ON ORDER-PRESERVING INTEGRATION(1)

BY

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Introduction. This paper is concerned with integrals of the form $\int_S f(s)\mu(ds)$, where f is a real-valued function defined on S and μ is a finitely-additive function whose domain is a ring Σ of subsets of S and whose range is contained in the class of positive elements of a Dedekind complete(2) partially ordered vector space $\mathfrak{X}$. It covers another aspect of the problem of integration with respect to a vector-valued measure, considered by McShane [8] and by Bartle, Dunford, and Schwartz [2].

The theory of integration is based on a theory of convergence of generalized sequences in the space $\mathfrak{X}$. This theory is presented in §I, with a brief discussion of other convergence theories.

§II treats μ-measurability and integrability. The central concept is that of convergence in measure. A generalized sequence $\{f_a\}$ *converges to* f *in measure* if, for every $\epsilon>0$,

$$\lim_a \mu^*\{s \mid |f_a(s) - f(s)| \geqq \epsilon\} = 0,$$

where

$$\mu^* D = \bigwedge (e \supseteq D \cdot e \in \Sigma)\mu e;$$

a function f is *μ-measurable* if for each $e\in\Sigma$ there exists a generalized sequence of simple functions converging to $\chi_e f$ in measure; the *integral* of a μ-measurable function $f\geqq 0$ is the supremum of the integrals of all simple functions g such that $0\leqq g\leqq f$; the integral of an arbitrary f is $\int_S f^+ - \int_S f^-$. If properly worded, the Vitali and Lebesgue convergence theorems for integrals are valid in the above-described situation: This is the principal result of §II. When μ is countably-additive, this result improves a special case of a theorem of McShane [8, pp. 59 and 81].

The main theorem of §III is an abstract form of the Riesz representation theorem: If S is a normal space and T is a positive linear transformation from $C(S)$ into $\mathfrak{X}$, then

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Received by the editors March 20, 1956.

(1) The research contained in this paper was done under contract NONR609(04) with the Office of Naval Research, and was submitted to the Faculty of the Graduate School of Yale University in 1954 in partial fulfillment of the requirements for the degree of Doctor of Philosophy. The author wishes to thank Professor N. Dunford for his advice and encouragement.

(2) Definitions are in the body of the paper.

$$Tf = \int_S f(s)\mu(ds)$$

for some finitely-additive function μ; if $\mathfrak{X}$ is an algebra and T is multiplicative, then

$$\mu(E_1 \cap E_2) = \mu E_1 \cdot \mu E_2.$$

This theorem is used as the basis of an alternative proof of the spectral theorem for self-adjoint operators in Hilbert space.

I

1. Partially ordered sets.

1.1. Throughout this section $(\mathfrak{X}, \leqq)$ will denote a partially ordered set. Whenever they exist, the supremum and infimum of a subset $\mathfrak{M} \subseteq \mathfrak{X}$ will be denoted by $\bigvee \mathfrak{M}$ and $\bigwedge \mathfrak{M}$, respectively. Alternate notations are $\bigvee_a X_a$, $\bigwedge_a X_a$, $X_1 \vee X_2 \vee \cdots \vee X_n$ (for a finite class), and $X_1 \wedge X_2 \wedge \cdots \wedge X_n$. The following statements are easy to prove: If $\mathfrak{M}$ and $\mathfrak{N}$ are subsets of $\mathfrak{X}$ such that $\bigvee \mathfrak{M}$ and $\bigwedge \mathfrak{N}$ exist, and if $M \leqq N$ for each $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$, then $\bigvee \mathfrak{M} \leqq \bigwedge \mathfrak{N}$. If $\{X_{ab}\} \subseteq \mathfrak{X}$ is a set such that $\bigwedge_b X_{ab}$, $\bigwedge_a X_{ab}$, $\bigwedge_a \bigwedge_b X_{ab}$, $\bigwedge_b \bigwedge_a X_{ab}$, and $\bigwedge_{a,b} X_{ab}$ all exist, then $\bigwedge_a \bigwedge_b X_{ab} = \bigwedge_b \bigwedge_a X_{ab} = \bigwedge_{a,b} X_{ab}$, and dually.

1.2. The theory of convergence on which the integration theory is based uses the concepts of *directed set*, *directed floor*, and *directed tower*. The following simple (artificial) example is given to illustrate the concepts involved.

Consider the following array:

$$\begin{array}{ccccccccc}
y_{11} & < & y_{12} & < & y_{13} & < & \cdots & < & y_1 \\
 & & & & & & & & \wedge \\
y_{21} & < & y_{22} & < & \cdots & & & < & y_2 \\
 & & & & & & & & \wedge \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & & & \wedge \\
y_{n1} & < & y_{n2} & < & \cdots & & & < & y_n \\
 & & & & & & & & \wedge \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & & & \wedge \\
 & & & & & & & & 0
\end{array}$$

Let $\mathfrak{A}$ be the partially ordered set consisting of all of the above elements y_{nk} and y_n, ordered as indicated in the diagram, together with the elements $x_1, x_2, \cdots$, which are related to the elements y_{nk}, y_n, and 0 as follows: $y_{nk} < x_k < 0$ for all n, but the x_i are not related to the y_n, nor is any x_i related to any x_j when $i \neq j$.

