phi_2^{(\nu)}$ is replaced by $\phi_2^{(\nu)} - \phi_2$.

Let s_ν be the functions of the class $\mathcal{E}(\mathfrak{B})$ which assume the distinguished boundary surface values \bar{s}_ν , $\nu = 1, 2, \dots, n$.

The function $(-s_\nu)$ is non-negative on the distinguished boundary surface and therefore also in \mathfrak{B} . At the point J_ν it becomes infinite.

We assume that for conveniently chosen constants p and q there exist domains \mathfrak{B}_ν , $\nu = 1, 2, \dots, n$, $\mathfrak{B}_\nu \cap \mathfrak{B}_\mu = 0$ for $\nu \neq \mu$, possessing the property that $\mathfrak{B}_\nu \supset (\mathfrak{B} \cap [-s_\nu \geq p])$ and such that the part of the boundary of \mathfrak{B} , which lies in \mathfrak{B}^3 includes $(\mathfrak{B}^3 \cap [|\partial s_\nu^* / \partial z_2| \geq q])$. Further we assume that zero and pole surfaces of f_1 lie outside of the \mathfrak{B}_ν , $\nu = 1, 2, \dots, n$.

HYPOTHESIS 4.V. After a function $s_\nu \in \mathcal{E}(\mathfrak{B})$ has been introduced for every intersection point $\{z_1^{(\nu)}, z_2^{(\nu)}\}$, $\nu = 1, 2, \dots, n$, of \mathfrak{D}^2 with a singularity surface of $\log |f_2(z_1, z_2)|$, we define a further function $s \in \mathcal{E}(\mathfrak{B})$, which on the distinguished boundary surface assumes the values

$$(17) \quad \bar{s}(\lambda, \phi_2) = \log |f_2(h(\lambda, \exp(i\phi_2)), \exp(i\phi_2))| + \sum_{\nu=1}^n \tau_\nu \bar{s}_\nu(\lambda, \phi_2),$$

$$\tau_\nu = -\operatorname{sg} [\alpha_{2\nu}(\exp(i\phi_2))].$$

⁽⁹⁾ For simplicity's sake we assume that the case in which (14) holds for $\phi_2 > \phi_2^{(\nu)}$ as well as for $\phi_2 < \phi_2^{(\nu)}$ does not occur. In the following we shall consider, as a rule, only the case 1. The modification of the formulas, when passing to the case 2, are obvious.

We assume that

$$(18) \quad |s| \leq S, \quad \left| \frac{\partial s}{\partial z_1} \right| \leq T_1 \quad \text{for } (z_1, z_2) \in \overline{\mathfrak{B}}$$

where S, T_1 are conveniently chosen constants.

THEOREM 4.1. *Let $f_k \equiv f_k(z_1, z_2)$, $k = 1, 2$, be two analytic functions which are meromorphic in $\overline{\mathfrak{B}}$ and such that $P_1 = -\log |f_1|$ and $P_2 = -\log |f_2| + e_2$, $e_2 = e(z_1, z_2; \log |f_2|)$ satisfy the hypotheses of Theorem 3.1.*

Let

$$z_1 = \alpha_{k\sigma}(z_2), \quad k = 1, 2; \sigma = 1, 2, \dots,$$

be the equations of the singularities of $\log |f_k(z_1, z_2)|$.

We assume that for a conveniently chosen $\rho < 1$ at every point of \mathfrak{B} , the hypotheses 4.I hold. Further, we assume that in the neighborhood of the intersection points J_ν of singularities of $\log |f_k|$ with the distinguished boundary surface \mathfrak{D}^2 the hypotheses 4.II–4.IV hold, and that the function s introduced in 4.V satisfies the inequality (18). Then

$$(19) \quad \begin{aligned} & |\mathcal{F}(-\log |f_1|, -\log |f_2| + e_2)| \\ & \leq \frac{\pi}{2} \left[\sum_{k=1}^2 \left(\rho + \frac{1}{2} B_k D_k \rho^2 \right) \cdot A_{3-k} C_{3-k} \right] \cdot \sum_{k=1}^2 \mathfrak{S}_2 \left(\mathfrak{S} \mathfrak{A}_{k\sigma}^2 \right) \\ & \quad + \left[\pi T_1 \left(\rho + \frac{1}{2} B_1 D_1 \rho^2 \right) \right] \cdot \sum_{\sigma} \mathcal{A}(\mathfrak{A}_{1\sigma}^2) \\ & \quad + \frac{1}{4} A_1 C_1 (A_2 C_2 + 2T_1) \cdot \mathcal{V}(\mathfrak{B}) \\ & \quad + A_1 C_1 H_3 \lambda_0 (-\log a - \log \lambda_0 + 1) \cdot \sum_{\nu=1}^n \mathfrak{S}_1(\mathfrak{B}_\nu) \\ & \quad + \pi n A_1 C_1 p P \lambda_0 + 2\pi m n \lambda_0 p P \left(\rho + \frac{1}{2} B_1 D_1 \rho^2 \right) + \frac{\pi}{2} n p \mathfrak{S}_2 \left(\mathfrak{S} \mathfrak{A}_{1\sigma}^2 \right), \end{aligned}$$

where $\mathcal{A}(a^2)$ denotes the area of a^2 , $\mathcal{V}(\mathfrak{B})$ denotes the volume of \mathfrak{B} , and $\mathfrak{S}_k(\mathfrak{G}^m)$, $k = 1, 2$, means the area of the projection $\mathcal{P}(\mathfrak{G}^m)$ of \mathfrak{G}^m , on the z_2 -plane. Here A_k , B_k , C_k , D_k have been introduced in (8)–(11), H_3 in 2.I, a in (12), T_1 in 4.V, λ_0 and p in 4.IV, n in 4.II, m is an upper bound for the number of segments of singularity lines

$$z_1 = h(\lambda, Z(\lambda)), \quad z_2 = Z(\lambda)$$

of $\log |f_1|$ in $\mathfrak{h}_\nu^3 = \mathfrak{S}_{\lambda-\lambda_\nu}^{\lambda_\nu+\lambda_0} \mathfrak{G}^2(\lambda)$ (see (2.7)) P an upper bound for $|dz_1/d\lambda|$, and when we move along these lines.

REMARK. $\mathfrak{S}_1(\mathfrak{G}^m)$ means the area of the domain of z_2 -plane consisting of all z_2 coordinates of the points (z_1, z_2) of \mathfrak{G}^m . $\mathfrak{S}_2(\mathfrak{N}^2)$ is the area of (in general multiply-sheeted) projection of \mathfrak{N}^2 on the z_2 -plane. ($\mathfrak{S}_2(\mathfrak{N}^2)$ is the sum of the areas of all sheets.)

Proof. Obviously

$$(20) \quad \begin{aligned} \mathcal{Y}_P(-\log |f_1|, -\log |f_2| + e_2) &= \mathcal{Y}_P(-\log |f_1|, -\log |f_2| + s) \\ &\quad - \sum_{\nu=1}^n \tau_\nu \mathcal{Y}_P(-\log |f_1|, s_\nu) \end{aligned}$$

(see (17)) and we proceed to the derivation of upper bounds for each term on the right-hand side of (20) separately. We consider the first integral in a tube, say $t_{1\sigma}(\rho)$. Using (5), (8), (9), (10), (11), (18) we obtain

$$(21) \quad \begin{aligned} &\int_{\mathfrak{G}_{1\sigma}^2(z_2, \rho) \cap \mathfrak{B}^2(z_2)} \left| \frac{\partial \log |f_1|}{\partial z_1} \cdot \frac{\partial(\log |f_2| - s)}{\partial \bar{z}_1} \right| d\omega_1 \\ &\leq \frac{1}{2} \int_{\mathfrak{G}_{1\sigma}^2(z_2, \rho) \cap \mathfrak{B}^2(z_2)} \left\{ |z_1 - \alpha_{1\sigma}(z_2)|^{-1} \cdot \left| \frac{\partial(z_1 - \alpha_{1\sigma}(z_2))}{\partial z_1} \right| \right. \\ &\quad \left. + \left| \frac{\partial \log [(z_1 - \alpha_{1\sigma}(z_2))^{\tau} f_1]}{\partial z_1} \right| \right\} \cdot \left| \frac{\partial \log |f_2|}{\partial \bar{z}_1} - \frac{\partial s}{\partial \bar{z}_1} \right| d\omega_1 \\ &= \frac{1}{4} \int_{\rho^*=0}^{\rho} \int_{\phi_2=0}^{2\pi} \left\{ \frac{1}{\rho^*} + \left| \frac{\partial \log [(z_1 - \alpha_{1\sigma}(z_2))^{\tau} f_1]}{\partial z_1} \right| \right\} \\ &\quad \cdot \left| \frac{1}{\bar{f}_2} \frac{\partial \bar{f}_2}{\partial \bar{z}_1} - 2 \frac{\partial s}{\partial \bar{z}_1} \right| d\omega_1 \\ &\leq \frac{1}{4} \int_{\rho^*=0}^{\rho} \int_{\phi_2=0}^{2\pi} \left(\frac{1}{\rho^*} + B_1 D_1 \right) (A_2 C_2 + 2T_1) \rho^* d\rho^* d\phi_2 \\ &= \frac{\pi}{2} \left(\rho + \frac{1}{2} B_1 D_1 \rho^2 \right) (A_2 C_2 + 2T_1). \end{aligned}$$

Thus

$$(22) \quad \begin{aligned} &\left| \int_{\sum_{k=1}^2 \sum_{\sigma} t_{k\sigma}(\rho)} \frac{\partial \log |f_1|}{\partial z_1} \cdot \frac{\partial(\log |f_2| - s)}{\partial \bar{z}_1} d\omega \right| \\ &\leq \frac{\pi}{2} \left[\sum_{k=1}^2 \left(\rho + \frac{1}{2} B_k D_k \rho^2 \right) A_{3-k} C_{3-k} \right] \cdot \sum_{k=1}^2 \sum_{\sigma} \mathcal{A}(\mathfrak{N}_{k\sigma}^2) \\ &\quad + \left[\pi T_1 \left(\rho + \frac{1}{2} B_1 D_1 \rho^2 \right) \right] \cdot \mathfrak{S}_2 \left(\mathfrak{S}_{\sigma} \mathfrak{N}_{1\sigma}^2 \right). \end{aligned}$$

When considering $\mathcal{Y}_P(-\log |f_1|, -\log |f_2| + s)$ in $\mathfrak{B}_P^* = \mathfrak{B}_P - \sum_{k=1}^2 \sum_{\sigma} t_{k\sigma}(\rho)$ we obtain, using (10), (11), (18),

$$\begin{aligned}
 (23) \quad & \left| \int_{\mathfrak{B}_P^*} \frac{\partial \log |f_1|}{\partial z_1} \cdot \frac{\partial (\log |f_2| - s)}{\partial \bar{z}_1} d\omega \right| \\
 & \leq \int_{\mathfrak{B}_P^*} \left| \frac{\partial \log |f_1|}{\partial z_1} \cdot \frac{\partial (\log |f_2| - s)}{\partial \bar{z}_1} \right| d\omega \\
 & = \int_{\mathfrak{B}_P^*} \left| \frac{1}{2} f_1^{-1} \frac{\partial f_1}{\partial z_1} \left(\frac{1}{2} \bar{f}_2^{-1} \frac{\partial \bar{f}_2}{\partial \bar{z}_1} - \frac{\partial s}{\partial \bar{z}_1} \right) \right| d\omega \\
 & \leq \frac{1}{4} A_1 C_1 (A_2 C_2 + 2T_1) \cdot \mathcal{U}(\mathfrak{B}).
 \end{aligned}$$

We proceed similarly to determine a bound for $|\mathcal{F}_P(-\log |f_1|, s_\nu)|$, but certain modifications are needed. They are caused by the fact that in the neighborhood of the intersection point $J, \{z_1^{(\nu)}, z_2^{(\nu)}\}$ of the distinguished boundary surface with the surface $z_1 = \alpha_{2\sigma}(z_2)$, s_ν becomes infinite.

We proceed now to determine the bounds for

$$\left| \int_{\mathfrak{B}_P} \frac{\partial \log |f_1|}{\partial z_1} \cdot \frac{\partial s_\nu}{\partial \bar{z}_1} d\omega \right|.$$

(We choose the constants⁽¹⁰⁾ $p < \infty$, $q < \infty$, see 4.IV, so large that $\log |f_1|$ is regular in every \mathfrak{B}_ν .)

