ON ASYMPTOTIC VALUES OF FUNCTIONS ANALYTIC IN A CIRCLE(1)

BY

MAKOTO OHTSUKA

1. The class of functions $f(z)$, which are analytic and bounded, $|f(z)|<1$, in the unit circle U: $|z|<1$ and which have radial limit values of modulus 1 for almost all points $e^{i\theta}$ of $|z|=1$ is well known; for literature and general properties of these functions we refer the reader to the papers of W. Seidel [16] and A. J. Lohwater [10]. Some of the results mentioned in these papers can be obtained from general theorems in the theory of cluster sets of functions analytic in U (cf. [4] and [13]). In recent papers Lohwater [9; 10; 11] has extended the concept of this class to functions which are meromorphic in U and whose moduli have radial limit 1 for almost all points of some arc A of $|z|=1$. In particular, we cite the following result [10; 11]: If $f(z)$ is meromorphic in $|z|<1$ with at most a finite number of zeros and poles and if $\lim_{r\to 1}|f(re^{i\theta})|=1$ for almost all $e^{i\theta}$ belonging to an arc A of $|z|=1$, then, unless $f(z)$ is analytic on A, there exists at least one curve (called an asymptotic path) terminating at a point of A along which $f(z)$ tends either to 0 or ∞. If, in addition, $f(z)$ is of bounded characteristic in $|z|<1$, there exists at least one radius having this property.

In the present paper, we are motivated by Lohwater's results to define new boundary cluster sets of functions analytic in U and taking values on an abstract Riemann surface $\mathfrak{R}$, and to establish relations between the cluster sets and the asymptotic values of the functions.

2. We begin with the definition of boundary points of an abstract Riemann surface $\mathfrak{R}$. Let $\mathfrak{F}$ be a class of *filters* such that each filter has a base consisting of open sets of $\mathfrak{R}$ which have no accumulation points on $\mathfrak{R}$. Furthermore we assume that of any two open sets of a base, one is contained in the other; that is, we have a nested base. We obtain a countable sub-base $\{G_n\}$ from the base if we take an exhaustion $\{\mathfrak{R}_n\}$, $\overline{\mathfrak{R}}_n\subset\mathfrak{R}_{n+1}$, with compact closures, and if we choose an element G_n of the base so that $G_n\cap\mathfrak{R}_n=\emptyset$ for each n. For, given any element G of the base, there is an $\mathfrak{R}_n$ such that $\mathfrak{R}_n\cap G\neq\emptyset$ and this shows $G_n\subset G$. Each filter of $\mathfrak{F}$ is defined to be *a boundary point of* $\mathfrak{R}$, and we denote the set of all such boundary points by $\mathfrak{F}_{\mathfrak{R}}$. Let P_F be a point of $\mathfrak{F}_{\mathfrak{R}}$ with a base $\{G_n\}$, and let $\{P_\nu\}$ be a sequence of points of $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$. If for each n there exists an integer ν_0 such that every P_ν, $\nu\geqq\nu_0$, or some domain of its base, is contained in G_n, we say that P_ν converges to P_F. We keep the original definition of the convergence of points of $\mathfrak{R}$. Thus we obtain a topol-

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ogy for the space $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$. Boundary points obtained by the completion with respect to a metric in $\mathfrak{R}$ can be reinterpreted in the way above. The ramified boundary points and geodesic boundary points in [2] are examples.

3. Let $f(z)$ be an analytic function defined in U: $|z|<1$ and taking values on an abstract Riemann surface $\mathfrak{R}$ (with boundary $\mathfrak{F}_{\mathfrak{R}}$ if $\mathfrak{R}$ is open). For any set $E\subset U$ and any point z_0 on C: $|z|=1$ we define the *cluster set* $S_{z_0}^{(E)}$ at z_0 along E to be the set of all values of $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$, for each point P of which there exists a sequence of points $\{z_n\}$ of E tending to z_0 such that $f(z_n)\rightarrow P$ as $n\rightarrow\infty$ (2). We shall write S_{z_0} for $S_{z_0}^{(U)}$, and T_{z_0} for the cluster set along the radius Oz_0.