1.3. Definitions. A set A is *directed* by a partial ordering $\prec$ if to each pair of its elements, a_1 and a_2, there exists $a_3 \in A$ such that $a_3 \prec a_1$ and $a_3 \prec a_2$. We symbolize such a directed set by $A^{\prec}$ or by $A(\prec)$. A directed *floor* $\Phi^{\leqq}$ (or $\Phi^{\geqq}$) is a class of directed subsets of a partially ordered set $\mathfrak{X}$, all of which are directed by the same order relation $\leqq$ (or $\geqq$). A floor $\Phi^{\leqq}$ (or $\Phi^{\leqq}$) is a directed *first floor with base B* if it consists of a single element $\mathfrak{M}^{\leqq}$ ($\mathfrak{M}^{\geqq}$) such that $\bigwedge \mathfrak{M} = B$ ($\bigvee \mathfrak{M} = B$). Let $\omega = \{\Phi_1, \cdots, \Phi_k\}$ be a finite ordered collection of floors, all of which are directed by $\leqq$ ($\geqq$). Then ω is a directed *tower with base B* if

(a) Φ_1 is a first floor with base B;

(b) if $M \in \mathfrak{M} \in \Phi_n$ and $n < k$, then there exists $\mathfrak{N}_M \in \Phi_{n+1}$ such that $\bigwedge \mathfrak{N}_M = M$ ($\bigvee \mathfrak{N}_M = M$).

We symbolize such a tower by $\omega^{\leqq}$ ($\omega^{\geqq}$). The number k will be called the *height* of the tower ω. A tower of height k is a *k-tower*. To simplify the notation we shall write $M \in \omega$ to mean $M \in \mathfrak{M} \in \Phi_k \in \omega$, where ω is a k-tower. Similarly, $M \in \Phi_i$ will mean $M \in \mathfrak{M} \in \Phi_i$.

1.4. In the preceding example, $\{y_{nk}\}^{\geqq}$ is a directed set for each n; $\{y_n\}$ is also a directed set. The class $\Phi^{\geqq} = \{\{y_n\}\}$ is a first floor with base 0; the class $\psi^{\geqq} = \{\{y_{1k}\}, \{y_{2k}\}, \cdots\}$ is also a floor. The class $\omega^{\geqq} = \{\Phi, \psi\}$ forms a 2-tower with base 0.

1.5. Definition. A tower $\omega_n = \{\Phi_1, \cdots, \Phi_n\}$ is an *extension* of a tower $\sigma_k = \{\psi_1, \cdots, \psi_k\}$ if $k \leqq n$ and $\psi_i = \Phi_i$ for $i = 1, \cdots, k$. Here all floors are directed by the same order relation.

1.6. Lemma. *Let $\omega^{\leqq}$ ($\omega^{\geqq}$) be a k-tower. Suppose that, for each $M \in \mathfrak{M} \in \Phi_k \in \omega$ there exists an n-tower $(\sigma_M)^{\leqq}$ $[(\sigma_M)^{\geqq}]$ with base M: $\sigma_M = \{\psi_1(M), \cdots, \psi_n(M)\}$. Let $\omega' = \{\Phi_1, \cdots, \Phi_{k+n}\}$, where $\Phi_{k+i} = \bigcup_M \psi_i(M)$, $i = 1, \cdots, n$. Then ω' is a tower which is an extension of ω.*

Proof. Obviously all Φ_m are floors, and Φ_1 is a first floor. Suppose $N \in \mathfrak{N} \in \Phi_j$. If $j < k$, then, since ω is a tower, there exists $\mathfrak{N}_N \in \Phi_{j+1}$ such that $\bigwedge \mathfrak{N}_N = N$ ($\bigvee \mathfrak{N}_N = N$). If $j = k$, then by hypothesis there exists a tower σ_N with base N. Hence $\mathfrak{N}_N \in \psi_1(N) \in \sigma_N$ has the property that $\bigwedge \mathfrak{N}_N = N$ ($\bigvee \mathfrak{N}_N = N$). If $k < j < k+n$, then $\mathfrak{N} \in \psi_j(M)$ for some M. Since σ_M is a tower, there exists $\mathfrak{N}_N \in \psi_{j+1}(M) \subseteq \Phi_{j+1}$ such that $\bigwedge \mathfrak{N}_N = N$ ($\bigvee \mathfrak{N}_N = N$). Hence ω' is a tower. The fact that ω' is an extension of ω follows immediately.

1.7. Definitions. The tower ω' of the preceding lemma will be called the *extension* of ω *by means of the n-towers* σ_M. An *elementary k-tower with base B* is a tower $\{\Phi_1, \cdots, \Phi_k\}$, where each Φ_i consists of the single-element set $\{B\}$. The direction of $\{B\}$ is considered to be the same as the direction of all Φ_i. An extension of a tower ω by means of elementary towers σ_M is called a *canonical extension* of ω.

