

THE FATOU THEOREM AND ITS CONVERSE

BY

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1. **Introduction.** Let H^+ denote the class of functions which are non-negative and harmonic in the upper half plane $y > 0$ and let H denote the class of functions which can be expressed as the difference of two functions in H^+ ; obviously $H^+ \subset H$. It is well known that a function $u(x, y)$ has the Poisson-Stieltjes representation

$$(1.1) \quad u(x, y) = Ky + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(s-x)^2 + y^2} d\alpha(s),$$

where K is a constant and

$$(1.2) \quad \int_{-\infty}^{\infty} \frac{|d\alpha(s)|}{s^2 + 1} < \infty,$$

if and only if $u(x, y)$ is in H (see [5]). Hence with each function $u(x, y)$ in H we can associate a function $\alpha(s)$ and a constant K such that (1.1) and (1.2) hold; we can further assume that $\alpha(0) = 0$. For each $u(x, y)$ in H^+ , $\alpha(s)$ is non-decreasing and $K \geq 0$.

In this paper we are concerned with the behaviour of a function $u(x, y)$, in H , and its derivatives as (x, y) approaches a point P on the x -axis. For convenience we take P to be the origin.

Our starting point is the following pair of results (see [4]).

FATOU THEOREM. *Suppose that $u(x, y)$ is in H and that $\alpha'(0) = A$. Then for each $0 < \theta < \pi$, $u(r \cos \theta, r \sin \theta) \rightarrow A$ as $r \rightarrow 0+$.*

The converse of this theorem is not true unless we restrict ourselves to the subclass H^+ . In this case we have the following result.

LOOMIS THEOREM. *Suppose that $u(x, y)$ is in H^+ , that $0 < a \neq b < \pi$, and that for $\theta = a$ and $\theta = b$,*

$$u(r \cos \theta, r \sin \theta) \rightarrow A$$

as $r \rightarrow 0+$. Then $\alpha'(0) = A$.

The Fatou theorem is an Abelian theorem, the Loomis theorem is the corresponding Tauberian theorem, and the Tauberian condition is the restriction that $u(x, y) \geq 0$.

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In §3 we prove several extensions of the Fatou theorem and in §4 we present the corresponding converses. The arguments in §4 are very simple and make use of a variant of the Wiener Tauberian theorem which is developed in §2. Finally in §5 we consider an extension of a well known result due to Lindelöf.

2. **A Tauberian theorem.** In this section we introduce a two dimensional form of the Wiener Tauberian theorem for Stieltjes integrals. We begin with some definitions and notation.

A function $f(x)$ is in L if it is Lebesgue integrable over the real line. For $f(x)$ in L we let

$$(2.1) \quad F = \int_{-\infty}^{\infty} f(x)dx, \quad F(y) = \int_{-\infty}^{\infty} f(x)e^{ixy}dx$$

for all real y .

A function $f(x)$ is in M if it is continuous for all x and if

$$(2.2) \quad \sum_{n=-\infty}^{\infty} \text{Max}_{n \leq x \leq n+1} |f(x)| < \infty.$$

Obviously $M \subset L$.

A function $\beta(x)$ is in V if it has bounded variation over each finite interval and if

$$(2.3) \quad \int_n^{n+1} |d\beta(x)|$$

is bounded in n .

LEMMA 1. *Suppose that $f(x)$ and $g(x)$ are in L . Then $k(x) = f * g(x)$, that is*

$$k(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy,$$

exists p.p. in x , $k(x)$ is in L , and $K(y) = F(y)G(y)$. Furthermore $k(x)$ is in M if $f(x)$ and/or $g(x)$ is in M .

Proof. The first part of this lemma follows from known results. For the last part assume that $f(x)$ is in M . Then the fact that $f(x)$ is bounded and continuous allows us to apply the Lebesgue "dominated convergence" theorem to conclude that $k(x)$ is continuous. Next we see that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \text{Max}_{n \leq x \leq n+1} |k(x)| &\leq \sum_{n=-\infty}^{\infty} \text{Max}_{n \leq x \leq n+1} \int_{-\infty}^{\infty} |f(x - y)| |g(y)| dy \\ &\leq \sum_{n=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} \text{Max}_{n-m-1 \leq x \leq n-m+1} |f(x)| \int_m^{m+1} |g(y)| dy \right\} \end{aligned}$$

and, inverting the order of summation, we conclude that

$$\sum_{n=-\infty}^{\infty} \text{Max}_{n \leq x \leq n+1} |k(x)| \leq \left(\int_{-\infty}^{\infty} |g(y)| dy \right) \left(\sum_{n=-\infty}^{\infty} \text{Max}_{n \leq x \leq n+2} |f(x)| \right) < \infty$$

which completes the argument.

The Wiener Tauberian Theorem for Stieltjes integrals can be stated, using (2.1), as follows (see [3, p. 294]).

WIENER THEOREM. *Suppose that $f(x)$ is in M , that $F(y) \neq 0$ for all y , that $\beta(x)$ is in V , and that*

$$\int_{-\infty}^{\infty} f(x - y) d\beta(y) \rightarrow B$$

as $x \rightarrow +\infty$. Then for each $h(x)$ in M ,

$$\int_{-\infty}^{\infty} h(x - y) d\beta(y) \rightarrow \frac{B}{F} H$$

as $x \rightarrow +\infty$.

With the aid of Lemma 1 we obtain the following result.