Let $\{K_n\}$ be an open base of $\mathfrak{R}$, and z_0 a point of C. If, for a given integer n, there exists at least one open arc C_n containing z_0 such that the inner linear measure of the set $\{z\in C_n;\ z\neq z_0,\ T_z\cap K_n\neq\emptyset\}=C_n'$ (which may be empty) is zero(3), we define K_n^* by setting it equal to K_n; otherwise we put $K_n^*=\emptyset$. Denote the set $C-\bigcup_\nu C_{n_\nu}'-z_0$ by C^*, where the summation $\bigcup_\nu$ is taken over all n_ν for which $K_{n_\nu}^*=K_{n_\nu}$. Next we take an open base $\{K_\alpha\}$ (this is not countable in general) of $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$ and define K_α^* in a similar way. We shall denote the set $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}-\bigcup_\alpha K_\alpha^*$ by ST_{z_0}. This set is clearly a closed set in $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$ and may be considered as a sort of *boundary cluster set*(4).

Let us denote the intersection of any set X with the circle $|z-z_0|<\rho$ by X_ρ. The cluster set ST_{z_0} has a minimal property in the following sense: Taking any set $H\subset C$, $z_0\in H$, of linear measure zero, forming the closure $M_\rho^{(C-H)}$ of $\bigcup_{z\in(C-H)_\rho}T_z$, and denoting $\bigcap_{\rho>0}M_\rho^{(C-H)}$ by $ST_{z_0}^{(C-H)}$, we have the relation $ST_{z_0}\subset ST_{z_0}^{(C-H)}$. The set $M_\rho^{(C^*)}$ will be used in the following Theorem 2.

If $f(z)\rightarrow P\in\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$ along a curve in U terminating at z_0, this curve is called an *asymptotic path* and the value P an *asymptotic value*. The set of points on $\mathfrak{R}$ taken in any neighborhood in U of z_0 is called the *range of values* and denoted by R_{z_0}.

4. We first prove the following lemma.

LEMMA. *Let T be a continuous transformation of U into a topological space X. Let Δ be a domain in U whose image under T is contained in a closed set F in X and, for almost all $e^{i\theta}\in\Delta^b\cap C$, where Δ^b is the boundary of Δ, let the image of some end-part of the radius $Oe^{i\theta}$ be contained in a closed set F', disjoint from F. If there exists a continuous real-valued function $g(P)$ in X which assumes the value* 0 *on F and* 1 *on F', then $m(\Delta^b\cap C)=0$.*

Proof. We denote by $G(z)$ the function obtained by composing the transformation T with $g(P)$. By our assumption $\lim_{r\rightarrow 1}G(re^{i\theta})=1$ at almost all points $e^{i\theta}$ of $\Delta^b\cap C$. By Egoroff's theorem, for any integer p there exists a

(2) If $\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}$ is compact, $S_{z_0}^{(E)}$ is never empty whenever z_0 belongs to the closure of E.

(3) By means of the theory of functions of real variables, it can be proved that the set C_n' is linearly measurable. However, the set corresponding to K_α may be nonmeasurable in general.

(4) We used an idea in [4] in the definition of ST_{z_0}.

closed subset E_p of $\Delta^b \cap C$ such that $m(\Delta^b \cap C - E_p) < 1/p$ and $G(re^{i\theta})$ tends to 1 uniformly for $e^{i\theta} \in E_p$. Thus we can find $r_1 < 1$ such that $G(z) > 1/2$ on the set $Y = \{re^{i\theta}; r_1 < r < 1, e^{i\theta} \in E_p\}$. We decompose the complement of Y with respect to the annulus: $r_1 < r < 1$ into components $\{B_n\}$ $(n = 1, 2, \cdots)$. Let $\{B_{n_i}\}$ be the components which have points in common with Δ. Then its number is finite: $i = 1, 2, \cdots, k$. To prove this, suppose that there are an infinite number of $\{B_{n_i}\}$ having points in common with Δ. Since Δ is a domain, we can connect a point of $B_{n_1} \cap \Delta$ with a point of each $B_{n_i} \cap \Delta$ $(i \geqq 2)$ by a curve inside Δ. This curve must cross the boundary arc of every B_{n_i} on the circle: $|z| = r_1$. Any point of accumulation of these points of intersection is a boundary point of Δ, and, at the same time, a point of Y. This is impossible because, by the continuity of $G(z)$, $G(z) = 0$ on the closure of Δ and $G(z) > 1/2$ on Y. Therefore $\Delta^b \cap C$ is contained in $(\bigcup_{i=1}^k B_{n_i})^b \cap C$. The linear measure of that part of $\Delta^b \cap C$ lying in the open intervals of $(\bigcup_{i=1}^k B_{n_i})^b \cap C$ has the same value as $m(\Delta^b \cap C)$. But this part is the set $\Delta^b \cap C - E_p$ which has linear measure less than $1/p$. Hence $m(\Delta^b \cap C) < 1/p$. Since p is an arbitrary integer we see that $m(\Delta^b \cap C) = 0$.