1.8. A k-tower has one and only one canonical extension to a $(k+n)$-tower, $n = 0, 1, 2, \cdots$.

1.9. DEFINITIONS. A *generalized sequence* of elements of a set $\mathfrak{M}\subseteq\mathfrak{X}$ is a function whose domain is a directed set and whose range is contained in $\mathfrak{M}$. A generalized sequence $\{X_a | a\in A(\prec)\}$ is *k-convergent* to X if there exist k-towers $\omega^{\geqq}$ and $\omega^{\leqq}$ with base X such that, for each $M\in\omega^{\leqq}$ and $N\in\omega^{\geqq}$, there exists $a(M, N)\in A$ such that $M\leqq X_a\leqq N$ for $a\prec a(M, N)$. The towers $\omega^{\geqq}$ and $\omega^{\leqq}$ are said to be *associated* with $\{X_a\}$ or with X. Symbolically, $X=k\text{-lim } X_a$.

The sequence $\{y_n\}$ of the preceding example is 1-convergent to 0; on the other hand, the sequence $\{x_k\}$ is 2-convergent to 0 but 1-lim x_k does not exist. Intuitively, a generalized sequence $\{X_a\}$ 1-converges to X if its elements are eventually "squeezed close" to X; the generalized sequence 2-converges to X if its elements are eventually squeezed close to elements which are squeezed close to X.

1.10. Note that 1-convergence is the same as *o*-convergence as defined by McShane [8, p. 15].

In the sequel, we shall often abbreviate statements such as "given M and N, there exists $a(M, N)\in A$ such that $M\leqq X_a\leqq N$ for $a\prec a(M, N)$" by writing "eventually, $M\leqq X_a\leqq N$."

1.11. THEOREM. *If $X=k\text{-lim } X_a$ and $Y=k\text{-lim } X_a$, then $X=Y$. Hence k-convergence of a generalized sequence is uniquely defined and independent of a particular choice of associated towers.*

Proof. Suppose that the towers $\omega^{\leqq}=\{\Phi_1(\leqq), \cdots, \Phi_k\}$ and $\omega^{\geqq}=\{\Phi_1(\geqq), \cdots, \Phi_k\}$ are associated with X, and that the towers $\sigma^{\leqq}=\{\psi_1(\leqq), \cdots, \psi_k\}$ and $\sigma^{\geqq}=\{\psi_1(\geqq), \cdots, \psi_k\}$ are associated with Y. If $k=1$, let $R=X$ and $S=Y$; if $k>1$, let $R\in\Phi_{k-1}(\geqq)$ and $S\in\psi_{k-1}(\leqq)$ be arbitrary. Then there exists $\mathfrak{M}_R\in\Phi_k(\geqq)$ and $\mathfrak{N}_S\in\psi_k(\leqq)$ such that $\bigvee\mathfrak{M}_R=R$ and $\bigwedge\mathfrak{N}_S=S$. If $M\in\mathfrak{M}_R$ and $N\in\mathfrak{N}_S$, then eventually $M\leqq X_a\leqq N$. Hence $\bigvee\mathfrak{M}_R\leqq\bigwedge\mathfrak{N}_S$. If $k>1$, it follows that $R\leqq S$ for every $R\in\Phi_{k-1}(\geqq)$ and $S\in\psi_{k-1}(\leqq)$. If §1.1 is applied $k-1$ times then $X\leqq Y$; if $k=1$, then the result $X\leqq Y$ was obtained directly. Similarly, $Y\leqq X$. Hence $X=Y$.

1.12. LEMMA. *If $X=k\text{-lim } X_a$, then $X=(k+n)\text{-lim } X_a$ for any non-negative integer n.*

Proof. Let $\omega^{\leqq}$ and $\omega^{\geqq}$ be k-towers associated with $\{X_a\}$. Let σ be the canonical extension of ω to a $(k+n)$-tower. Then $\sigma^{\leqq}$ and $\sigma^{\geqq}$ are $(k+n)$-towers associated with $\{X_a\}$.

1.13. DEFINITION. On the basis of §1.12, we say that $\{X_a\}$ is *convergent* to the *limit* X, and write $X=\lim X_a$, if $X=k\text{-lim } X_a$ for some k. From §1.11 it is clear that a generalized sequence can converge to at most one limit.

1.14. THEOREM. (a) *If $X_a=X$, then* $\lim X_a=X$; (b) *if $X=\lim X_a$ and eventually $Y\leqq X_a\leqq Z$, then $Y\leqq X\leqq Z$;* (c) *if $Y=k\text{-lim } X_a=k\text{-lim } Z_a$ and eventually $X_a\leqq Y_a\leqq Z_a$, then $k\text{-lim } Y_a$ exists and equals Y.*

Proof. Part (a) is clear from the fact that $\{\{\{X\}\}\}^{\leqq}$ is a tower associated with X. To prove (b) let $\omega^{\leqq}$ be a k-tower associated with $\{X_a\}$. If $M\in\omega$, then eventually $Y\leqq X_a\leqq M$. By §1.1 it follows that $Y\leqq N$ for arbitrary $N\in\omega$. Hence $Y\leqq X$. The inequality $X\leqq Z$ is proved in similar fashion. To prove (c), let $\omega^{\geqq}$ be associated with $\{X_a\}$ and $\omega^{\leqq}$ be associated with $\{Z_a\}$. Then $\omega^{\geqq}$ and $\omega^{\leqq}$ are associated with $\{Y_a\}$.

