ON SOME SIMPLE GROUPS DEFINED BY C. CHEVALLEY

BY

RIMHAK REE

Introduction. For any semi-simple Lie algebra $\mathfrak{g}$ over the field of complex numbers and an arbitrary field K, C. Chevalley [2] defined a group $\mathfrak{G}$ by a uniform method. The group $\mathfrak{G}$ turns out to be simple when $\mathfrak{g}$ is simple (except for a few exceptional cases), and yields some new classes of simple groups not contained in the theory of classical groups, if, in particular, $\mathfrak{g}$ is one of the exceptional simple Lie algebras.

In this paper first we consider the case where $\mathfrak{g}$ is one of the classical algebras A_n, B_n, C_n, D_n, and identify the group $\mathfrak{G}$ with classical groups, thus answering a question raised in the last section of [2]. Our results are as follows:

(a) if $\mathfrak{g}$ is of the type A_n, then $\mathfrak{G}$ is the special projective group $\mathrm{PSL}(n+1, K)$;

(b) if $\mathfrak{g}$ is of the type C_n, then $\mathfrak{G}$ is the quotient group of the symplectic group $\mathrm{Sp}(2n, K)$ over its center;

(c) if $\mathfrak{g}$ is of the type D_n, then $\mathfrak{G}$ is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=1}^{n} \xi_i\xi_{-i}$ $(3 \leqq n)$;

(d) if $\mathfrak{g}$ is of the type B_n and if K is not of characteristic 2 then $\mathfrak{G}$ is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=0}^{n} \xi_i\xi_{-i}$ $(2 \leqq n)$; if $\mathfrak{g}$ is of type B_n and if K is of characteristic 2 then $\mathfrak{G}$ is a subgroup of Sp $(2n, K)$; if, moreover, K is perfect(1) then $G = \mathrm{Sp}(2n, K)$. The author has been unable to identify $\mathfrak{G}$ in case K is not perfect and of characteristic 2.

Secondly, we consider the case where $\mathfrak{g}$ is the exceptional algebra G_2, and identify the group $\mathfrak{G}$ with the groups defined by L. E. Dickson [4; 5].

1. **The group defined by Chevalley.** Let $\mathfrak{g}$ be a semi-simple Lie algebra over the field of complex numbers and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Denote by (X, Y) the Killing form of $\mathfrak{g}$. For a root(2) r of $\mathfrak{g}$, we set $H_r = 2(r, r)^{-1}r$, which is called the co-weight (co-poids) attached to r. It is known that the additive group $\mathfrak{H}$ generated by all co-weights H_r is a free abelian group of rank n, where n is the dimension of $\mathfrak{h}$. Chevalley proves the existence of a system $\{X_r\}$ of root vectors satisfying the following conditions (1.1)–(1.2):

$$(1.1) \qquad [X_r, X_{-r}] = H_r \qquad \text{for all roots } r;$$

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(1) A field K of characteristic $p>0$ is called *perfect* if every element in K is a pth power of an element in K.

(2) Roots of a semi-simple algebra over the field of complex numbers can be identified with elements in the Cartan subalgebra by which the roots are defined. See [7].

(1.2) if r, s, and $r+s$ are roots then $[X_r, X_s]=N_{r,s}X_{r+s}$ with $N_{r,s}=\pm(p+1)$, where p is the greatest integer $i\geqq 0$ such that $s-ir$ is a root.

If a system $\{X_r\}$ of root vectors satisfies (1.1) and (1.3) below then it it also satisfies (1.2) (see [2, pp. 22–23]).

(1.3) $\quad N_{-r,-s} = N_{r,s}$ for any two roots r, s such that $r+s$ is a root.

Let now $\{\tilde{H}_1, \cdots, \tilde{H}_n\}$ be a basis of the group $\mathfrak{H}$ and $\{X_r\}$ a system of root vectors satisfying (1.1)–(1.2). Then the set $\{\tilde{H}_1, \cdots, \tilde{H}_n\}$ together with $\{X_r\}$ forms a basis $\{\tilde{H}_i, X_r\}$ of $\mathfrak{g}$. It is easily seen that the coefficients appearing in the multiplication table of the basis $\{\tilde{H}_i, X_r\}$ are all integers. Chevalley proves that the coefficients of the matrix $A_r(t)$ representing the automorphism