THEOREM 1. *Suppose that $f_1(x), f_2(x), g_1(x), g_2(x)$ are in M , that $F_1(y)G_2(y) - F_2(y)G_1(y) \neq 0$ for all y , that $\beta(x), \gamma(x)$ are in V , and that*

$$(2.4) \quad \begin{aligned} \int_{-\infty}^{\infty} f_1(x - y) d\beta(y) + \int_{-\infty}^{\infty} f_2(x - y) d\gamma(y) &\rightarrow B, \\ \int_{-\infty}^{\infty} g_1(x - y) d\beta(y) + \int_{-\infty}^{\infty} g_2(x - y) d\gamma(y) &\rightarrow C \end{aligned}$$

as $x \rightarrow +\infty (-\infty)$. Then for each $h(x)$ in M

$$(2.5) \quad \begin{aligned} \int_{-\infty}^{\infty} h(x - y) d\beta(y) &\rightarrow \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1} H, \\ \int_{-\infty}^{\infty} h(x - y) d\gamma(y) &\rightarrow \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1} H \end{aligned}$$

as $x \rightarrow +\infty (-\infty)$.

Proof. We consider the case where $x \rightarrow +\infty$; the case where $x \rightarrow -\infty$ then follows with trivial modifications. Set

$$\begin{aligned} b(x) &= \int_{-\infty}^{\infty} f_1(x - y) d\beta(y) + \int_{-\infty}^{\infty} f_2(x - y) d\gamma(y), \\ c(x) &= \int_{-\infty}^{\infty} g_1(x - y) d\beta(y) + \int_{-\infty}^{\infty} g_2(x - y) d\gamma(y). \end{aligned}$$

Both $b(x)$ and $c(x)$ are bounded by virtue of the hypotheses, (2.2), and (2.3). Hence with (2.4) and the Lebesgue theorem we conclude that

$$(2.6) \quad \begin{aligned} b * g_2(x) &= \int_{-\infty}^{\infty} b(x - y)g_2(y)dy \rightarrow BG_2, \\ c * f_2(x) &= \int_{-\infty}^{\infty} c(x - y)f_2(y)dy \rightarrow CF_2 \end{aligned}$$

as $x \rightarrow +\infty$. Moreover, from the Fubini theorem it follows that

$$(2.7) \quad \begin{aligned} b * g_2(x) &= \int_{-\infty}^{\infty} f_1 * g_2(x - y)d\beta(y) + \int_{-\infty}^{\infty} f_2 * g_2(x - y)d\gamma(y), \\ c * f_2(x) &= \int_{-\infty}^{\infty} g_1 * f_2(x - y)d\beta(y) + \int_{-\infty}^{\infty} g_2 * f_2(x - y)d\gamma(y) \end{aligned}$$

and, with $k(x) = f_1 * g_2(x) - f_2 * g_1(x)$, we conclude from (2.6) and (2.7) that

$$(2.8) \quad \int_{-\infty}^{\infty} k(x - y)d\beta(y) \rightarrow BG_2 - CF_2$$

as $x \rightarrow +\infty$. By Lemma 1, $k(x)$ is in M ,

$$K(y) = F_1(y)G_2(y) - F_2(y)G_1(y) \neq 0$$

for all y , and we can apply the Wiener theorem to (2.8) to obtain the first part of (2.5). The second part follows in exactly the same way.

To obtain a more convenient form for our Tauberian theorem we perform a familiar exponential change of variable and shift from the interval $-\infty < x < \infty$ to the interval $0 < t < \infty$. (For the details see [3, pp. 295-296].) The analogues for L, M , and V are as follows.

A function $f(t)$ is in L' if it is Lebesgue integrable over $0 < t < \infty$. For $f(t)$ in L' we let

$$F = \int_0^{\infty} f(t)dt, \quad F(y) = \int_0^{\infty} f(t)t^{iy}dt$$

for all real y .

A function $f(t)$ is in M' if $f(t)$ is continuous over $0 < t < \infty$ and if

$$\sum_{n=-\infty}^{\infty} \text{Max}_{e^n \leq t \leq e^{n+1}} |tf(t)| < \infty.$$

Again $M' \subset L'$. Observe that $f(t)$ is in M' if $f(t)$ is continuous over $0 \leq t < \infty$ and if, for some $\epsilon > 0$,

$$(2.9) \quad f(t) = O(t^{-1-\epsilon})$$

for large t (see [3, p. 299]).

A function $\beta(t)$ is in V' if it has bounded variation over each finite interval in $0 < t < \infty$ and if

$$\int_t^{et} \frac{|d\beta(s)|}{s}$$

is bounded for all $t > 0$.

THEOREM 2. *Suppose that $f_1(t), f_2(t), g_1(t), g_2(t)$ are in M' , that $F_1(y)G_2(y) - F_2(y)G_1(y) \neq 0$ for all y , that $\beta(t), \gamma(t)$ are in V' , and that*

$$\begin{aligned} \frac{1}{t} \int_0^\infty f_1\left(\frac{s}{t}\right) d\beta(s) + \frac{1}{t} \int_0^\infty f_2\left(\frac{s}{t}\right) d\gamma(s) &\rightarrow B, \\ \frac{1}{t} \int_0^\infty g_1\left(\frac{s}{t}\right) d\beta(s) + \frac{1}{t} \int_0^\infty g_2\left(\frac{s}{t}\right) d\gamma(s) &\rightarrow C \end{aligned}$$

as $t \rightarrow +\infty (0+)$. Then for each $h(t)$ in M' ,

$$(2.10) \quad \begin{aligned} \frac{1}{t} \int_0^\infty h\left(\frac{s}{t}\right) d\beta(s) &\rightarrow \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1} H, \\ \frac{1}{t} \int_0^\infty h\left(\frac{s}{t}\right) d\gamma(s) &\rightarrow \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1} H \end{aligned}$$

as $t \rightarrow +\infty (0+)$.