1.15. Note that the method given in §1.9 for defining limits in a partially ordered set $\mathfrak{X}$ imposes a topology t_k on $\mathfrak{X}$: A set $\mathfrak{M}\subseteq\mathfrak{X}$ is closed in this topology if k-lim $X_a\in\mathfrak{M}$ whenever $\{X_a\}\subseteq\mathfrak{M}$ and k-lim X_a exists: Similarly, a topology t can be defined by means of the convergence defined in §1.13. If $k\leqq n$, then by §1.12 every closed set in t_n is closed in t_k. That is, t_k is stronger than t_n. Similarly, the topology t is weaker than every t_k. Since every one-point set is closed relative to t, it is clear that $\mathfrak{X}$ is a T_1-space relative to t and all t_k.

1.16. If we take the class of all closed intervals $\mathfrak{J}_0^+=\{X\mid X\geqq X_0\}$ and $\mathfrak{J}_0^-=\{X\mid X\leqq X_0\}$—where X_0 is an arbitrary element of $\mathfrak{X}$—as a subbasis for a topology on $\mathfrak{X}$ we obtain the *interval topology*. The set $\mathfrak{X}$ is a T_1-space relative to the interval topology [4, p. 570] and [3, p. 61]. Since every closed interval is closed in any of the topologies of §1.9, it follows that the interval topology is weaker than the topology t and all topologies t_k.

1.17. Definition. A lattice $\mathfrak{X}$ is *complete* if every subset of $\mathfrak{X}$ which is bounded above has a supremum in $\mathfrak{X}$ and every subset of $\mathfrak{X}$ which is bounded below has an infimum in $\mathfrak{X}$.

1.18. Theorem. *If $\mathfrak{X}$ is a complete lattice and $\{X_a\}$ is a generalized sequence of elements in $\mathfrak{X}$, then the following statements are equivalent*:

(a) $\{X_a\}$ *1-converges to X*;
(b) $\{X_a\}$ *converges to X*;
(c) lim inf X_a *and* lim sup X_a *both exist and equal X*.

Proof. Obviously (a) implies (b). If (b) holds, let $\omega^{\leqq}=\{\Phi_1(\leqq),\cdots,\Phi_k(\leqq)\}$ and $\omega^{\geqq}=\{\Phi_1(\geqq),\cdots,\Phi_k(\geqq)\}$ be towers associated with $\{X_a\}$, and let $M^{\leqq}\in\omega^{\leqq}$ and $M^{\geqq}\in\omega^{\geqq}$ be arbitrary. Then eventually $M^{\geqq}\leqq\bigwedge(a\prec b)X_a\leqq\bigvee(a\prec b)X_a\leqq M^{\leqq}$. Hence $M^{\geqq}\leqq\liminf X_a\leqq\limsup X_a\leqq M^{\leqq}$. By §1.1, this implies that $N^{\geqq}\leqq\liminf X_a\leqq\limsup X_a\leqq N^{\leqq}$ for arbitrary $N^{\leqq}\in\Phi_i(\leqq)$ and $N^{\geqq}\in\Phi_i(\geqq)$, $i=1,\cdots,k$. It follows that $X=\liminf X_a=\limsup X_a$. Now assume (c). Then eventually $M_b=\bigwedge(a\prec b)X_a$ and $N_b=\bigvee(a\prec b)X_a$ exist. Now $\mathfrak{M}^{\geqq}=\{M_b\}$ and $\mathfrak{N}^{\leqq}=\{N_b\}$ are obviously directed sets, and eventually $M_b\leqq X_a\leqq N_b$ for arbitrary b. Part (a) now follows, thus completing the proof.

1.19. According to MacNeille [7, pp. 443–444], any partially ordered set $\mathfrak{X}$ can be imbedded in a complete lattice $\mathfrak{L}$ having a greatest element and a least element in such a way that infima and suprema are preserved. Birkhoff [3, p. 60] suggests that this device could be used to define a topology in $\mathfrak{X}$: If X_a, $X\in\mathfrak{X}$, then $X=\lim X_a$ in $\mathfrak{X}$ if and only if $X=\liminf X_a=\limsup X_a$ in

$\mathfrak{L}$. With Birkhoff, we call this topology the *relative topology*. Let $(\mathfrak{X}, \leqq)$ be a partially ordered set topologized with the interval topology. Let $\mathfrak{J}_0^+ = \{X \mid X \geqq X_0\}$ be a closed interval in $\mathfrak{X}$. If $X_a \in \mathfrak{J}_0^+$ and $X = \lim X_a$ in the relative topology, then since $X \geqq \bigwedge_a X_a \geqq X_0$, it is clear that $X \in \mathfrak{J}_0^+$. Hence $\mathfrak{J}_0^+$ is closed in the relative topology. Similarly, $\mathfrak{J}_0^- = \{X \mid X \leqq X_0\}$ is closed in the relative topology. It follows that the relative topology is stronger than the interval topology. We now show that the relative topology is weaker than the topology t mentioned in §1.15.

