ON BIEBERBACH-EILENBERG FUNCTIONS. II

BY

JAMES A. JENKINS

1. Let us denote by C the class of functions $f(z)$ regular for $|z| < 1$ which have power series expansion about $z = 0$ beginning

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

and which satisfy the condition

$$f(z_1)f(z_2) \neq 1, \qquad |z_1| < 1, \qquad |z_2| < 1. \tag{1}$$

Let $C(\lambda)$ denote the subclass of C consisting of functions $f(z)$ for which $|a_1| = \lambda$, $0 < \lambda \leqq 1$. If $f \in C(\lambda)$ it is subordinate to a function $g(z) \in C$ univalent in $|z| < 1$ [9] for which then $|g'(0)| \geqq \lambda$. Thus the image of $|z| < 1$ by $w = g(z)$ covers the circle $|w| < \lambda/4$ and hence by condition (1), for $|z| < 1$, $|g(z)| < 4/\lambda$. Consequently, for $|z| < 1$, $|f(z)| < 4/\lambda$ so that the functions in $C(\lambda)$ are uniformly bounded. Rogosinski [9] raised the question of the best possible bound $P(\lambda)$ for the functions in $C(\lambda)$. He gave certain upper estimates for $P(\lambda)$ but not the precise value. The object of the present paper is to give the complete solution of this problem by a method used earlier to solve certain other questions for the same class of functions [5].

2. We begin by solving first the related problem of the closest boundary point to the origin of the image of $|z| < 1$ by a univalent function in $C(\lambda)$. In the ζ-plane ($\zeta = \xi + i\eta$) we regard the domain $\Delta(t)$ defined by

$$\begin{aligned} 0 < \eta < \pi, &\qquad \xi < 0, \\ 0 < \eta < \pi - t, &\qquad \xi = 0, \\ -t < \eta < \pi - t, &\qquad \xi > 0, \end{aligned}$$

where $0 \leqq t < \pi$. Let us denote the following boundary points of $\Delta(t)$ in the manner indicated: 0, B; $-it$, C; $i(\pi - t)$, E; πi, F. Further let the boundary point of $\Delta(t)$ at infinity between F and B in the natural cyclic order be denoted by A, the corresponding point between C and E by D. We map $\Delta(t)$ conformally onto the left-hand half-plane $\mathfrak{I}w < 0$ in such a way that A goes into $w = 0$, D into $w = \infty$. Let us denote by P the point $\zeta = (\pi - t)i/2$. Rotation of $\Delta(t)$ through $180°$ about P corresponds in the w-plane to a linear transformation of $\mathfrak{I}w < 0$ onto itself interchanging 0 and ∞. This transformation has the form $w^* = a/w$, a real and positive. The fixed point $-a^{1/2}$ (positive root) of the latter transformation is the image of P. We adjust the original mapping from the ζ-plane to the w-plane so that this becomes the point -1 and the linear transformation becomes $w^* = 1/w$.

Received by the editors May 10, 1954.

Let the images of B and C be i/l and i/m, $l \geqq m > 0$. Then the images of E and F are $-il$, $-im$. If we extend ζ as a (non-single-valued) function of w to the whole w-plane by reflection in various segments of the imaginary axis we see at once that $d\zeta^2$ is a quadratic differential on the w-sphere with double poles at 0, ∞, simple poles at i/m, $-im$ and simple zeros at i/l, $-il$ (except in the case $t=0$ when the simple poles and zeros coincide in pairs and cancel). Indeed we can write

$$d\zeta^2 = K\frac{(w + il)(w - i/l)}{w^2(w + im)(w - i/m)}\,dw^2$$

with K a suitable positive constant. In case $t=0$ this assumes a simple limiting form. The curves on which $d\zeta^2 > 0$ will be called trajectories, those on which $d\zeta^2 < 0$ will be called orthogonal trajectories.