We require the following corollary in §4.

COROLLARY 1. *Suppose, in addition to the hypotheses of Theorem 2, that $\beta(t)$ and $\gamma(t)$ are monotone and that $\beta(0) = \gamma(0) = 0$. Then*

$$(2.11) \quad \frac{\beta(t)}{t} \rightarrow \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1}, \quad \frac{\gamma(t)}{t} \rightarrow \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1}$$

as $t \rightarrow +\infty (0+)$.

Proof. If we set $h(t) = e^{-t}$, then $h(t)$ is in M' and we conclude, from (2.10), that

$$\begin{aligned} \frac{1}{t} \int_0^\infty e^{-(s/t)} d\beta(s) &\rightarrow \frac{BG_2 - CF_2}{F_1G_2 - F_2G_1}, \\ \frac{1}{t} \int_0^\infty e^{-(s/t)} d\gamma(s) &\rightarrow \frac{CF_1 - BG_1}{F_1G_2 - F_2G_1} \end{aligned}$$

as $t \rightarrow +\infty (0+)$. Now (2.11) follows from a familiar Tauberian theorem for the Laplace transform. (See [8, p. 192].)

3. Extensions of the Fatou theorem. Here we develop some Abelian results related to the Fatou theorem. It will be convenient to make use of the

complex notation, writing $u(x+iy)$ for $u(x, y)$ and letting $R(w)$ and $I(w)$ denote the real and imaginary parts, respectively, of the complex number w .

For $u(z)$, harmonic, and $n \geq 1$ we let

$$D_{\psi}^n u = \cos \psi \frac{\partial^n u}{\partial x^n} + \sin \psi \frac{\partial^n u}{\partial y \partial x^{n-1}}$$

denote the n th directional derivative of $u(z)$ in the direction ψ . Observe that

$$D_{\psi}^n u = R \left(e^{i\psi} \frac{d^n f}{dz^n} \right),$$

where $f(z)$ is analytic with $u = R(f)$, and that any linear combination of n th derivatives of $u(z)$ can be expressed as $CD_{\psi}u(z)$, where C is a constant and ψ is an appropriate angle.

For $u(z)$ in H we have the Poisson-Stieltjes representation (1.1) which, in complex notation, reduces to

$$(3.1) \quad u(z) = Ky + \frac{1}{\pi} \int_{-\infty}^{\infty} I \left\{ \frac{1}{s-z} \right\} d\alpha(s).$$

The Lebesgue theorem and (1.2) allow us to differentiate underneath the integral sign in (3.1) to obtain

$$(3.2) \quad D_{\psi}^1 u(z) = K \sin \psi + \frac{1}{\pi} \int_{-\infty}^{\infty} I \left\{ \frac{e^{i\psi}}{(s-z)^2} \right\} d\alpha(s)$$

and, for $n > 1$,

$$(3.3) \quad D_{\psi}^n u(z) = \frac{n!}{\pi} \int_{-\infty}^{\infty} I \left\{ \frac{e^{i\psi}}{(s-z)^{n+1}} \right\} d\alpha(s).$$

Next for any $c > 0$, it follows from (1.2) that

$$\int_{|s| \geq c} \frac{|d\alpha(s)|}{|s-z|^2} = O(1)$$

as $z \rightarrow 0$ and hence

$$(3.4) \quad \int_{|s| \geq c} I \left\{ \frac{1}{s-z} \right\} d\alpha(s) = O(y),$$

$$\int_{|s| \geq c} I \left\{ \frac{e^{i\psi}}{(s-z)^{n+1}} \right\} d\alpha(s) = O(1)$$

as $z \rightarrow 0$.

We list some formulas which are required later on.

LEMMA 2⁽¹⁾. *Suppose that $-1 < R(\zeta) < 1$, that $0 < \theta < \pi$, and that $n \geq 1$. Then*

$$(3.5) \quad \frac{1}{\pi} \int_0^\infty I \left\{ \frac{1}{s - e^{i\theta}} \right\} s^\zeta ds = \frac{\sin(\pi - \theta)\zeta}{\sin \pi\zeta},$$

$$(3.6) \quad -\frac{1}{\pi} \int_0^\infty I \left\{ \frac{1}{s + e^{i\theta}} \right\} s^\zeta ds = \frac{\sin \theta\zeta}{\sin \pi\zeta},$$

$$(3.7) \quad \frac{n!}{\pi} \int_0^\infty I \left\{ \frac{e^{i\psi}}{(s - e^{i\theta})^{n+1}} \right\} s^\zeta ds = \frac{(\zeta)_n \sin [(\pi - \theta)\zeta - (\psi - n\theta)]}{\sin \pi\zeta},$$

$$(3.8) \quad (-1)^{n+1} \frac{n!}{\pi} \int_0^\infty I \left\{ \frac{e^{i\psi}}{(s + e^{i\theta})^{n+1}} \right\} s^\zeta ds = \frac{(\zeta)_n \sin [\theta\zeta + (\psi - n\theta)]}{\sin \pi\zeta}$$

where we set

$$(3.9) \quad (\zeta)_n = \zeta(\zeta - 1) \cdots (\zeta - n + 1).$$

Proof. (3.6) and (3.8) can be established for $-\pi < \theta < \pi$ by contour integration. (Alternatively, they can be deduced from known Mellin transforms. See [2, pp. 307–308].) Then (3.5) and (3.7) follow from (3.6) and (3.8) by substituting $\theta - \pi$ for θ .