Let $(\mathfrak{X}, \leqq)$ be a partially ordered set. In $\mathfrak{X}$, suppose that $X = \lim X_a$. Let $\omega^{\leqq}$ and $\omega^{\geqq}$ be towers associated with $\{X_a\}$. Then eventually $M^{\geqq} \leqq X_a \leqq M^{\leqq}$ for any $M \in \omega$. If $\mathfrak{X}$ is imbedded in a complete lattice $\mathfrak{L}$ then, as in §1.18, $M^{\geqq} \leqq \liminf X_a \leqq \limsup X_a \leqq M^{\leqq}$ in $\mathfrak{L}$. Hence $X = \liminf X_a = \limsup X_a$, or $\{X_a\}$ converges to X in the relative topology. It follows that the tower topology of §1.15 is stronger than the relative topology. Summarizing, we have the following theorem.

1.20. Theorem. *If we symbolize the relation "stronger than" by $\gg$, then* $t_1 \gg t_2 \gg \cdots \gg t \gg$relative topology $\gg$ interval topology.

1.21. Definitions. If $(\mathfrak{X}, \leqq)$ and $(\mathfrak{Y}, \leqq)$ are partially ordered sets, a mapping $T: \mathfrak{X} \rightarrow \mathfrak{Y}$ is *positive* if $TX_1 \geqq TX_2$ for $X_1 \geqq X_2$. Positive mappings are also called *order-preserving*. A positive mapping T is *continuous* if $T(\bigvee \mathfrak{M}) = \bigvee (T\mathfrak{M})$ and $T(\bigwedge \mathfrak{N}) = \bigwedge (T\mathfrak{N})$ whenever $\mathfrak{M}$ is directed by $\geqq$ and $\mathfrak{N}$ is directed by $\leqq$.

1.22. Let $T: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a positive mapping. The following statements are clear from the definitions involved:

(a) T maps directed sets and floors onto similarly-directed sets and floors;

(b) if T is continuous, then T maps directed towers with base B onto similarly-directed towers with base TB, and hence maps convergent generalized sequences onto convergent generalized sequences.

1.23. Definition. A partially ordered set $(\mathfrak{X}, \leqq)$ is *Dedekind complete* if every directed subset $\mathfrak{M}^{\geqq}$ of $\mathfrak{X}$ which is bounded above has a supremum in $\mathfrak{X}$, and every directed subset $\mathfrak{N}^{\leqq} \subseteq \mathfrak{X}$ which is bounded below has an infimum in $\mathfrak{X}$.

2. Partially ordered vector spaces.

2.1. Definition. If $\mathfrak{X}$ is a real vector space which is at the same time a partially ordered set $(\mathfrak{X}, \leqq)$, then $\mathfrak{X}$ is a *partially ordered vector space* if

(a) $\alpha X \geqq 0$ whenever $X \geqq 0$ and $\alpha \geqq 0$;

(b) $X + Z \geqq Y + Z$ for all Z whenever $X \geqq Y$.

2.2. Throughout the remaining portion of this section $\mathfrak{X}$ will denote a Dedekind complete partially ordered vector space; elements of $\mathfrak{X}$ will be symbolized by capital Roman letters, and subsets of $\mathfrak{X}$ by capital German letters. The small Roman letters a and b will denote elements of some (usually unspecified) index (directed) set, and the small Greek letter α will stand for some real number.

2.3. Lemma. (a) *If* $\bigwedge_a X_a$ *exists, then* $\bigwedge_a X_a = -\bigvee_a (-X_a)$, *and dually*;

(b) *if* $\bigwedge_a X_a$ *exists and* $\alpha \geqq 0$, *then* $\bigwedge_a \alpha X_a = \alpha \bigwedge_a X_a$; *if* $\alpha \leqq 0$, *then* $\bigvee_a \alpha X_a = \alpha \bigwedge_a X_a$;

(c) *if* $\bigvee_a X_a$ *exists and* $\alpha \geqq 0$, *then* $\bigvee_a \alpha X_a = \alpha \bigvee_a X_a$; *if* $\alpha \leqq 0$, *then* $\bigwedge_a \alpha X_a = \alpha \bigvee_a X_a$;

(d) *if* $X \geqq 0$, *then* $\bigwedge(\alpha > 0)\alpha X = \bigvee(\alpha < 0)\alpha X = 0$;

(e) *if* $X \leqq 0$, *then* $\bigvee(\alpha > 0)\alpha X = \bigwedge(\alpha < 0)\alpha X = 0$;