Let us regard now the mapping from the domain $|z| < 1$, $\Re z < 0$ in the z-plane onto the domain $0 < \eta < \pi$, $\xi < 0$ in the ζ-plane by the function $\zeta = \log(z/i)$ (with a suitable determination). Combining this with the above mapping from the ζ-plane into the w-plane and extending it by reflection across the segments joining i and $-i$ in the z-plane and i/l and $-im$ in the w-plane we obtain a function regular and univalent for $|z| < 1$. We denote this function by $f(z; t)$ and the image of $|z| < 1$ under it by $D(t)$. The latter domain is bounded by orthogonal trajectories joining i/l and $-il$ and a rectilinear slit along an orthogonal trajectory from $-im$ to $-il$. The latter slit degenerates to a point when $t=0$. Further $f(z; t) \in C$ since the transformation $w^* = 1/w$ carries $D(t)$ into its exterior; also $f'(0; t) > 0$ and $f(z; 0) \equiv z$.

The quantities m and l depend on the parameter t and when appropriate will be denoted by $m(t)$ and $l(t)$.

3. THEOREM 1. *Let $g(w)$ be regular and univalent in $D(t)$ with $g(0)=0$, $|g'(0)| = 1$ and such that $g(w_1)g(w_2) \neq 1$, $w_1, w_2 \in D(t)$. Let μ denote the modulus of the boundary point of the image of $D(t)$ under $w' = g(w)$ closest to $w' = 0$. Then $\mu \geqq m(t)$ and equality can be obtained only for $g(w) = \pm\phi(w)$, provided $t > 0$, where $\phi(0) = 0$ and ϕ is a conformal mapping of $D(t)$ upon itself. When $t=0$, equality is attained only for $g(w) = e^{i\theta}\phi(w)$, θ real, ϕ as before.*

The proof of this result depends on the consideration of the following module problem. Let L be a Jordan curve in the w-plane enclosing $w=0$ and having reflectional symmetry in the imaginary w-axis. Let L^* be the image of L under the transformation $w^* = 1/w$. We suppose that L^* is exterior to L and denote by $\mathfrak{D}$ the doubly-connected domain bounded by L and L^*. Let G be a point on the positive imaginary axis in $\mathfrak{D}$ and G^* its image under the transformation $w^* = 1/w$. Let C_1 denote the class of rectifiable Jordan curves lying in $\mathfrak{D}$ and separating L from G, G^*, and L^*. Let C_2 denote the class of rectifiable Jordan curves lying in $\mathfrak{D}$ and separating L^* from L, G and G^*.

Let ρ be a real-valued non-negative function of integrable square over $\mathfrak{D}$ and such that, for $c \in C_i$ $(i=1, 2)$, $\int_c \rho|dw|$ exists and that $\int_c \rho|dw| \geqq 1$. Then let the greatest lower bound of $\iint_{\mathfrak{D}} \rho^2 du dv$ $(w=u+iv)$ for all such functions ρ be denoted by $M(L, G)$. This actually is a minimum attained for a particular function ρ. This can be proved by reducing the problem to a hexagon problem by a method similar to that of [2] or by a general construction method, but this result is not needed here.

Let us return now to the quadratic differential $d\zeta^2$ which we denote by $Q(w)dw^2$. We have seen that $D(t)$ is bounded by the union of certain orthogonal trajectories. The orthogonal trajectories interior to $D(t)$ are Jordan curves with reflectional symmetry in the imaginary w-axis. The orthogonal trajectory of this set which meets the positive imaginary axis at the point ir $(r>0)$ will be denoted by $H(r)$. As r tends to zero $H(r)$ tends to circular form [10; 6]. Let $H(r)$ for r sufficiently small play the role of L in the preceding module problem and let the point $w=i/m$ play the role of G. Let the doubly-connected domain bounded by $H(r)$ and the boundary of $D(t)$ be denoted by $E(r)$ and its image under the transformation $w^*=1/w$ be denoted by $E^*(r)$. These two domains have equal module (for the class of curves separating the boundaries) which we denote by $M(r)$. We then verify readily that for a suitable constant ν the metric $\nu|Q(w)|^{1/2}|dw|$ provides the extremal metric in the module problem defining $M(L, G)$ and this independently of the value of r. Further, in this case, $M(L, G)=2M(r)$.

With again the choice $L=H(r)$ but for a point H with affix ih, $h>1/m$, we have $M(L, G)\geqq M(L, H)+d$ where d (>0) is independent of r but depends on h. This can be seen in various ways, perhaps most easily by observing that it is possible to modify the function $|Q(w)|^{1/2}$ by setting it equal to zero in a sufficiently small neighborhood of G to obtain a function admissible in the competition for the greatest lower bound $M(L, H)$, independent of the choice of r.