Our first Abelian theorem is the following result.

THEOREM 3. *Suppose that $u(z)$ is in H , that $-1 < \delta < 1$, and that*

$$\alpha(t) = O(|t|^{1+\delta})$$

as $t \rightarrow 0$. Then for all $0 < \theta < \pi$, ψ , and $n \geq 1$,

$$u(re^{i\theta}) = O(r^\delta), D_\psi^n u(re^{i\theta}) = \overset{n}{O}(r^{\delta-n})$$

as $r \rightarrow 0+$. We can replace “ O ” by “ o ” in both the hypotheses and the conclusions.

Proof. We will consider the “ o ” case. By hypothesis we can select $c > 0$ such that

$$(3.10) \quad |\alpha(t)| \leq \epsilon |t|^{1+\delta}$$

for $|t| \leq c$ and by (3.4) we can assume that $\alpha(t)$ is constant outside of this interval. Next we can assume that $K=0$ in (3.1) and (3.2) and, with an integration by parts, we obtain

$$u(z) = \frac{1}{\pi} \int_{-\infty}^\infty I \left\{ \frac{1}{(s - z)^2} \right\} \alpha(s) ds,$$

$$D_\psi^n u(z) = \frac{(n + 1)!}{\pi} \int_{-\infty}^\infty I \left\{ \frac{e^{i\psi}}{(s - z)^{n+2}} \right\} \alpha(s) ds.$$

(1) When $\zeta=0$ in (3.5), (3.6), (3.7), (3.8), (4.12) and similarly when $\delta=0$ in (3.13), (4.5), (4.7), the left hand side of each equation is equal to the limit of the right hand side as the relevant variable, ζ or δ , approaches 0.

Then with (3.10) we have

$$|u(re^{i\theta})| \leq \frac{2r \sin \theta}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon |s|^{1+\delta}}{|s - re^{i\theta}|^3} ds,$$

$$|D_{\psi}^n u(re^{i\theta})| \leq \frac{(n+1)!}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon |s|^{1+\delta}}{|s - re^{i\theta}|^{n+2}} ds$$

and the desired conclusions follow immediately since, for $n \geq 1$,

$$\int_{-\infty}^{\infty} \frac{|s|^{1+\delta}}{|s - re^{i\theta}|^{n+2}} ds = r^{\delta-n} \int_{-\infty}^{\infty} \frac{|\sigma|^{1+\delta}}{|\sigma - e^{i\theta}|^{n+2}} d\sigma = Cr^{\delta-n}$$

where $C = C(\theta, \delta, n) < \infty$.

THEOREM 4. *Suppose that $u(z)$ is in H , that $-1 < \delta < 1$, and that*

$$(3.11)^{(2)} \quad \begin{aligned} \alpha(t) &\sim At^{1+\delta}, \\ -\alpha(-t) &\sim Bt^{1+\delta} \end{aligned}$$

as $t \rightarrow 0+$. Then for all $0 < \theta < \pi$, ψ , and $n \geq 1$,

$$(3.12) \quad \begin{aligned} u(re^{i\theta}) &= u^*(re^{i\theta}) + o(r^\delta), \\ D_{\psi}^n u(re^{i\theta}) &= D_{\psi}^n u^*(re^{i\theta}) + o(r^{\delta-n}) \end{aligned}$$

as $r \rightarrow 0+$, where

$$(3.13) \quad u^*(re^{i\theta}) = (1 + \delta) \frac{A \sin(\pi - \theta)\delta + B \sin \theta\delta}{\sin \pi\delta} r^\delta.$$

Proof. This last result follows almost directly from Theorem 3. First set

$$(3.14) \quad \alpha^*(t) = At^{1+\delta}, \quad -\alpha^*(-t) = Bt^{1+\delta}$$

for $0 \leq t < \infty$; then $\alpha^*(t)$ satisfies (1.2) and

$$u^*(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} I \left\{ \frac{1}{s-z} \right\} d\alpha^*(s)$$

is in H . From (3.11) and (3.14) it follows that

$$\alpha(t) - \alpha^*(t) = o(|t|^{1+\delta})$$

as $t \rightarrow 0$ and hence (3.12) follows from Theorem 3. Finally we see that $u^*(re^{i\theta})$ is simply

$$(1 + \delta) \left\{ \frac{A}{\pi} \int_0^\infty I \left(\frac{1}{\sigma - e^{i\theta}} \right) \sigma^\delta d\sigma - \frac{B}{\pi} \int_0^\infty I \left(\frac{1}{\sigma + e^{i\theta}} \right) \sigma^\delta d\sigma \right\} r^\delta$$

and (3.13) follows from (3.5) and (3.6) of Lemma 2 with $\zeta = \delta$.