$$x_r(t) = \exp t(\mathrm{ad}\, X_r),$$

where t is a complex variable, of $\mathfrak{g}$ with respect to the basis $\{\tilde{H}_i, X_r\}$ are all polynomials in t with integral coefficients. Therefore for any field K we can define a Lie algebra $\mathfrak{g}_K$ over K by using the basis $\{\tilde{H}_i, X_r\}$, and also automorphisms $x_r^*(t)$, where $t\in K$, of $\mathfrak{g}_K$ by using the matrix $A_r(t)$. Then the group considered by Chevalley is the group $\mathfrak{G}(\mathfrak{g}; K)$ (denoted by the symbol $\mathfrak{G}'$ in [2]) generated by all $x_r^*(t)$, where r runs over all roots of $\mathfrak{g}$ while t takes on all elements in K. This group was proved to be simple except for a few exceptional cases.

Since every $\tilde{H}_i$ is a linear combination of the co-weights H_r with integral coefficients, it follows that an element in $\mathfrak{G}(\mathfrak{g}; K)$ is uniquely determined by its effect on H_r^* and X_r^*, where H_r^* and X_r^* are elements in $\mathfrak{g}_K$ corresponding to H_r and X_r in $\mathfrak{g}$, respectively. In particular an element in $\mathfrak{G}$ is the identity if and only if it leaves every H_r^* and every X_r^* invariant.

2. **The classical algebras.** For the classical algebras A_n, B_n, C_n, D_n the co-weights H_r and the system $\{X_r\}$ of root vectors satisfying (1.1) and (1.3) can be computed easily (cf. [7]).

The algebra A_n is the algebra of all $(n+1)\times(n+1)$ matrices of trace zero. Denote by E_{ij} the $(n+1)\times(n+1)$ matrix whose (i, j)-entry is 1 and whose other entries are all zero. Then

(2.1) $$\{H_r\} = \{E_{ii} - E_{jj} \mid i \neq j\}, \qquad \{X_r\} = \{E_{ij} \mid i \neq j\},$$

and if $H_r=E_{ii}-E_{jj}$ then $X_r=E_{ij}$, $X_{-r}=E_{ji}$. The mapping ϕ which maps a matrix T to $-T'$, where T' is the transpose of T, is an automorphism of A_n. We have $\phi(E_{ij})=-E_{ji}$, and hence it is easily seen that the system $\{X_r\}$ satisfies the condition (1.3).

The algebra D_n is the algebra of all $2n\times 2n$ matrices X which satisfy $f_D(X\xi, \eta)+f_D(\xi, X\eta)=0$ identically, where

$$f_D(\xi, \eta) = \xi_{-n}\eta_n + \cdots + \xi_{-1}\eta_1 + \xi_1\eta_{-1} + \cdots + \xi_n\eta_{-n}.$$

Denote by E_{ij} the $2n\times 2n$ matrix such that $E_{ij}\xi=\xi'$, where $\xi_i'=\xi_j$, $\xi_k'=0$ for $k\neq i$, and set

$$F_{ij} = E_{-ij} - E_{-ji}, \qquad H_{ij} = F_{i,-i} + F_{j,-j}.$$

Then we have

$$\begin{aligned} \{H_r\} &= \{H_{ij} \mid i < j,\ i \neq -j\}, \\ \{X_r\} &= \{F_{ij} \mid i < j,\ i \neq -j\}, \end{aligned} \tag{2.2}$$

where if $H_r=H_{ij}$ then $X_r=F_{ij}$, $X_{-r}=F_{-j,-i}$. The mapping $\phi(T)=-T'$ is also an automorphism of D_n, and hence the system $\{X_r\}$ satisfies (1.3).

The algebra B_n may be regarded as the algebra of all $(2n+1)\times(2n+1)$ matrices X satisfying $f_B(X\xi,\eta)+f_B(\xi,X\eta)=0$ identically, where

$$f_B(\xi,\eta) = 2\xi_0\eta_0 + \sum_{i=1}^{n} (\xi_{-i}\eta_i + \xi_i\eta_{-i}).$$

