VARIATIONAL MEASURE

BY

MAURICE SION(1)

Introduction. The concept of a set which is ν-measurable for all ν in some class M of measures is possibly due to J. D. Tamarkin, who communicated some of his ideas to A. P. Morse around 1936. It is the latter, however, who first studied such sets extensively and, in a Real Variables course at the University of California, Berkeley, had them play a role in measure theory traditionally assumed by Borel sets.

Let $\mathfrak{M}_0$ be the set of outer measures ν on the real line such that open sets are ν-measurable. A. P. Morse suggested to the author the problem of determining if a set, which is ν-measurable for all ν in $\mathfrak{M}_0$, is mapped into a similar set by a continuous function of bounded variation. It is while attempting to solve this problem that the writer was led to introduce a measure in the domain of a function, which is related to a given measure in the range and gives some indication of the extent to which the function fluctuates. This we have called variational measure, and in this paper we study some of its properties. As an application of the theory we have developed here, we get conditions on the function f and class M of measures in order that f map a set, which is ν-measurable for all ν in M, into a set of the same kind.

The problem suggested by A. P. Morse, however, remains open.

The approach to measure theory in this paper is that of C. Carathéodory and we refer the reader to [3] and [7] for the general background material.

1. **Notation.**

1.1. $\mathbf{E}x(\cdots)$ will denote the set of all x such that $(\cdots)$.

1.2. $\{x\}$ will denote the set consisting of the point x only.

1.3. $\sigma F = \bigcup A \in F\ A$.

1.4. F is disjointed if and only if $A \in F$, $B \in F$, $A \neq B$ imply $AB = 0$.

1.5. dmn f and rng f will denote respectively the domain and range of f.

1.6. ${}_*fA = \mathbf{E}y(y = f(x)$ for some $x \in A)$.

1.7. ${}^*fA = \mathbf{E}x(f(x) = y$ for some $y \in A)$.

1.8. When dealing with set functions ν, we shall use the notation $.\nu A$ to stand for $\nu(A)$.

1.9. ω will denote the set of all non-negative integers.

1.10. A non-negative integer is the set of all non-negative integers preced-

Presented to the Society, February 25, 1956; received by the editors March 10, 1956.

(1) This research was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

ing it, that is: if $n \in \omega$ then $m \in n$ if and only if $m \in \omega$ and $m < n$. Thus, the number 0 is also the empty set.

2. Fundamental definitions.

2.1. A measure ν on A is a function on the family of all subsets of A to the non-negative real numbers such that $.\nu 0 = 0$ and if B is a sequence of subsets of A, $C \subset \bigcup n \in \omega\ B_n$, then

$$.\nu C \leq \sum n \in \omega\ .\nu B_n.$$

2.2. B is ν-measurable if and only if for every $T \in \operatorname{dmn} \nu$,

$$.\nu T = .\nu(TB) + .\nu(T - B).$$

2.3. ν is an outer measure if and only if ν is a measure and for every $B \in \operatorname{dmn} \nu$ there is a B' such that $B \subset B'$, B' is ν-measurable, and $.\nu B = .\nu B'$.

2.4. Given a family F of sets, we define:

$$\mathcal{P}(F) = \mathbf{E} P\,(P \subset F,\ P \text{ is finite, } P \text{ is disjointed, } \sigma P = \sigma F).$$

2.5. P' is a refinement of P if and only if for every $\alpha' \in P'$ there is an $\alpha \in P$ such that $\alpha' \subset \alpha$, and $\sigma P = \sigma P'$.

2.6. If F is such that for every P' and P'' in $\mathcal{P}(F)$ there is a P in $\mathcal{P}(F)$ which is a refinement of both, if for every $A \in F$, there is a $P \in \mathcal{P}(F)$ with $A \in P$, if f is a function on σF, and if ν is a measure on $\operatorname{rng} f$, we define the variational measure $V(F, f, \nu)$ to be the function μ on the subsets of σF such that for every $A \subset \sigma F$:

$$.\mu A = \sup_{P \in \mathcal{P}(F)} \sum \alpha \in P\ .\nu_* f(A\alpha).$$

If the variables F, f, ν do not satisfy the above conditions, we define $V(F, f, \nu)$ to be the function μ such that $.\mu 0 = 0$ and $.\mu A = \infty$ for $0 \neq A \subset \sigma F$.

3. General properties of variational measure.

3.1. THEOREM. *$V(F, f, \nu)$ is a measure.*

Proof. Let $\mu = V(F, f, \nu)$, B a sequence of subsets of σF, $A = \bigcup n \in \omega\ B_n$, $P \in \mathcal{P}(F)$. Then if ν is a measure:

$$\begin{aligned} &\sum \alpha \in P\ .\nu_* f(A\alpha) \\ &\qquad = \sum \alpha \in P\ .\nu \bigcup n \in \omega\ {}_* f(B_n \alpha) \leq \sum \alpha \in P\ \ \sum n \in \omega\ .\nu_* f(B_n \alpha) \\ &\qquad = \sum n \in \omega\ \sum \alpha \in P\ .\nu_* f(B_n \alpha) \leq \sum n \in \omega\ .\mu B_n. \end{aligned}$$

Hence:

$$.\mu A \leq \sum n \in \omega\ .\mu B_n.$$

All the other conditions are easily checked.

3.2. LEMMA. *If P' is a refinement of P, P is disjointed, f is a function on σP, ν a measure on* $\operatorname{rng} f$, $A \subset \sigma P$, *then*:

$$\sum \alpha \in P \ .\nu_* f(A\alpha) \leqq \sum \alpha' \in P' \ .\nu_* f(A\alpha').$$

3.3. THEOREM. *If* $\mu = V(F, f, \nu)$, $A \in F$, *then* A *is* μ*-measurable.*

Proof. Let $.\mu T < \infty$. Then $.\mu(TA) < \infty$ and $.\mu(T-A) < \infty$. Given $\epsilon > 0$, in view of the condition on F (2.6) and Lemma 3.2, we can choose $P \in \mathcal{P}(F)$ such that

(1) for every $\alpha \in P$, if $\alpha A \neq 0$, then $\alpha \subset A$,

(2) $.\mu(TA) \leqq \sum \alpha \in P \ .\nu_* f(TA\alpha) + \epsilon$,

(3) $.\mu(T-A) \leqq \sum \alpha \in P \ .\nu_* f((T-A)\alpha) + \epsilon$.

Let

$$P_1 = \mathbf{E}\alpha \, (\alpha \in P \text{ and } \alpha \subset A),$$
$$P_2 = \mathbf{E}\alpha \, (\alpha \in P \text{ and } \alpha A = 0).$$

Notice that $P = P_1 \cup P_2$, and:

$$\begin{aligned} .\mu(TA) + .\mu(T-A) &\leqq \sum \alpha \in P_1 \ .\nu_* f(TA\alpha) + \sum \alpha \in P_2 \ .\nu_* f((T-A)\alpha) + 2\epsilon \\ &= \sum \alpha \in P_1 \ .\nu_* f(T\alpha) + \sum \alpha \in P_2 \ .\nu_* f(T\alpha) + 2\epsilon \\ &= \sum \alpha \in P \ .\nu_* f(T\alpha) + 2\epsilon \leqq .\mu T + 2\epsilon. \end{aligned}$$

Letting $\epsilon \to 0$, since μ is a measure, we see that

$$.\mu T = .\mu(TA) + .\mu(T-A).$$

If $.\mu T = \infty$, the above equality is trivial.

3.4. LEMMA. *If* $\mu = V(F, f, \nu)$, $.\nu_* fA$ *is a real number, and* $.\nu_0 = 0$, *then*

$$.\nu_* fA \leqq .\mu A.$$

4. Properties of $V(F, f, \nu)$ when ν is an outer measure.

4.1. LEMMA. *If* f *is a function on* σF, ν *is an outer measure on* rng f, $P \in \mathcal{P}(F)$, $A \subset B \subset \sigma F$, $.\nu_* fB < \infty$, *then*

$$\sum \alpha \in P \ .\nu_* f(A\alpha) - .\nu_* fA \leqq \sum \alpha \in P \ .\nu_* f(B\alpha) - .\nu_* fB.$$

Proof. Let $A^* = {}_*fA$, $B^* = {}_*fB$, A' be a ν-measurable set such that $A^* \subset A'$ and $.\nu A^* = .\nu A'$. Since $.\nu B^* = .\nu A^* + .\nu(B^* - A')$, we need only show:

$$\sum \alpha \in P \ .\nu_* f(A\alpha) + .\nu(B^* - A') \leqq \sum \alpha \in P . \nu_* f(B\alpha).$$

We use induction. If P has only one element, the statement reduces to $.\nu_* fA + .\nu(B^* - A') \leqq .\nu_* fB$ and hence is trivially true. Assume then that the inequality holds whenever P has n elements, and suppose that P has $n+1$ elements: $\alpha_0, \cdots, \alpha_n$. Let

$$A_1^* = \bigcup k \in n \ {}_*f(A\alpha_k), \ A_2^* = {}_*f(A\alpha_n), \ B_1^* = \bigcup k \in n \ {}_*f(B\alpha_k), \ B_2^* = {}_*f(B\alpha_n).$$

Note that: $A^* = A_1^* \cup A_2^*$ and $B^* = B_1^* \cup B_2^*$. Next choose ν-measurable sets A_1' and A_2' such that: $A_1^* \subset A_1' \subset A'$, $.\nu A_1^* = .\nu A_1'$, $A_2^* \subset A_2' \subset A'$, $.\nu A_2^* = .\nu A_2'$.