According to (37b), page 57 of [B. 8], we have

$$\begin{aligned}
 (24) \quad & \int_{\mathfrak{B}^{t_2}(z_2)} \frac{\log |f_1|}{\partial z_1} \frac{\partial s_\nu}{\partial \bar{z}_1} d\omega_1 = \frac{1}{2i} \int_{b^1(z_2)} \frac{\partial \log |f_1|}{\partial z_1} s_\nu dz_1 \\
 & - \frac{1}{2i} \sum_{\sigma} \int_{c_{1\sigma}^{-1}(z_2, \rho_2)} \frac{\partial \log |f_1|}{\partial z_1} s_\nu dz_1.
 \end{aligned}$$

Here $\mathfrak{B}^{t_2}(z_2) = \mathfrak{B}^2(z_2) - \sum_{\sigma} \mathfrak{C}_{1\sigma}^2(z_2, \rho_2)$. Concerning $\mathfrak{C}_{1\sigma}^2$ see (5), $c_{1\sigma}^1$ are boundary curves of $\mathfrak{C}_{1\sigma}^2$. Since outside the interval $\{\lambda_\nu - \lambda_0, \lambda_\nu + \lambda_0\}$, s_ν vanishes on $b^1(z_2)$, we have for $z_2 \in \mathcal{P}(\mathfrak{B}_\nu)$, see (16), (12), (2.1),

$$\begin{aligned}
 (25) \quad & \left| \frac{1}{2i} \int_{b^1(z_2)} \frac{\partial \log |f_1|}{\partial z_1} s_\nu dz_1 \right| \\
 & \leq \frac{1}{2} \int_{\lambda = \lambda_\nu - \lambda_0}^{\lambda_\nu + \lambda_0} \left| \frac{\partial \log |f_1|}{\partial z_1} \right| |\log a + \log |\lambda - \lambda_\nu|| |h_\lambda(\lambda, z_2)| d\lambda \\
 & \leq A_1 C_1 H_3 \lambda_0 |\log a + \log \lambda_0 - 1| \quad (\lambda_0 \text{ sufficiently small}),
 \end{aligned}$$

$$\begin{aligned}
 (26) \quad & \left| \frac{1}{2i} \int_{\mathfrak{S}_1(\mathfrak{B}_\nu)} \int_{b^1(z_2)} \frac{\partial \log |f_1|}{\partial z_1} s_\nu d\omega_2 dz_1 \right| \\
 & \leq A_1 C_1 H_3 \lambda_0 (-\log a - \log \lambda_0 + 1) \mathfrak{S}_1(\mathfrak{B}_\nu).
 \end{aligned}$$

⁽¹⁰⁾ We note that the inequality $|\partial s_\nu^*/\partial z_2| > q$ is used only in §5.

We proceed now to the derivation of bounds for

$$\int_{\mathfrak{E}_{P,r}^2} \int_{\mathfrak{b}^1(z_2)} \frac{\partial \log |f_1|}{\partial z_1} s_r dz_1 d\omega_2 = \frac{1}{2} \int_{\mathfrak{E}_{P,r}^2} \int_{\mathfrak{b}^1(z_2)} \frac{f_{1,s_1}}{f_1} s_r dz_1 d\omega_{z_2},$$

$$\mathfrak{E}_{P,r}^2 = \mathfrak{E}_P^2 - \mathfrak{S}_1(\mathfrak{B}_r).$$

According to the hypothesis 4.III, one can cover the neighborhood of $\mathfrak{b}^3 \cap ([f_1=0] + [f_1=\infty])$ in \mathfrak{b}^3 by finitely many tubes $\mathfrak{T}_{\nu\mu}^3$ each possessing the "axis" $i_\nu^1 = [z_1 = h(\lambda, Z_2(\lambda)), z_2 = Z_2(\lambda)]$, $-\lambda_\nu \leq \lambda \leq \lambda_{\nu+1}$. In each $\mathfrak{T}_{\nu\mu}^3$ we can represent f_1 in the form

$$\log |f_1| = \log |(z_1 - \alpha_{1\sigma}(z_2))^{-\tau}| + \log |(z_1 - \alpha_{1\sigma}(z_2))^{\tau} f_1|,$$

$$\tau \equiv -\operatorname{sg}(\alpha_{1\sigma}(z_2)),$$

$$(27) \quad \frac{\partial \log |f_1|}{\partial z_1} = \frac{1}{2} \left\{ \frac{\partial \log (z_1 - \alpha_{1\sigma}(z_2))^{-\tau}}{\partial z_1} + \frac{\frac{\partial}{\partial z_1} [(z_1 - \alpha_{1\sigma}(z_2))^{\tau} f_1]}{(z_1 - \alpha_{1\sigma}(z_2))^{\tau} f_1} \right\},$$

$$\left| \frac{\partial \log |f_1|}{\partial z_1} \right| \leq \frac{1}{2} \left\{ \frac{1}{|z_1 - \alpha_{1\sigma}(z_2)|} + B_1 D_1 \right\},$$

by 4.I. Consequently, since according to 4.IV in the tube $\mathfrak{T}_{\nu\mu}^3$ with the axis i_ν^1 , $|s_r| \leq p$, and since $|dz_1| \leq P d\lambda$,

$$\left| \int_{\mathfrak{T}_{\nu\mu}^3} \frac{\partial \log |f_1|}{\partial z_1} s_r d\omega_2 dz_1 \right|$$

$$\equiv \left| \int_{\lambda=\lambda_\nu-\lambda_0}^{\lambda=\lambda_\nu+\lambda_0} \int_{|z_2-Z_2(\lambda)|<\rho} \frac{\partial \log |f_1|}{\partial z_1} s_r d\omega_2 dz_1(\lambda) \right|$$

$$\leq \frac{1}{2} \left| \int_{\lambda_\nu-\lambda_0}^{\lambda_\nu+\lambda_0} \int_0^{2\pi} \int_0^\rho \left[\frac{1}{\rho^*} + B_1 D_1 \right] p P \rho^* d\rho^* d\phi_2 d\lambda \right|$$

$$= 2\pi \lambda_0 p P \left(\rho + \frac{1}{2} B_1 D_1 \rho^2 \right).$$

Since we assumed that in $\mathfrak{S}_{\lambda_\nu-\lambda_0}^{\lambda_\nu+\lambda_0} \mathfrak{S}^2(\lambda)$ there are at the most m tubes $\mathfrak{T}_{\nu\mu}^3$ and there are only n points $\{z_1^{(p)}, z_2^{(p)}\}$, the sum of these integrals

$$(28) \quad \sum_{\nu=1}^n \sum_{\mu} \left| \int_{\mathfrak{T}_{\nu\mu}^3} \frac{\partial \log |f_1|}{\partial z_1} s_r d\omega_2 dz_1 \right| \leq 2\pi m n \lambda_0 p P \cdot \left(\rho + \frac{1}{2} B_1 D_1 \rho^2 \right).$$

On the remaining part of \mathfrak{b}^3 accordingly, (10), (11),

$$\left| \frac{\partial \log |f_1|}{\partial z_1} \right| < \frac{1}{2} C_1 A_1$$

and therefore

$$\left| \int_{\mathfrak{A}_p^3} \frac{\partial \log |f_1|}{\partial z_1} s_p d\omega_2 dz_1 \right| \leq \int_{\lambda_p - \lambda_0}^{\lambda_p + \lambda_0} \frac{1}{2} A_1 C_1 p P \pi d\lambda = \pi A_1 C_1 p P \lambda_0.$$

(The integration on the left-hand side is taken over $\mathfrak{A}_p^3 = \mathfrak{h}^3 - \mathcal{P}(\mathfrak{B}_p) \times \mathfrak{h}^1(z_2) - \sum_{\nu=1}^n \sum_{\mu} \mathfrak{A}_{p\mu}^3$.) Since we have n intersection points $\{z_1^{(\nu)}, z_2^{(\nu)}\}$,

$$\sum_{\nu=1}^n \left| \int_{\mathfrak{A}_p^3} \cdots \right| \leq \pi n A_1 C_1 p P \lambda_0.$$

It still remains to determine bounds for

$$\left| \int_{\mathfrak{E}_p^2} \left(\frac{1}{2} \int_{c_1^{-1}(z_2, \rho_2)} \left| \frac{\partial \log |f_1|}{\partial z_1} \right| s_p dz_1 \right) d\omega_2 \right|$$

Now

$$\lim_{\rho_2 \rightarrow 0} \left| \frac{1}{2} \int_{c_1^{-1}(z_2, \rho_2)} \frac{\partial \log |f_1|}{\partial z_1} s_p dz_1 \right| = \frac{\pi}{2} s_p(\alpha_{1\sigma}(z_2), z_2)$$

where $z_1 = \alpha_{1\sigma}(z_2^0)$ are infinity points of $-\text{sg}(\alpha_{1\sigma}(z_2^0)) \log |f_1(z_1, z_2^0)|$. But for $z_2 \in \mathcal{P}(\mathfrak{B}_p)$ these points lie outside \mathfrak{B}_p , for $z_2 \in \mathfrak{E}_p^2 - \mathcal{P}(\mathfrak{B}_p)$, $|s_p| \leq p$ on the boundary curve $\mathfrak{h}^1(z_2)$ of $\mathfrak{B}^2(z_2)$, and therefore also in $\mathfrak{B}^2(z_2)$. Consequently

$$0 \leq \frac{\pi}{2} s_p(\alpha_{1\sigma}(z_2)) \leq \frac{\pi}{2} p$$

everywhere in \mathfrak{B} , so that

$$(30) \quad \lim_{\rho_2 \rightarrow 0} \left| \int_{\mathfrak{E}_p^2} \sum_{\sigma} \left(\frac{1}{2} \int_{c_1^{-1}(z_2, \rho_2)} \frac{\partial \log |f_1|}{\partial z_1} s_p dz_1 \right) d\omega_2 \right| \leq \frac{\pi}{2} p \mathfrak{S}_2 \left(\mathfrak{S}_{\sigma} \mathfrak{A}_{1\sigma}^2 \right)$$

where $\mathfrak{S}_2(\mathfrak{S}_{\sigma} \mathfrak{A}_{1\sigma}^2)$ is the area of the projections of the zero and the pole surface of f_1 on the z_2 -plane.

Let us finally mention that the integrals

$$\int_{\mathfrak{E}_{1\sigma}^2(z_2, \rho)} \frac{\partial \log |f_1|}{\partial z_1} \frac{\partial(\log |f_2| + s_p)}{\partial \bar{z}_1} d\omega_2$$

tend to zero for $\rho \rightarrow 0$ since the integrand is infinite of the first order and the integration is taken over a two-dimensional domain whose area tends to zero.

5. A bound for $|\mathcal{B}_P(-\log |f_1|, -\log |f_2| + e_2)|$. In determining an upper bound for the absolute value of the integral $\mathcal{B}_P(P_1, P_2)$ (over \mathfrak{h}_2^3), see (3.2), the line following (3.10), and (4.1), we write again $e_2 = s + \sum_{\nu=1}^n s_{\nu}$, and shall determine a bound at first for $|\mathcal{B}_P(-\log |f_1|, -\log |f_2| + s)|$ and later for $\sum_{\nu=1}^n |\mathcal{B}_P(-\log |f_1|, s_{\nu})|$.

In addition to the hypotheses made in §§3 and 4, we make now the following further assumptions.

HYPOTHESIS 5.I. In the part \mathfrak{b}^3 (see (2.4), (2.9), and p. 416) of the boundary of \mathfrak{B}

$$\frac{\partial f_2^*}{\partial z_2}, \quad \log |f_k^*|, \quad k = 1, 2, \quad \left(\frac{\partial f_1}{\partial z_2} \right)_{z_1=h(\lambda, z_2)}$$

become infinite along lines $z_2 = \gamma_{2\sigma}(\lambda)$, $z_2 = \beta_{k\sigma}(\lambda)$, $k = 1, 2$, $z_2 = \gamma_{1\sigma}(\lambda)$, respectively. As before we assume that in a conveniently chosen (three-dimensional) tube \mathfrak{t}^3 with the above mentioned line as axis and intersection with $\mathfrak{E}^2(\lambda)$, see (2.7), being a circle of radius ρ ,

$$(1) \quad \left| (z_2 - \gamma_{2\sigma}(\lambda)) \frac{\partial f_2^*}{\partial z_2} \right| \leq \tilde{c}_2 = \tilde{c}_2(\rho),$$

$$(2) \quad \frac{1}{Q_k} \leq |(z_2 - \beta_{k\sigma}(\lambda))^\tau f_k^*| \leq Q_k \equiv Q_k(\rho), \quad \tau = -\operatorname{sg}(\beta_{k\sigma}(\lambda)),$$

$$(3) \quad \left| (z_2 - \gamma_{1\sigma}(\lambda)) \left(\frac{\partial f_1}{\partial z_2} \right)_{z_1=h(\lambda, z_2)} \right| \leq L_1 \equiv L_1(\rho)$$

holds, while outside of the corresponding tube \mathfrak{t}^3 in \mathfrak{b}^3 the inequalities

$$(4) \quad \left| \frac{\partial f_2^*}{\partial z_2} \right| \leq \tilde{c}_4 = \tilde{c}_4(\rho),$$

$$(4.10) \quad \frac{1}{A_k} \leq |f_k^*| \leq A_k = A_k(\rho),$$

$$(5) \quad \left| \left(\frac{\partial f_1}{\partial z_2} \right)_{z_1=h(\lambda, z_2)} \right| \leq \delta_1$$

are valid.