It is readily seen that a matrix $X=(x_{ij})$ belongs to B_n if and only if

$$x_{-ij} + x_{-ji} = 0, \qquad x_{-i0} + 2x_{0i} = 0, \qquad x_{00} = 0$$

for all $i\neq 0$ and $j\neq 0$. Define the matrices E_{ij} as in the case of algebra D_n, and set

$$\begin{aligned} &F_{ij} = E_{-ij} - E_{-ji}, \qquad F_{i0} = 2E_{-i0} - E_{0i}, \qquad F_{0j} = -F_{j0}, \\ &H_{ij} = F_{i,-i} + F_{j,-j}, \qquad H_{i0} = 2F_{i,-i}, \qquad H_{0j} = 2F_{j,-j} \end{aligned}$$

for $i\neq 0$, $j\neq 0$. Then we see easily that

$$\{H_r\} = \{H_{ij} \mid i < j,\ i \neq -j\} \tag{2.3}$$

is the set of all co-weights (with respect to a certain Cartan subalgebra) and that

$$\{X_r\} = \{F_{ij} \mid i < j,\ i \neq -j\} \tag{2.4}$$

is a system of root vectors satisfying (1.1), where, $X_r=F_{ij}$ and $X_{-r}=F_{-j,-i}$ if $H_r=H_{ij}$. This system also satisfies (1.3). This is seen as follows: Let $P=2E_{00}+\sum_{i=1}^{n}(E_{ii}+E_{-i,-i})$. Then the mapping $\phi(X)=-P^{-1}X'P$ is an automorphism of B_n such that $\phi(F_{ij})=-F_{-j,-i}$, that is, $\phi(X_r)=-X_{-r}$ for all roots r, and hence (1.3) follows immediately.

The algebra C_n is the algebra of all $2n\times 2n$ matrices X which satisfy $f_C(X\xi,\eta)+f_C(\xi,X\eta)=0$ identically, where

$$f_C(\xi,\eta) = \xi_{-n}\eta_n + \cdots + \xi_{-1}\eta_1 - \xi_1\eta_{-1} - \cdots - \xi_n\eta_{-n}.$$

Define E_{ij} as in the case of the algebra D_n and set

$$F_{ij} = \begin{cases} E_{-ij} + \operatorname{sign}(ij)E_{-ji}, & \text{if } i \neq j, \\ E_{-ii} & \text{if } i = j, \end{cases} \tag{2.5}$$

where sign k denotes $+1$ if $k>0$, and -1 if $k<0$, and

$$H_{ij} = \begin{cases} F_{i,-i} + F_{j,-j}, & \text{if } i \neq j, \\ F_{i,-i} & \text{if } i = j. \end{cases}$$

Then we have

$$(2.6)\qquad \begin{aligned} \{H_r\} &= \{H_{ij} \mid i \leqq j,\ i + j \neq 0\}, \\ \{X_r\} &= \{F_{ij} \mid i \leqq j,\ i + j \neq 0\}, \end{aligned}$$

and if $H_r = H_{ij}$ then $X_r = F_{ij}$, $X_{-r} = F_{-j,-i}$. The mapping $\phi: X \to -X'$ is an automorphism of C_n such that $\phi(X_r) = -X_{-r}$ for all roots. Thus (1.3) is also satisfied by the system $\{X_r\}$.

3. **The group $\overline{\mathfrak{G}}$ ($\mathfrak{g}$, K).** For any two matrices X, Y with complex coefficients we have the formula

$$(3.1)\qquad \exp(\operatorname{ad} X) \cdot Y = (\exp X) Y (\exp(-X)),$$

where $(\operatorname{ad} X) \cdot Y = XY - YX$. This can be verified easily if we use the identity

$$\frac{(\operatorname{ad} X)^k}{k!} \cdot Y = \sum_{i+j=k} \frac{X^i}{i!} Y \frac{(-X)^j}{j!}.$$

For the matrix algebras $\mathfrak{g} = A_n$, C_n, D_n, every root vector $X_r = F_{ij}$ (we set $F_{ij} = E_{ij}$, $i \neq j$, for the algebra A_n) given in the preceding section satisfies $X_r^2 = 0$, and hence

$$(3.2)\qquad \exp(tX_r) = I + tF_{ij},$$

where I is the unit matrix. For the algebra $\mathfrak{g} = B_n$, every root vector $X_r = F_{ij}$ such that $i \neq 0$, $j \neq 0$ satisfies $X_r^2 = 0$, and hence we have (3.2) for this X_r also. For the root vector $X_r = F_{i0}$, of the algebra B_n we have $X_r^2 = -2E_{-ii}$, $X_r^3 = 0$. Therefore

$$(3.3)\qquad \exp(tX_r) = I + t(2E_{-i0} - E_{0i}) - t^2 E_{-ii},$$

for $i = \pm 1, \pm 2, \cdots, \pm n$. From (3.2) and (3.3) we see that $\exp(tX_r)$ can always be represented as a matrix whose entries are polynomials in t with integral coefficients. Now, the matrix $A_r(t)$ representing the automorphism $x_r(t)$ of $\mathfrak{g}$ with respect to the basis $\{\tilde{H}_i, X_r\}$ can be obtained from the formula

$$(3.4)\qquad x_r(t) \cdot Y = (\exp(tX_r)) Y (\exp(-tX_r)), \qquad Y \in \mathfrak{g},$$

which follows from (3.1), and the expression for $\exp(tX_r)$ given in (3.2)–(3.3).