Note that: $B^* - A' \subset (B_1^* - A_1') \cup (B_2^* - A_2')$ and $.\nu A_2^* + .\nu(B_2^* - A_2') = .\nu B_2^*$. We then have:

$$\begin{aligned}
&\sum\alpha \in P\ .\nu_* f(A\alpha) + .\nu(B^* - A') \\
&\qquad \leqq \sum k \in n\ .\nu_* f(A\alpha_k) + .\nu A_2^* + .\nu(B_1^* - A_1') + .\nu(B_2^* - A_2') \\
&\qquad \leqq \sum k \in n\ .\nu_* f(B\alpha_k) + .\nu B_2^* = \sum \alpha \in P\ .\nu_* f(B\alpha).
\end{aligned}$$

4.2. THEOREM. *If $\mu = V(F, f, \nu)$, ν is an outer measure, P is a sequence of elements in $\mathcal{P}(F)$ such that*

$$\lim_{n\to\infty} \sum \alpha \in P_n\ .\nu_* f(B\alpha) = .\mu B < \infty,$$

then, for any $A \subset B$

$$\lim_{n\to\infty} \sum \alpha \in P_n\ .\nu_* f(A\alpha) = .\mu A.$$

Proof. Let $A \subset B$. Given $\delta > 0$, let $2\epsilon = \delta$ and choose N so that whenever $N < n \in \omega$ we have

$$.\mu B < \sum \alpha \in P_n\ .\nu_* f(B\alpha) + \epsilon.$$

Take $Q \in \mathcal{P}(F)$, a refinement of P_n, such that

$$.\mu A < \sum \alpha \in Q\ .\nu_* f(A\alpha) + \epsilon.$$

For each $\alpha \in P_n$, let $S_\alpha = \mathbf{E}\alpha'(\alpha' \in Q$ and $\alpha' \subset \alpha)$. Then, using 4.1:

$$\begin{aligned}
&.\mu A - \sum \alpha \in P_n\ .\nu_* f(A\alpha) \\
&\qquad < \sum \alpha \in Q\ .\nu_* f(A\alpha) - \sum \alpha \in P_n\ .\nu_* f(A\alpha) + \epsilon \\
&\qquad = \sum \alpha \in P_n [\sum \alpha' \in S_\alpha\ .\nu_* f(A\alpha') - .\nu_* f(A\alpha)] + \epsilon \\
&\qquad \leqq \sum \alpha \in P_n [\sum \alpha' \in S_\alpha\ .\nu_* f(B\alpha') - .\nu_* f(B\alpha)] + \epsilon \\
&\qquad \leqq \sum \alpha \in Q\ .\nu_* f(B\alpha) - \sum \alpha \in P_n\ .\nu_* f(B\alpha) + \epsilon < 2\epsilon = \delta.
\end{aligned}$$

4.3. LEMMA. *If $P \in \mathcal{P}(F)$, f is a function on σF, ν an outer measure on* rng f, *$B \subset \sigma F$, then there exists a $C \subset B$ such that:*

$$.\nu_* f(B\alpha) = .\nu_* f(C\alpha) + .\nu_* f((B - C)\alpha)$$

for each $\alpha \in P$ and

$$\sum \alpha \in P\ .\nu_* f(C\alpha) = .\nu_* f C = .\nu_* f B.$$

Proof. Denote the elements of P by $\alpha_0, \cdots, \alpha_{n-1}$. Let $B_k^* = {}_* f(B\alpha_k)$. Choose ν-measurable sets B_k' such that:

$$B_k^* \subset B_k',\quad .\nu B_k^* = .\nu B_k',\quad .\nu \textstyle\bigcup k \in r\ B_k^* = .\nu \bigcup k \in r\ B_k' \qquad \text{for } r \in n + 1.$$

We define

$$C_0^* = B_0^* \quad \text{and} \quad C_k^* = B_k^* - \bigcup i \in k\ B_i'.$$

We then let

$$C_k = B\alpha_k^* f C_k^* \quad \text{and} \quad C = \bigcup k \in n\ C_k.$$

We note first that the C_k^* are disjoint, ${}_*fC_k = C_k^*$, and $C\alpha_k = C_k$. Also:

$$C_k^*{}_*f((B - C)\alpha_k) = 0 \text{ so that } {}_*f((B - C)\alpha_k) = B_k^* \bigcup i \in k\ B_i'.$$

Because of the ν-measurability of the B_i', we conclude

$$.\nu B_k^* = .\nu C_k^* + .\nu_* f((B - C)\alpha_k).$$

On the other hand we easily check by induction on m that:

$$.\nu \bigcup k \in m\ C_k^* = \sum k \in m\ .\nu C_k^* = .\nu \bigcup k \in m\ B_k^* \qquad \text{for } m \in n + 1.$$

4.4. THEOREM. *If $\mu = V(F, f, \nu)$, ν is an outer measure, $.\mu B < \infty$, then there exists an $A \subset B$ such that*:

$$.\mu A = .\nu_* fA = .\nu_* fB.$$

Proof. Let P be a sequence of elements in $\mathcal{P}(F)$ such that, for each $n \in \omega$, P_{n+1} is a refinement of P_n and

$$.\mu B < \sum \alpha \in P_n\ .\nu_* f(B\alpha) + 2^{-n}.$$

By 4.3, we can choose a sequence C such that, for all $n \in \omega$, $C_{n+1} \subset C_n \subset B$ and

$$\sum \alpha \in P_n\ .\nu_* f(C_n\alpha) = .\nu_* fC_n = .\nu_* f(B),$$

$$.\nu_* f(C_n\alpha) = .\nu_* f(C_{n+1}\alpha) + .\nu_* f((C_n - C_{n+1})\alpha) \qquad \text{for } \alpha \in P_{n+1}.$$

Let $A = \bigcap n \in \omega\ C_n$ and complete the proof in 5 steps.

.1 $.\mu A = (\text{by 4.2}) = \lim_{n\to\infty} \sum \alpha \in P_n\ .\nu_* f(A\alpha) \leqq \lim_{n\to\infty} \sum \alpha \in P_n\ .\nu_* f(C_n\alpha) = .\nu_* fB$

.2 $\sum \alpha \in P_{n+1}\ .\nu_* f(C_n\alpha) - \sum \alpha \in P_n\ .\nu_* f(C_n\alpha) \leqq 2^{-n}$.

Proof. For $\alpha \in P_n$, let $S_\alpha = E\alpha'$ ($\alpha' \in P_{n+1}$ and $\alpha' \subset \alpha$). Then, using 4.1:

$$\begin{aligned} &\sum \alpha \in P_n [\sum \alpha' \in S_\alpha\ .\nu_* f(C_n\alpha\alpha') - .\nu_* f(C_n\alpha)] \\ &\qquad \leqq \sum \alpha \in P_n [\sum \alpha' \in S_\alpha\ .\nu_* f(B\alpha\alpha') - .\nu_* f(B\alpha)] \\ &\qquad = \sum \alpha \in P_{n+1}\ .\nu_* f(B\alpha) - \sum \alpha \in P_n\ .\nu_* f(B\alpha) \leqq 2^{-n}. \end{aligned}$$

.3 $.\nu_* f(C_n - C_{n+1}) \leqq 2^{-n}$.