HYPOTHESIS 5.II. The functions s_ν introduced in §4, IV satisfy in \mathfrak{n}_ν^3 : $[z_1 = h(\lambda, z_2), |z_2 - \delta(\lambda_\nu)| < \rho_0 |\lambda - \lambda_\nu| < \lambda_0]$ the inequality

$$(6) \quad \left| \frac{\partial s_\nu^*}{\partial z_2} \right| \leq \frac{K}{|\lambda - \lambda_\nu|^\mu + |z_2 - \delta(\lambda_\nu)|^\eta}, \quad \delta(\lambda) = \exp(i\lambda),$$

where $K < \infty$, $\mu < \infty$, $\eta \leq 2$ are conveniently chosen constants.

REMARK. Here we assume for simplicity's sake that $\delta(\lambda) = \exp(i\lambda)$, but we note that it would be possible to choose for $\delta(\lambda)$ some other function.

HYPOTHESIS 5.III. For the function s introduced in §4, Hypothesis V, see (4.17), the inequality

$$(7) \quad \left| \frac{\partial s(h(\lambda, z_2), z_2)}{\partial z_2} \right| = \left| \frac{\partial s^*}{\partial z_2} \right| \leq T_2 < \infty$$

holds.

Let $i_{k0}^1, i_{k\infty}^1, k=1, 2, i_{2\infty}^1, \mathcal{G}_{1\infty}^1$ denote the respective intersections of $[|f_k|=0], [|f_k|=\infty], [|\partial f_2^*/\partial z_2|=\infty], [(\partial f_1/\partial z_2)_{z_1=h(\lambda, z_1)}=\infty]$ with b_2^3 , see p. 422. Let further $j_1^1 = i_{10}^1 + i_{1\infty}^1$.

Finally we assume that p and q have been chosen so large that $\mathcal{G}_{1\infty}^1, i_1^1$ lie outside of the $n_\nu^3, \nu=1, 2, \dots, n$. See page 432.

We remark also that we can assume that $j_{2\infty}^1$ does not intersect the lines i_{11}^1, i_{20}^1 , nor $\mathcal{G}_{1\infty}^1$ intersect i_{10}^1, i_2^1 in b^3 . (If necessary we can exclude these intersection points assuming that their projections belong to \mathfrak{S}_P^3 , see 2.IV.b.)

THEOREM 5.1. *Let $f_k, k=1, 2$, be two analytic functions which are meromorphic in \mathfrak{B} and satisfy hypotheses of the Theorem 4.1. If in addition they satisfy the hypotheses 5.I-5.III, then*

$$(8) \quad \mathcal{B}_P(-\log |f_1|, -\log |f_2| + e_2) \leq B_1 + B_2$$

where

$$(9) \quad \begin{aligned} B_1 = & \frac{\pi}{4} \left\{ 2H |\log A_1| \rho [(\tilde{c}_2 A_2 + T_2 \rho) \mathcal{L}(i_{2\infty}^1) + (\tilde{c}_4 Q_2 + T_2 \rho) \mathcal{L}(i_{20}^1)] \right. \\ & + 4(|\log A_2| + S) \rho H [A_1 L_1 \mathcal{L}(\mathcal{G}_{1\infty}^1) + \delta_1 Q_1 \mathcal{L}(i_{10}^1)] \\ & + \sum_{k=1}^2 \left\{ \frac{\pi}{4} \rho^2 [H R_k + 2G(|\log A_{3-k}| + S(2-k))] \right. \\ & \times \left[\frac{1}{2} - \log \rho + |\log Q_k| + S(k-1) \right] \mathcal{L}(i_k^1) \\ & \left. + \frac{H}{4} [(\log |A_k| + (k-1)S) U_k] \mathcal{V}(b^3) \right\}, \end{aligned}$$

$$R_1 = (A_2 \tilde{c}_4 + 2T_2), \quad R_2 = \delta_1 A_1, \quad U_1 = \tilde{c}_4 A_2 + 2T_2,$$

$$U_2 = A_1 \delta_1 + 2GH^{-1} |\log A_1|, \quad i_k^1 = i_{k0}^1 + i_{k\infty}^1,$$

and B_2 is given in (34a). $\mathcal{L}(i^1)$ denotes the "length" of the line segment i^1 , see (17).

Proof. I. In this part we derive an upper bound for $|\mathcal{B}_P(-\log |f_1|, -\log |f_2| + s)|$ while in II we shall compute a bound for $|\mathcal{B}_P(-\log |f_1|, s_\nu)|$.

According to (3.2)

$$(10) \quad \mathcal{B}_P(-\log |f_1|, -\log |f_2| + s) = \int_0^{2\pi} \int_{\mathfrak{S}_P^3} (\tau_1 + \tau_2 + \tau_3) d\omega_2 d\lambda,$$

where

$$(11) \quad \tau_1 = \frac{1}{2i} \left[\log |f_1^*| \frac{\partial(\log |f_2^*| - s^*)}{\partial z_2} \frac{h_\lambda}{h_{z_2}} \right],$$

$$(12) \quad \tau_2 = \frac{1}{2i} \left[\left(\frac{\partial \log |f_1|}{\partial z_2} \right)_{z_1=h(\lambda, z_2)} (\log |f_2^*| - s^*) \frac{h_\lambda}{h_{z_2}} \right],$$

$$(13) \quad \tau_3 = \frac{1}{2i} \left[\log |f_1^*| \cdot (\log |f_2^*| - s^*) \frac{\partial}{\partial z_2} \left(\frac{h_\lambda}{h_{z_2}} \right) \right].$$

Since

$$\frac{\partial \log |f_2^*|}{\partial z_2} = \frac{1}{2} \frac{1}{f_2^*} \frac{\partial f_2^*}{\partial z_2},$$

τ_1 becomes infinite along the curves $j_1^1 \equiv j_{10}^1 + j_{1\infty}^1$, $j_{2\infty}^1$ and j_{20}^1 . The second integrand τ_2 becomes infinite along j_{10}^1 , $j_2^1 \equiv j_{20}^1 + j_{2\infty}^1 \equiv j_{20}^1 + [|f_2^*| = \infty]$, and $j_{1\infty}^1$. Finally, the third integrand τ_3 becomes infinite along j_k^1 , $k=1, 2$.

Around every point (z_1^0, z_2^0) , $z_1^0 = h(\lambda, z_2^0)$, of a singularity curve, say j^1 , we form in the lamina $\mathfrak{S}^2(\lambda)$ a *disc* $[z_1 = h(\lambda, z_2), |z_2 - z_2^0| < \rho] = \mathfrak{b}^2(z_1^0, z_2^0)$ and form a (three-dimensional) *tube*

$$(13a) \quad \mathfrak{t}^3(j^1) = \bigcup_{(z_1, z_2) \in j^1} \mathfrak{b}^2(z_1, z_2).$$

The curve j^1 will be denoted as the *axis of the tube*. Determining bounds for the absolute value of the left side of (10), as before, we derive them separately at first in the tubes, and later in the remaining part of \mathfrak{b}_2^3 . Considering τ_1 in $\mathfrak{t}^3(j_{2\infty}^1)$, see p. 433, we write

$$(14) \quad \begin{aligned} \tau_1 &\leq \frac{1}{4} \left| \log |f_1^*| \right| \left| \frac{1}{f_2^*} \frac{\partial f_2^*}{\partial z_2} - 2 \frac{\partial s^*}{\partial z_2} \right| \cdot \left| \frac{h_\lambda}{h_{z_2}} \right| \\ &= \frac{1}{4} \left| \log |f_1^*| \right| \left| \frac{(z_2 - \gamma_{2\sigma}(\lambda))(\partial f_2^* / \partial z_2)}{(z_2 - \gamma_{2\sigma}(\lambda))f_2^*} - 2 \frac{\partial s^*}{\partial z_2} \right| \cdot \left| \frac{h_\lambda}{h_{z_2}} \right|. \end{aligned}$$

Thus by (4.10), (1), (4.18), (2.11), (7)

$$(15) \quad \begin{aligned} \left| \int_{\mathfrak{b}^2(z_1, z_2)} \tau_1 d\omega_2 \right| &\leq \frac{1}{4} |\log A_1| H \int_0^{2\pi} \int_0^\rho \left\{ \frac{\tilde{c}_2 A_2}{\rho^*} + 2T_2 \right\} \rho^* d\rho^* d\phi_2 \\ &= \frac{\pi}{2} H |\log A_1| (\tilde{c}_2 A_2 \rho + T_2 \rho^2). \end{aligned}$$

Consequently

$$(16) \quad \left| \int_{j_{2\infty}^1} d\lambda \int_{\mathfrak{b}^2(z_1, z_2)} \tau_1 d\omega_2 \right| \leq \frac{\pi}{2} H |\log A_1| (\tilde{c}_2 A_2 \rho + T_2 \rho^2) \mathcal{L}(j_{2\infty}^1)$$

where

$$(17) \quad \mathcal{L}(j_{2\infty}^1) = \int_{j_{2\infty}^1} d\lambda.$$

In the case of \mathfrak{J}_{20}^1 we proceed exactly in the same way. Using (4.10), (2.11), (4), (2), (4.18), we obtain

$$(18) \quad \left| \int_{\mathfrak{b}^2(z_1, z_2)} \tau_1 d\omega_2 \right| \leq \frac{1}{4} |\log A_1| H \int_0^{2\pi} \int_0^\rho \left\{ \frac{\tilde{c}_4 Q_2}{\rho^*} + 2T_2 \right\} \rho^* d\rho^* d\phi_2 \\ = \frac{\pi}{2} H |\log A_1| (\tilde{c}_4 Q_2 \rho + T_2 \rho^2),$$

$$(19) \quad \left| \int_{\mathfrak{J}_{20}^1} \int_{\mathfrak{b}^2(z_1, z_2)} \tau_1 d\omega_2 d\lambda \right| \leq \frac{\pi}{2} H |\log A_1| (\tilde{c}_4 Q_2 \rho + T_2 \rho^2) \mathcal{L}(\mathfrak{J}_{20}^1).$$

Considering τ_1 in the tube $\mathfrak{t}^3(\mathfrak{J}_1^1)$ (around the zero and the pole lines of f_1) we use (2), (7), (4), (4.10), (2.11), (4.8) to obtain

$$(20) \quad \left| \int_{\mathfrak{b}^2(z_1, z_2)} \tau_1 d\omega_2 \right| \\ \leq \frac{1}{4} (A_2 \tilde{c}_4 + 2T_2) H \int_0^{2\pi} \int_0^\rho (|\log \rho^*| + |\log Q_1|) \rho^* d\rho^* d\phi_2 \\ = \frac{\pi}{4} H \rho^2 (A_2 \tilde{c}_4 + 2T_2) \left(\frac{1}{2} - \log \rho + |\log Q_1| \right),$$

and therefore

$$(21) \quad \left| \int_{\mathfrak{J}_1^1} \int_{\mathfrak{b}^2(z_1, z_2)} \tau_1 d\omega_2 d\lambda \right| \\ \leq \frac{\pi}{4} H \rho^2 (A_2 \tilde{c}_4 + 2T_2) \left(\frac{1}{2} - \log \rho + |\log Q_1| \right) \mathcal{L}(\mathfrak{J}_1^1).$$

Considering the next term τ_2 in the tubes around singularities, we observe that by 5.I in the tube $\mathfrak{t}^3(\mathfrak{J}_{1\infty}^1)$

$$(22) \quad \left(\frac{\partial \log |f_1|}{\partial z_2} \right)_{-\mathfrak{h}(\lambda, z_2) z_1} = \frac{1}{2} \left(\frac{\partial f_1}{\partial z_2} \right)_{z_1 = \mathfrak{h}(\lambda, z_2)} \cdot \frac{1}{f_1^*} \\ = \frac{1}{2} \left\{ \frac{1}{z_2 - \gamma_{1\sigma}(\lambda)} \left[\left(\frac{\partial f_1}{\partial z_2} \right)_{z_1 = \mathfrak{h}(\lambda, z_2)} (z_2 - \gamma_{1\sigma}(\lambda)) \right] \right\} \frac{1}{f_1^*}.$$