(f) *if* $\bigwedge_a X_a$ *and* $\bigwedge_b Y_b$ *exist*,
then $\bigwedge_a X_a + \bigwedge_b Y_b = \bigwedge_a \bigwedge_b (X_a + Y_b) = \bigwedge_b \bigwedge_a (X_a + Y_b)$, *and dually.*

2.4. Definitions. (a) If $\mathfrak{M}$ and $\mathfrak{N}$ are subsets of $\mathfrak{X}$, then $\mathfrak{M}+\mathfrak{N}$ is defined as the set $\{M+N \mid M \in \mathfrak{M} \cdot N \in \mathfrak{N}\}$;

(b) if Φ and ψ are two similarly-directed floors, then $\Phi+\psi$ is defined as the collection $\{\mathfrak{M}+\mathfrak{N} \mid \mathfrak{M} \in \Phi \cdot \mathfrak{N} \in \psi\}$;

(c) if $\sigma = \{\Phi_1, \cdots, \Phi_k\}$ and $\omega = \{\psi_1, \cdots, \psi_k\}$ are two similarly-directed k-towers, then $\sigma+\omega$ is defined as the collection $\{\Phi_1+\psi_1, \cdots, \Phi_k+\psi_k\}$;

(d) if a k-tower σ and an n-tower ω are similarly-directed and $k \geqq n$, then $\sigma+\omega$ is defined as $\sigma+\omega'$, where ω' is the canonical extension of ω to a k-tower.

2.5. Lemma. (a) *The sum of two similarly-directed subsets of* $\mathfrak{X}$ *is a directed subset of* $\mathfrak{X}$;

(b) *the sum of two similarly-directed floors is a directed floor*;

(c) *the sum of two similarly-directed towers is a directed tower*; *the base of this sum is the sum of the bases of the component towers*;

(d) *the indicated sums in* (a), (b), *and* (c) *are directed in the same way as the components.*

Proof. We may suppose the order relation to be $\leqq$. By §2.1, $M_i+N_i \leqq M_3+N_i \leqq M_3+N_3$ whenever $M_i \leqq M_3$, $i=1, 2$. From this, (a) and part of (d) follow. This in turn implies (b) and another part of (d). If Φ and ψ are first floors with bases X and Y, respectively, then by (a) and §2.3(j), the floor $\Phi+\psi$ is a first floor with base $X+Y$. The rest of the proof is similar.

2.6. Theorem. *The limit of the sum of two convergent generalized sequences is the sum of the limits of the individual generalized sequences.*

Proof. If σ^* is a tower with base X associated with a convergent $\{X_a\}$ and ω^* is a tower with base Y associated with a convergent $\{Y_b\}$, then $\sigma^*+\omega^*$ is a tower with base $X+Y$ associated with $\{X_a+Y_b\}$. Here * represents $\leqq$ or $\geqq$, and $\{X_a+Y_b\}$ is a function of the directed set $\{(a, b)\}$.

2.7. Corollary. *The generalized sequence* $\{X_a\}$ *converges to* X *if and only if* $\{X_a - X\}$ *converges to zero.*

2.8. Theorem. *For each* $a \in A(\prec)$ *and* $c \in C(\prec)$ *let* $\{Z(a, b) \mid b \in B(\prec)\}$ *and* $\{W(c, b)\}$ *be generalized sequences which k-converge to zero. Suppose that*

$V(c)+W(c, b) \leqq X(b) \leqq Y(a)+Z(a, b)$ for each a, b, and c, where $\{V(c)\}$ and $\{Y(a)\}$ are generalized sequences which h-converge to zero. Then

$$(h+k)\text{-}\lim X(b)=0.$$

Proof. We shall show the existence of a tower, directed by $\leqq$ and having base zero, which is associated with $\{X(b)\}$. In doing this, we consider only towers which are directed by $\leqq$. Similar arguments hold for $\geqq$.

Let σ be an h-tower with base zero associated with $\{Y(a)\}$. For each $H\in\sigma$ there exists an $a(H)$ such that $Y(a)\leqq H$ for $a\prec a(H)$. Let $\omega(a(H))$ be a k-tower with base zero associated with $\{Z(a(H), b)\}$ and let $\rho(H)=\{\{\{H\}\}\}+\omega(a(H))$, where $\{\{\{H\}\}\}$ is the elementary k-tower with base H. If σ' is the extension of σ by means of the towers $\rho(H)$, then σ' is an $(h+k)$-tower with base zero. Let $N\in\sigma'$. Then $N=H+K(a(H))$ for some $H\in\sigma$ and $K(a(H))\in\omega(a(H))$. Since k-lim $Z(a(H), b)=0$, there exists $b(H)=b(K(a(H)))$ such that $Z(a(H), b)\leqq K(a(H))$ for $b\prec b(H)$. Since

$$X(b) \leqq Y(a(H))+Z(a(H), b) \leqq H+K(a(H))=N$$

for $b\prec b(H)$, it follows that σ' is associated with $\{X(b)\}$. Hence $(h+k)$-lim $X(b)=0$.