Let K be an arbitrary field and $\mathfrak{g}$ one of the algebras A_n, B_n, C_n, D_n. In the preceding section all co-weights H_r and root vectors X_r of $\mathfrak{g}$ are represented by matrices with integral coefficients, and consequently, every element in the basis $\{\tilde{H}_i, X_r\}$ is also represented as a matrix with integral coefficients.

Therefore we see that $\mathfrak{g}_K$ is represented in this way as a matrix algebra over K. For $t \in K$, define the matrix exp (tX_r^*), by the right hand side of (3.3) if X_r is one of the root vectors F_{i0}, F_{0j} (in case $g = B_n$), and by that of (3.2) if X_r is not, and consider the automorphism $x_r^*(t)$ defined by $A_r(t)$. Then (3.4) yields

$$x_r^*(t) \cdot Y^* = (\exp(tX_r^*))Y^*(\exp(-tX_r^*)) \tag{3.5}$$

for any Y^* in $\mathfrak{g}_K$. Denote by $\overline{\mathfrak{G}}(\mathfrak{g}, K)$ the multiplicative group of matrices generated by all exp (tX_r^*), where r runs over all roots of $\mathfrak{g}$ while t runs over all elements in K. For any $S \in \overline{\mathfrak{G}}$ define the mapping S^σ of $\mathfrak{g}_K$ into itself by $S^\sigma \cdot Y^* = SY^*S^{-1}$, where $Y^* \in \mathfrak{g}_K$. Then (3.5) shows that S^σ is a product of certain automorphisms of $\mathfrak{g}_K$ of the form $x_r^*(t)$, and hence belongs to the group $\mathfrak{G}(\mathfrak{g}, K)$ defined in §1. The mapping $\sigma: S \to S^\sigma$ is clearly a homomorphism of $\overline{\mathfrak{G}}$ onto $\mathfrak{G}$. Denote by $\mathfrak{Z}$ the kernel of the homomorphism σ. By the remark given at the end of §1, we see that an element S in $\overline{\mathfrak{G}}$ belongs to $\mathfrak{Z}$ if and only if S commutes with all X_r^* and H_r^*. If $S \in \mathfrak{Z}$ commutes with all X_r^* then S belongs to the center of $\overline{\mathfrak{G}}$, for $\overline{\mathfrak{G}}$ is generated by elements of the form exp (tX_r^*). Conversely, suppose that S belongs to the center of $\overline{\mathfrak{G}}$. Then S commutes with all exp (tX_r^*). If $\mathfrak{g}$ is one of the algebras A_n, C_n, D_n, then exp $(tX_r^*) = I + tX_r^*$. Hence S commutes with all X_r^*. Since $[X_r^*, X_{-r}^*] = H_r^*$, we see that S also commutes with all H_r^*. Therefore S belongs to $\mathfrak{Z}$. Thus we have proved that $\mathfrak{Z}$ coincides with the center of $\overline{\mathfrak{G}}(\mathfrak{g}; K)$ if $\mathfrak{g}$ is one of the algebras A_n, C_n, D_n. Consider now the case $\mathfrak{g} = B_n$. If S belongs to the center of $\overline{\mathfrak{G}}(B_n, K)$ then S commutes with exp X_r^*, and hence with (exp $X_r^* - I)^2 = (X_r^*)^2$. Therefore, if K is of characteristic $\neq 2$, then S commutes with all X_r^* and hence belongs to $\mathfrak{Z}$, as before. If K is of characteristic 2, then from (3.3) we see that S commutes with all $E_{0i} + E_{-ii}$, $i = \pm 1, \pm 2, \cdots, \pm n$. Then it follows easily that S is of the form aI, with $a \in K$. Therefore S commutes with all X_r^* and H_r^*, and consequently it belongs to $\mathfrak{Z}$. Thus we have proved

$$\mathfrak{G}(\mathfrak{g}, K) \cong \overline{\mathfrak{G}}(\mathfrak{g}, K)/\mathfrak{Z}, \tag{3.6}$$

where $\mathfrak{Z}$ is the center of $\overline{\mathfrak{G}}(\mathfrak{g}, K)$.