5. Our theorems are

THEOREM 1. *Let $f(z)$ be an analytic function defined in U and taking values on an abstract Riemann surface $\mathfrak{R}$ (with boundary $\mathfrak{F}_{\mathfrak{R}}$ if $\mathfrak{R}$ is open). Then a point P_0 of $S_{z_0} - ST_{z_0} - R_{z_0}$ is an asymptotic value at z_0 or at points z_n of C tending to z_0 if there exists a path in $\mathfrak{R} \cap S_{z_0}$ converging to P_0.*

THEOREM 2. *Let $f(z)$ be the same function as in Theorem 1. A point P_0 of the set $ST_{z_0} - R_{z_0}$ is an asymptotic value at z_0 or at points z_n tending to z_0 if*

(i) *there exists a number $\rho > 0$ such that there is a path in $\mathfrak{R} \cap (S_{z_0} - M_\rho^{(C^*)})$ converging to P_0, and if*

(ii) *the set of points on $|z| = 1$ where the radial cluster sets T_z do not contain P_0 is everywhere dense in a certain open arc $C' \subset C$ containing z_0.*

We shall prove Theorem 2 for $P \in \mathfrak{F}_{\mathfrak{R}}$. The proof for the case $P \in \mathfrak{R}$ and the proof of Theorem 1 are easily obtained by modifying the proof given below.

Let L be the path in $\mathfrak{R} \cap (S_{z_0} - M_\rho^{(C^*)})$, converging to P_0. We form two paths on each side of L and close enough to L that the domain D between them is contained in $\mathfrak{R} - M_\rho^{(C^*)}$. Let $\{G_n\}$ be a nested countable base of the filter defining P_0 and let D_n be that component of the intersection of G_n with D which contains an end-part of L. Obviously, $D_1 \supset D_2 \supset \cdots \rightarrow P_0$.

We take two points z_1 and z_2 on C near z_0 so that $\arg z_1 < \arg z_0 < \arg z_2$, $|z_0 - z_1| < \rho$ and $|z_0 - z_2| < \rho$, and such that $T_{z_1} \cup T_{z_2}$ does not contain P_0. Let $r' < 1$ be a number sufficiently near 1. Denote the sector $\{re^{i\theta}; r' < r < 1, \arg z_1 < \theta < \arg z_2\}$ by Q and its boundary inside U by q. We may assume that the image of q lies outside some neighborhood of P_0. The inverse image of D_n in Q is not empty since $L \subset S_{z_0}$. For n sufficiently large, some component, say Δ_n, together with its closure, has no common point with q.

Suppose that, in Δ_n, $f(z)$ does not assume values of D_{n+1}. Then the closure of the image $f(\Delta_n)$ of Δ_n is compact in $\mathfrak{R}$, and for almost all z of C_ρ the radial cluster sets T_z lie outside the closure of $f(\Delta_n)$. Then by our lemma, the measure $m(\Delta_n^b \cap C)=0$, the continuous function $g(P)$ of the lemma being defined by the aid of a metric in $\mathfrak{R}$. Therefore the harmonic measure of $\Delta_n^b \cap C$ with respect to Δ_n is zero. We take a small compact Jordan domain K_0 inside D_{n+1} and form a harmonic measure function of the boundary of K_0 in the domain $D_n - K_0$. If we regard this function as a function defined in Δ_n, it has boundary value 0 except for points of $\Delta_n^b \cap C$ which has harmonic measure zero. By the maximum principle this function must be the constant zero, which is a contradiction.

Thus we have shown that $f(\Delta_n) \cap D_{n+1} \neq \emptyset$. Consider the inverse image of D_{n+1} in Δ_n and let Δ_{n+1} be any component of the image. We can show as above that $f(\Delta_{n+1}) \cap D_{n+2} \neq \emptyset$. In this manner we obtain a sequence of domains $\Delta_n \supset \Delta_{n+1} \supset \cdots$ where $f(\Delta_k) \subset D_k$ $(k=n, n+1, \cdots)$. Taking a point z_k in Δ_k and connecting it with any point z_{k+1} of Δ_{k+1} by a curve in Δ_k, we get a path l in Q along which $f(z) \to P_0$. By assumption (ii) (we may suppose that the arc $z_1 z_2$ is contained in C'), l terminates at a single point of $|z|=1$. Since Q may be taken arbitrarily near z_0 the conclusion of Theorem 2 is obtained.