Proof.

$$\begin{aligned} .\nu_* f(C_n - C_{n+1}) &\leqq \sum \alpha \in P_{n+1} . \nu_* f((C_n - C_{n+1})\alpha) \\ &= \sum \alpha \in P_{n+1}\ .\nu_* f(C_n\alpha) - \sum \alpha \in P_{n+1}\ .\nu_* f(C_{n+1}\alpha) \\ &= \sum \alpha \in P_{n+1} . \nu_* f(C_n\alpha) - \sum \alpha \in P_n\ .\nu_* f(C_n\alpha) \leqq 2^{-n}. \end{aligned}$$

.4 $.\mu(C_n - C_{n+1}) \leqq .\nu_* f(C_n - C_{n+1}) + 2^{-n} \leqq 2^{-n+1}$.

Proof. Let $S_\alpha^{(m)} = \mathbf{E}\alpha'$ ($\alpha' \in P_{n+m}$ and $\alpha' \subset \alpha$). Then

$$\begin{aligned}
&\sum \alpha \in P_{n+m} \; .\nu_* f((C_n - C_{n+1})\alpha) \\
&\quad = \sum \alpha \in P_n \; \sum \alpha' \in S_\alpha^{(m)} \; .\nu_* f((C_n - C_{n+1})\alpha') \leqq \text{(by 4.1)} \\
&\quad \leqq \sum \alpha \in P_n [.\nu_* f((C_n - C_{n+1})\alpha) + \sum \alpha' \in S_\alpha^{(m)} \; .\nu_* f(B\alpha') - .\nu_* f(B\alpha)] \\
&\quad = .\nu_* f(C_n - C_{n+1}) + \sum \alpha \in P_{n+m} \; .\nu_* f(B\alpha) - \sum \alpha \in P_n \; .\nu_* f(B\alpha) \\
&\quad \leqq .\nu_* f(C_n - C_{n+1}) + 2^{-n}.
\end{aligned}$$

Using 4.2 and letting $m \to \infty$ we have the desired conclusion.

.5 $.\nu_* fB = .\mu A$.

Proof. $.\mu A \geqq .\mu C_n - \sum m \in \omega \; .\mu(C_{n+m} - C_{n+m+1}) \geqq .\nu_* fB - \sum m \in \omega \; 2^{-n-m+1} \geqq .\nu_* fB - 2^{-n+2}$.

Letting $n \to \infty$ and using Step 1, we have the desired result.

4.5. Theorem. *If $\mu = V(F, f, \nu)$, ν is an outer measure, A is ν-measurable, then $^*fA$ is μ-measurable.*

Proof. Let $.\mu T < \infty$. Choose a sequence P of elements in $\mathcal{P}(F)$ such that $.\mu T = \lim_{n\to\infty} \sum \alpha \in P_n \; .\nu_* f(T\alpha)$. Letting $B = {}^*fA$, we see by 4.2 that:

$$\begin{aligned}
.\mu(TB) + .\mu(T - B) &= \lim_{n\to\infty} \sum \alpha \in P_n [.\nu_* f(TB\alpha) + .\nu_* f((T - B)\alpha)] \\
&= \lim_{n\to\infty} \sum \alpha \in P_n \; .\nu_* f(T\alpha) = .\mu T.
\end{aligned}$$

The next to last equality is due to the ν-measurability of $A = {}_*fB$ and the fact that ${}_*f(TB\alpha) \subset A$ and ${}_*f((T - B)\alpha) \subset {}_*f(T\alpha) - A$.

4.6. Theorem. *If $\mu = V(F, f, \nu)$, ν is an outer measure, $.\mu A = .\nu_* fA < \infty$, ${}_*fA$ is ν-measurable, $\alpha \in F$, then ${}_*f(A\alpha)$ is ν-measurable.*

Proof. Choose a ν-measurable set B such that:

$$ {}_*f(A\alpha) \subset B \subset {}_*fA \text{ and } .\nu_* f(A\alpha) = .\nu B. $$

Since $.\mu A = .\nu_* fA$, we must have: $.\nu_* fA = .\nu_* f(A\alpha) + .\nu_* f(A - \alpha)$. Since B is ν-measurable, we also have:

$$\begin{aligned}
.\nu_* f(A - \alpha) &= .\nu({}_*f(A - \alpha)B) + .\nu({}_*f(A - \alpha) - B), \\
.\nu_* fA &= .\nu B + .\nu({}_*fA - B) = .\nu_* f(A\alpha) + .\nu({}_*fA - B).
\end{aligned}$$

Now, ${}_*f(A - \alpha) - B = {}_*fA - B$. Hence, comparing the two expressions above for $.\nu_* fA$, we conclude that $.\nu({}_*f(A - \alpha)B) = 0$ and hence $.\nu(B - {}_*f(A\alpha)) \leqq .\nu({}_*f(A - \alpha)B) = 0$.

5. **Properties of $V(F, f, \nu)$ when F is fine.**

5.1. Definition. P is a *mesh* of F if and only if P is a sequence of elements in $\mathcal{P}(F)$ such that: P_{n+1} is a refinement of P_n and, for every $x\in\sigma F$, $y\in\sigma F$, $x\neq y$, there exists an $n\in\omega$, $\alpha\in P_n$, $\beta\in P_n$, $\alpha\beta=0$ such that $x\in\alpha$ and $y\in\beta$.

5.2. DEFINITION. F is *fine* if and only if: $F\neq 0$; for every $A\in F$ there is a $P\in\mathcal{P}(F)$ with $A\in P$; for every P' and P'' in $\mathcal{P}(F)$ there is a mesh P of F such that for every $n\in\omega$, P_n is a refinement of both P' and P''.

5.3. DEFINITION. $N(f, A, y)$ denotes the number of points in $A^*f\{y\}$ if this set is finite and ∞ otherwise.

5.4. DEFINITION. $Cr(A, y)$ is 1 if $y\in A$ and 0 otherwise.

5.5. LEMMA. *If f is a function on σF, P is a mesh of F, then*

$$N(f, A, y) = \sup_{n\in\omega} \sum\alpha \in P_n\ Cr({}_*f(A\alpha), y).$$

Proof. Clearly, for every $n\in\omega$:

$$\sum\alpha \in P_n\ Cr({}_*f(A\alpha), y) \leqq N(f, A, y).$$

Let $m\leqq N(f, A, y)$, $m\in\omega$. Take m distinct points $x_1, \cdots, x_m$ in A with $f(x_i)=y$. Then choose $n\in\omega$ such that no α in P_n contains more than one of these points. We must therefore have

$$m \leqq \sum\alpha \in P_n\ Cr({}_*f(A\alpha), y).$$

5.6. THEOREM. *If $\mu=V(F, f, \nu)$, F is fine, ${}_*f(A\alpha)$ is ν-measurable for every $\alpha\in F$, then $.\mu A=\int N(f, A, y)d\nu y$.*

Proof. Let P be a mesh of F such that

$$.\mu A = \lim_{n\to\infty} \sum\alpha \in P_n\ .\nu_*f(A\alpha).$$

Since ${}_*f(A\alpha)$ is ν-measurable for every $\alpha\in P_n$, we see from 5.5 that $N(f, A, y)$ is ν-measurable in y and:

$$\begin{aligned} .\mu A &= \lim_{n\to\infty} \sum\alpha \in P_n \int Cr({}_*f(A\alpha), y)d\nu y \\ &= \int \lim_{n\to\infty} \sum\alpha \in P_n\ Cr({}_*f(A\alpha), y)d\nu y = \int N(f, A, y)d\nu y. \end{aligned}$$

5.7. COROLLARY. *If $\mu=V(F, f, \nu)$, F is fine, ν is an outer measure, $.\mu A = .\nu_*fA<\infty$, ${}_*fA$ is ν-measurable, then $N(f, A, y)\leqq 1$ for ν-almost all y.*

Proof. Follows immediately from 4.6 and 5.6.

5.8. THEOREM. *If $\mu=V(F, f, \nu)$, F is fine, ν is an outer measure, $.\mu A = .\nu_*fA<\infty$, ${}_*fA$ is ν-measurable, then A is μ-measurable.*

Proof. Let P be a mesh of F. For each $n\in\omega$ and $\alpha\in P_n$, let $C_\alpha=\alpha^*f_*f(A\alpha)$ and $D_n=\bigcup\alpha\in P_n\ C_\alpha$.

Since ${}_*f(A\alpha)$ is ν-measurable, then by 3.3 and 4.5 we see that the C_α and hence D_n are μ-measurable. Also: $D_n\alpha = C_\alpha$, ${}_*f(D_n\alpha) = {}_*f(A\alpha)$, and $A \subset D_n$.