Let now $E'(r)$ be the image of $E(r)$ under the function g of Theorem 1. We shall suppose it to lie again in the w-plane. Let K_1 be the bounded continuum complementary to $E'(r)$ bounded by the image L' of L. Let K_2 be the other continuum complementary to $E'(r)$. We obtain from $E'(r)$ a circularly symmetrized domain $\tilde{E}(r)$ in the following manner. Let the intersections of $|w|=R$ with K_1 and K_2 have respectively angular Lebesgue measure $l_1(R)$ and $l_2(R)$. Let $\tilde{K}_1$ be the set defined by $\pi/2-l_1(R)/2\leqq\Phi\leqq\pi/2+l_1(R)/2$ for those values of R for which K_1 meets $|w|=R$ where R, Φ are polar coordinates in the w-plane. Let $\tilde{K}_2$ be the set defined by $-\pi/2-l_2(R)/2\leqq\Phi\leqq-\pi/2+l_2(R)/2$ for those values of R for which K_2 meets $|w|=R$. The complement of $\tilde{K}_1\cup\tilde{K}_2$ is a doubly-connected domain which we denote by $\tilde{E}(r)$.

Let $E'(r)$, $\tilde{E}(r)$ have modules $M'(r)$, $\tilde{M}(r)$. Clearly $M'(r)=M(r)$ while by a standard symmetrization argument $M'(r)\leqq\tilde{M}(r)$ [8]. Under the assumption of Theorem 1 the point of K_2 closest to $w=0$ has modulus μ, thus the point $-i\mu$ is a point of $\tilde{K}_2$ and a boundary point of $\tilde{E}(r)$. We verify at once that $\tilde{E}(r)$ does not overlap with its image $\tilde{E}^*(r)$ under the transformation $w^*=1/w$ (as a consequence of the condition $g(w_1)g(w_2)\neq 1$, $w_1, w_2\in D(t)$). Let us now denote the point i/μ by H and let $\mathfrak{E}(r)$ be the intersection of $\tilde{E}(r)$ with the exterior of L. Let this domain, which is doubly-connected at least for r small enough, have module $\mathfrak{M}(r)$ (for the class of curves separating its boundaries as usual). Since $H(r)$ tends to circular form as r tends to zero we have that $\tilde{M}(r)-\mathfrak{M}(r)$ approaches zero as r tends to zero. We see also that $\mathfrak{E}(r)$ separates L from H, H^* and L^*, and $\mathfrak{E}^*(r)$, its image by the transformation $w^*=1/w$, separates L^* from L, H and H^*.

Suppose now that $\mu<m(t)$. Then, by an earlier remark, $M(L, G) \geqq M(L, H)+d$ where d (>0) is independent of r. On the other hand, by a standard argument [4], $M(L, H)\geqq 2\mathfrak{M}(r)$. Thus, combining this with the statements $M(L, G)=2M(r)$, $\tilde{M}(r)-\mathfrak{M}(r)=o(1)$, we have

$$2M(r) \geqq 2\tilde{M}(r) + d + o(1),$$

a contradiction to the result $\tilde{M}(r)\geqq M(r)$. Thus $\mu\leqq m(t)$.

Suppose next that $\mu=m(t)$. If $\tilde{K}_2$ did not coincide with the complement of $D(t)$ it would follow by a standard form of argument [3] that we would have $M(r)\geqq \tilde{M}(r)+\delta+o(1)$ for δ (>0) independent of r for r sufficiently small. This would be in contradiction to the inequality $\tilde{M}(r)\geqq M(r)$. Further, unless $\tilde{K}_2$ is obtained from K_2 by a rigid rotation about $w=0$, we shall have $\tilde{M}(r)\geqq M(r)+p$ where p is a positive constant independent of r, for r sufficiently small [7, Theorem 3]. In this case we would have $M(L, G)\geqq 2\mathfrak{M}(r)$ and we would get

$$2M(r) \geqq 2\tilde{M}(r) + o(1)$$

contrary to the inequality $\tilde{M}(r)\geqq M(r)+p$.