REMARK. If we allow a path to oscillate, we may infer the existence of such a path with asymptotic value P_0 in any neighborhood of z_0, with the following condition replacing (ii):

(ii′) there exist points ζ on $|z|=1$ on both sides of z_0 and arbitrarily close to z_0 such that P_0 does not belong to T_ζ. If we assume only (i), then we know that either there is a path (which may oscillate) in any neighborhood of z_0, or there is a sequence of curves which accumulate on a closed arc containing z_0, such that $f(z) \to P_0$ uniformly along these curves[5].

6. In the theory of cluster sets the difference between a cluster set such as S_{z_0} and a boundary cluster set such as ST_{z_0} is an open set. In the case of an abstract Riemann surface, it is not generally true that $S_{z_0} - ST_{z_0}$ is an open set in $\mathfrak{R} + \mathfrak{F}_{\mathfrak{R}}$. On the one hand, suppose that there is a point $P_0 \in \mathfrak{R} \cap ((S_{z_0})^b - ST_{z_0})$. Since ST_{z_0} is a closed set, there is a domain G on $\mathfrak{R}$, containing P_0 and with compact closure corresponding to a parametric circle $|\omega| \leqq 1$, such that $\overline{G} \cap ST_{z_0} = \emptyset$. We denote the open set which is the inverse image in U of G by Δ, and its boundary by δ. It follows from the lemma that, if we take a part H of $\delta \cap C$ sufficiently near z_0, its linear measure, and hence its relative harmonic measure with respect to Δ, is zero. The cluster set $S_{z_0}^{(\Delta)}$ of the composed function $\omega(f(z))$ contains the point $\omega=0$ but does not coincide with the whole $|\omega| \leqq 1$, and the boundary cluster set $S_{z_0}^{(\delta-H)}$ (this is defined by setting

(5) This fact is expressed by the notation $P_0 \in \Phi(f)$ in [6].

$E=\delta-H$ in $S_{z_0}^{(E)}$ of §3) is contained in $|\omega|=1$. This fact contradicts the following theorem which is easily deduced from a theorem in M. Brelot [1]: Let Δ be an open set with boundary δ in the z-plane, z_0 a nonisolated boundary point of Δ, H a subset of δ, containing z_0, of relative harmonic measure zero with respect to Δ, and $f(z)$ analytic in Δ and on $\delta-H$. Then the difference between the cluster set $S_{z_0}^{(\Delta)}$ and the boundary cluster set $S_{z_0}^{(\delta-H)}$ is an open set.

We state our result in

THEOREM 3(6). *Under the assumption of Theorem* 1, $\mathfrak{R}\cap(S_{z_0}-ST_{z_0})$ *is an open set.*

On the other hand, however, we shall show that $S_{z_0}-ST_{z_0}$ is not necessarily open. Let $\mathfrak{R}$ be the circle $|w|<1$ and suppose that $\mathfrak{F}_{\mathfrak{R}}$ consists of only one point w_0 on $|w|=1$. Let $f(z)$ be the identity function: $w=z$ $(w_0=z_0)$. Then $S_{z_0}=\{w_0\}$ but $ST_{z_0}=\emptyset$, so that $S_{z_0}-ST_{z_0}=\{w_0\}$ is a closed set.

7. We shall discuss next some special cases of Theorems 1 and 2. Suppose first that $f(z)$ possesses a radial limit almost everywhere on an open arc A of $|z|=1$ containing z_0. Then ST_{z_0} is the intersection of the closures of certain sets of such limit values. For instance, if the range of values R_{z_0} is compact relatively in $\mathfrak{R}$ and does not cover a set of positive logarithmic capacity, $f(z)$ has this property(7).

We have next the following corollary to Theorem 1:

COROLLARY 1(8). *Let R_{z_0} be conformally equivalent neither to the Riemann sphere punctured at most two points nor to a torus. Then any value P_0 of $\mathfrak{R}$ belonging to $S_{z_0}-ST_{z_0}-R_{z_0}$ is a radial limit either at z_0 or at points z_n tending to z_0.*

First we observe that there is a path in $\mathfrak{R}\cap S_{z_0}$ terminating at P_0 since a neighborhood of P_0 is contained in S_{z_0} by Theorem 3. By Theorem 1 we can then obtain a path terminating at a point z' of C near z_0, with the asymptotic value P_0. We can prove by the generalization of Lindelöf's theorem that $f(z)$ has the same asymptotic value P_0 along the radius with the end point z'.