(²) Here and in what follows $\phi(x) \sim Cx^\rho$ means $x^{-\rho}\phi(x) \rightarrow C$ as $x \rightarrow 0+$.

Theorem 4 establishes a relation between the behaviour of $\alpha(t)$ near $t=0$ and the behaviour of $u(z)$ near $z=0$ for $|\delta| < 1$. (In particular we obtain the Fatou theorem when $\delta=0$ and $A=B$.) It is natural to inquire about the situation when $|\delta| = 1$.

The following two results discuss the cases when $\delta = -1$ and $\delta = 1$ respectively.

THEOREM 5. *Suppose that $u(z)$ is in H and that*

$$\alpha(t) - \alpha(-t) \rightarrow A\pi$$

as $t \rightarrow 0+$. Then for all $0 < \theta < \pi$, ψ , and $n \geq 1$,

$$\begin{aligned} u(re^{i\theta}) &= u^*(re^{i\theta}) + o(r^{-1}), \\ D_{\psi}^n u(re^{i\theta}) &= D_{\psi}^n u^*(re^{i\theta}) + o(r^{-1-n}) \end{aligned}$$

as $r \rightarrow 0+$, where $u^*(x, y) = Ay(x^2 + y^2)$.

THEOREM 6. *Suppose that $u(z)$ is in H and that the integral*

$$(3.15) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(s)}{s^2} = A$$

is absolutely convergent. Then for all $0 < \theta < \pi$, ψ , and $n \geq 1$,

$$(3.16) \quad \begin{aligned} u(re^{i\theta}) &= u^*(re^{i\theta}) + o(r), \\ D_{\psi}^n u(re^{i\theta}) &= D_{\psi}^n u^*(re^{i\theta}) + o(r^{1-n}) \end{aligned}$$

as $r \rightarrow 0+$, where $u^*(x, y) = (A + Ky)$.

Proofs. First note that Theorem 3 holds for $\delta = -1$; Theorem 5 is then established by an argument similar to the one used in the proof of Theorem 4.

The hypotheses of Theorem 6 imply that $1/s^2$ is integrable, in the Lebesgue-Stieltjes sense, over the real line. By (3.1),

$$\frac{u(re^{i\theta})}{r \sin \theta} = K + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(s)}{|s - re^{i\theta}|^2}.$$

For $s \neq 0$, $1/|s - re^{i\theta}|^2$ is bounded by C/s^2 and converges to $1/s^2$ as $r \rightarrow 0+$, and the first part of (3.16) follows by the "dominated convergence" theorem. The second part follows similarly.

When $u(z)$ is in H , there is no significant Fatou theorem with $|\delta| > 1$. For on the one hand, $\alpha(t)$ has left and right hand limits at each point and hence the conclusion of Theorem 5 *always* holds for some value of A . On the other hand, Theorem 6 shows us that for $\delta \geq 1$ the behaviour of $u(z)$ near $z=0$ depends upon the values $\alpha(t)$ assumes along the *entire* real line. For functions in H^+ this last fact is strikingly illustrated in Theorem 11.

4. **Extensions of the Loomis theorem.** We begin with a result which, for functions in H^+ , is a converse for Theorem 3 (cf. [1, Lemma 1]).

THEOREM 7. *Suppose that $u(z)$ is in H^+ , that δ is any real number, that $0 < a < \pi$, and that for $\theta = a$,*

$$(4.1) \quad u(re^{i\theta}) = O(r^\delta)$$

as $r \rightarrow 0+$. Then (4.1) holds for all $0 < \theta < \pi$ and

$$\alpha(t) = O(|t|^{1+\delta})$$

as $t \rightarrow 0$. We can replace "O" by "o" in the hypotheses and the conclusions.

Proof. It is not difficult to see that we may take $K = 0$ in (3.1). Then, with

$$C = \frac{\sin \theta}{\sin a} \text{LUB}_{|\sigma| < \infty} \left| \frac{\sigma - e^{ia}}{\sigma - e^{i\theta}} \right|^2 < \infty,$$

we obtain

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{r \sin \theta}{|s - re^{i\theta}|^2} d\alpha(s) \\ &\leq \frac{C}{\pi} \int_{-\infty}^{\infty} \frac{r \sin a}{|s - re^{ia}|^2} d\alpha(s) = Cu(re^{ia}). \end{aligned}$$

The second part follows since

$$\frac{\alpha(t) - \alpha(-t)}{t} \leq 2 \int_{-t}^t \frac{t}{s^2 + t^2} d\alpha(s) \leq 2\pi u(0, t)$$

for each $t > 0$.

The next three theorems are converses for Theorem 4. In these results we assume that $0 < a, b < \pi$, that $-1 < \delta < 1$, and that $m, n \geq 1$. Furthermore we set

$$(4.2) \quad \mu = c - ma, \quad \eta = d - nb,$$

and we adopt the notation (3.9).

THEOREM 8. *Suppose that $u(z)$ is in H^+ , that $a \neq b$, and that*

$$(4.3) \quad u(re^{ia}) \sim Ar^\delta, \quad u(re^{ib}) \sim Br^\delta$$

as $r \rightarrow 0+$. Then

$$(4.4) \quad \alpha(t) \sim Ct^{1+\delta}, \quad -\alpha(-t) \sim Dt^{1+\delta}$$

as $t \rightarrow 0+$, where

$$(4.5) \quad \begin{aligned} (1 + \delta)C &= \frac{A \sin b\delta - B \sin a\delta}{\sin(b - a)\delta}, \\ (1 + \delta)D &= \frac{B \sin(\pi - a)\delta - A \sin(\pi - b)\delta}{\sin(b - a)\delta}. \end{aligned}$$

Theorem 8 was first proved by Allen and Kerr by a different method (see [1, Theorem III]).