Let $B = \bigcap n \in \omega\ D_n$. Then B is μ-measurable, $A \subset B$, and for each $\alpha \in P_n$: ${}_*f(A\alpha) \subset {}_*f(B\alpha) \subset {}_*f(D_n\alpha) = {}_*f(A\alpha)$. Thus, by 5.5 and 5.7, $N(f, B, y) = N(f, A, y) \leqq 1$ for ν-almost all y. Hence $N(f, B-A, y) = 0$ for ν-almost all y, so that $.\nu_*f(B-A) = 0$. Therefore $.\mu(B-A) = 0$ and A is μ-measurable.

Remark. We are unable to prove, in general, that $\mu = V(F, f, \nu)$ is an outer measure when F is fine and ν is an outer measure. Since we are interested in the converse of Theorem 5.8, we use the same conditions to prove both results. It may be of interest to note, however, that the two are not naturally related. For example, if F is the family of all intervals on the real line and ν is an outer measure, then μ is also an outer measure. On the other hand, it is quite clear that the converse of 5.8 is not true for all functions f and all outer measures ν on rng f.

6. **Properties of $V(F, f, \nu)$ when σF is a topological space.** Throughout this section we assume a fixed topology for σF. All the topological concepts refer to this topology.

6.1. Definition. For P a mesh of F, we denote by $Sq(P)$ the set of sequences α such that $0 \neq \alpha_{n+1} \subset \alpha_n \in P_n$, for every $n \in \omega$.

6.2. Definition. P is a *topological mesh* of F if and only if: P is a mesh of F; for every $x \in \sigma F$ and neighborhood U of x, there is an $n \in \omega$ and $\alpha \in P_n$ such that $x \in \alpha \subset U$.

6.3. Definition. F is *almost complete* if and only if: F is fine; there exists a topological mesh P of F; $\mathbf{E}\alpha$ $(\alpha \in Sq(P),\ \bigcap n \in \omega\ \alpha_n = 0)$ is countable for all topological meshes P of F.

6.4. Definition. $\mathfrak{M}(F, f) = \mathbf{E}\nu$ (ν is an outer measure on rng f; dmn $f \subset \sigma F$; ${}_*f\alpha$ is ν-measurable for all $\alpha \in F$; ${}^*f\{y\}$ is closed for ν-almost all y).

6.5. Lemma. *If P is a mesh of F, for every $n \in \omega: 0 \neq A_{n+1} \subset A_n \subset \sigma F$, then for some $\alpha \in Sq(P)$ and every $n \in \omega: A_n\alpha_n \neq 0$.*

Proof. We define α by recursion. Since P_n is finite and $\sigma P_n = \sigma F$ for every $n \in \omega$, we can choose $\alpha_0 \in P_0$ so that $\alpha_0 A_k \neq 0$ for an infinite number of, and hence all, $k \in \omega$. Having chosen α_n so that $\alpha_n A_k \neq 0$ for all $k \in \omega$, we take $\alpha_{n+1} \in P_{n+1}$ so that $\alpha_{n+1} \subset \alpha_n$ and $\alpha_{n+1}A_k \neq 0$ for all $k \in \omega$.

6.6. Lemma. *If F is almost complete, $A \in F$, P is a topological mesh of F, and $S = \mathbf{E}\alpha$ $(\alpha \in Sq(P);\ A\alpha_n \neq 0$ for all $n \in \omega;\ \bigcap n \in \omega\ (A\alpha_n) = 0)$, then S is countable.*

Proof. Let $A \in Q \in \mathcal{P}(F)$ and choose P' a mesh of F such that, for every $n \in \omega$, P_n' is a refinement of both Q and P_n. Let $S' = \mathbf{E}\alpha'$ $(\alpha' \in Sq(P');\ A\alpha_n' \neq 0$ for all $n \in \omega;\ \bigcap n \in \omega\ (A\alpha_n') = 0)$. Since, for $\alpha_n' \in P_n'$, $A\alpha_n' \neq 0$ implies $A\alpha_n' = \alpha_n'$, we see that S' is countable. By 6.5, for each $\alpha \in S$ there exists $\alpha' \in S'$ such that $\alpha_n'\alpha_n \neq 0$ and hence $\alpha_n' \subset \alpha_n$ for all $n \in \omega$. This means that for two different α's in S, the corresponding α''s in S' are also different. Thus S is also countable.

6.7. Lemma. *If* $B \subset {}_*f\alpha$, $A = \alpha^*fB$, $\alpha' \subset \alpha$, *then* ${}_*f(A\alpha') = {}_*fA{}_*f\alpha' = B{}_*f\alpha'$.

6.8. Lemma. *If F has a topological mesh P,* $\nu \in \mathfrak{M}(F, f)$, $\mu = V(F, f, \nu)$, $x \in \sigma F$, *then* $\{x\}$ *is μ-measurable and* ${}_*f\{x\}$ *is ν-measurable.*

Proof. Choose a sequence α so that for every $n \in \omega$, $\alpha_n \in P_n$ and $\{x\} = \bigcap n \in \omega\ \alpha_n$. By 3.3 the α_n are μ-measurable and hence $\{x\}$ is μ-measurable. Let $A = \bigcap n \in \omega\ {}_*f\alpha_n$. Then A is ν-measurable. Let $y = f(x)$ and $A^* = \mathbf{E}z(z \in A$ and $^*f\{z\}$ is closed). We note that: $.\nu(A - A^*) = 0$; if $z \in A^*$, then x is in the closure of $^*f\{z\}$, hence $x \in {}^*f\{z\}$ and $y = z$. Thus, if $.\nu\{y\} \neq 0$, then $\{y\} = A^*$ and in either case $\{y\}$ is ν-measurable.

6.9. Lemma. *If P is a mesh of F,* $\alpha \in Sq(P)$, $\lim_{n\to\infty} \sum \alpha' \in P_n\ .\nu_*f(A\alpha') < \infty$, *$\nu$ is an outer measure, and* $.\nu\{y\} = 0$ *for all* $y \in {}_*fA$, *then* $\lim_{n\to\infty} .\nu_*f(A\alpha_n) = 0$.

Proof. Let $B = \bigcap n \in \omega\ \alpha_n$. Then B contains at most one point and $.\nu_*f(AB) = 0$. Since ${}_*f(A\alpha_n) = \bigcup k \in \omega\ {}_*f(A\alpha_n - \alpha_{n+k}) \cup {}_*f(AB)$, we must have

$$\lim_{k\to\infty} .\nu_*f(A\alpha_n - \alpha_{n+k}) = .\nu_*f(A\alpha_n)$$

for all $n \in \omega$. Now, given $\epsilon > 0$ we can choose n so large that for all $k \in \omega$:

$$.\nu_*f(A\alpha_n - \alpha_{n+k}) + .\nu_*f(A\alpha_{n+k}) \leqq .\nu_*f(A\alpha_n) + \epsilon.$$

Hence: $\lim_{k\to\infty} .\nu_*f(A\alpha_{n+k}) \leqq .\nu_*f(A\alpha_n) - \lim_{k\to\infty} .\nu_*f(A\alpha_n - \alpha_{n+k}) + \epsilon = \epsilon$. Since ϵ is arbitrary we have the desired conclusion.

6.10. Theorem. *If F is almost complete,* $\nu \in \mathfrak{M}(F, f)$, $\mu = V(F, f, \nu)$, *P is a topological mesh of F, then for any* $A \subset \sigma F$ *there is a μ-measurable A' such that* $A \subset A'$, ${}_*fA'$ *is ν-measurable, and*

$$.\mu A = .\mu A' = \lim_{n\to\infty} \sum \alpha \in P_n\ .\nu_*f(A'\alpha).$$

Proof. If $\lim_{n\to\infty} \sum \alpha \in P_n\ .\nu_*f(A\alpha) = \infty$, take $A' = \sigma F$. Otherwise, set $A_1 = \mathbf{E}x\ (x \in A$ and $.\nu_*f\{x\} > 0)$. Then A_1 is countable and, by 6.8, ${}_*fA_1$ is ν-measurable. Thus we need only prove the theorem for $A_2 = A - A_1$. We define the sets $B_\alpha^{(n)}$, $C_\alpha^{(n)}$, D_n by recursion on n. For $\alpha \in P_0$, let $B_\alpha^{(0)}$ be ν-measurable, ${}_*f(A_2\alpha) \subset B_\alpha^{(0)} \subset {}_*f\alpha$, and $.\nu_*f(A_2\alpha) = .\nu B_\alpha^{(0)}$. Then we set

$$C_\alpha^{(0)} = \alpha\, {}^*fB_\alpha^{(0)} \quad \text{and} \quad D_0 = \bigcup \alpha \in P_0\, C_\alpha^{(0)}.$$