Thus by (3), (4.10), (4.18), (2.11)

$$(23) \quad |\tau_2| \leq \frac{1}{2} \frac{L_1}{\rho^*} A_1 (|\log A_2| + S) H,$$

$$(24) \quad \left| \int_{\mathfrak{J}_{1\infty}^1} \int_{\mathfrak{b}^2(z_1, z_2)} \tau_2 d\omega_2 d\lambda \right| \leq \pi (|\log A_2| + S) H A_1 L_1 \rho \mathcal{L}(\mathfrak{J}_{1\infty}^1).$$

Considering τ_2 in $t^3(i_{10}^1)$ we have, according to 5.I, (5), (2),

$$(25) \quad \left| \left(\frac{\partial \log |f_1|}{\partial z_2} \right)_{z_1=h(\lambda, z_2)} \right| = \frac{1}{2} \left| \left(\frac{\partial f_1}{\partial z_2} \right)_{z_1=h(\lambda, z_2)} \frac{(z_2 - \beta_{1\sigma}(\lambda))^{-1}}{[(z_2 - \beta_{1\sigma}(\lambda))^{-1} f_1^*]} \right| \leq \frac{1}{2} \frac{\delta_1 Q_1}{\rho^*}$$

and therefore

$$(26) \quad \begin{aligned} & \left| \int_{i_{10}^1} \int_{b^2(z_1, z_2)} \tau_2 d\omega_2 d\lambda \right| \\ & \leq \frac{1}{2} \left| \int_{i_{10}^1} \int_0^{2\pi} \int_0^\rho \frac{\delta_1 Q_1}{\rho^*} (|\log A_2| + S) H \rho^* d\rho^* d\omega_2 d\lambda \right| \\ & = \pi \delta_1 Q_1 (|\log A_2| + S) H \rho \mathcal{L}(i_{10}^1). \end{aligned}$$

Similarly in the tubes $t^3(i_{12}^1)$ around zero and pole surface j_2^1 using 5.I, (2) we have

$$(27) \quad \begin{aligned} |\log |f_2^*|| &= |\log |(z_2 - \beta_{2\sigma}(\lambda))^{-\tau} (z_2 - \beta_{2\sigma}(\lambda))^{\tau} f_2^*|| \\ &= |\log |(z_2 - \beta_{2\sigma}(\lambda))^{\tau}| + \log |(z_2 - \beta_{2\sigma}(\lambda))^{\tau} f_2^*|| \\ &\leq -\log \rho^* + |\log Q_2| \end{aligned}$$

and therefore by (5), (4.10), (2.11), (4.18)

$$(28) \quad \begin{aligned} & \left| \int_{i_2^1} \int_{b^2(z_1, z_2)} \tau_2 d\omega_2 d\lambda \right| \\ & \leq \frac{1}{2} \left| \int_{i_2^1} \int_0^{2\pi} \int_0^\rho \frac{\delta_1 A_1}{2} (-\log \rho^* + |\log Q_2| + S) H \rho^* d\rho^* d\phi_2 d\lambda \right| \\ & = \frac{\pi}{4} \delta_1 A_1 H \rho^2 \left(\frac{1}{2} - \log \rho + |\log Q_2| + S \right) \mathcal{L}(i_2^1). \end{aligned}$$

Finally, determining bounds for the integral of τ_3 we note that in $t^3(i_1^1)$ by 5.I and (2), $|\log |f_k^*|| \leq -\log \rho^* + |\log Q_k|$ and therefore by (8), (4.10), (4.18), (2), (2.11)

$$(29) \quad \begin{aligned} & \left| \int_{i_1^1} \int_{b^2(z_1, z_2)} \tau_3 d\omega_2 d\lambda \right| \\ & \leq \left| \int_{i_1^1} \int_0^{2\pi} \int_0^\rho \frac{G}{2} (|\log A_2| + S) (-\log \rho^* + |\log Q_1|) \rho^* d\rho^* d\phi_2 d\lambda \right| \\ & = \frac{\pi}{2} G (|\log A_2| + S) \rho^2 \left(\frac{1}{2} - \log \rho + |\log Q_1| \right) \mathcal{L}(i_1^1), \end{aligned}$$

$$(30) \quad \left| \int_{i_2^1} \int_{b^s(z_1, z_2)} \tau_3 d\omega_2 d\lambda \right| \\ \leq \frac{\pi}{2} G \rho^2 |\log A_1| \left(\frac{1}{2} - \log \rho + |\log Q_2| + S \right) \mathcal{L}(i_2^1).$$

Considering the integrands τ_1, τ_2, τ_3 in the complementary parts of \mathfrak{h}_2^3 we have by (4.10), (4), (4.18), (2.11), (5), (7)

$$(31) \quad |\tau_1| \leq \frac{1}{4} |\log A_1| (\tilde{c}_4 A_2 + 2T_2)H,$$

$$(32) \quad |\tau_2| \leq \frac{1}{4} A_1 \delta_1 (|\log A_2| + S)H,$$

$$(33) \quad |\tau_3| \leq \frac{1}{2} |\log A_1| (|\log A_2| + S)G,$$

so that the integral over the complementary parts

$$(34) \quad \left| \int \tau_1 d\omega_2 d\lambda + \int \tau_2 d\omega_2 d\lambda + \int \tau_3 d\omega_2 d\lambda \right| \\ \leq \left\{ \frac{1}{4} |\log A_1| (\tilde{c}_4 A_2 + 2T_3)H + \frac{1}{4} A_1 \delta_1 (|\log A_2| + S)H \right. \\ \left. + \frac{1}{2} |\log A_1| (|\log A_2| + S)G \right\} \mathcal{V}(\mathfrak{h}_1^3).$$

Combining (16), (19), (21), (24), (26), (28), (29), (30), and (34), we obtain B_1 .

II. In this part we shall show that

$$(34a) \quad \left| \sum_{r=1}^n \mathfrak{B}_F(-\log |f_1|, s_r) \right| \\ \leq B_2 \equiv \frac{\pi}{2} \rho^2 \left(\frac{1}{2} - \log \rho + |\log Q_1| \right) \left(qH \sum_{r=1}^n \mathcal{L}_r(i_1^1) + pnG \mathcal{L}(i_1^1) \right) \\ + \left(\frac{1}{2} A_1 \delta_1 H + |\log A_1| G \right) \lambda_0 (-\log a - \log \lambda_0 + 1) \sum_{r=1}^n \mathcal{S}_2(\mathfrak{N}_r^2) \\ + \frac{\pi}{2} \left\{ 2n |\log A_1| HK(\mu \lambda_0 (1 - \log \lambda_0) + \psi) + \sum_{r=1}^n (\rho p L_1 A H \mathcal{L}_r(\mathfrak{J}_{1\infty}^1) \right. \\ \left. + \rho H \delta_1 Q_1 p \mathcal{L}_r(i_{10}^1) \right) + \frac{1}{2} [A_1 \delta_1 H p + |\log A_1| (Hq + Gp)] \mathcal{V}(\mathfrak{h}_r^3) \right\}$$

where ψ is a constant for which the inequality

$$2\lambda_0 \log \rho_0 \leq \psi \leq (\lambda_0/2) \log (\lambda_0^2 + \rho_0^2)$$

holds, and $\mathcal{L}_\nu(j^1)$ denotes the length of that part of j^1 which lies in

$$(34b) \quad \mathfrak{h}_\nu^3 = \sum_{\lambda=\lambda_\nu-\lambda_0}^{\lambda_\nu+\lambda_0} \mathfrak{G}^2(\lambda).$$

In order to prove (34a) we proceed to derive bounds for the absolute value of

$$(35) \quad \mathcal{B}_P(-\log |f_1|, s_\nu) = - \int_{\lambda=0}^{2\pi} \int_{\mathfrak{E}_P^2} (\tau_4^{(\nu)} + \tau_5^{(\nu)} + \tau_6^{(\nu)}) d\omega_2 d\lambda,$$

$$(36) \quad \tau_4^{(\nu)} = \frac{1}{2i} \left(\frac{\partial \log |f_1|}{\partial z_2} \right)_{z_1=\mathfrak{h}(\lambda, z_2)} s_\nu^* \frac{h_\lambda}{h_{z_2}},$$

$$(37) \quad \tau_5^{(\nu)} = \frac{1}{2i} \log |f_1^*| \frac{\partial s_\nu^*}{\partial z_2} \frac{h_\lambda}{h_{z_2}},$$

$$(38) \quad \tau_6^{(\nu)} = \frac{1}{2i} \log |f_1^*| s_\nu^* \frac{\partial}{\partial z_2} \left(\frac{h_\lambda}{h_{z_2}} \right).$$

We repeat the same considerations as before. The only modification is that in this case instead of $(-\log |f_2| + s)$ the function s_ν appears which is regular in \mathfrak{B} except in a neighborhood of the point $\{z_1^{(\nu)}, z_2^{(\nu)}\}$ so that we have to derive separately bounds at first in the neighborhood \mathfrak{n}_ν^3 of $\{z_1^{(\nu)}, z_2^{(\nu)}\}$ (see 5.II) and then in the complementary part $\mathfrak{h}_2^3 - \mathfrak{n}_\nu^3$ of \mathfrak{h}_2^3 . See p. 422.

In the case of the term (36) we consider $\tau_4^{(\nu)}$ at first in $\mathfrak{t}^3(\mathfrak{B}_{1\infty}^1)$, $\mathfrak{t}^3(\mathfrak{t}_{110}^1)$, \mathfrak{n}_ν^3 and then in the complementary part.

Considering $\tau_5^{(\nu)}$ and $\tau_6^{(\nu)}$ we shall determine bounds in the tubes $\mathfrak{t}^3(\mathfrak{t}_{11}^1)$ and \mathfrak{n}_ν^3 , and then in the complementary part $\mathfrak{h}_2^3 - \mathfrak{t}^3(\mathfrak{t}_{11}^1) - \mathfrak{n}_\nu^3$.

We consider at first the term $\tau_4^{(\nu)}$ in $\mathfrak{t}^3(\mathfrak{B}_{1\infty}^1)$ and $\mathfrak{t}^3(\mathfrak{t}_{110}^1)$. See p. 433 and (13a). Using (22), (23), 4.II we obtain

$$(39) \quad \left| \int_{\mathfrak{B}_{1\infty}^1} \int_{\mathfrak{b}^3(z_1, z_2)} \tau_4^{(\nu)} d\omega_2 d\lambda \right| \leq \frac{1}{2} \int_{\mathfrak{B}_{1\infty}^1} \int_0^{2\pi} \int_0^\rho \frac{1}{2} H \frac{L_1}{\rho^*} A_1 S_\nu(\lambda) \rho^* d\rho^* d\phi_2 d\lambda$$

where $S_\nu(\lambda) = \rho$ for $\lambda_\nu - \lambda_0 \leq \lambda \leq \lambda_\nu + \lambda_0$ and $S_\nu(\lambda) = 0$ for the remaining values of λ . Consequently

$$(40) \quad \left| \int_{\mathfrak{B}_{1\infty}^1} \int_{\mathfrak{b}^3(z_1, z_2)} \tau_4^{(\nu)} d\omega_2 d\lambda \right| \leq \frac{\pi}{2} \rho p L_1 A_1 H \mathcal{L}_\nu(\mathfrak{B}_{1\infty}^1).$$

($\mathcal{L}_\nu(\mathfrak{B}_{1\infty}^1)$ denotes the length of that part of $\mathfrak{B}_{1\infty}^1$ which lies in \mathfrak{h}_ν^3 . See (34b)). Similarly, in $\mathfrak{t}^3(\mathfrak{t}_{110}^1)$ using (25) we obtain

$$\begin{aligned}
 (41) \quad & \left| \int_{i_{10}^1} \int_{b^2(z_1, z_2)} \tau_4^{(r)} d\omega_2 d\lambda \right| \\
 & \leq \frac{1}{2} \int_{i_{10}^1} \int_0^{2\pi} \int_0^{\rho} \frac{1}{2} H \delta_1 \frac{Q_1}{\rho^*} S_r(\lambda) \rho^* d\rho^* d\phi_2 d\lambda \\
 & \leq \frac{\pi}{2} \rho H \delta_1 Q_1 p \mathcal{L}_r(i_{10}^1).
 \end{aligned}$$

We proceed now to the determination of the bounds in n_r^3 . See p. 432.

In accordance with the hypothesis 5.III, $t^3(\mathfrak{B}_{1\infty}^1)$ and $t^3(i_{10}^1)$ lie outside of n_r^3 so that by (4.10), (5), we have

$$(42) \quad \left| \frac{\partial \log |f_1|}{\partial z_2} \right|_{z_1=h(\lambda, z_2)} \leq \frac{1}{2} A_1 \delta_1$$

for $(z_1, z_2) \in n_r^3 \subset b^3 - t^3(\mathfrak{B}_{1\infty}^1) - t^3(i_{10}^1)$.