THEOREM 9. *Suppose that $u(z)$ is in H^+ , that $(b-a)\delta + \eta \not\equiv 0 \pmod{\pi}$, and that*

$$(4.6) \quad \begin{aligned} u(re^{ia}) &\sim Ar^\delta, \\ D_a^n u(re^{ib}) &\sim Br^{\delta-n} \end{aligned}$$

as $r \rightarrow 0+$. Then (4.4) holds with

$$(4.7) \quad \begin{aligned} (1 + \delta)C &= \frac{(\delta)_n A \sin [b\delta + \eta] - B \sin a\delta}{(\delta)_n \sin [(b-a)\delta + \eta]}, \\ (1 + \delta)D &= \frac{B \sin (\pi - a)\delta - (\delta)_n A \sin [(\pi - b)\delta - \eta]}{(\delta)_n \sin [(b-a)\delta + \eta]}. \end{aligned}$$

THEOREM 10. *Suppose that $u(z)$ is in H^+ , that $(b-a)\delta + R - \mu \not\equiv 0 \pmod{\pi}$, that $\delta \neq 0$, and that*

$$\begin{aligned} u(re^{ia}) &= O(r^\delta), \\ D_c^m u(re^{ia}) &\sim Ar^{\delta-m}, \\ D_a^n u(re^{ib}) &\sim Br^{\delta-m} \end{aligned}$$

as $r \rightarrow 0+$. Then (4.4) holds with

$$\begin{aligned} (1 + \delta)C &= \frac{(\delta)_n A \sin [b\delta + \eta] - (\delta)_m B \sin [a\delta + \mu]}{(\delta)_m (\delta)_n \sin [(b-a)\delta + \eta - \mu]}, \\ (1 + \delta)D &= \frac{(\delta)_m B \sin [(\pi - a)\delta - \mu] - (\delta)_n A \sin [(\pi - b)\delta - \eta]}{(\delta)_m (\delta)_n \sin [(b-a)\delta + \eta - \mu]}. \end{aligned}$$

Proofs. The proofs for these three theorems follow along similar lines. We give only the details for Theorem 9.

By (3.4) we can assume that $\alpha(t)$ is constant for large t ; by (4.6) and Theorem 7 we see that

$$\alpha(t) = O(|t|^{1+\delta})$$

for small t . From these two observations it follows that

$$(4.8) \quad \begin{aligned} \beta(t) &= \int_0^t \frac{d\alpha(s)}{s^\delta}, \\ \gamma(t) &= - \int_0^t \frac{d\alpha(-s)}{s^\delta} \end{aligned}$$

are defined and that

$$\int_t^{et} \frac{|d\beta(s)|}{s} = \int_t^{et} \frac{d\alpha(s)}{s^{1+\delta}} \leq \frac{\alpha(et) - \alpha(t)}{t^{1+\delta}} = O(1),$$

$$\int_t^{et} \frac{|d\gamma(s)|}{s} = - \int_t^{et} \frac{d\alpha(-s)}{s^{1+\delta}} \leq \frac{\alpha(-t) - \alpha(-et)}{t^{1+\delta}} = O(1)$$

for all $t > 0$. Hence $\beta(t)$ and $\gamma(t)$ are in V' . Furthermore, by a Tauberian theorem due to Allen and Kerr,

$$(4.9) \quad \begin{aligned} \frac{\alpha(t)}{t^{1+\delta}} \rightarrow C \quad \text{iff} \quad \frac{\beta(t)}{t} \rightarrow (1 + \delta)C, \\ -\frac{\alpha(-t)}{t^{1+\delta}} \rightarrow D \quad \text{iff} \quad \frac{\gamma(t)}{t} \rightarrow (1 + \delta)D \end{aligned}$$

as $t \rightarrow 0+$ (see [1, Lemma 2]).

Once again we can assume that $K = 0$ in (3.1) and (3.2). Then with (4.8) we obtain

$$u(z) = \frac{1}{\pi} \int_0^\infty I \left\{ \frac{s^\delta}{s-z} \right\} d\beta(s) - \frac{1}{\pi} \int_0^\infty I \left\{ \frac{s^\delta}{s+z} \right\} d\gamma(s),$$

$$D_\psi^n u(z) = \frac{n!}{\pi} \int_0^\infty I \left\{ \frac{e^{i\psi} s^\delta}{(s-z)^{n+1}} \right\} d\beta(s) + (-1)^{n+1} \frac{n!}{\pi} \int_0^\infty I \left\{ \frac{e^{i\psi} s^\delta}{(s+z)^{n+1}} \right\} d\gamma(s),$$

and with

$$(4.10) \quad \begin{aligned} f_1(t) &= \frac{1}{\pi} I \left\{ \frac{t^\delta}{t - e^{ia}} \right\}, & f_2(t) &= -\frac{1}{\pi} I \left\{ \frac{t^\delta}{t + e^{ia}} \right\}, \\ g_1(t) &= \frac{n!}{\pi} I \left\{ \frac{e^{id} t^\delta}{(t - e^{ib})^{n+1}} \right\}, & g_2(t) &= (-1)^{n+1} \frac{n!}{\pi} I \left\{ \frac{e^{id} t^\delta}{(t + e^{ib})^{n+1}} \right\}, \end{aligned}$$