We consider the class of functions studied by Lohwater [11]: A function $f(z)$, meromorphic in $|z|<1$, is said to belong to class (U^*) if there exists an arc A of C such that $\lim_{r\to 1}|f(re^{i\theta})|=1$ for almost all $e^{i\theta}$ of A.

COROLLARY 2 [10; 11]. *Let $w=f(z)$ be a function of class (U^*). If $f(z)$ is not analytic on the arc A and if it possesses at most a finite number of zeros and poles in a neighborhood of A, then there exists at least one curve, terminating at a point of A, along which $f(z)\to 0$ or ∞.*

From an extension of Schwarz's symmetry principle [4] it follows that, at any singular point z_0 of A, at least one of 0 and ∞ belongs to S_{z_0}. Since

(6) This was proved in [4] and [13] in the case when $\mathfrak{R}$ is the extended w-plane.

(7) See Theorem 3.3 in [14].

(8) The case when R is the whole plane and S_{z_0} is not was discussed in [13].

$ST_{z_0} \subset \{|w|=1\}$, and since $S_{z_0} - ST_{z_0}$ is an open set by Theorem 3, the assumptions of Theorem 1 are satisfied for $P_0=0$ or ∞; hence Corollary 2 is established.

In connection with Theorem 2, we remark that if R_{z_0} satisfies the conditions of Corollary 1, it cannot happen that $f(z)$ tends to a value in $\mathfrak{R}$ uniformly on a sequence of curves accumulating on an arc of C near z_0. Therefore for such a function condition (ii) in Theorem 2 is not necessary if the point P_0 belongs to $\mathfrak{R}$. However, we do not know whether condition (ii) is necessary in general, even for the functions of class (U^*)(9), although condition (i) is fulfilled for these functions.

We shall prove

COROLLARY 3 (CALDERÓN-DOMÍNGUES-ZYGMUND [3], *cf. also* [8]). *Let $w=f(z)$ be a bounded analytic function defined in $|z|<1$. Let $f(z)$ have a radial limit of modulus one almost everywhere on an arc A of C. Then if $f(z)$ is not analytic on A, every value of $|w|=1$ is a radial limit at infinitely many points of A.*

Let $z_0 \in A$ be a singular point. By Theorem 2 and Lindelöf's theorem any point w of $|w|=1$ is the radial limit at z_0 or at z_n tending to z_0. If such $\{z_n\}$ exists, the corollary is already proved. Also if there are singular points on C tending to z_0, then our corollary follows. Hence suppose that $f(z)$ were analytic on C near z_0, except at z_0, and $f(z) \neq w$ there. In this situation $f(z)$ would have limit values w_1 and w_2 respectively as $z \in C$ moves toward z_0 from both sides. Since $f(z)$ is bounded, $w_1 = w_2$ and $f(z)$ would tend to this value uniformly as z approaches z_0 from the inside of U by Lindelöf's theorem. Then $f(z)$ would be analytic at z_0 and this is a contradiction. Thus the corollary is proved.

8. In this section we shall consider mermorphic functions of bounded characteristic. Such a function has radial limit almost everywhere, and the set of points of C where the function has the same radial limit has linear measure zero by Riesz-Nevanlinna's theorem. Therefore condition (ii) of Theorem 2 is not necessary. However, we can show by an example that at a point where an asymptotic path terminates, the function does not always have a radial limit of the same value. We know that this is true for a meromorphic function with at least three exceptional values. Hence let us remark that the following theorem is not a special case of Corollary 1 if and only if $f(z)$ omits only two values.

THEOREM 4. *Let $w=f(z)$ be a meromorphic function of bounded characteristic in U, and suppose that it does not take two values w_0 and w_1 near a point z_0 of $|z|=1$. If w_0 belongs to $S_{z_0} - ST_{z_0}$, then w_0 is a radial limit at z_0 or at points z_n tending to z_0.*

(9) In a recent letter Professor Lohwater remarked to the author that his unpublished proof of the first part of [9] is not complete and that that aspect of the question is still open.