According to (4.12) and the definition 4.IV.2

$$\begin{aligned}
 (43) \quad & -s_r(\lambda, \phi_2) \leq -\log a - \log |\lambda - \lambda_r| \\
 & \text{for } |\lambda - \lambda_r| < \lambda_0, 0 < \phi_2 - \phi_2^{(r)} \leq \phi,
 \end{aligned}$$

$s_r(\lambda, \phi_2) = 0$ on the remaining part of the distinguished boundary surface.

Since $s_r^*(\lambda, z_2)$ is for every fixed $\lambda, \lambda \neq \lambda_r$, an analytic function of a complex variable regular in $|z_2| < 1$, it assumes the maximum value of its absolute value on the boundary and therefore in every $\mathfrak{S}^2(\lambda)$ and consequently in $n_r^3 \cap \mathfrak{S}^2(\lambda)$

$$\begin{aligned}
 (44) \quad & |s_r^*(\lambda, z_2)| \leq -\log a - \log |\lambda - \lambda_r|, \\
 (45) \quad & \left| \int_{\lambda=\lambda_r-\lambda_0}^{\lambda_r+\lambda_0} \int_{n_r^3 \cap \mathfrak{S}^2(\lambda)} \tau_4^{(r)} d\omega_2 d\lambda \right| \\
 & \leq -\frac{1}{4} \int_{\lambda=\lambda_r-\lambda_0}^{\lambda_r+\lambda_0} \int_{n_r^3 \cap \mathfrak{S}^2(\lambda)} A_1 \delta_1 (\log a + \log |\lambda - \lambda_r|) H d\omega_2 d\lambda \\
 & \leq \frac{1}{2} A_1 \delta_1 H \lambda_0 [-\log a - \log \lambda_0 + 1] \mathfrak{S}_1(n_r^3).
 \end{aligned}$$

In the remaining part, $b_r^3 - n_r^3$, we have by hypothesis 4.IV (see (4.16), (34b)) $|s_r| \leq p$. Therefore by (42) and (2.11)

$$(46) \quad \left| \int_{b_r^3 - n_r^3} \tau_4^{(r)} d\omega_2 d\lambda \right| \leq \frac{1}{2} A_1 \delta_1 H p \mathcal{V}(b_r^3),$$

where $\mathcal{V}(b_r^3)$ denotes the volume of b_r^3 .

In deriving bounds for $\tau_6^{(r)}$ we observe at first that by (2), 4.IV (p. 426), (2.11) in $t^3(i_{11}^1)$

$$\begin{aligned}
 & \left| \int_{i_1^1} \int_{b^2(z_1, z_2)} \tau_5^{(\nu)} d\omega_2 d\lambda \right| \\
 (47) \quad &= \frac{1}{2} \left| \int_{i_1^1} \int_{b^2(z_1, z_2)} \log |f_1^*(\lambda, z_2)| \frac{\partial s_\nu^*(\lambda, z_2)}{\partial z_2} \frac{h_\lambda}{h_{z_2}} d\omega_2 d\lambda \right| \\
 &\leq \frac{1}{2} \int_{i_1^1} \int_0^{2\pi} \int_0^\rho (-\log \rho^* + |\log Q_1|) q H \rho^* d\rho^* d\phi_2 d\lambda \\
 &= \frac{\pi}{2} q H \rho^2 \left(\frac{1}{2} - \log \rho + |\log Q_1| \right) \mathcal{L}_\nu(i_1^1)
 \end{aligned}$$

where $\mathcal{L}_\nu(i_1^1)$ denotes the length of that part of j_1^1 which lies in

$$(34b) \quad \mathfrak{h}_\nu^3 \equiv \sum_{\lambda=\lambda_\nu-\lambda_0}^{\lambda_\nu+\lambda_0} \mathfrak{G}^2(\lambda).$$

We proceed now to the derivation of the bounds in \mathfrak{n}_ν^3 , see p. 432. According to Hypothesis 5.II in \mathfrak{n}_ν^3 ,

$$\left| \frac{\partial s_\nu^*(\lambda, z_2)}{\partial z_2} \right| \leq \frac{K}{|\lambda - \lambda_\nu|^\mu + \rho^{*2}}$$

where $K < \infty$, $\mu < \infty$. Therefore according to (2.11), (4.10)

$$(48) \quad |\tau_5^{(\nu)}| \leq \frac{1}{2} |\log A_1| H \frac{K}{|\lambda - \lambda_\nu|^\mu + \rho^{*2}},$$

and

$$\begin{aligned}
 & \left| \int_{\mathfrak{n}_\nu^3} \tau_5^{(\nu)} d\omega_2 d\lambda \right| \equiv \left| \int_{\lambda_\nu-\lambda_0}^{\lambda_\nu+\lambda_0} \int_{\mathfrak{R}_\nu^2(\lambda)} \tau_5^{(\nu)} d\omega_2 d\lambda \right| \\
 (49) \quad &\leq \frac{1}{2} |\log A_1| H K \int_{\lambda_\nu-\lambda_0}^{\lambda_\nu+\lambda_0} \int_0^{2\pi} \int_0^{\rho_0} \frac{\rho^* d\rho^* d\phi_2 d\lambda}{|\lambda - \lambda_\nu|^\mu + \rho^{*2}} \\
 &= \pi |\log A_1| H K [\mu \lambda_0 (1 - \log \lambda_0) + \psi]
 \end{aligned}$$

where $\psi = 2 \int_0^{\lambda_0} d\lambda^* [\log (\lambda^* \mu + \rho_0^2)]$ satisfies the inequality

$$(50) \quad 2\lambda_0 \log \rho_0 \leq \psi \leq \frac{\lambda_0}{2} \log (\lambda_0^\mu + \rho_0^2).$$

In the remaining part, $\mathfrak{b}_2^3 - \mathfrak{n}_\nu^3$, by 4.IV, $|\partial s_\nu^*/\partial z_2| \leq q$ and therefore⁽¹¹⁾

$$(51) \quad \left| \int_{\mathfrak{b}_2^3 - \mathfrak{i}^3(i_1^1) - \mathfrak{n}_\nu^3} \tau_5^{(\nu)} d\omega_2 d\lambda \right| \leq \frac{1}{2} |\log A_1| H q \mathcal{V}(\mathfrak{h}_\nu^3).$$

⁽¹¹⁾ We remind the reader that $s_\nu^* = 0$ in $\mathfrak{b}_2^3 - \mathfrak{h}_\nu^3$.

Considering $\tau_6^{(\nu)}$ we proceed exactly in the same way as in the case of $\tau_4^{(\nu)}$, and using (2), (2.11), 4.IV we obtain in $t^3(i_1^1)$

$$(52) \quad \left| \int_{t^3(i_1^1)} \tau_6^{(\nu)} d\omega_2 d\lambda \right| \leq \frac{1}{2} \left| \int_{t^3(i_1^1)} Gp(-\log \rho^* + |\log Q_1|) d\omega_2 d\lambda \right| \\ \leq \frac{\pi}{2} pG\rho^2 \left[\frac{1}{2} - \log \rho + |\log Q_1| \right] \mathcal{L}(i_1^1),$$

and in n_ν^3 using (4.10), (43)

$$(53) \quad \left| \int_{\lambda=\lambda_\nu-\lambda_0}^{\lambda_\nu+\lambda_0} \int_{n_\nu^3 \cap \mathfrak{S}^2(\lambda)} \tau_6^{(\nu)} d\omega_2 d\lambda \right| \\ \leq |\log A_1| G\lambda_0 [-\log a - \log \lambda_0 + 1]_2 (\mathfrak{S} \mathfrak{N}_\nu^2).$$

Finally, integrating in the remaining part, $\mathfrak{b}_2^3 - n_\nu^3 - t^3(i_1^1)$, we obtain

$$(54) \quad \left| \int_{\mathfrak{b}_2^3 - t^3(i_1^1) - n_\nu^3} \tau_6^{(\nu)} d\omega_2 d\lambda \right| \leq \frac{1}{2} |\log A_1| Gp \mathcal{V}(\mathfrak{h}_\nu^3).$$

Combining (40), (41), (45), (47), (49), (52), (53), (46), (51), and (54) we obtain B_2 .

6. A geometric interpretation of terms on the right-hand side of (3.1).

In §§4 and 5 we derived upper bounds for the absolute value of $\mathcal{F}_P(-\log |f_1|, -\log |f_2| + e_2) + \mathcal{B}_P(-\log |f_1|, -\log |f_2| + e_2)$. According to (3.1), this last sum equals

$$(1) \quad \frac{1}{4} \int_{\mathfrak{A}^2} (\log |f_1|)(\log |f_2| - e_2) h_{z_2}^{-1} dz_1 dz_2 \\ + \frac{\pi}{2} \int_{\mathfrak{S}^2} \sum_{v=1}^{N_1(z_2)} \{ \text{sg}(\alpha_{1v}(z_2)) [\log |f_2(\alpha_{1v}(z_2), z_2)| - e_2(\alpha_{1v}(z_2), z_2)] \} d\omega_2,$$

see p. 420, so that the sum of expressions obtained in (4.19) and (5.8) (see also (5.9), (5.34a)) represents an upper bound for (1). In this section we shall give a geometrical interpretation of the terms appearing in (1). $P_2 \equiv -\log |f_2| + e_2$ vanishes on the distinguished boundary surface, so that

$$(2) \quad \int_{\mathfrak{A}^2} P_1 P_2 h_{z_2}^{-1} dz_1 dz_2 = \sum_v \int_{\mathfrak{A}_v^2} (\log |f_1|)(\log |f_2| - e_2) h_{z_2}^{-1} dz_1 dz_2$$

where $\mathfrak{A}_v^2 = [\mathfrak{S}b^1(z_2), z_2 \in c_v^1]$, see (2.2), are distinguished boundary surfaces of subdomains of \mathfrak{B} which include the exceptional points Q_v of the pair f_1, f_2 . Here $c_v^1 = [|z_2 - z_2^{(\nu)}| = P]$ (see Hypothesis 2.IV, 4.II) are boundary curves of the domains \mathfrak{b}_v^2 , see Theorem 3.1. The double integrals (2) can be expressed by lines integrals over the curves $\mathfrak{b}^3(z_2 = z_2^{(\nu)})$ and some additional terms. The

behavior of these integrals, as we shall discuss elsewhere, depends to a large extent upon certain properties of the product $\log |f_1| \cdot (\log |f_2| - e_2)$ at exceptional points. (In particular one can derive for these integrals upper bounds, which depend essentially upon the local behavior of f_1 and f_2 in the neighborhood of exceptional points.)

We proceed now to the geometric interpretation of the second term in (1). We write

$$(3) \quad \log |f_2| - e_2 = [\log |f_2| - \mathcal{E}_2] + [\mathcal{E}_2 - e_2].$$

Here $\mathcal{E}_2(z_1, z_2)$, $z_2 = \text{const.}$, is that harmonic function of x_1, y_1 which on the boundary curve $\mathfrak{h}^1(z_2)$ of $\mathfrak{B}^2(z_2)$ assumes the values

$$\log |f_2(z_1, z_2)| \equiv \log |f_2^*(\lambda, z_2)|.$$

The second term on the right-hand side of (3.1) will assume the form $\mathcal{M}_1(P) + \mathcal{M}_2(P)$ where

$$(4a) \quad \mathcal{M}_1(P) = \frac{\pi}{2} \int_{\mathfrak{S}^2} \sum_{r=1}^{N_1(z_2)} \{ \text{sg}(\alpha_{1r}(z_2)) [\log |f_2(\alpha_{1r}(z_2), z_2)| - \mathcal{E}_2(\alpha_{1r}(z_2), z_2)] \} d\omega_2,$$

$$(4b) \quad \mathcal{M}_2(P) = \frac{\pi}{2} \int_{\mathfrak{S}^2} \sum_{r=1}^{N_1(z_2)} \{ \text{sg}(\alpha_{1r}(z_2)) [\mathcal{E}_2(\alpha_{1r}(z_2), z_2) - e_2(\alpha_{1r}(z_2); z_2)] \} d\omega_2.$$

If $z_1 = \alpha_{2n}(z_2)$ are the zeros and the poles of $f_2(z_1, z_2)$ in $\mathfrak{B}^2(z_2)$ and $g(z_1, Z_1, z_2)$ is the Green's function of $\mathfrak{B}^2(z_2)$

$$(5) \quad \begin{aligned} \log |f_2(z_1, z_2)| - \mathcal{E}_2(z_1, z_2) &= - \sum \text{sg}(\alpha_{2n}(z_2)) g(z_1, \alpha_{2n}(z_2); z_2) \\ &\equiv \sum_{n=1}^{N_2(z_2)} \text{sg}(\alpha_{2n}(z_2)) \log(z_1, \alpha_{2n}(z_2); \mathfrak{B}^2(z_2)). \end{aligned}$$

Here

$$(z_1, Z_1; \mathfrak{B}^2(z_2)) = \left| \frac{w(z_1; z_2) - w(Z_1; z_2)}{1 - \overline{w(Z_1; z_2)} w(z_1; z_2)} \right|,$$

and

$$w(z_1; z_2) = w(z_1; z_2, \bar{z}_2), \quad w(0; z_2) = 0, \quad \left. \frac{dw(z_1; z_2)}{dz_1} \right|_{z_1=0} > 0,$$

denotes that analytic function which maps $\mathfrak{B}^2(z_2)$ onto the unit circle.