Using 6.7, for $\alpha \in P_{n+1}$, let $B_\alpha^{(n+1)}$ be ν-measurable, ${}_*f(A_2\alpha) \subset B_\alpha^{(n+1)} \subset {}_*f(D_n\alpha)$, and $.\nu_*f(A_2\alpha) = .\nu B_\alpha^{(n+1)}$, then we set:

$$C_\alpha^{(n+1)} = \alpha\, {}^*fB_\alpha^{(n+1)} \quad \text{and} \quad D_{n+1} = \bigcup\alpha \in P_{n+1}\, C_\alpha^{(n+1)}.$$

Finally, let $A' = \bigcap n\in\omega\ D_n$. Clearly, $A_2 \subset A'$, A' is μ-measurable since the $C_\alpha^{(n)}$ are μ-measurable by 3.3 and 4.5, and for $n\in\omega$ and $\alpha\in P_n$:

$$.\nu_* f(A_2\alpha) \leqq .\nu_* f(A'\alpha) \leqq .\nu_* f(D_n\alpha) = .\nu B_\alpha^{(n)} = .\nu_* f(A_2\alpha).$$

Now, we wish to show that ${}_*f(A'\beta)$ is ν-measurable for any $\beta\in F$ and $.\mu A_2 = .\mu A'$. To this end, we let $B' = \bigcap n\in\omega\ {}_*f(D_n\beta)$, $B^* = \mathbf{E}y$ ($y\in B'$ and ${}^*f\{y\}$ is closed), and, for $\alpha\in Sq(P)$, $G_\alpha = \mathbf{E}y$ ($y\in B^*$ and, for $n\in\omega$, $\beta D_n\alpha_n\ {}^*f\{y\}\neq 0$), and complete the proof in 7 steps.

Part 1. B' is ν-measurable, $.\nu(B'-B^*)=0$, and B^* is ν-measurable.

Proof. $\nu\in\mathfrak{M}(F,f)$ and ${}_*f(D_n\beta)$ is ν-measurable.

Part 2. $B^* = \bigcup\alpha\in Sq(P)\ G_\alpha$.

Proof. Clearly $G_\alpha\subset B^*$ for all $\alpha\in Sq(P)$. If $y\in B^*$, then $D_n\beta{}^*f\{y\}\neq 0$ for all $n\in\omega$ and hence by 6.5 there is an $\alpha\in Sq(P)$ such that $\beta D_n\alpha_n\ {}^*f\{y\}\neq 0$, i.e., $y\in G_\alpha$.

Part 3. If $\alpha\in Sq(P)$ and $\bigcap n\in\omega\ (\beta\alpha_n)\neq 0$, then $G_\alpha\subset {}_*f(A'\beta)$.

Proof. Let $x\in\bigcap n\in\omega\ (\beta\alpha_n)$ and $y\in G_\alpha$. Then x is in the closure of ${}^*f\{y\}$, hence $x\in{}^*f\{y\}$. Since $D_n\alpha_n\ {}^*f\{y\}\neq 0$ we have $\alpha_n\ {}^*f\{y\}\subset C_{\alpha_n}^{(n)}\subset D_n$. Thus $x\in D_n$ for all $n\in\omega$ and therefore $x\in A'\beta{}^*f\{y\}$, so that $y\in{}_*f(A'\beta)$.

Part 4. $\mathbf{E}\alpha$ ($\alpha\in Sq(P)$; $G_\alpha\neq 0$; $\bigcap n\in\omega\ (\beta\alpha_n)=0$) is countable.

Proof. Follows immediately from 6.6.

Part 5. $.\nu G_\alpha = 0$ for all $\alpha\in Sq(P)$.

Proof. $G_\alpha\subset{}_*f(D_n\alpha_n)$ for all $n\in\omega$, and hence by 6.9

$$.\nu G_\alpha \leqq \lim_{n\to\infty}\ .\nu_* f(D_n\alpha_n) = \lim_{n\to\infty}\ .\nu_* f(A_2\alpha_n) = 0.$$

Part 6. $.\nu(B^* - {}_*f(A'\beta)) = 0$ and ${}_*f(A'\beta)$ is ν-measurable.

Proof. Clearly ${}_*f(A'\beta)\subset B^*\cup(B'-B^*)$. Let S be the countable set of Part 4. Then, in view of Parts 2 and 3:

$$B^* - {}_*f(A'\beta) \subset \bigcup\alpha\in S\, G_\alpha$$

and by Part 5:

$$.\nu(B^* - {}_*f(A'\beta)) \leqq \sum\alpha\in S\ .\nu G_\alpha = 0.$$

Part 7. $.\mu A_2 = .\mu A' = \lim_{n\to\infty}\sum\alpha\in P_n\ .\nu_* f(A'\alpha)$.

Proof. By 5.5 and 5.6 we have:

$$.\mu A_2 \leqq .\mu A' = \int N(f, A', y)d\nu y = \lim_{n\to\infty}\ \sum\alpha\in P_n\ .\nu_* f(A'\alpha)$$

$$= \lim_{n\to\infty}\ \sum\alpha\in P_n\ .\nu_* f(A_2\alpha) \leqq .\mu A_2.$$

6.11. Corollary. *If F is almost complete, $\nu\in\mathfrak{M}(F,f)$, $\mu = V(F,f,\nu)$, then μ is an outer measure.*

6.12. THEOREM. *If F is almost complete, $\nu \in \mathfrak{M}(F, f)$, $\mu = V(F, f, \nu)$, $.\mu A = .\nu_* fA < \infty$, then A is μ-measurable if and only if $_*fA$ is ν-measurable.*

Proof. *If A is μ-measurable, by 6.10 we can take A' μ-measurable so that: $A \subset A'$, $.\mu A' = \mu A$ and $_*fA'$ is ν-measurable. Since A is μ-measurable*:

$$.\nu(_*fA' - {_*fA}) \leqq .\nu_* f(A' - A) \leqq .\mu(A' - A) = .\mu A' - .\mu A = 0.$$

The converse is given by 5.8.

6.13. THEOREM. *If F is almost complete, $\nu \in \mathfrak{M}(F, f)$, $\mu = V(F, f, \nu)$, A is μ-measurable and, for some $B \subset A$, $.\nu_* fB = .\nu_* fA$ and $.\mu B < \infty$, then $_*fA$ is ν-measurable.*

Proof. By 4.4 choose $C \subset B$ so that $.\mu C = .\nu_* fC = .\nu_* fA$. By 6.10, let C' be μ-measurable, $C \subset C'$, $.\mu C = .\mu C' = .\nu_* fC'$, and $_*fC'$ is ν-measurable. Then:

$$\begin{aligned} .\nu(_*fC' - {_*fA}) &\leqq .\nu_* f(C' - A) \leqq .\mu(C' - A) = .\mu C' - .\mu(C'A) \\ &\leqq .\mu C' - .\mu C = 0. \end{aligned}$$

Thus $_*fC'{_*fA}$ is ν-measurable and $.\nu(_*fC'{_*fA}) \geqq .\nu_* fC = .\nu_* fA$ so that $_*fA$ is also ν-measurable.

6.14. COROLLARY. *If F is almost complete, $\nu \in \mathfrak{M}(F, f)$, $\mu = V(F, f, \nu)$, A is μ-measurable, and for every $B \subset A$ with $.\mu B > 0$ there is a $C \subset B$ with $0 < .\mu C < \infty$, then $_*fA$ is ν-measurable.*

6.15. REMARK. In view of 6.14 above, we next turn our attention to sets which have no subset of finite positive variational measure, and see what conditions this imposes on f and ν.

7. **Sets having no subsets of finite positive variational measure.** Throughout this section we assume that F is almost complete, $\nu \in \mathfrak{M}(F, f)$, $\mu = V(F, f, \nu)$, $T \subset \sigma F$, $0 < .\nu_* fT < \infty$, and, for every $T' \subset T$, if $.\mu T' < \infty$, then $.\mu T' = 0$.

Our aim is to show that f and ν must satisfy the conditions stated in Theorems 7.3 and 7.4. To this end, we let P be a topological mesh of F such that: $P_0 = \{\sigma F\}$ and, for each $n \in \omega$ and $\alpha \in P_n$, there are at most two sets β_1, β_2 in P_{n+1} with $\beta_1 \cup \beta_2 \subset \alpha$. This is no added restriction on σF since, starting with any topological mesh P' of F, we can define a P with the above property by regrouping the elements in $\bigcup n \in \omega\, P_n'$. We choose such a P only to simplify the formulas involved in the construction of certain sets.