2.9. Definition. A class $\{X(a, b)\}$ of generalized sequences $[a\in A(\prec)]$ is *uniformly k-convergent to X^b* if there exist k-towers $\omega^{\leqq}$ and $\omega^{\geqq}$ with base zero which are associated with all $\{X(a, b)-X^b\}$. The class $\{X(a, b)\}$ is *uniformly convergent to X^b* if it is uniformly k-convergent to X^b for some k.

2.10. Theorem. *Let $F(n, a)$ be an element of $\mathfrak{X}$ defined for every positive integer n and every element a of a directed set $A^{\prec}$. If k-$\lim_a F(n, a)=0$ for each n and $\lim_n F(n, a)=0$ uniformly in a, then $\lim_a F(n, a)=0$ uniformly in n.*

Proof. Let $\omega^{\leqq}$ be a tower with base zero associated with $\{F(n, a) \mid a$ arbitrary$\}$, and let $\sigma^{\leqq}(n)$ be a tower with base zero associated with $\{F(n, a) \mid n$ fixed$\}$. Then for each $N\in\omega$ there is an integer $P(N)$ such that $F(n, a)\leqq N$ for $n\geqq P(N)$, uniformly in a. Similarly, for each $S(n)\in\sigma(n)$ eventually $F(n, a)\leqq S(n)$. Since $\{1, 2, \cdots, P(N)\}$ is a finite set of integers, eventually $F(n, a)\leqq S(1)+\cdots+S(P(N))$ for $n=1, 2, \cdots, P(N)$.

For each $N\in\omega$, let $\rho(N)=\omega(N)+\sigma(1)+\cdots+\sigma(P(N))$ where $\omega(N)$ is the elementary k-tower with base N. Then $\rho(N)$ is a tower with base N. Let ω' be the extension of ω by means of the towers $\rho(N)$. If $M\in\omega'$, then M is of the form $M=N+S(1)+\cdots+S(P(N))$. From the above, eventually $F(n, a)\leqq M$ for $n=1, 2, \cdots, P(N)$, and $F(n, a)\leqq M$ for $n\geqq P(N)$ uniformly in a. This implies that eventually $F(n, a)\leqq M$ uniformly in n. Hence ω' is a tower with base zero associated with all $\{F(n, a) \mid n$ arbitrary$\}$. Similar considerations apply to the case of direction $\geqq$.

2.11. Definitions. (a) If $\mathfrak{M}\subseteq\mathfrak{X}$, then $\alpha\mathfrak{M}$ is defined as the set

$$\{\alpha M \mid M\in\mathfrak{M}\};$$

(b) if Φ is a floor, then $\alpha\Phi = \{\alpha\mathfrak{M} \mid \mathfrak{M} \in \Phi\}$;
(c) if $\omega = \{\Phi_1, \cdots, \Phi_k\}$ is a tower, then $\alpha\omega = \{\alpha\Phi_1, \cdots, \alpha\Phi_k\}$.

2.12. LEMMA. (a) *Let $\mathfrak{M}$ be a subset of $\mathfrak{X}$ which is directed by $\leqq$ ($\geqq$). If $\alpha > 0$, then $\alpha\mathfrak{M}$ is directed by $\leqq$ ($\geqq$); if $\alpha < 0$, then $\alpha\mathfrak{M}$ is directed by $\geqq$ ($\leqq$);*

(b) *if Φ is a floor directed by $\leqq$ ($\geqq$) and $\alpha > 0$, then $\alpha\Phi$ is a floor which is directed by $\leqq$ ($\geqq$); if $\alpha < 0$, then $\alpha\Phi$ is a floor which is directed by $\geqq$ ($\leqq$);*

(c) *If ω is a tower with base X which is directed by $\leqq$ ($\geqq$) and $\alpha > 0$, then $\alpha\omega$ is a tower with base αX which is directed by $\leqq$ ($\geqq$); if $\alpha < 0$, then $\alpha\omega$ is a tower with base αX which is directed by $\geqq$ ($\leqq$).*

Proof. Immediate from §§2.3 and 2.11.

2.13. THEOREM. *If $\{X_a\}$ converges to X, then $\{\alpha X_a\}$ converges to αX.*

Proof. If $\omega^{\leqq}$ is a tower with base X associated with $\{X_a\}$ and $\alpha > 0$, then $(\alpha\omega)^{\leqq}$ is a tower with base αX associated with $\{\alpha X_a\}$; if $\alpha < 0$, then $(\alpha\omega)^{\geqq}$ is a tower with base αX associated with $\{\alpha X_a\}$; if $\alpha = 0$, the result is trivial. Similar arguments apply to towers directed by $\geqq$.

2.14. DEFINITIONS. If $\mathfrak{Y}$ is a partially ordered vector space such that the partially ordered set $(\mathfrak{Y}, \leqq)$ is a lattice, then $\mathfrak{Y}$ is called a *vector lattice*. A *partially ordered algebra* is a partially ordered vector space which is at the same time an algebra, such that $XY \geqq 0$ whenever $X \geqq 0$ and $Y \geqq 0$. A partially ordered algebra is *Dedekind complete* if it is a Dedekind complete partially ordered vector space such that the mappings $Y \to XY$ and $Y \to YX$ are continuous for each $X \geqq 0$. A *lattice ordered algebra* whose partially ordered vector space is a vector lattice.