We shall denote $(z, Z; \mathfrak{B}^2) = (Z, z; \mathfrak{B}^2)$ as the $\mathcal{C}(\mathfrak{B}^2)$ -distance between the points z and Z . $(z, Z; \mathfrak{B}^2)$ is invariant with respect to conformal transformations. If \mathfrak{C}^2 is the unit circle with center at Z , then $(z, Z; \mathfrak{C}^2)$ is the Euclidean distance between z and Z . We note that

$$(z, Z; \mathfrak{B}^2) = \frac{\exp [z, Z; \mathfrak{B}^2] - 1}{\exp [z, Z; \mathfrak{B}^2] + 1}$$

where $[z, Z; \mathfrak{B}^2]$ is the non-Euclidian distance between the points z and Z measured with the invariant metric defined in \mathfrak{B}^2 . Thus

$$(6) \quad \log |f_2(z_1, z_2)| - \mathcal{E}_2(z_1, z_2) = \log b(z_1, f_2; z_2),$$

where $b(z_1, f_2; \mathfrak{B}^2(z_2)) = \prod_{m=1}^{N_2(z_2)} [(z_1, \alpha_{2m}(z_2); \mathfrak{B}^2(z_2))]^{\text{sg}(\alpha_{2n}(z_2))}$, is a quotient, $b = b_1/b_2$, of generalized Blaschke products b_k , $k=1, 2$ (b_1 vanishes at zero points, b_2 at poles of f_2 in $\mathfrak{B}^2(z_2)$). Functions b will be denoted as *Blaschke functions*. (See p. 415 and N. 1, p. 137, B.3.)

It is clear that the integral

$$(7) \quad \int_{\mathfrak{E}_P^2} \sum_{n=1}^{N_1(z_2)} \text{sg}(\alpha_{1n}(z_2)) \sum_{m=1}^{N_2(z_2)} \text{sg}(\alpha_{2m}(z_2)) \log \left| \frac{w(\alpha_{1n}(z_2); z_2) - w(\alpha_{2m}(z_2); z_2)}{1 - \overline{w(\alpha_{1n}(z_2); z_2)} w(\alpha_{2m}(z_2); z_2)} \right| d\omega_2$$

exists. Indeed, for $z_2 \in \mathfrak{E}_P^2$,

$$0 < a \leq \left| \frac{w(\alpha_{1n}(z_2); z_2) - w(\alpha_{2m}(z_2); z_2)}{1 - \overline{w(\alpha_{1n}(z_2); z_2)} w(\alpha_{2m}(z_2); z_2)} \right| < 1,$$

since according to Hypothesis 2.IVa the intersection points of zero and pole surfaces of f_1 and f_2 lie outside $[\mathfrak{S}\mathfrak{B}^2(z_2), z_2 \in \mathfrak{E}_P^2]$. If we approach to values of z_2 for which $w(\alpha_{k\nu}(z_2), z_2) \rightarrow 1$,

$$\left| \frac{w(\alpha_{1n}(z_2); z_2) - w(\alpha_{2m}(z_2); z_2)}{1 - \overline{w(\alpha_{1n}(z_2); z_2)} w(\alpha_{2m}(z_2); z_2)} \right| \rightarrow 1.$$

Thus we see that for $z_2 \in \mathfrak{E}_P^2$,

$$0 \leq |\log(\alpha_{1n}(z_2), \alpha_{2m}(z_2); \mathfrak{B}^2(z_2))| \leq |\log a|$$

uniformly in z_2 as it can be easily shown by classical arguments. (Concerning a , see Hypothesis 4.II.)

Since according to our hypotheses, see Theorem 4.1, for the number, $N_k(z_2)$, of zeros and poles $\alpha_{kn}(z_2)$ of $f_k(z_1, z_2)$ we have $N_k(z_2) \leq \tilde{N} < \infty$, the integral (7) exists. Thus $\mathcal{M}_1(\mathbf{P}) = (\pi/2) \int_{\mathfrak{E}_P^2} \log A(z_2) d\omega_2$ where $A(z_2)$ is the product of $\mathcal{C}(\mathfrak{B}^2(z_2))$ -distances between the poles of f_1 and f_2 and their zeros, divided by the products of $\mathcal{C}(\mathfrak{B}^2(z_2))$ -distances between the poles of f_k and zeros of f_{3-k} , $k=1, 2$; $f_k = f_k(z_1, z_2)$, $z_2 = \text{const.}$

$A(z_2)$ can also be interpreted as the product of Blaschke functions

$$\prod_{n=1}^{N_1(z_2)} [b(\alpha_{1n}(z_2), f_2; \mathfrak{B}^2(z_2))]^{\text{sg}(\alpha_{1n}(z_2))}$$

where $\alpha_{1n}(z_2)$ are zero and poles of $f_1(z_1, z_2)$, $z_1 \in \mathfrak{B}^2(z_2)$.

In the present paper we consider functions f_k , $k=1, 2$, which for every fixed z_2 , $|z_2| \leq 1$, possess finitely many, say $N(z_2)$, zeros and $N(z_2) \leq \tilde{N}$, where $\tilde{N} < \infty$, is independent of z_2 . The methods applied here can be easily generalized to the case where $\lim_{r_2 \rightarrow 1} N(r_2 \exp(i\phi_2^{(v)})) = \infty$, for finitely many $\phi_2^{(v)}$ or even (under some additional assumption) in the case where $\phi_2^{(v)}$ have accumulation points. See also N.1, p. 137, B.3, etc.

We proceed now to the interpretation of the second term in (4). For every z_2 for which the zero or pole surface of f_2 does not intersect $b^1(z_2)$ (see (2.2)), $\mathcal{E}_2(z_1, z_2) - e_2(z_1, z_2)$ is a regular harmonic function of x_1, y_1 in $\mathfrak{B}^2(z_2)$. Therefore

$$(8) \quad \mathcal{E}_2 - e_2 = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} [\mathcal{E}_2^*(\lambda, z_2) - e_2^*(\lambda, z_2)] \left[\frac{e^{i\theta}}{e^{i\theta} - w(z_1, z_2)} + \frac{\overline{w(z_1, z_2)}}{e^{-i\theta} - \overline{w(z_1, z_2)}} \right] d\theta$$

where $\exp(i\theta) = w^*(\lambda, z_2) = w(h(\lambda, z_2), z_2)$, and $w(z_1, z_2)$ is the function which maps $\mathfrak{B}^2(z_2)$ onto the unit circle. (See p. 416.) For every fixed λ

$$(9) \quad \begin{aligned} \mathcal{E}_2^* - e_2^* &= \sum_{\nu=1}^{M_2(\lambda)} \operatorname{sg}(\beta_{2\nu}(\lambda)) \log \left| \frac{z_2 - \beta_{2\nu}(\lambda)}{1 - z_2 \overline{\beta_{2\nu}(\lambda)}} \right| \\ &= \log \left\{ \prod_{\nu=1}^{M_2(\lambda)} ((z_2, \beta_{2\nu}(\lambda))^{\operatorname{sg}(\beta_{2\nu}(\lambda))}) \right\} \end{aligned}$$

where $(z_2, Z_2) = (z_2, Z_2; \mathbb{C}^2) = |(z_2 - Z_2)/(1 - z_2 \overline{Z_2})|$ denotes the $\mathcal{C}(\mathbb{C}^2)$ -distance between z_2 and Z_2 , and $M_2(\lambda)$ the number of zeros and poles of $f_2^*(\lambda, z_2)$ in $|z_2| < 1$. Substituting (9) into (8) (multiplying by $\pi/2$ and integrating over \mathfrak{S}_P^2) we obtain

$$(10) \quad \begin{aligned} \mathcal{M}_2 &= -\frac{i}{4} \int_{\mathfrak{S}_P^2} \left\{ \sum_{\mu=1}^{N_1(z_2)} \operatorname{sg}(\alpha_{1\mu}(z_2)) \right. \\ &\quad \cdot \int_{\lambda=0}^{2\pi} \left[\sum_{\nu=1}^{M_2(\lambda)} \operatorname{sg}(\beta_{2\nu}(\lambda)) \log \left| \frac{z_2 - \beta_{2\nu}(\lambda)}{1 - z_2 \overline{\beta_{2\nu}(\lambda)}} \right| \right] \\ &\quad \times \mathcal{P}(w(\alpha_{1\mu}(z_2)), w^*(\lambda, z_2)) \left. \right\} \frac{\partial w^*(\lambda, z_2)}{\partial \lambda} \frac{d\lambda d\omega_2}{w^*(\lambda, z_2)} \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}(w(\alpha_{1\mu}(z_2)), w^*(\lambda, z_2)) &= \frac{w^*(\lambda, z_2)}{w^*(\lambda, z_2) - w(\alpha_{1\mu}(z_2), z_2)} + \frac{\overline{w(\alpha_{1\mu}(z_2), z_2)}}{\overline{w^*(\lambda, z_2)} - \overline{w(\alpha_{1\mu}(z_2), z_2)}} \\ &= \frac{1 - |w(\alpha_{1\mu}(z_2), z_2)|^2}{1 + |w(\alpha_{1\mu}(z_2), z_2)|^2 - 2|w(\alpha_{1\mu}(z_2), z_2)| \cos \Phi}, \end{aligned}$$

$$\Phi = \arccos w^*(\lambda, z_2) - \arccos w(\alpha_{1\mu}(z_2), z_2).$$

The integrand of (10) can become infinite only if $w^*(\lambda, z_2) = w(\alpha_{1\mu}(z_2), z_2)$ (i.e., at a point where a zero or pole surface of f_1 intersects \mathfrak{b}^3) or if $z_2 = \beta_{2\nu}(\lambda)$.

1. Let $z_1 = h(\lambda, \zeta_2(\lambda))$, $z_2 = \zeta_2(\lambda)$ be a segment of the intersection line of $\log |f_1(z_1, z_2)| = \pm \infty$ with $[z_1 = h(\lambda, z_2), |z_2| \leq 1]$. For every fixed λ in the neighborhood of $z_2 = \zeta_2(\lambda)$, $\bar{z}_2 = \bar{\zeta}_2(\lambda)$

$$(11) \quad w(h(\lambda, z_2); z_2) - w(\alpha_{1\mu}(z_2); z_2) \\ = A_{10}(\lambda)(z_2 - \zeta_2(\lambda)) + A_{01}(\lambda)(\bar{z}_2 - \bar{\zeta}_2(\lambda)) + \dots$$

We make now the additional HYPOTHESIS 6.I. For $z_2 \in \mathfrak{C}_P^2$,

$$|A_{10}(\lambda)| \neq |A_{01}(\lambda)|.$$

If we introduce polar coordinates $z_2 - \zeta_2(\lambda) = r_2 e^{i\phi_2}$, $d\omega_2 = r_2 dr_2 d\phi_2$, then for a fixed λ

$$A_{10}(\lambda)(z_2 - \zeta_2(\lambda)) + A_{01}(\lambda)(\bar{z}_2 - \bar{\zeta}_2(\lambda)) + \dots \\ = r_2 [A_{10}(\lambda)e^{i\phi_2} + A_{01}(\lambda)e^{-i\phi_2}] + \dots$$

and

$$(12) \quad \frac{d\omega_2}{w(h(\lambda, z_2), z_2) - w(\alpha_{1\mu}(z_2), z_2)} = \frac{dr_2 d\phi_2}{[A_{10}(\lambda)e^{i\phi_2} + A_{01}(\lambda)e^{-i\phi_2}] + r_2 [\dots]}$$

Consequently (12) remains bounded also in the neighborhood of intersection points of \mathfrak{b}^3 with the zero and pole surfaces of f_1 for $z_2 \in \mathfrak{C}_P^2$.