Proof(10). Without loss of generality we may suppose that $w_0=0$, $w_1=\infty$. and ST_{z_0} lies outside $|w|\leqq 1$. Then $f(z)$ has a representation (see [12])

$$f(z) = \frac{\Omega_1(z)}{\Omega_2(z)} \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\,d\mu(\phi) + i\lambda\right],$$

where $\Omega_1(z)$ and $\Omega_2(z)$ are finite Blaschke products, λ is a real constant, and $\mu(\theta)$ is a function of bounded variation in $[0, 2\pi]$ such that $\mu(\theta)=\{\mu(\theta+)+\mu(\theta-)\}/2$. In order to prove our theorem it is sufficient to prove that

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}\,d\mu(\phi) \qquad (z = re^{i\theta})$$

tends to $-\infty$ along a radius at z_0 or at a point arbitrarily near z_0.

Let C_ρ be an arc such that the set $M_\rho^{(C^*)}$ defined in §3 lies outside $|w|\leqq 1$, and let us decompose $u(z)$ into the integral $u_1(z)$ on C_ρ and the integral $u_2(z)$ on its complement.

We denote the set function corresponding to $\mu(\theta)$ by $\mu^*(X)$. Then by de la Vallée Poussin's decomposition theorem (see [15, p. 127]) we have

$$\mu^*(X) = \mu^*(X \cap E_{+\infty}) + \mu^*(X \cap E_{-\infty}) + \int_X \mu'(\phi)d(\phi)$$

for any Borel set X consisting of points of continuity of $\mu(\theta)$, where $E_{+\infty}$ and $E_{-\infty}$ represent the sets of points at which $\mu(\theta)$ has derivatives equal to $+\infty$ and $-\infty$ respectively. According to Fatou's theorem, $u(re^{i\theta})\to\mu'(\theta)$ as $r\to 1$ for almost all θ. By hypothesis, $|f(re^{i\theta})| = \exp[u(re^{i\theta})]$ tends to $\exp[\mu'(\theta)]>1$ as $r\to 1$ for almost all $e^{i\theta}$ of C_ρ. Hence $\mu'(\theta)>0$ for almost all $e^{i\theta}$ of C_ρ. Now it is a result of Lohwater [10, Lemma] that if there is a negative jump of $\mu(\theta)$ on C, $u(z)$ tends to $-\infty$ radially at this point. We now suppose that $\mu(\theta)$ has no negative jump on C_ρ. Let Y be any Borel subset of C_ρ which does not contain points of discontinuity of $\mu(\theta)$. If $E_{-\infty}\cap C_\rho=\varnothing$, then the positiveness of $\mu^*(Y)$ follows from the above equality because $\mu^*(Y\cap E_{+\infty})$ is always non-negative (see lemma in [15, p. 126]). Let X be any Borel subset of C_ρ, and $\{a_n\}$ be the points of discontinuity of $\mu(\theta)$ on X; $\{a_n\}$ coincide with the jumps of $\mu(\theta)$. Then $\mu^*(X)=\mu^*(X-\{a_n\})+\sum_n\mu^*(a_n)$ and $\mu^*(a_n)$ is equal to the saltus at a_n. Since both terms of the right side are positive, $\mu^*(X)>0$. Therefore $u_1(z)>0$ and hence $u(z)>m>-\infty$ near z_0. Hence $|f(z)|>m_1>0$ near z_0. This contradicts the assumption that $0\in S_{z_0}$. Thus there is at least one point $e^{i\theta}$ of $E_{-\infty}$ on C_ρ. At this point $u(z)$ has a radial limit $-\infty$. On account of the arbitrariness of C_ρ, the theorem is concluded.

As mentioned in §1, a special case of this theorem was proved in [10]. Whether the existence of $w_1\in R_{z_0}$ in our theorem is necessary or not is not yet determined. Also notice that we have no such result corresponding to Theo-

(10) The writer owes the idea of the proof to [10].

rem 2.

9. In order to determine more completely the asymptotic values of an analytic function, we shall define a smaller boundary cluster set. Up to now we have used the radial cluster set T_{z_0} to define the boundary cluster set ST_{z_0}. In many cases the set T_{z_0} is too large; we shall find it more suitable to use, instead, the set Γ_{z_0} of all asymptotic values at z_0, in some cases.