Thus equality is possible at most when $g(w)$ has the form $e^{i\theta}\phi(w)$, θ real, with $\phi(w)$ a conformal mapping of $D(t)$ onto itself and $\phi(0)=0$. The function $e^{i\theta}\phi(w)$ can satisfy the conditions of Theorem 1 only if neither 1 or -1 is interior to the image of $D(t)$ by this function. Let us assume first that $t>0$. Then the only boundary points of $D(t)$ on $|w|=1$ are the points $w=\pm 1$. This can be verified by observing that $|w|=1$ has the direction of an orthogonal trajectory only at the points $w=\pm i$ and that this situation will occur between any two boundary points on $|w|=1$. That $d\zeta^2<0$ on $|w|=1$ only at $w=\pm i$ is the consequence of a simple direct numerical calculation. Thus the open arc $|w|=1$, $\mathfrak{I}w>0$ is interior to $D(t)$, the open arc $|w|=1$, $\mathfrak{I}w<0$ is exterior to $D(t)$ and the only values of $e^{i\theta}$ for which $e^{i\theta}\phi(w)$ satisfies the conditions on $g(w)$ in Theorem 1 are ± 1. When $t=0$ all values $e^{i\theta}$, θ real, are clearly admissible. This completes the proof of Theorem 1.

4. The uniqueness part of Theorem 1 implies that no two values $f'(0; t)$ are equal. By an argument involving the theory of normal families we see that

$f'(0;\, t)$ tends to zero as t approaches π. Also for $t=0$, $f'(0;\, 0)=1$. Thus as t takes the values $0 \leqq t < \pi$, $f'(0;\, t)$ takes the values $1 \geqq f'(0;\, t) > 0$. In particular for $1 \geqq \lambda > 0$ there is a unique such function with $f'(0;\, t)=\lambda$. We denote this function by $F(z;\, \lambda)$ and the modulus of the closest boundary point to the origin of the image of $|z|<1$ under this function by $\mu(\lambda)$. Clearly $F(z;\, \lambda) \in C(\lambda)$. We then have the following result.

THEOREM 2. *If $f(z)$ is a univalent function in $C(\lambda)$, then for the modulus μ of the boundary point of the image of $|z|<1$ under f closest to the origin we have $\mu \geqq \mu(\lambda)$ and equality can be attained only for $f(z)=\pm F(e^{i\psi}z;\, \lambda)$, ψ real, when $\lambda<1$ and for $f(z)=e^{i\theta}z$, θ real, when $\lambda=1$.*

This is an immediate consequence of Theorem 1.

5. We now turn to the determination of $P(\lambda)$, the best possible uniform bound for functions in $C(\lambda)$. We observe first that we obtain the same result if we restrict ourselves to univalent functions in $C(\lambda)$. Indeed $f(z) \in C(\lambda)$ is subordinate to a univalent function $g(z) \in C$ [9]. The function $g(z)$ will not in general be in $C(\lambda)$ but $|g'(0)| \geqq \lambda$. Thus, producing a slit from the boundary of the image under g of $|z|<1$ toward the origin and letting $h(z)$ be a function mapping $|z|<1$ on the slit domain with $h(0)=0$, for a suitable length of slit we shall have $|h'(0)|=\lambda$. Further the least upper bound of $|h(z)|$ for $|z|<1$ will be the same as the least upper bound of $|g(z)|$ for $|z|<1$ and this is at least as large as the same quantity for $f(z)$.

Now, if for $f(z)$ a univalent function in C we have

$$\operatorname*{l.u.b.}_{|z|<1} |f(z)| = M,$$

for the modulus μ of the boundary point of the image of $|z|<1$ under f closest to the origin we have $\mu \leqq 1/M$ so $M \leqq 1/\mu$. Thus $P(\lambda) \leqq (\mu(\lambda))^{-1}$. We shall now show that the precise value is $P(\lambda)=(\mu(\lambda))^{-1}$.

First we observe by an argument similar to that of the first paragraph of this section that for $\lambda' \geqq \lambda$, $P(\lambda') \leqq P(\lambda)$.