2. For every fixed λ ,

$$\log \left| \frac{z_2 - \beta_{2\nu}(\lambda)}{1 - z_2 \bar{\beta}_{2\nu}(\lambda)} \right|$$

becomes infinite at finitely many points, say $\beta_{2\nu}(\lambda)$, $\nu = 1, 2, \dots, M_2(\lambda)$. If $\beta_{2\nu}(\lambda)$ lies inside \mathfrak{C}^2 obviously

$$\int_{|z_2 - \beta_{2\nu}(\lambda)| \leq \rho_2} \log \left| \frac{z_2 - \beta_{2\nu}(\lambda)}{1 - z_2 \bar{\beta}_{2\nu}(\lambda)} \right| dr_2 d\phi_2$$

is bounded. (If $\beta_{2\nu}(\lambda)$ lies on the boundary of \mathfrak{C}^2 the integrand vanishes.) Thus \mathcal{M}_2 represents the following integral: For every value λ , $0 \leq \lambda \leq 2\pi$, $z_2 \in \mathfrak{C}_P^2$, we form the sum of the products of the logarithm of the Blaschke function

$$b(z_2, f_1^*(\lambda, z_2); \mathfrak{C}^2) = \prod_{\nu} \left| \frac{z_2 - \beta_{2\nu}(\lambda)}{1 - z_2 \bar{\beta}_{2\nu}(\lambda)} \right|^{\text{sg}(\beta_{2\nu}(\lambda))}$$

by

$$\text{sg}(\alpha_{1\mu}(z_2)) \mathcal{P}(w(\alpha_{1\mu}(z_2), z_2), w^*(\lambda, z_2)).$$

Here $\beta_{2\nu}(\lambda)$ are zeros and poles of $f_2^*(\lambda, z_2)$ in $|z_2| \leq 1$, \mathcal{P} is the Poisson kernel, and $\alpha_{1\mu}(z_2)$ are the zeros and poles of $f_1(z_1, z_2)$ in $\mathfrak{B}^2(z_2)$. \mathcal{M}_2 is the integral of this sum multiplied by the weighting function

$$\frac{1}{4iw^*(\lambda, z_2)} \frac{\partial w^*(\lambda, z_2)}{\partial \lambda}$$

over $[0 \leq \lambda \leq 2\pi, z_2 \in \mathfrak{C}_P^2]$.

7. **The kernel function of the extended class $\mathcal{E}(\mathfrak{B})$.** In [B.4, 5, 6, 10] as well as in §§2–6 of the present paper it has been shown that the extended class represents a useful tool in the study of various problems of the theory of functions of two complex variables. It is therefore of interest to investigate further properties of these functions. In the present section we shall consider functions of the extended class in domains described in §2. These domains are bounded by two segments of analytic hypersurfaces \mathfrak{a}^3 and \mathfrak{b}^3 (see (2.4) and (2.5)). In addition to the hypotheses of §2, we are making in the present section the following assumption 7.I about the structure of the domains \mathfrak{B} .

By a transformation⁽¹²⁾

$$(1) \quad z_1^* = w(z_1, z_2) = w(z_1; z_2, \bar{z}_2), \quad z_2^* = z_2,$$

where w is a conveniently chosen function which is analytic in z_1 (but not necessarily in z_2) and which satisfies the conditions

$$(2) \quad w[h(0, z_2), z_2] = w[h(2\pi, z_2), z_2] = 1, \quad w(0, z_2) = 0.$$

\mathfrak{B} is mapped onto the bicylinder $\mathfrak{B}^* = [|z_1^*| < 1, |z_2| < 1]$. Every segment of the analytic surface $[z_1 = h_1(e^{i\lambda}, z_2), |z_2| \leq 1, \lambda = \text{const.}]$, goes over into the segment

$$\begin{aligned} [\arccos z_1^* = \arccos W(e^{i\lambda}, z_2), |z_1^*| = 1, |z_2| \leq 1], \\ W(e^{i\lambda}, z_2) = W(e^{i\lambda}, z_2, \bar{z}_2) \equiv w[h_1(e^{i\lambda}, z_2), z_2, \bar{z}_2] \end{aligned} \quad (\text{see p. 416}).$$

We shall assume in this section that the domain \mathfrak{B} has the property that

$$(3) \quad 0 < a \leq \frac{\arccos W(e^{i\lambda}, z_2) - \arccos W(e^{i\mu}, z_2)}{\lambda - \mu} \leq A < \infty$$

where a and A are conveniently chosen constants. (Hypothesis 7.I.)

NOTATION. The functions $H^\dagger(x_1, y_1, x_2, y_2)$ of $\mathcal{E}(\mathfrak{B})$ which are continuous on the distinguished boundary surface and satisfy the condition

$$(4) \quad H^\dagger(0, 0, x_2, y_2) = 0$$

will be said to belong to $\mathcal{E}_0(\mathfrak{B})$.

THEOREM 7.1. *There exists a system of real functions $\phi_p^\dagger(x_1, y_1, x_2, y_2)$*

⁽¹²⁾ Since $z_2^* = z_2$, we omit the asterisk at z_2^* .

$\equiv T_p(z_1, z_2)$, $p = 1, 2, \dots$, of the class $\mathcal{E}_0(\mathfrak{B})$ which are orthonormal in the sense that

$$(5) \quad \int_{\mathfrak{B}} \left[\frac{\partial \phi_\nu^\dagger}{\partial x_1} \frac{\partial \phi_\mu^\dagger}{\partial x_1} + \frac{\partial \phi_\nu^\dagger}{\partial y_1} \frac{\partial \phi_\mu^\dagger}{\partial y_1} \right] d\omega = 4 \operatorname{Re} \left[\int_{\mathfrak{B}} \frac{\partial T_\nu}{\partial z_1} \frac{\partial T_\mu}{\partial \bar{z}_1} d\omega \right] \\ = \delta_{\nu\mu} = \begin{cases} 1 & \text{if } \nu = \mu, \\ 0 & \text{if } \nu \neq \mu, \end{cases}$$

and possess the following properties:

1°. The system $\{T_p\}$ is (uniformly) complete on the distinguished boundary surface \mathfrak{D}^2 , i.e., possesses the property that every continuous function defined on \mathfrak{D}^2 can be uniformly approximated by a linear combination

$$\sum_{p=1}^N a_p T_p(z_1, z_2), \quad (z_1, z_2) \in \mathfrak{D}^2, \quad N < \infty, \quad (a_p \equiv a_p^{(N)} \text{ are constants}).$$

2°. The kernel function of the system $\{T_p\}$ converges in \mathfrak{B} , i.e., we have

$$(6) \quad \sum_{p=1}^{\infty} [\phi_p^\dagger(x_1, y_1; x_2, y_2)]^2 < \infty \quad \text{for } (x_1, y_1, x_2, y_2) \in \mathfrak{B}.$$

Proof. The proof of the theorem follows in the usual way (see also pp. 451-452) from the following

LEMMA. If $\mathcal{F} = \sum_{\nu=1}^m \alpha_\nu T_\nu$ satisfies the condition

$$(7) \quad \left(\frac{\partial \mathcal{F}}{\partial z_1} \right)_{z_1=\zeta_1, z_2=\zeta_2} = 1, \quad (\zeta_1, \zeta_2) \in \mathfrak{B},$$

then

$$(8) \quad \int_{\mathfrak{B}} \left| \frac{\partial \mathcal{F}}{\partial z_1} \right|^2 d\omega \geq C(\zeta_1, \zeta_2) > 0$$

where $C(\zeta_1, \zeta_2)$ is a fixed positive constant which depends only on \mathfrak{B} and ζ_1, ζ_2 .

Proof. We consider at first the case of $\mathfrak{B} = \mathfrak{C} = [|z_1| < 1, |z_2| < 1]$. In this case the real and imaginary parts of the set

$$(9) \quad z_1^m, z_1^m z_2^n, z_1^m \bar{z}_2^n, \quad m = 0, 1, 2, 3, \dots, n = 1, 2, 3, \dots,$$

form a complete system. Using polar coordinates, i.e., writing $z_k = x_k + iy_k = r_k(\cos \phi_k + i \sin \phi_k)$, $k = 1, 2$, the real and imaginary parts of functions of the system (9) can be written in the form⁽¹³⁾

⁽¹³⁾ We note that in (10) and in the following we use the double index $\{m, n\}$ instead of the index $[p]$, $p = 1, 2, \dots$. The functions of the set (10) are assumed to be ordered in such a way that $\{1, 0\}$, $\{2, 0\}$, $\{1, 1\}$, $\{1, -1\}$, \dots , correspond to $[1]$, $[2]$, $[3]$, $[4]$, \dots . We omit the terms corresponding to $m = 0$, since the derivatives of these terms with respect to z_1 vanish.

$$(10) \quad r_1^m \cos m\phi_1, r_2^m \sin m\phi_1, r_1^m r_2^{|n|} \cos (m\phi_1 + n\phi_2), r_1^m r_2^{|n|} \sin (m\phi_1 + n\phi_2),$$

$m=1, 2, \dots; n=1, -1, 2, -2, \dots$. The functions (10) are orthogonal to each other in the sense (5) (with $\mathfrak{B}=\mathfrak{C}$). Dividing them by conveniently chosen constants we obtain a system which is orthonormal in \mathfrak{C} .

Using the well-known development of a (real) continuous function F defined on the distinguished boundary surface [see e.g. [B.9], [B.-Marcinkiewicz 1], [Bers 1]], we conclude that every F can be developed in \mathfrak{C} in a series

$$(11) \quad F = \sum_{m=1}^{\infty} \left\{ \alpha_{m0}(z_1^m + \bar{z}_1^m) - i\beta_{m0}(z_1^m - \bar{z}_1^m) \right. \\ \left. + \sum_{n=1}^{\infty} [\alpha_{mn}(z_1^m z_2^n + \bar{z}_1^m \bar{z}_2^n) - i\beta_{mn}(z_1^m z_2^n - \bar{z}_1^m \bar{z}_2^n) \right. \\ \left. + \gamma_{mn}(z_1^m \bar{z}_2^n + \bar{z}_1^m z_2^n) - i\delta_{mn}(z_1^m \bar{z}_2^n - \bar{z}_1^m z_2^n)] \right\}.$$

The condition (7) with $\zeta_1=0, \zeta_2=0$ yields $\alpha_{10}=1, \beta_{10}=0$. On the other hand $F_1=z_1$ is orthogonal in the sense (5) to each of the functions $z_1^m, m=2, 3, \dots, \bar{z}_1^m \bar{z}_2^n, \bar{z}_1^m z_2^n, m=1, 2, 3, \dots, n=1, 2, 3, \dots$, and consequently

$$(12) \quad \int_{\mathfrak{C}} \left| \frac{\partial F}{\partial z_1} \right|^2 d\omega = \int_{\mathfrak{C}} d\omega + \int_{\mathfrak{C}} |S|^2 d\omega \geq \int_{\mathfrak{C}} d\omega = \pi^2 = C,$$

$$(13) \quad S = \sum_{m=2}^{\infty} m(\alpha_{m0} - i\beta_{m0}) z_1^{m-1} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [m(\alpha_{mn} - i\beta_{mn}) z_1^{m-1} z_2^n \\ + m(\gamma_{mn} - i\delta_{mn}) z_1^{m-1} \bar{z}_2^n]$$

and C is independent of F (satisfying (7) with $\zeta_1=\zeta_2=0$). If instead of the origin we choose an arbitrary interior point (ζ_1, ζ_2) of \mathfrak{C} , an analogous consideration leads to an inequality (12) with $C=\pi^2(1-|\zeta_1|^2)(1-|\zeta_2|^2)$.

We proceed now to prove an inequality analogous to (8) for general domains \mathfrak{B} , as described in §2 and satisfying 7.I. By the transformation (1) satisfying (2) \mathfrak{B} is mapped onto the bicylinder $\mathfrak{B}^*=[|z_1^*|<1, |z_2|<1]$. Every segment of the analytic surface $[z_1^*=h_1(e^{i\lambda}, z_2), |z_2|\leq 1], \lambda=\text{const.}$, goes over into a segment

$$(14) \quad [\text{arc } z_1^* = \text{arc } W(e^{i\lambda}, z_2), |z_1^*|=1, |z_2|\leq 1], W(e^{i\lambda}, z_2) \equiv w(h_1(e^{i\lambda}, z_2), z_2).$$

The class of functions $\mathcal{E}(\mathfrak{B})$ goes into the class $\mathcal{S}(\mathfrak{B}^*)$. Every function $K_p(z_1, z_2)$ of the system (10) assumes continuous values, say $K_p(e^{i\lambda}, e^{i\psi})$ on $[|z_1|=1, |z_2|=1]$. Let now $\tau_n(z_1^*, z_2) \in \mathcal{S}(\mathfrak{B}^*)$ be so determined that

$$(15) \quad \tau_p[w(h_1(e^{i\lambda}, e^{i\phi_2}), e^{i\phi_2}), e^{i\phi_2}] = K_p(e^{i\lambda}, e^{i\phi_2}), \quad h_1(e^{i\lambda}, e^{i\phi_2}) \equiv h(\lambda, e^{i\phi_2}), \\ 0 \leq \lambda \leq 2\pi, 0 \leq \phi_2 \leq 2\pi, p=1, 2, \dots$$

In the following, the lamina $[z_1^* = \exp(iw(h(\lambda, z_2), z_2)), |z_2| < 1, \lambda \text{ constant}]$ will be denoted by $\mathfrak{g}^2(\lambda)$.