Let $f(z)$, $\{K_n\}$, $\{K_\alpha\}$ and z_0 be the same as in §3. We shall use the same notation as in §3, whenever it causes no confusion. If, for a given integer n, there is an open arc C_n containing z_0 such that the inner linear measure of the set $\{z\in C_n;\ z\neq z_0,\ \Gamma_z\cap K_n\neq\emptyset\}=C_n'$ is zero, we define K_n^* by K_n; otherwise we set $K_n^*=\emptyset$. We denote the set $C-\bigcup_\nu C_{n_\nu}'-z_0$ by C^*, where the summation $\bigcup_\nu$ is taken over all n_ν for which $K_{n_\nu}^*=K_{n_\nu}$. We define K_α^* in a similar fashion for $\{K_\alpha\}$ and set $S\Gamma_{z_0}=\mathfrak{R}+\mathfrak{F}_{\mathfrak{R}}-\bigcup_\alpha K_\alpha^*$. This set $S\Gamma_{z_0}$ is the smallest set in the following sense: Let $H\subset C$, $z_0\in H$, be a set of linear measure zero, and denote the closure of $\bigcup_{z\in(C-H)_\rho}'\Gamma_z$ by $N_\rho^{(C-H)}$ and the intersection $\bigcap_{\rho>0}N_\rho^{(C-H)}$ by $S\Gamma_{z_1}^{(C-H)}$. Then $S\Gamma_{z_0}\subset S\Gamma_{z_1}^{(C-H)}$.

The following theorems correspond respectively to Theorems 1 and 2:

THEOREM 5. *Let $f(z)$ be an analytic function defined in $|z|<1$ and taking values of an abstract Riemann surface $\mathfrak{R}$ (with boundary $\mathfrak{F}_{\mathfrak{R}}$ if $\mathfrak{R}$ is open). Then a point P_0 of $S_{z_0}-S\Gamma_{z_0}-R_{z_0}$ is an asymptotic value either at z_0 or at points z_n tending to z_0 if* (i) *there exists a path in $\mathfrak{R}\cap S_{z_0}$ converging to P_0, and if* (ii) *there exists a set E of points on $|z|=1$, dense in some neighborhood of z_0, such that, for each $\zeta\in E$, there is a path l_ζ on $|z|<1$ terminating at ζ with the property that the cluster set of $f(z)$ along l_ζ does not contain P_0.*

THEOREM 6. *Let $f(z)$ be the same as in Theorem* 5. *A point P_0 of $S\Gamma_{z_0}-R_{z_0}$ is an asymptotic value either at z_0 or at points z_n tending to z_0 if* (i) *there exists a path in $\mathfrak{R}\cap(S_{z_0}-N_\rho^{(C^*)})$ converging to P_0, where ρ is a certain positive number, and if condition* (ii) *of Theorem* 5 *is satisfied.*

Notice that condition (ii) is required even in Theorem 5. If we lift this requirement in Theorems 5 and 6, then the same remark as in §5 is given.

The proofs for these theorems are similar to but simpler than those for Theorems 1 and 2 and are omitted here. We shall explain Theorem 5 in a special case.

Let $w=f(z)$ be a meromorphic function defined in $|z|<1$ and suppose that $f(z)$ omits at least three values in a certain neighborhood of a point z_0 on $|z|=1$. Then the points of $|z|=1$ which have radial limits are everywhere dense in an open arc containing z_0 (see [5] and [13]). Therefore condition (ii) of Theorem 5 is not necessary. If S_{z_0} is the whole w-plane, condition (i) in Theorem 5 is clearly fulfilled, while, if S_{z_0} is not the whole w-plane, Theorem 5 is contained in Theorem 1. Therefore, for our function $f(z)$, Theorem 5 may be stated without conditions (i) and (ii) (this theorem was stated in [13]). The modular function shows that Theorem 5 is not contained in Theorem 1.

We remark that $\mathfrak{R}\cap(S_{z_0}-S\Gamma_{z_0})$ is an open set. This is trivial if $\mathfrak{R}\subset S_{z_0}$, and if $\mathfrak{R}\not\subset S_{z_0}$ then $\mathfrak{R}\cap S\Gamma_{z_0}=\mathfrak{R}\cap ST_{z_0}$(11) and our assertion follows from Theorem 3.

10. Finally we shall examine the assumptions of Theorems 1 and 2. Let $\mathfrak{R}$ be a simply-connected domain in the w-plane which spirals down on $|w|=1$ from the outside and suppose that $\mathfrak{F}_{\mathfrak{R}}$ consists of only one point w_0 of $|w|=1$ with the ordinary topology. We map $\mathfrak{R}$ onto $|z|<1$ and denote by z_0 the point on $|z|=1$ which corresponds to w_0. Then $S_{z_0}=\{w_0\}$ and $ST_{z_0}=\emptyset$. Clearly w_0 is never an asymptotic value; here, the condition in Theorem 1 is not satisfied. However, if we take $P_0\in S_{z_0}-ST_{z_0}$ in $\mathfrak{R}$, then the required curve is obtained by Theorem 3.