7.1. LEMMA. *There exist two sets A and B such that:*
(1) *$AB = 0$ and A, B are μ-measurable.*
(2) *$_*fA = {_*fB}$ and $_*fA$ is ν-measurable.*
(3) *$.\nu_* f(T - (A \cup B)) = 0$.*
(4) *$.\nu_* f(A\alpha) = .\nu_* f(TA\alpha)$ and $.\nu_* f(B\alpha) = .\nu_* f(TB\alpha)$ for $\alpha \in P_1$.*
(5) *For any T', $\sum \alpha \in P_1\ .\nu_* f(T'A\alpha) = .\nu_* f(T'A)$ and $\sum \alpha \in P_1 .\nu_* f(T'B\alpha) = .\nu_* f(T'B)$.*

(6) *If* $A^*f\{y\}\neq 0$, *then for some* $k\in\omega: \alpha\in P_k$, $\alpha A^*f\{y\}\neq 0$ *imply* $\alpha^*f\{y\}\subset A$; *and* $\alpha\in P_k$, $\alpha B^*f\{y\}\neq 0$ *imply* $\alpha^*f\{y\}\subset B$.

Proof. Let M^* be ν-measurable, ${}_*fT\subset M^*$, $.\nu_*fT=.\nu M^*$. We shall define the sets M_α, for $\alpha\in P_n$, A_n^*, B_n^* by recursion on n. For $\alpha\in P_0$ let $M_\alpha=M^*$, $A_0^*=0$, $B_0^*=0$. Next, if $\alpha'\in P_n$, let α and β be in P_{n+1} with $\alpha\cup\beta\subset\alpha'$. We choose M_α and M_β ν-measurable and such that

$$
\begin{aligned}
&{}_*f(T\alpha - \cup k\in n+1\ (A_k^*\cup B_k^*))\subset M_\alpha\subset (M_{\alpha'} - \cup k\in n+1\ {}_*fA_k^*){}_*f\alpha,\\
&{}_*f(T\beta - \cup k\in n+1\ (A_k^*\cup B_k^*))\subset M_\beta\subset (M_{\alpha'} - \cup k\in n+1\ {}_*fA_k^*){}_*f\beta,\\
&.\nu_*f(T\alpha - \cup k\in n+1\ (A_k^*\cup B_k^*)) = .\nu M_\alpha,\\
&.\nu_*f(T\beta - \cup k\in n+1\ (A_k^*\cup B_k^*)) = .\nu M_\beta.
\end{aligned}
$$

Then we set

$$A_{n+1}^{(\alpha')} = \alpha^*f(M_\alpha M_\beta),\qquad B_{n+1}^{(\alpha')} = \beta^*f(M_\alpha M_\beta)\qquad\text{if } \alpha\beta = 0$$

and $A_{n+1}^{(\alpha')}=B_{n+1}^{(\alpha')}=0$ if $\alpha=\beta=\alpha'$.

Finally, we let

$$
\begin{aligned}
A_{n+1}^* &= \cup\alpha'\in P_nA_{n+1}^{(\alpha')}, & B_{n+1}^* &= \cup\,\alpha'\in P_nB_{n+1}^{(\alpha')}.\\
A &= \cup n\in\omega\ A_n^*, & B &= \cup n\in\omega\ B_n^*.
\end{aligned}
$$

Now, we observe the following facts.

(a) ${}_*fA_n^*={}_*fB_n^*$ for all $n\in\omega$.

(b) $A_n^*\cup k\in n\ (A_k^*\cup B_k^*)=B_n^*\cup k\in n\ (A_k^*\cup B_k^*)=A_n^*B_n^*=0$ for all $n\in\omega$.

(c) A_n^*, B_n^* are μ-measurable and ${}_*fA_n^*$ is ν-measurable for all $n\in\omega$.

(d) ${}_*f(A_n^*\alpha)\ {}_*f(A_n^*\beta)={}_*f(B_n^*\alpha)\ {}_*f(B_n^*\beta)=0$ for all $n\in\omega$ and α, $\beta\in P_n$, $\alpha\beta=0$.

Proof. It is trivially true for $n=0$. So, let $1\leqq n\in\omega$. In view of the choice of P, we can find α', $\beta'\in P_k$ and $\gamma\in P_{k-1}$, for some $k\leqq n$, such that $\alpha\subset\alpha'$, $\beta\subset\beta'$, $\alpha'\beta'=0$, $\alpha'\cup\beta'\subset\gamma$. If $k=n$, then $\alpha'=\alpha$, $\beta'=\beta$ and the statement is clearly true. If $k\in n$, then ${}_*f(B_n^*\alpha)\cup{}_*f(A_n^*\alpha)\subset M_\alpha\subset M_{\alpha'}-{}_*fA_k^{(\gamma)}=M_{\alpha'}-M_{\beta'}$, ${}_*f(B_n^*\beta)\cup{}_*f(A_n^*\beta)\subset M_\beta\subset M_{\beta'}$.

(e) $.\nu_*f(TA_n^*\alpha)=.\nu_*f(A_n^*\alpha)$ and $.\nu_*f(TB_n^*\alpha)=.\nu_*f(B_n^*\alpha)$ for $n\in\omega$, $a\in P_j$, $j\in n+1$.

Proof. Clearly true for $n=0$. In view of (c) and (d) we may assume $n=m+1$, $\alpha'\in P_m$, $\alpha\in P_{m+1}$, $\beta\in P_{m+1}$, $\alpha\beta=0$, and $\alpha\cup\beta\subset\alpha'$. Suppose $A_{m+1}^*\alpha=A_{m+1}^{(\alpha')}=\alpha^*f(M_\alpha M_\beta)$. Let $T_1=T\alpha-\cup k\in m+1\ (A_k^*\cup B_k^*)$. Then:

$$
\begin{aligned}
.\nu_*fT_1 &= .\nu({}_*fT_1M_\beta) + .\nu({}_*fT_1 - M_\beta) = .\nu_*f(TA_{m+1}^*\alpha) + .\nu({}_*fT_1 - M_\beta),\\
.\nu M_\alpha &= .\nu(M_\alpha M_\beta) + .\nu(M_\alpha - M_\beta).
\end{aligned}
$$

Since ${}_*fT_1\subset M_\alpha$ and $.\nu_*fT_1=.\nu M_\alpha$, we must have:

$$.\nu_* f(T\overset{*}{A}_{m+1}\alpha) = .\nu(M_\alpha M_\beta) = .\nu f(\overset{*}{A}_{m+1}\alpha).$$

Similarly for the other equality.

(f) $.\nu_* f(A\alpha) = .\nu_* f(TA\alpha)$ and $.\nu_* f(B\alpha) = .\nu_* f(TB\alpha)$ for $\alpha \in P_1$.

Proof. Since the sets ${}_*fA_n^*$ are ν-measurable and disjoint, $A_0^* = B_0^* = 0$, we get from (e):

$$.\nu_* f(A\alpha) = \sum n \in \omega\ .\nu_* f(A_n^*\alpha) = \sum n \in \omega\ .\nu_* f(TA_n^*\alpha) = .\nu_* f(TA\alpha) \text{ for } \alpha \in P_1.$$

(g) If $T' = T - \bigcup k \in n+1\ (A_k^* \cup B_k^*)$, then $.\nu_* fT' = \sum \alpha \in P_n\ .\nu_* f(T'\alpha)$.

Proof. It is clearly true for $n = 0$. So let $1 \leqq n \in \omega$, $\alpha\beta = 0$, $\alpha, \beta \in P_n$. Then for some $k \leqq n$, $\alpha', \beta' \in P_k$, $\gamma \in P_{k-1}$ we must have $\alpha \subset \alpha'$, $\beta \subset \beta'$, $\alpha'\beta' = 0$, $\alpha' \cup \beta' \subset \gamma$. Then: ${}_*f(T'\alpha) \subset {}_*f(T'\alpha') \subset M_{\alpha'} - M_{\beta'}$ and ${}_*f(T'\beta) \subset {}_*f(T'\beta') \subset M_{\beta'}$. Since $M_{\alpha'}$ and $M_{\beta'}$ are ν-measurable we have the desired result.

(h) $.\nu_* f(T - (A \cup B)) = 0$.

Proof. Let $T' = T - (A \cup B)$. Then by part (g) and 6.10 we see that $.\mu T' = .\nu_* fT' < \infty$. Hence, $\nu_* fT' = 0$.

(i) For any T', $\sum \alpha \in P_1\ .\nu_* f(T'A\alpha) = .\nu_* f(T'A)$ and $\sum \alpha \in P_1\ .\nu_* f(T'B\alpha) = .\nu_* f(T'B)$.