4. **The group** $\mathfrak{G}(A_n, K)$. It is well known that the matrices of the form $I + tE_{ij}$, where $t \in K$, $i \neq j$, generate the special linear group $\mathrm{SL}(n+1, K)$. Therefore $\overline{\mathfrak{G}}(A_n, K) = \mathrm{SL}(n+1, K)$. By (3.6) it follows that $\mathfrak{G}(A_n, K)$ is isomorphic to the special projective group $\mathrm{PSL}(n+1, K)$.

5. **The group** $\mathfrak{G}(C_n, K)$. We show that the group $\overline{\mathfrak{G}}(C_n, K)$ coincides with the symplectic group $\mathrm{Sp}(2n, K)$ (in its matric representation). It is known [6] that the group $\mathrm{Sp}(2n, K)$ is generated by the matrices

$$I + tE_{-ii}, \; I + t(E_{-ij} - E_{-ji}),$$

and

$$S_i = I - E_{ii} - E_{-i,-i} + E_{-ii} - E_{i,-i},$$

where $t \in K$, $i, j = 1, 2, \cdots, n$, except when $n=1$ and K is of order 2 or 3. In view of (2.5)–(2.6), the first two sets of matrices belong to $\overline{\mathfrak{G}}(C_n, K)$. In order to obtain the equality $\overline{\mathfrak{G}} = \mathrm{Sp}(2n, K)$, it thus suffices to show that the matrices S_i also belong to $\overline{\mathfrak{G}}$. As is mentioned in the preceding section, however, the group $\mathrm{SL}(2, K)$ is generated by matrices of the form $I + tE_{ij}$, where $t \in K$, $i \neq j$. Therefore S_i can be written as a product of matrices of the forms $I + tE_{-ii}$ and $I + tE_{i,-i}$, which belong to $\overline{\mathfrak{G}}$ by definition. Except when $n=1$ and K is of order 2 or 3, therefore we obtain $\overline{\mathfrak{G}} = \mathrm{Sp}(2n, K)$. Hence, by (3.6), $\mathfrak{G}(C_n, K)$ is isomorphic to the quotient of the symplectic group $\mathrm{Sp}(2n, K)$ over its center. It should be noted that $\mathrm{Sp}(2n, K)$ is the group of all $2n \times 2n$ matrices X which satisfy

$$f_C(X\xi, X\eta) = f_C(\xi, \eta) \text{ identically.}$$

6. **The group** $\mathfrak{G}(D_n, K)$. First we shall show that the group $\overline{\mathfrak{G}}(D_n, K)$ contains the commutator subgroup $\mathfrak{O}'$ of the orthogonal group $\mathfrak{O}$ consisting of all $2n \times 2n$ matrices which leave the quadratic form $\sum_{i=1}^{n} \xi_i \xi_{-i}$ invariant. By (2.2), the group $\overline{\mathfrak{G}}$ is generated by the matrices

$$W_{i,j,t} = I + t(E_{i,-j} - E_{j,-i}),$$

where $t \in K$ and $i, j = \pm 1, \cdots, \pm n$; $i \neq \pm j$. Denote by P_{ij}, Q_i and $R_{i,t}$ the matrices corresponding to the permutations $(\xi_i\ \xi_j)(\xi_{-i}\xi_{-j})$, $(\xi_i \xi_{-i})$ and the transformations $\xi_i' = t\xi_i$, $\xi_{-i}' = t^{-1}\xi_{-i}$, $\xi_k' = \xi_k$ $(k \neq i, -i)$, respectively. It is known [3] that the group $\mathfrak{O}$ is generated by the matrices $W_{i,j,t}$, P_{ij}, Q_i, and $R_{i,t}$. Therefore, $\mathfrak{O}' \leqq \overline{\mathfrak{G}}$ is proved if we can show that for any distinct $|i|$, $|j|$, $|k|$ the matrices $(P_{ik}P_{ij})^2$, Q_iQ_j, $R_{i,t}R_{j,t}^{-1}$, and $R_{i,t}^2$ are in $\overline{\mathfrak{G}}$. By elementary computations we have