we conclude that

$$(4.11) \quad \begin{aligned} r^{-\delta} u(re^{ia}) &= \frac{1}{r} \int_0^\infty f_1\left(\frac{s}{r}\right) d\beta(s) + \frac{1}{r} \int_0^\infty f_2\left(\frac{s}{r}\right) d\gamma(s), \\ r^{n-\delta} D_a^n u(re^{ib}) &= \frac{1}{r} \int_0^\infty g_1\left(\frac{s}{r}\right) d\beta(s) + \frac{1}{r} \int_0^\infty g_2\left(\frac{s}{r}\right) d\gamma(s) \end{aligned}$$

for $r > 0$. By virtue of (2.9), all the functions in (4.10) are in M' and, by Lemma 2 and (4.2), we have

$$\begin{aligned} F_1(y) &= \frac{\sin(\pi - a)\zeta}{\sin \pi\zeta}, & F_2(y) &= \frac{\sin a\zeta}{\sin \pi\zeta}, \\ G_1(y) &= \frac{(\zeta)_n \sin[(\pi - b)\zeta - \eta]}{\sin \pi\zeta}, & G_2(y) &= \frac{(\zeta)_n \sin[b\zeta + \eta]}{\sin \pi\zeta}, \end{aligned}$$

where $\zeta = \delta + iy$. It follows that

$$(4.12) \quad F_1(y)G_2(y) - F_2(y)G_1(y) = \frac{(\zeta)_n \sin [(b - a)\zeta + \eta]}{\sin \pi \zeta} \neq 0$$

for all real y and we appeal to (4.6), (4.11), Corollary 1, and (4.9) to complete the proof.

Unfortunately, the restriction that $\delta \neq 0$ is essential in Theorem 10. For consider

$$f(z) = (\log z)^i = e^{i \log (\log z)}$$

where, for $z = \rho e^{i\phi}$ and $0 < \phi < \pi$, we define

$$\log z = \log \rho + i\phi.$$

Now $f(z)$ is analytic and $|f(z)| \leq 1$ in $y > 0$. Furthermore it is not difficult to show that for all $0 < \theta < \pi$ and $n \geq 1$,

$$\frac{d^n}{dz^n} f(re^{i\theta}) = o(r^{-n})$$

as $r \rightarrow 0+$. With $u = R(f) + 1$ it follows that

$$(4.13) \quad \limsup_{r \rightarrow 0+} u(re^{i\theta}) - \liminf_{r \rightarrow 0+} u(re^{i\theta}) = 2e^{-\pi},$$

for each $0 < \theta < \pi$, and that $u(z)$ satisfies the hypotheses of Theorem 10 with $\delta = 0$ and $A = B = 0$. However, (4.4) cannot hold for any C and D for otherwise Theorem 4 would contradict (4.13).

Finally consider the situation when $|\delta| = 1$. Though Theorem 5 has no converse we can invert Theorem 6 as follows (cf. [7, p. 208]).

THEOREM 11. *Suppose that $u(z)$ is in H^+ and that $0 < a < \pi$. If*

$$u(re^{ia}) = O(r)$$

as $r \rightarrow 0+$, then the integral (3.15) is absolutely convergent. If

$$(4.14) \quad u(re^{ia}) = o(r)$$

as $r \rightarrow 0+$, then $u(z) \equiv 0$.

Proof. By Theorem 7 we see that

$$(4.15) \quad \frac{u(0, t)}{t} = K + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(s)}{s^2 + t^2} = O(1)$$

as $t \rightarrow 0+$ and, by Fatou's lemma, we conclude that

$$(4.16) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(s)}{s^2} \leq \liminf_{t \rightarrow 0+} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\alpha(s)}{s^2 + t^2} < \infty.$$

With (4.14) we can replace "O" by "o" on the right side of (4.15) and we conclude that K and the left side of (4.16) are equal to zero; hence $\alpha(t) \equiv 0$.

5. An application. Here we use Theorem 9 to obtain an extension of the following result due to Lindelöf (see [6, p. 70]).

LINDELÖF THEOREM. *Suppose that $f(z)$ is analytic and bounded in $y > 0$, that $0 < a < \pi$, and that for $\theta = a$,*

$$(5.1) \quad f(re^{i\theta}) \rightarrow A + iB$$

as $r \rightarrow 0+$. Then (5.1) holds for all $0 < \theta < \pi$.

LEMMA 3. *Suppose that $u(z)$ is in H , that $v(z)$ is a conjugate for $u(z)$, that $-1 < \delta < 1$, and that*

$$(5.2) \quad \alpha(t) = O(|t|^{1+\delta})$$

as $t \rightarrow 0$. If $0 < a < \pi$ and

$$(5.3) \quad u(re^{ia}) \sim Ar^\delta$$

as $r \rightarrow 0+$, then

$$\frac{\partial}{\partial r} u(re^{ia}) \sim \delta Ar^{\delta-1}$$

as $r \rightarrow 0+$. If $0 < b < \pi$ and

$$v(re^{ib}) \sim Br^\delta$$

as $r \rightarrow 0+$, then

$$\frac{\partial}{\partial r} v(re^{ib}) \sim \delta Br^{\delta-1}$$

as $r \rightarrow 0+$.