Proof. From Part (d), it follows that

$$\sum \alpha \in P_n\ .\nu_* f(T'A_n^*\alpha) = .\nu_* f(T'A_n^*).$$

Hence, since $A_0^* = B_0^* = 0$, we have:

$$\begin{aligned}
&\sum \alpha \in P_1\ .\nu_* f(T'A\alpha) \\
&\quad = \sum \alpha \in P_1\ \sum n \in \omega\ .\nu_* f(T'A_n^*\alpha) = \sum n \in \omega\ \sum \alpha \in P_1\ .\nu_* f(T'A_n^*\alpha) \\
&\quad = \sum n \in \omega\ .\nu_* f(T'A_n^*) = .\nu_* f(T'A).
\end{aligned}$$

Similarly for B.

(j) If $A^*f\{y\} \neq 0$, then for some $k \in \omega: \alpha \in P_k$, $\alpha A^*f\{y\} \neq 0$ imply $\alpha^* f\{y\} \subset A$; and $\alpha \in P_k$, $\alpha B^* f\{y\} \neq 0$ imply $\alpha^* f\{y\} \subset B$.

Proof. If $A^*f\{y\} \neq 0$, then there is one and only one $k \in \omega$ such that $A_k^*\ {}^*f\{y\} \neq 0$. Since ${}_*fA_k^* = {}_*fB_k^*$, we also have $B_k^*\ {}^*f\{y\} \neq 0$. Now, let $\alpha \in P_k$ and $\alpha A_k^*\ {}^*f\{y\} \neq 0$. Since $A_k^*\alpha = \alpha^* f_* f(A_k^*\alpha)$, we have $\alpha^* f\{y\} \subset A_k^*\alpha \subset A$. Similarly for B.

7.2. Lemma. *There exists a sequence G such that, for every $n \in \omega$, G_{n+1} is a refinement of G_n, G_n has 2^n elements, and*

(1) *If $A, B \in G_n$, $A \neq B$, then $AB = 0$ and A is μ-measurable.*

(2) *If $A, B \in G_n$, then ${}_*fA = {}_*fB$ and ${}_*fA$ is ν-measurable.*

(3) $.\nu_* f(T - \sigma G_n) = 0$.

(4) *If $A \in G_n$, $\alpha \in P_n$, then $.\nu_* f(A\alpha) = .\nu_* f(TA\alpha)$.*

(5) *If $A \in G_n$, then $\sum \alpha \in P_n\ .\nu_* f(T'A\alpha) = .\nu_* f(T'A)$, for any T'.*

(6) *If $A \in G_n$ and $A^*f\{y\} \neq 0$, then for some $k \in \omega: \alpha \in P_k$ and $\alpha A^*f\{y\} \neq 0$ imply $\alpha^* f\{y\} \subset A$.*

Proof. We define G by recursion. Let M be ν-measurable, ${}_*fT\subset M$ and $.\nu_*fT=.\nu M$. We set $G_0=\{{}^*fM\}$. For any $n\in\omega$, if G_n is defined, $A\in G_n$ and $\alpha\in P_n$, we define $C_\alpha(A)$ and $C'_\alpha(A)$ as follows: if $.\nu_*f(TA\alpha)=0$, let $C_\alpha(A)=C'_\alpha(A)=0$; if $.\nu_*f(TA\alpha)>0$ let $C_\alpha(A)$ and $C'_\alpha(A)$ be two sets satisfying all the conditions of Lemma 7.1 with 'T' replaced by '$TA\alpha$' and 'P' replaced by 'P'', where P' is the sequence defined by $P'_k=\mathbf{E}\alpha'$ ($\alpha'\in P_{n+k}$ and $\alpha'\subset\alpha$), and such that: $C_\alpha(A)\cup C'_\alpha(A)\subset A\alpha$. Then we set:

$$D(A) = \bigcup\alpha \in P_n\, C_\alpha(A) \quad\text{and}\quad D'(A) = \bigcup\alpha \in P_n\, C'_\alpha(A).$$

We note that, since

$$.\nu_*fT = .\nu_*f(T\sigma G_n) \leqq .\nu_*f(\sigma G_n) = .\nu_*fA = .\nu_*f(TA) \leqq .\nu_*fT,$$

we must have for some $\alpha\in P_n$, $.\nu_*f(TA\alpha)>0$. Thus, $D(A)\neq D'(A)$ and they both satisfy all conditions of 7.1 with 'P_1' replaced by 'P_{n+1}' in (4) and (5). Finally, to insure property (2), we note that $.\nu_*fA=.\nu_*f(TA)=.\nu_*f(TD(A))=.\nu_*f(D(A))$ so that the sets

$$\overline{D}(A) = D(A) - \bigcup B \in G_n\, {}^*f({}_*fA - {}_*f(D(B))),$$
$$\overline{D}'(A) = D'(A) - \bigcup B \in G_n\, {}^*f({}_*fA - {}_*f(D(B)))$$

differ from $D(A)$ and $D'(A)$ respectively by a set of μ-measure zero, and ${}_*f(\overline{D}(A))={}_*f(\overline{D}(B))$ for $B\in G_n$. Therefore, we let

$$G_{n+1} = \bigcup A \in G_n\, (\{\overline{D}(A)\} \cup \{\overline{D}'(A)\})$$

and see that properties (1) through (6) are satisfied with 'n' replaced by '$n+1$'.

7.3. THEOREM. *If F is almost complete, $\nu\in\mathfrak{M}(F,f)$, $\mu=V(F,f,\nu)$, $T\subset\sigma F$, $0<.\nu_*fT<\infty$, and, for every $A\subset T$, $.\mu A<\infty$ implies $.\mu A=0$, then there exists a sequence S such that, for every $n\in\omega$, S_{n+1} is a refinement of S_n and*

(1) $\sigma S_n=\sigma S_0\subset {}_*fT$,
(2) $A\in S_n$, $B\in S_n$, $A\neq B$ *imply* $AB=0$,
(3) $A\in S_n$ *implies* $.\nu A=.\nu_*fT$,
(4) $B\subset\sigma S_0$ *and* $.\nu B\neq 0$ *imply* $\lim_{n\to\infty}\sum A\in S_n\ .\nu(BA)=\infty$.

Proof. Let $T'\subset T$ be such that ${}_*fT'={}_*fT$ and $f(x)\neq f(y)$ whenever $x\in T'$, $y\in T'$, $x\neq y$. Let G be a sequence satisfying the conditions in 7.2 with 'T' replaced by 'T''. Let $T_0={}_*f\bigcap n\in\omega\ (T'\sigma G_n)$. We define S by setting

$$S_n = \bigcup A \in G_n\, \{T_0\, {}_*f(T'A)\} \qquad \text{for } n \in \omega$$

Properties (1) and (2) are clearly satisfied. Also

$$.\nu({}_*fT' - T_0) \leqq \sum n \in \omega\ .\nu_*f(T' - \sigma G_n) = 0.$$

Hence:

$$.\nu(T_{0*}f(T'A)) = .\nu_* f(T'A) = .\nu_* fT' \qquad \text{for } A \in G_n.$$

Finally, if $B \subset T_0$ and $.\nu B \neq 0$, let $C = \bigcap n \in \omega\ (T'\sigma G_n\ {}^*fB)$. Then:

$$\begin{aligned} \infty &= \lim_{n\to\infty} \sum \alpha \in P_n\ .\nu_* f(C\alpha) \leqq \lim_{n\to\infty} \sum \alpha \in P_n\ \sum A \in G_n\ .\nu_* f(CA\alpha) \\ &= \lim_{n\to\infty} \sum A \in G_n\ .\nu_* f(CA) = \lim_{n\to\infty} \sum A \in G_n\ .\nu({}_*fC_*f(T'A)) \\ &= \lim_{n\to\infty} \sum A \in S_n\ .\nu(BA). \end{aligned}$$

7.4. Theorem. *If F is almost complete, $\nu \in \mathfrak{M}(F, f)$, $\mu = V(F, f, \nu)$, $T \subset \sigma F$, $0 < .\nu_* fT < \infty$, and, for every $A \subset T$, $.\mu A < \infty$ implies $.\mu A = 0$, then there exists T^*, ν-measurable and $.\nu T^* > 0$, such that, for every $y \in T^*$, ${}^*f\{y\}$ is noncountable.*

Proof. Let G be a sequence satisfying 7.2 $T' = \bigcap n \in \omega\ {}_*f\sigma G_n$, and $T^* = \mathbf{E}y$ ($y \in T'$ and ${}^*f\{y\}$ is closed). Clearly T^* is ν-measurable and $.\nu T^* = .\nu T' = .\nu_* fT > 0$.