Let t be the value corresponding to λ as in §4. We may assume $t>0$, $\lambda<1$ since in the special case excluded the result is evident. Let us denote by $D(t,\, \epsilon)$ the domain obtained from $D(t)$ by shortening the slit from $-il$ to $-im$ to a slit from $-il$ to $-i(m+\epsilon)$, $0<\epsilon<l-m$. Let $f(z;\, t,\, \epsilon)$ be the function mapping $|z|<1$ onto $D(t,\, \epsilon)$ such that $f(0;\, t,\, \epsilon)=0$, $f'(0;\, t,\, \epsilon)>0$. Clearly $f'(0;\, t,\, \epsilon)>\lambda$. Let $U(\sigma)$ denote the open set of points whose distance from the segment joining i/l to $i/(m+\epsilon)$ is less than σ. Let $U^*(\sigma)$ be the image of $U(\sigma)$ by the transformation $w^*=1/w$. Let

$$D(t,\, \epsilon,\, \sigma) = (D(t,\, \epsilon) \cup U(\sigma)) - \overline{U}^*(\sigma).$$

For σ sufficiently small this is a simply-connected domain containing the point $w=0$. Let $f(z;\, t,\, \epsilon,\, \sigma)$ be the function with $f(0;\, t,\, \epsilon,\, \sigma)=0$, $f'(0;\, t,\, \epsilon,\, \sigma)>0$

mapping $|z|<1$ onto $D(t, \epsilon, \sigma)$. By the general theory of kernels of domains [1], $f(z; t, \epsilon, \sigma)$ converges to $f(z; t, \epsilon)$ as σ tends to zero. Thus $f'(0; t, \epsilon, \sigma)>\lambda$ for σ sufficiently small. However

$$\operatorname*{l.u.b.}_{|z|<1} \left| f(z; t, \epsilon, \sigma) \right| = (m+\epsilon)^{-1} + \sigma.$$

Thus by the monotone property given above

$$P(\lambda) \geqq (m+\epsilon)^{-1} + \sigma$$

for ϵ, σ positive, sufficiently small. Hence

$$P(\lambda) \geqq m^{-1} = (\mu(\lambda))^{-1}.$$

We summarize these statements in our principal result.

THEOREM 3. *If $P(\lambda)$ is the best possible uniform bound for functions in $C(\lambda)$, $P(\lambda)=(\mu(\lambda))^{-1}$. There is no function $f(z)$ in $C(\lambda)$ having $P(\lambda)$ as least upper bound of $|f(z)|$, $|z|<1$, except for $\lambda=1$.*

The final statement of Theorem 3 follows from the uniqueness part of Theorem 2, since for such a function $f(z)$ the modulus μ of the boundary point of the image of $|z|<1$ under f closest to the origin would satisfy $\mu \leqq \mu(\lambda)$, thus $\mu=\mu(\lambda)$ and $f(z)$ would be $e^{i\theta}F(e^{i\psi}z; \lambda)$, θ, ψ real. However $|F(e^{i\psi}z; \lambda)|$, for $|z|<1$, has least upper bound strictly less than $(\mu(\lambda))^{-1}$ except when $\lambda=1$.

BIBLIOGRAPHY

1. C. Carathéodory, *Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten*, Math. Ann. vol. 72 (1912) pp. 107–144.

2. James A. Jenkins, *Some problems in conformal mapping*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 327–350.

3. ———, *Remarks on "Some problems in conformal mapping,"* Proc. Amer. Math. Soc. vol. 3 (1952) pp. 147–151.

4. ———, *Symmetrization results for some conformal invariants*, Amer. J. Math. vol. 75 (1953) pp. 510–522.

5. ———, *On Bieberbach-Eilenberg functions*, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 389–396.

6. ———, *On the local structure of the trajectories of a quadratic differential*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 357–362.

7. ———, *Some uniqueness results in the theory of symmetrization*, Ann. of Math. vol. 61 (1955) pp. 106–115.

8. G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Annals of Mathematics Studies, No. 27, Princeton University Press, 1951.

9. W. Rogosinski, *On a theorem of Bieberbach-Eilenberg*, J. London Math. Soc. vol. 14 (1939) pp. 4–11.

10. A. C. Schaeffer and D. C. Spencer, *Coefficient regions for schlicht functions*, Amer. Math. Soc. Colloquium Publications, vol. 35, 1950.

JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MD.