$$\begin{aligned} (P_{ik}P_{ij})^2 &= W_{j,-k,1}W_{-j,k,1}W_{j,-k,1}W_{i,-j,1}W_{-i,j,1}W_{i,-j,1}, \\ Q_iQ_j &= W_{j,-i,1}W_{-j,i,1}W_{j,-i,1}W_{i,j,1}W_{-i,-j,1}W_{i,j\,1}, \\ R_{i,t}R_{j,t}^{-1} &= W_{-i,j,t-1}W_{-j,i,1}W_{-i,j,s}W_{-j,i,-t}, \\ R_{i,t}R_{j,t} &= W_{-i,-j,t-1}W_{j,i,1}W_{-i,-j,s}W_{j,i,-t}, \end{aligned} \tag{6.1}$$

where $s = t^{-1} - 1$. Thus the matrices $(P_{ik}P_{ij})^2$, Q_iQ_j, $R_{i,t}R_{j,t}^{-1}$ are shown to be in $\overline{\mathfrak{G}}$. Since $R_{i,t}$ and $R_{j,t}$ commute, we have $R_{i,t}^2 = (R_{i,t}R_{j,t}^{-1})(R_{i,t}R_{j,t})$. Hence $R_{i,t}^2$ is also in $\overline{\mathfrak{G}}$. Therefore $\mathfrak{O}' \leqq \overline{\mathfrak{G}}$ is proved.

Now we shall show $\overline{\mathfrak{G}} \leqq \mathfrak{O}'$, assuming that $3 \leqq n$. For any $W_{i,j,t}$ take k such that $|k| \neq |i|, |j|$. Then we have

$$W_{i,j,t} = W_{-k,j,-1}^{-1} W_{i,k,t}^{-1} W_{-k,j,-1} W_{i,k,t},$$

which shows that $\overline{\mathfrak{G}} \leqq \mathfrak{O}'$. Therefore we have $\overline{\mathfrak{G}} = \mathfrak{O}'$ provided that $3 \leqq n$, and hence $\mathfrak{G}(D_n, K)$ is the commutator group of projective orthogonal group defined by the quadratic form $\sum_{i=1}^{n} \xi_i \xi_{-i}$.

7. **The group** $\mathfrak{G}(B_n, K)$. First we consider the case where K is not of characteristic 2. We shall show that the group $\overline{\mathfrak{G}}(B_n, K)$ is isomorphic to the commutator group $\mathfrak{O}'$ of the orthogonal group $\mathfrak{O}$ defined by the quadratic form $\sum_{i=0}^{n} \xi_i\xi_{-i}$, provided that $2 \leqq n$. By (2.4), the group $\overline{\mathfrak{G}}$ is generated by the matrices

$$W_{i,j,t} = I + t(E_{i,-j} - E_{j,-i}),$$
$$V_{i,t} = I + t(2E_{-i0} - E_{0i}) - t^2 E_{-ii},$$

where $t \in K$, $i, j = \pm 1, \cdots, \pm n$; $i \neq \pm j$. Define the matrices P_{ij}, Q_i, $R_{i,t}$ as in §6 for $i, j \neq 0$. It is known [4] that the group $\mathfrak{O}$ is generated by the matrices $W_{i,j,t}$, $V_{i,t}$, P_{ij}, Q_i, $R_{i,t}$. The formula (6.1) shows $\mathfrak{O}' \leqq \overline{\mathfrak{G}}$.

We shall show $\overline{\mathfrak{G}} \leqq \mathfrak{O}'$ under the assumption that $2 \leqq n$. By an elementary computation we have

$$W_{i,j,2t} = V_{i,t} V_{j,1} V_{i,t}^{-1} V_{j,1}^{-1},$$

which shows that every $W_{i,j,t}$ is in $\mathfrak{O}'$, since K is not of characteristic 2. Also we have

$$V_{i,t} = W_{ji,t^2}(W_{-ji,1}^{-1} V_{j,-t}^{-1} W_{-ji,1} V_{j,-t}).$$

Hence we see that $V_{i,t}$ is also in $\mathfrak{O}'$. Thus $\overline{\mathfrak{G}} \leqq \mathfrak{O}'$, and hence $\overline{\mathfrak{G}} = \mathfrak{O}'$ is proved. Therefore if K is not of characteristic 2 then $\mathfrak{G}(B_n, K)$ is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=0}^{n} \xi_i\xi_{-i}$.