The following examples show that condition (i) is necessary in Theorem 2 even if we take the point P_0 in $\mathfrak{R}$.

EXAMPLE 1. Take the circles U_w: $|w|<1$ and V_n: $|w-1|\leqq 1/n$ $(n=1, 2, \cdots)$. Set $U_w-V_n=G_n$ and connect G_n and G_{n+1} by a small strip domain S_n near the point $w=-1$ so that $S_n\to -1$ as $n\to\infty$ (12) and $G_1\cup S_1\cup G_2\cup S_2\cup\cdots$ is a simply-connected Riemann surface $\mathfrak{R}_1$. Map $\mathfrak{R}_1$ onto U: $|z|<1$ conformally. Then by Koebe's theorem the image of S_n and hence the image of G_n tends to a point, say z_0, of C: $|z|=1$, and z_0 is the only point which is not an image of any boundary point of $\{G_n\}$ and $\{S_n\}$. For the function $w=f(z)$ mapping U into the w-plane through $\mathfrak{R}_1$, the cluster sets are $S_{z_0}=\{|w|\leqq 1\}$ and $ST_{z_0}=S\Gamma_{z_0}=\{|w|=1\}$. The point $w=1$ is neither taken by $f(z)$ nor is an asymptotic value; the path which is required in (i) of Theorem 2 actually does not exist.

EXAMPLE 2(13). We shall construct a similar example in which $w=1$ is the only one exceptional value (i.e. $R_{z_0}=\{w\neq 1\}$). Let $\{V_n\}$ be the same as in the first example. Let us set $A_n=\{|w|<1+1/n\}-V_n$ and $B_n=\{|w|>1-1/n\}-V_n$, and let us connect A_n with B_n by a strip S_n and B_n with A_{n+1} by a strip S_n' so that these strips S_n and S_n' tend to $w=-1$ as $n\to\infty$ and $A_1\cup S_1\cup B_1\cup S_1'\cup A_2\cup S_2\cup B_2\cdots$ is a simply-connected Riemann surface $\mathfrak{R}_1$. We map $\mathfrak{R}_1$ onto U conformally and denote the function corresponding to the mappings $U\to\mathfrak{R}_1\to$the w-plane by $f(z)$. We shall construct $\{S_n\}$ so that the z-images of A_n, B_n, S_n, and S_n' tend to a point of C as $n\to\infty$.

Suppose that each S_n contains a part S_n^* which is mapped conformally onto a rectangle: $0<u<a_n$, $0<v<b_n$ such that the sides with length a_n correspond to a part of the boundary of $\mathfrak{R}_1$. Consider, on S_n^*, the function which maps $\mathfrak{R}_1$ onto U, we transform it into the function defined on the rectangle and denote it by $z=g(u+iv)$. This is a schlicht function, and we have, by Schwarz's inequality,

(11) We can prove this as for Theorem 3.3 of [14].

(12) For instance, take the part of $1/n+1<|w+1|<1/n$ outside U_w as S_n.

(13) The writer owes some technique in the construction of this example to [7].

$$\left\{\int_0^{a_n}\int_0^{b_n} | g'(u+iv) | dudv\right\}^2$$
$$\leqq \left\{\int_0^{a_n}\int_0^{b_n} | g'(u+iv) |^2 dudv\right\} \left\{\int_0^{a_n}\int_0^{b_n} dudv\right\}.$$

If the images of A_n, S_n, B_n, and S'_n do not tend to a point, they must tend to some arc, say z_1z_2. Then, denoting the area of the image of S_n by s_n, we get from the above inequality

$$(| z_1 - z_2 | \cdot a_n)^2 \leqq s_n \cdot a_n b_n$$

whence

$$0 < | z_1 - z_2 |^2 \leqq s_n b_n / a_n.$$

Since $s_n \to 0$ as $n \to \infty$, there arises a contradiction if we assume $b_n/a_n < M < \infty$. Therefore under the assumption that S_n contains such a part S_n^* (this means that S_n^* is "narrow") it is proved that A_n, S_n, B_n, and S'_n tend to a point, say z_0, of C. In this example $w=1$ is the only one exceptional value and $ST_{z_0} = S\Gamma_{z_0}$ coincides with $|w| = 1$. The point $w=1$ is not an asymptotic value; actually condition (i) in Theorem 2 is not fulfilled(14).

Whether condition (ii) in Theorem 2 is really necessary or not is not yet known, as already stated in §7.

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Harvard University,
Cambridge, Mass.
Nagoya University,
Nagoya, Japan.