Through every point $\{\exp(i\mu), \zeta_2\}$ of $[|z_1^*| = 1, |z_2| \leq 1]$ goes a lamina $\mathfrak{g}^2(\lambda)$ where λ is the solution of the equation

$$(16) \quad \exp(i\mu) = w[h(\lambda, \zeta_2), \zeta_2].$$

Since in every lamina $\mathfrak{g}^2(\lambda)$, the function $\tau_p[w(h(\lambda, z_2), z_2), z_2]$ is a harmonic function of x_2, y_2 it follows from (15) that

$$(17) \quad \tau_p[w(h(\mu, z_2), z_2), z_2] = K_p(e^{i\lambda}, z_2) \quad \text{for } |z_2| < 1$$

where

$$(18) \quad \mu \equiv \mu(\lambda, z_2), \quad \mu(0, z_2) = 0.$$

By orthogonalizing the system $\{\tau_n\}$ we obtain the system $\{T_p\}$, $p = 1, 2, \dots$, of functions of the class $\mathfrak{S}(\mathfrak{B}^*)$ which are orthonormal in the sense (5). We proceed now to prove the inequality (8) for functions F of the class $\mathfrak{S}(\mathfrak{B}^*)$ satisfying the condition (7). Since $(\zeta_1, \zeta_2) \in \mathfrak{B}^*$, $|\zeta_1| < 1, |\zeta_2| < 1$. Let us form a function

$$\mathcal{F}_N(z_1, z_2) = \sum_{n=1}^N \beta_n T_n(z_1^*, z_2)$$

possessing the property that \mathcal{F}_N satisfies the condition (7).

As we mentioned before through every point $\{e^{i\mu}, \zeta_2\}$, $|\zeta_2| \leq 1$, goes the lamina $\mathfrak{g}^2(\lambda)$, where λ has to be determined from (16). In every segment $\mathfrak{g}^2(\lambda)$, \mathcal{F}_N assumes the values $\mathcal{F}_N[\exp(i\mu(\lambda, z_2)), z_2]$. $\mathcal{F}_N[\exp(i\mu(\lambda, z_2)), z_2]$, $\lambda \text{ const.}$, is a harmonic function of x_2, y_2 .

We determine now a function $P_N(z_1, z_2)$ of the class $\mathcal{E}(\mathbb{C})$ by requiring that⁽¹⁴⁾

$$(19) \quad P_N[\exp(i\mu(\lambda, \zeta_2)), z_2] = \mathcal{F}_N[\exp(i\mu(\lambda, z_2)), z_2].$$

$P_N(z_1, \zeta_2)$ and $\mathcal{F}_N(z_1^*, \zeta_2)$ are harmonic functions of x_1, y_1 in $|z_1| < 1$, and of x_1^*, y_1^* in $|z_1^*| < 1$. Therefore from $P_N(e^{i\lambda}, \zeta_2) = \mathcal{F}_N(e^{i\lambda}, \zeta_2)$ follows $P_N(z_1, \zeta_2) = \mathcal{F}_N(z_1^*, \zeta_2)$, $|z_1| \leq 1$. Therefore, in particular

$$(20) \quad \left(\frac{\partial P_N}{\partial z_1} \right)_{z_1=\zeta_1, z_2=\zeta_2} = \left(\frac{\partial \mathcal{F}_N}{\partial z_1^*} \right)_{z_1^*=\zeta_1, z_2=\zeta_2} = 1.$$

According to (11) and (12), we have

$$(21) \quad \int_{\mathbb{C}} \left| \frac{\partial P_N}{\partial z_1} \right|^2 d\omega \geq C(\zeta_1, \zeta_2) \equiv \pi^2(1 - |\zeta_1|)^2(1 - |\zeta_2|)^2.$$

⁽¹⁴⁾ We note that the mapping $z_1^* = z_1^*(z_1, z_2)$, $z_2 = z_2$ defined in $[|z_1| = 1, |z_2| \leq 1]$ has the property that $z_1^*(z_1, \zeta_2) = z_1$.

Let us now compare

$$(22a) \quad \mathfrak{I}(z_2) = \int_{|z_1| < 1} \left| \frac{\partial P_N(z_1^*, z_2)}{\partial z_1^*} \right|^2 d\omega_1^*$$

and

$$(22b) \quad \mathfrak{I}^\dagger(z_2) = \int_{|z_1^*| < 1} \left| \frac{\partial \mathfrak{I}_N(z_1, z_2)}{\partial z_1} \right|^2 d\omega_1.$$

For $z_2 = \zeta_2$, as we mentioned, $\mathfrak{I}(\zeta_2) = \mathfrak{I}^\dagger(\zeta_2)$.

According to a formula of Douglass [D. 1, 2], [Sh. 1], [M.-T. 1],

$$(23) \quad \mathfrak{I}(z_2) = \frac{1}{16\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{2\pi} \left| \frac{P_N(\exp(i\phi), z_2) - P_N(\exp(i\theta), z_2)}{\sin[\frac{1}{2}(\phi - \theta)]} \right|^2 d\theta d\phi,$$

$$(24) \quad \mathfrak{I}^\dagger(z_2) = \frac{1}{16\pi} \int_{\phi^*=0}^{2\pi} \int_{\theta^*=0}^{2\pi} \left| \frac{\mathfrak{I}_N(\exp(i\phi^*), z_2) - \mathfrak{I}_N(\exp(i\theta^*), z_2)}{\sin[\frac{1}{2}(\phi^* - \theta^*)]} \right|^2 d\theta^* d\phi^*.$$

Here $\phi^*(z_2) = \arccos z_1^* = \arccos [w(h(e^{i\phi}, z_2), z_2)]$ and the same formula holds for $\theta^*(z_2)$ as a function of θ . Since by (19), $P_N(e^{i\phi}, z_2) = \mathfrak{I}_N(e^{i\phi^*}, z_2)$,

$$(25) \quad \begin{aligned} \mathfrak{I}^\dagger(z_2) &= \frac{1}{16\pi} \int_{\phi^*=0}^{2\pi} \int_{\theta=0}^{2\pi} \left| \frac{P_N(\exp(i\phi), z_2) - P_N(\exp(i\theta), z_2)}{\sin(\frac{1}{2}(\phi - \theta))} \right|^2 d\phi d\theta \\ &\times \frac{\sin(\frac{1}{2}(\phi - \theta))}{\sin(\frac{1}{2}(\phi^* - \theta^*))} \frac{d\phi^*}{d\phi} \frac{d\theta^*}{d\theta}. \end{aligned}$$

For every $|z_2| \leq 1$, according to (3)

$$(26) \quad 0 < a^2 \leq \frac{d\phi^*}{d\phi} \frac{d\theta^*}{d\theta} \leq A^2 < \infty.$$

In order to obtain bounds for $|\sin((\phi - \theta)/2)/\sin((\phi^* - \theta^*)/2)|^2$ we distinguish two cases (a) and (b), assuming

$$(27a) \quad 0 \leq |\phi - \theta| \leq \frac{\pi}{2A} \quad \text{or} \quad (27'a) \quad 0 \leq 2\pi - |\phi - \theta| \leq \frac{\pi}{2A}$$

and

$$(27b) \quad \frac{\pi}{2A} < |\phi - \theta| < 2\pi \left(1 - \frac{1}{4A}\right).$$

In the case (a), we have

$$(28) \quad \left| \frac{\sin(s/2)}{\sin(s^*/2)} \right|^2 \equiv \left| \frac{2 \sin(s/2)/s}{2 \sin(s^*/2)/s^*} \right|^2 \left| \frac{s}{s^*} \right|^2.$$

Here $s = |\phi - \theta|$, $s^* = |\phi^* - \theta^*|$ in the case (27a) and $s = 2\pi - |\phi - \theta|$, $s^* = 2\pi - |\phi^* - \theta^*|$ in the case (27'a), respectively. From (3) follows in the case (a)

$$(29) \quad 0 \leq s^* \leq \pi/2.$$

Since $2/\pi \leq [(\sin \alpha)/\alpha] \leq 1$ for $|\alpha| < \pi/2$, in the case (a) we have

$$(30) \quad \left(\frac{2}{\pi}\right)^2 a^2 \leq \left| \frac{\sin(s/2)/s}{\sin(s^*/2)/s^*} \right|^2 \left| \frac{s}{s^*} \right|^2 \leq A^2 \left(\frac{\pi}{2}\right)^2.$$

In the case (b) according to (26)

$$(31) \quad \frac{\pi a}{2A} \leq |\phi^* - \theta^*| \leq 2\pi - \frac{\pi a}{2A}$$

and therefore

$$(32) \quad [\sin(\pi a/4A)]^2 \leq \left| \frac{\sin(s/2)}{\sin(s^*/2)} \right|^2 \leq \frac{1}{[\sin(\pi a/4A)]^2}.$$

If therefore

$$(33) \quad b = \min [(\sin(\pi a/4A^2)), (2a/\pi)], \quad B = \max [1/\sin(\pi a/4A), \pi A/2],$$

we see that

$$(34) \quad 0 < b^2 a^2 \leq \frac{\Im^*(z_2)}{\Im(z_2)} \leq B^2 A^2 < \infty.$$

Using (34), (20), and (13), we see that

$$(35) \quad \int_{\mathfrak{B}^*} \left| \frac{\partial F_N}{\partial z_1^*} \right|^2 d\omega = \int_{|z_2| < (1-|z_1|)} \Im^*(z_2) d\omega_2 \geq b^2 a^2 \int_{|z_2| < (1-|z_1|)} \Im(z_2) d\omega_2 \\ = \pi b^2 a^2 (1 - |z_1|)^2 (1 - |z_2|)^2.$$

After the inequality (35) for functions F_N^* satisfying (7) have been derived the proof that the kernel function is finite follows in the usual way. (See [8, pp. 21 ff.].)

We determine the α_n , $n = 1, 2, \dots, N$, so that the expression

$$(36) \quad 4 \int_B \left| \sum_{n=1}^N \alpha_n T_n' \right|^2 d\omega = \sum_{n=1}^N |\alpha_n|^2, \quad T_n' \equiv \frac{\partial T_n}{\partial z_1},$$

becomes minimum⁽¹⁵⁾ under the condition that

$$(37) \quad \sum_{h=1}^N \alpha_h T_h'(\xi_1, \xi_2) = 1.$$

If we write

⁽¹⁵⁾ Since the function $w(z_1, z_2)$ in (1) is analytic in z_1 the value of the integral (36) remains unchanged when we pass from the domain \mathfrak{B}^* back to the domain \mathfrak{B} .

$$\alpha_n = \left[\overline{(T'_n(\zeta_1, \zeta_2) + A_n)} / \sum_{r=1}^N |T'_r(\zeta_1, \zeta_2)|^2 \right],$$

the condition (37) assumes the form

$$(38) \quad \sum_{n=1}^N A_n T'_n(\zeta_1, \zeta_2) = 0$$

and

$$(39) \quad \sum_{n=1}^N |\alpha_n|^2 = \frac{\sum_{n=1}^N |\overline{T'_n(\zeta_1, \zeta_2)} + A_n|^2}{\left(\sum_{n=1}^N |T'_n(\zeta_1, \zeta_2)|^2 \right)^2} = \frac{\sum_{n=1}^N |T'_n(\zeta_1, \zeta_2)|^2 + \sum_{n=1}^N |A_n|^2}{\left(\sum_{n=1}^N |T'_n(\zeta_1, \zeta_2)|^2 \right)^2}.$$

The minimum of (39) will be attained if and only if $A_n = 0$, $n = 1, 2, \dots, N$, and equals

$$(40) \quad \frac{1}{\sum_{n=1}^N |T'_n(\zeta_1, \zeta_2)|^2} \geq 4C(\zeta_1, \zeta_2) > 0$$

by (8). Since the right-hand side is independent of N it follows that

$$(41) \quad \sum_{n=1}^{\infty} |T'_n(\zeta_1, \zeta_2)|^2 \leq \frac{1}{4C(\zeta_1, \zeta_2)} < \infty.$$

(6) follows from (41) and from the relation

$$(42) \quad \sum_{n=1}^N |T_n(z_1, z_2)|^2 = \int_{\zeta_1=0}^{z_1} \int_{\zeta_1^*=0}^{z_1} \sum_{n=1}^N T'_n(\zeta_1, z_2) \overline{T'_n(\zeta_1^*, z_2)} d\zeta_1 d\zeta_1^*$$

which holds for the functions of the class $\mathcal{E}_0(\mathfrak{B})$, see (4).

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