Consider now the case where K is of characteristic 2. Denote by $\mathfrak{M}$ a vector space over K spanned by the indeterminates ξ_i, $i = 0, \pm 1, \cdots, \pm n$, and by $\mathfrak{M}'$ the subspace of $\mathfrak{M}$ spanned by ξ_i, $i = \pm 1, \cdots, \pm n$. The matrix $W_{i,j,t}$ is the matrix of the linear transformation of $\mathfrak{M}$:

$$\xi_i' = \xi_i + t\xi_{-j}, \quad \xi_j' = \xi_j + t\xi_{-i}, \quad \xi_k' = \xi_k \qquad (k \neq i, j), \tag{7.1}$$

while $V_{i,t} = I + tE_{0i} + t^2 E_{-ii}$ is the matrix of the linear transformation:

$$\xi_0' = \xi_0 + t\xi_i, \quad \xi_{-i}' = \xi_{-i} + t^2\xi_i, \quad \xi_k' = \xi_k \qquad (k \neq 0, -i). \tag{7.2}$$

Therefore the subspace $\mathfrak{M}'$ reduces the group $\overline{\mathfrak{G}}$. Moreover, (7.1) and (7.2) shows that the linear transformations in $\overline{\mathfrak{G}}$ leave the bilinear form $f_C(\xi, \eta) = \sum_{i=1}^{n} (\xi_i\eta_{-i} - \xi_{-i}\eta_i)$ invariant. It was noted in §3 that the center of $\overline{\mathfrak{G}}$ consists of elements of the form aI with $a \in K$. Since aI leaves $f_C(\xi, \eta)$ invariant, we have $a^2 = 1$ and hence $a = 1$. Then $\mathfrak{G} = \overline{\mathfrak{G}}$ and hence $\overline{\mathfrak{G}}$ is simple. (We exclude the exceptional cases. See [2, p. 65]). Therefore the representation of $\overline{\mathfrak{G}}$ induced in the space $\mathfrak{M}'$ is faithful. Thus we may regard $\mathfrak{G}$ as a multiplicative group of $2n \times 2n$ matrices generated by $W_{i,j,t} = I + t(E_{i,-j} - E_{j,-i})$ and $V_{i,t} = I + t^2 E_{-ii}$, where $t \in K$ and $i, j = \pm 1, \pm 2, \cdots, \pm n$. We have mentioned in §5 that the symplectic group $\mathrm{Sp}(2n, K)$ is generated by $W_{i,j,t}$ and $I + tE_{-ii}$. Thus the group $\mathfrak{G}(B_n, K)$, in the case when K is of characteristic 2, is a subgroup of $\mathrm{Sp}(2n, K)$. If, however, the field K is perfect then $\mathfrak{G} = \mathrm{Sp}(2n, K)$.

8. **The algebra G_2 as a subalgebra of B_3.** E. Cartan [1, p. 146] gave a representation of G_2 as a subalgebra of the algebra B_3, which we shall derive here in a form convenient for our purposes. The algebra B_3 is defined as the algebra of all 7×7 matrices X satisfying $f_B(X\xi, \eta) + f_B(\xi, X\eta) = 0$ identically, where

$$f_B(\xi, \eta) = 2\xi_0\eta_0 + \sum_{i=1}^{3} (\xi_{-i}\eta_i + \xi_i\eta_{-i}).$$

Denote by E_{ij} the 7×7 matrix such that $\xi' = E_{ij}\xi$ where $\xi_i' = \xi_j$, $\xi_k' = 0$ for $k \neq i$, and set

$$F_{ij} = E_{-ij} - E_{-ji}, \qquad F_{i0} = 2E_{-i0} - E_{0i}, \qquad F_{0i} = -F_{i0},$$
$$H_{ij} = F_{i,-i} + F_{j,-j}, \qquad H_{i0} = 2F_{i,-i}, \qquad H_{0i} = 2F_{i,-i}$$

for $i, j = 1, 2, 3$. Then $\{H_r\} = \{H_{ij} \mid i < j,\ i \neq -j\}$ is the set of all co-weights of B_3 with respect to the Cartan subalgebra $\mathfrak{h}$ spanned by all H_r, and $\{X_r\} = \{F_{ij} \mid i < j,\ i \neq -j\}$ is a system of root vectors such that if $X_r = F_{ij}$ then $H_r = H_{ij}$ and $X_{-r} = F_{-j,-i}$. It was shown in [5] that there is an automorphism ϕ of B_3 such that $\phi(X_r) = -X_{-r}$ for all roots r of B_3. Cartan showed that the six elements