Let $y \in T^*$ and $S = \mathbf{E}A$ (A is a sequence and $A_{n+1} \subset A_n \in G_n$ for all $n \in \omega$). If $A \in S$, we have $A_n\ {}^*f\{y\} \neq 0$ for all $n \in \omega$, and hence by 6.5 there is an $\alpha \in Sq(P)$ such that $\alpha_n A_n\ {}^*f\{y\} \neq 0$. By property (6) in 7.2, this means $0 \neq \alpha_{n+k}\ {}^*f\{y\} \subset A_n$ for some $k \in \omega$. Hence if $x \in \bigcap n \in \omega\ \alpha_n$, then x is in the closure of ${}^*f\{y\}$, therefore $x \in {}^*f\{y\}$ and $x \in \bigcap n \in \omega\ A_n$. If $B \in S$, $B \neq A$, and $\alpha' \in Sq(P)$, $\alpha_n' B_n\ {}^*f\{y\} \neq 0$ for all $n \in \omega$, then $\alpha' \neq \alpha$. Since there are a countable number of $\alpha \in Sq(P)$ such that $\bigcap n \in \omega\ \alpha_n = 0$, and S is noncountable, it follows that $S' = \mathbf{E}A$ ($A \in S$ and $T^* = {}_*f \bigcap n \in \omega\ A_n$) is noncountable, and since $A \in S'$, $B \in S'$, $A \neq B$ implies $A_n B_n = 0$ for some $n \in \omega$, this means ${}^*f\{y\}$ is noncountable for all $y \in T^*$.

7.5. Remark. Since F has a topological mesh, there can be at most a countable number of disjoint neighborhoods in σF. From this it follows that any closed subset of σF is noncountable if and only if it contains a nontrivial perfect subset. The fact that, in 7.4, ${}^*f\{y\}$ contains a nontrivial perfect subset for all $y \in T^*$ can also be seen directly by making a small change in the proof.

8. **Applications.** In this section we indicate some immediate consequences of the theory developed so far when it is applied to functions on the real line. Extensions to other spaces can easily be seen.

8.1. Definition. $\langle a, b\rangle = \mathbf{E}t$ (t is irrational and $a\langle t\rangle b$).

8.2. Definition. $F_0 = \mathbf{E}\alpha$ ($\alpha = \langle a, b\rangle$ for some rational a, b with $0 \leqq a < b \leqq 1$).

8.3. Definition. $\mathfrak{M}_0 = \mathbf{E}\nu$ (ν is an outer measure on $[0, 1]$ and intervals are ν-measurable).

8.4. Definition. $\mathfrak{M}_1 = \mathbf{E}\nu$ ($\nu \in \mathfrak{M}_0$; $.\nu[0, 1] < \infty$; $.\nu\{y\} = 0$ for $0 \leqq y \leqq 1$).

8.5. Definition. $\mathfrak{M}_2 = \mathbf{E}\nu$ ($\nu \in \mathfrak{M}_0$; there exists no sequence S such that, for $n \in \omega : S_{n+1}$ is a refinement of S_n; $\sigma S_n = \sigma S_0 \subset [0, 1]$; $A \in S_n$, $B \in S_n$, $A \neq B$ imply $AB = 0$; $0 < .\nu\sigma S_0 < \infty$; $B \subset \sigma S_0$ and $.\nu B \neq 0$ imply $\lim_{n\to\infty} \sum A \in S_n$ $.\nu(AB) = \infty$).

8.6. DEFINITION. A is measurable M if and only if A is ν-measurable for all $\nu \in M$.

8.7. DEFINITION. A has absolutely M measure zero if and only if $.\nu A = 0$ for all $\nu \in M$.

8.8. DEFINITION. $\mathfrak{F}(M) = \mathbf{E}f$ (f is a function on σF_0 to $[0, 1]$; $_*f\alpha$ is measurable M for all $\alpha \in F_0$; $^*f\{y\}$ is closed for all $y \in \text{rng}\, f - Z$ where Z has absolutely M measure zero).

We observe that: σF_0 is the set of irrationals in $[0, 1]$, and F_0 is almost complete; A is measurable $\mathfrak{M}_0$ if and only if A is measurable $\mathfrak{M}_1$; if $f \in \mathfrak{F}(\mathfrak{M}_1)$ and $\nu \in \mathfrak{M}_1$ then $\nu \in \mathfrak{M}(F_0, f)$ and $\mu = V(F_0, f, \nu) \in \mathfrak{M}_0$ (if we extend μ to all of $[0, 1]$ by setting $.\mu A = 0$ for all sets A of rationals); if $f \in \mathfrak{F}(\mathfrak{M}_2)$ and $\nu \in \mathfrak{M}_2$ then $\nu \in \mathfrak{M}(F_0, f)$ and $V(F_0, f, \nu) \in \mathfrak{M}_2$. Then, using 6.13, 7.3 and 7.4 we conclude:

8.9. THEOREM. *If $f \in \mathfrak{F}(\mathfrak{M}_2)$ and A is measurable $\mathfrak{M}_2$ then $_*fA$ is measurable $\mathfrak{M}_2$.*

8.10. THEOREM. *If $f \in \mathfrak{F}(\mathfrak{M}_1)$ and $^*f\{y\}$ is countable for $y \in \text{rng}\, f - Z$, for some Z of absolutely $\mathfrak{M}_1$ measure zero, and A is measurable $\mathfrak{M}_0$, then $_*fA$ is measurable $\mathfrak{M}_0$.*

8.11. COROLLARY. *If f is continuous on the irrationals to $[0, 1]$ and A is measurable $\mathfrak{M}_2$, then $_*fA$ is measurable $\mathfrak{M}_2$.*

Proof. $_*f\alpha$ is analytic for $\alpha \in F_0$. Thus $f \in \mathfrak{F}(\mathfrak{M}_0)$.

8.12. COROLLARY. *If f is continuous on the irrationals to $[0, 1]$, $^*f\{y\}$ is countable for $y \in \text{rng}\, f - Z$, for some Z of absolutely $\mathfrak{M}_1$ measure zero, and A is measurable $\mathfrak{M}_0$, then $_*fA$ is measurable $\mathfrak{M}_0$.*

8.13. REMARK. A set Z of absolutely $\mathfrak{M}_1$ measure zero need not be countable (see [1]).

8.14. REMARK. An immediate consequence of 8.11 is that all projective sets are measurable $\mathfrak{M}_2$. Since K. Gödel has indicated (see [2]) that the existence of a projective set P_2 which is not Lebesgue-measurable is consistent with the usual axioms of set theory if the latter are consistent, it follows that we cannot have Lebesgue measure in $\mathfrak{M}_2$. It would be interesting to show directly that Lebesgue measure is not in $\mathfrak{M}_2$, i.e., if one could produce a sequence satisfying conditions (1) through (4) in Theorem 7.3 with ν taken to be Lebesgue measure.

On the other hand, M. Kondô (see [4]) has shown that for every projective set P_2 of class 2 (see [6] for general definitions) there is a set A, whose complement is analytic, and a function f, continuous and one-to-one on A, such that $_*fA = P_2$. It is well known that f can be extended to a G_δ containing A, with continuity preserved. In view of K. Gödel's result mentioned above and 8.12, it follows that such an extension cannot always assume every value y only a countable number of times.

It may also be of interest to note that the results of this section may be extended immediately to complete, separable metric spaces since any such space that is also noncountable is the image within a countable set, of the set of irrationals by a one-to-one continuous function.

Bibliography

1. A. S. Besicovitch, *Concentrated and rarified sets of points*, Acta Math. vol. 62 (1934) pp. 289–300.
2. K. Gödel, *The consistency of the axiom of choice and of the generalized continuum-hypothesis*, Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 556–557.
3. H. Hahn and A. Rosenthal, *Set functions*, 1948.
4. M. Kondô, *Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de la seconde classe*, Japanese Journal of Mathematics vol. 15 (1939) pp. 197–230.
5. M. Lavrentieff, *Contribution à la théorie des ensembles homéomorphes*, Fund. Math. vol. 6 (1924) pp. 149–160.
6. N. Lusin, *Leçons sur les ensembles analytiques et leurs applications*, 1930.
7. M. E. Munroe, *Introduction to measure and integration*, 1953.
8. M. W. Sierpinski, *Les ensembles projectifs et analytiques*, Mémorial des Sciences Mathématiques, no. 112, 1950.

The Institute for Advanced Study,
Princeton, N. J.