Proof. For the first part set $\phi(r) = u(re^{ia})$ for $r > 0$. By (5.3), (5.2), and Theorem 3 we see that

$$\begin{aligned} \phi(r) &\sim Ar^\delta, \\ \phi''(r) &= O(r^{\delta-2}) \end{aligned}$$

as $r \rightarrow 0+$, and we conclude that

$$\phi'(r) \sim \delta Ar^{\delta-1}$$

as $r \rightarrow 0+$ by a well known Tauberian theorem due to Hardy and Littlewood (see [8, p. 194]). The second part follows similarly.

Observe that Lemma 3 allows us to deduce Theorem 8 from Theorem 9. For the hypotheses (4.3) imply that (5.2) holds, by virtue of Theorem 7, and hence that

$$\begin{aligned} u(re^{ia}) &\sim Ar^\delta, \\ D_a^1 u(re^{ib}) &\sim \delta Br^{\delta-1} \end{aligned}$$

as $r \rightarrow 0+$, by the first part of Lemma 3. Setting $d=b$ and $n=1$ in Theorem 9 completes the argument.

Now let \mathfrak{A}^+ denote the class of functions $f(z)$, analytic with $R(f) \geq 0$ in $y > 0$, and assume that $0 < a, b < \pi$ and that $-1 < \delta < 1$.

THEOREM 12. *Suppose that $f(z)$ is in \mathfrak{A}^+ , that $(b-a)\delta \not\equiv (\pi/2) \pmod{\pi}$, and that*

$$(5.4) \quad \begin{aligned} R\{f(re^{ia})\} &\sim Ar^\delta, \\ I\{f(re^{ib})\} &\sim Br^\delta \end{aligned}$$

as $r \rightarrow 0+$. Then for all $0 < \theta < \pi$ and $n \geq 1$,

$$(5.5) \quad \begin{aligned} f(re^{i\theta}) &= f^*(re^{i\theta}) + o(r^\delta), \\ \frac{d^n}{dz^n} f(re^{i\theta}) &= \frac{d^n}{dz^n} f^*(re^{i\theta}) + o(r^{\delta-n}) \end{aligned}$$

as $r \rightarrow 0+$, where

$$(5.6) \quad f^*(z) = \frac{Ae^{-ib\delta} + iBe^{-ia\delta}}{\cos(b-a)\delta} z^\delta.$$

Proof. Set $u = R(f)$ and $v = I(f)$; $u(z)$ is in H^+ . By (5.4) we see that

$$(5.7) \quad \begin{aligned} u(re^{ia}) &\sim Ar^\delta, \\ v(re^{ib}) &\sim Br^\delta \end{aligned}$$

as $r \rightarrow 0+$. With Theorem 7 and the second part of Lemma 3 we obtain

$$\frac{\partial}{\partial r} v(re^{ib}) \sim \delta Br^{\delta-1}$$

as $r \rightarrow 0+$, and appealing to the Cauchy-Riemann equations we conclude that

$$(5.8) \quad D_a^1 u(re^{ib}) \sim \delta Br^{\delta-1}$$

as $r \rightarrow 0+$, where $d=b-\pi/2$. Now apply Theorem 9 to the first part of (5.7) and to (5.8); we obtain (4.4) with

$$\begin{aligned} (1+\delta)C &= \frac{A \cos b\delta + B \sin a\delta}{\cos(b-a)\delta}, \\ (1+\delta)D &= \frac{A \cos(\pi-b)\delta - B \sin(\pi-a)\delta}{\cos(b-a)\delta}. \end{aligned}$$

From Theorem 4 it follows that for all $0 < \theta < \pi$, ψ , and $n \geq 1$

$$(5.9) \quad \begin{aligned} u(re^{i\theta}) &= u^*(re^{i\theta}) + o(r^\delta), \\ D_\psi^n u(re^{i\theta}) &= D_\psi^n u^*(re^{i\theta}) + o(r^{\delta-n}) \end{aligned}$$

as $r \rightarrow 0+$, where $u^* = R(f^*)$ and $f^*(z)$ is as defined in (5.6). Set $v^* = I(f^*)$ and observe, by the Cauchy-Riemann equations and the second part of (5.9), that

$$(5.10) \quad \frac{\partial}{\partial \phi} v(re^{i\phi}) = \frac{\partial}{\partial \phi} v^*(re^{i\phi}) + o(r^\delta)$$

as $r \rightarrow 0+$. We conclude⁽³⁾ that for all $0 < \theta < \pi$,

$$\begin{aligned} v(re^{i\theta}) &= \int_b^\theta \frac{\partial}{\partial \phi} v(re^{i\phi}) d\phi + v(re^{ib}) \\ &= \int_b^\theta \frac{\partial}{\partial \phi} v^*(re^{i\phi}) d\phi + v^*(re^{ib}) + o(r^\delta) \\ &= v^*(re^{i\theta}) + o(r^\delta) \end{aligned}$$

as $r \rightarrow 0+$ and the proof is complete.

Finally if $|f(z)| \leq M$ we can replace $f(z)$ by $f(z) + M$ and obtain the Lindelöf theorem from Theorem 12 with $a = b$ and $\delta = 0$.

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⁽³⁾ It is not difficult to show that (5.10) holds *uniformly* in ϕ for ϕ between b and θ .