$$U_{ij} = F_{i,-j}, \qquad (0 < i,\ 0 < j,\ i \neq j),$$

together with the six elements

$$U_{0i} = F_{i0} + F_{i'i''}, \qquad U_{i0} = F_{0,-i} + F_{-i'',-i'},$$

where $i = 1, 2, 3$ and where $(i\, i'\, i'')$ is a even permutation of 1, 2, 3, form a system of root vectors of a subalgebra $\mathfrak{g}$ of B_3 isomorphic to G_2. The Cartan subalgebra of $\mathfrak{g}$ is given by $\mathfrak{g} \cap \mathfrak{h}$. We have

$$\begin{aligned} [U_{ij}, U_{ji}] &= H_{i,-j}, \\ [U_{0i}, U_{i0}] &= H_{i0} + H_{i'i''}, \end{aligned} \tag{8.1}$$

where $i \neq 0, j \neq 0$, since $[F_{i0}, F_{-i'',-i'}] = 0$, $[F_{0,-i}, F_{i'i''}] = 0$. We have also the identities

$$[H_{i,-j}, U_{ij}] = 2U_{ij}, \qquad [H_{i0} + H_{i'i''}, U_{0i}] = 2U_{0i}.$$

Hence we see that the elements on the right hand side of (8.1) are co-weights of the algebra $\mathfrak{g}$. Moreover, it is easily seen that the automorphism ϕ of B_3 induces an automorphism of the subalgebra $\mathfrak{g}$ such that $\phi(U_{ij}) = -U_{ji}$, $\phi(U_{0i}) = -U_{i0}$. Therefore the system $\{U_{ij}, U_{0i}, U_{i0}\}$ of root vectors can be used to define the group $\mathfrak{G}(G_2, K)$.

9. **The group $\mathfrak{G}(G_2, K)$.** Let K be an arbitrary field. By the same argument as in §3 we see that $\mathfrak{G}(G_2, K)$ is isomorphic to the quotient group over

its center of the multiplicative group $\overline{\mathfrak{G}}(G_2, K)$ generated by the matrices

$$(9.1)\qquad \begin{cases} \exp\,(tU_{ij}) = I + tU_{ij}, \\ \exp\,(tU_{0i}) = I + tU_{0i} - t^2E_{-ii}, \\ \exp\,(tU_{i0}) = I + tU_{i0} - t^2E_{-ii}, \end{cases}$$

where $t \in K$ and $i, j = 1, 2, 3$.

Dickson [4, §9; 5], guided by the above mentioned representation of the algebra G_2 given by Cartan, considered the multiplicative group $\mathfrak{H}$ consisting of all 7×7 matrices X with entries in K which satisfy the conditions (9.2)–(9.4):

(9.2) $\det X = 1$;

(9.3) X leaves the quadratic form

$$q(\xi) = \xi_0^2 + \xi_1\xi_{-1} + \xi_2\xi_{-2} + \xi_3\xi_{-3}$$

invariant, that is, $q(X\xi) = q(\xi)$ identically;

(9.4) X leaves invariant the system of equations

$$\xi_0\eta_i - \xi_i\eta_0 + \xi_{-i'}\eta_{-i''} - \xi_{-i''}\eta_{-i'} = 0$$

where $(i\, i'\, i'')$ are cyclic even permutations of 1, 2, 3, or of -1, -2, -3, when X operates cogrediently upon the two sets of variables (ξ_i), (η_i), $-3 \leqq i \leqq 3$.

Dickson proved, assuming that K has more than two elements, that the group $\mathfrak{H}$ is simple and generated by the matrices in (9.1) and the matrices $R_{i,t}R_{j,t}^{-1}$, $i \neq j$, $i \neq 0$, $j \neq 0$, $t \in K$, where $R_{i,t}$ is the matrix of the transformation

$$\xi_i' = t\xi_i, \qquad \xi_{-i}' = t^{-1}\xi_{-i}, \qquad \xi_k' = \xi_k \qquad (k \neq i, -i).$$

In view of the formulas (6.1), the matrix $R_{i,t}R_{j,t}^{-1}$ can be represented as a product of matrices in (9.1), and hence it follows that the group $\mathfrak{H}$ is generated by the matrices in (9.1). Therefore we have $\mathfrak{H} = \overline{\mathfrak{G}}(G_2, K)$. Since $\mathfrak{H}$ is simple, we have also $\mathfrak{G}(G_2, K) = \overline{\mathfrak{G}}(G_2, K) = \mathfrak{H}$.

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University of British Columbia,
Vancouver, B. C.