2.15. EXAMPLES. A. Let $\mathfrak{X}$ denote the class of all bounded Hermitian operators in Hilbert space $\mathfrak{H}$. The set $\mathfrak{X}$ may be partially ordered by: $A \leqq B$ if and only if $(Ax, x) \leqq (Bx, x)$ for all $x \in \mathfrak{H}$ [8, p. 108]. Then [8, p. 111] $\mathfrak{X}$ si a Dedekind complete partially ordered vector space. Further [8, p. 109], 1-convergence of a generalized sequence $\{X_a\}$ to X is equivalent to eventually bounded (in norm) strong convergence of $\{X_a\}$ to X. For each $x \in \mathfrak{H}$ the mapping $X \to (Xx, x)$ is a continuous positive linear functional on X.

B. Let $\mathfrak{A}$ be a strongly closed C^*-algebra (i.e. uniformly closed self-adjoint operator algebra in Hilbert space) which is commutative and which contains the identity operator I. Then $\mathfrak{A}$ may be ordered in the same way as the class $\mathfrak{X}$ of the preceding paragraph. If $\{Y_a\}^{\geqq}$ is a bounded directed set in $\mathfrak{A}$, then for each $Y_0 \in \{Y_a\}$ the set $\{Y_a - Y_0 \mid Y_a \geqq Y_0\}^{\geqq}$ is a bounded directed set of Hermitian elements. Hence $\bigvee_a (Y_a - Y_0) = X \in \mathfrak{X}$ exists. Since Y is the strong limit of the directed set $\{Y_a - Y_0\}$, it follows from the preceding paragraph that $Y = X + Y_0 = \bigvee_a Y_a$ exists in $\mathfrak{A}$. Hence (a similar argument holds for $\bigwedge$) A is a Dedekind complete partially ordered vector space. If $X \geqq 0$ then $X(\bigvee_a Y_a) \geqq XY_a$. If $x \in \mathfrak{H}$ then $|XY_ax - XYx| \leqq |X| \cdot |Y_ax - Yx|$ so $XYx = \lim XY_ax$. Hence $\bigvee_a (XY_a)$ exists and equals $X(\bigvee_a Y_a)$. We thus see that

$\mathfrak{A}$ is a Dedekind complete partially ordered algebra. This result is related to that of §III.3.

C. Let $\mathfrak{P}$ be the class of all polynomial functions on an unbounded set of real numbers, ordered in the natural way. We shall show that $\mathfrak{P}$ is a Dedekind complete partially ordered vector space.

Suppose $\{P_a(t)\}^{\geqq}$ is a directed set of polynomials which is bounded above by a polynomial. Without loss of generality (observe the behavior for large values of t) we may assume that the degrees of the P_a are bounded by n.

Let $P_a(t) = \alpha_0^a + \alpha_1^a t + \cdots + \alpha_n^a t^n$ and for each t let $\phi(t) = \bigvee_a P_a(t) = \lim_a P_a(t)$. Then $P_a(t) = \phi(t) - \epsilon_a(t)$, where $\lim_a \epsilon_a(t) = 0$. Solving the equations

$$\sum_{p=0}^{n} \alpha_p^a t_k^p = \phi(t_k) - \epsilon_a(t_k)$$

(where $\{t_k\}$ is a collection of $n+1$ distinct real points) for the coefficients α_p^a by determinants and using the addition property of determinants shows that $\lim_a \alpha_p^a = \alpha_p$ exists for each $p = 0, 1, \cdots, n$. Hence ϕ is a polynomial $\sum \alpha_p t^p$ which is obviously $\bigvee_a P_a$. A similar argument holds for $\bigwedge$. Hence $\mathfrak{P}$ is a Dedekind complete partially ordered vector space. This proof was suggested by M. D. Marcus after examining the induction proof of the author. Using a similar argument, it is clear that k-convergence in $\mathfrak{P}$ is equivalent to dominated pointwise convergence.

II

NOTATION. Throughout this section $\mathfrak{X}$ will denote a Dedekind complete partially ordered vector space. The complement of a subset e of a set S will be denoted by $S-e$, or by $\tilde{e}$ when the set S is understood. The characteristic function of e will be denoted by χ_e. The class of all continuous real-valued functions defined on a topological space S will be denoted by $C(S)$. This class is ordered in a natural way by: $f \leqq g$ if and only if $f(s) \leqq g(s)$ for each $s \in S$. The space $C(S)$ is then a lattice ordered algebra. If $f \in C(S)$ then $f^+ = f \vee 0$, $f^- = (-f)^+$, and $|f| = f^+ + f^-$. Well-known are the facts that $f = f^+ - f^-$, $2(f \wedge g) = f + g - |f-g|$, and $2(f \vee g) = f + g + |f-g|$. If $f \in C(S)$ and $\alpha \geqq 0$, then the α-truncate of f is the function $f^\alpha = (f^+ \wedge \alpha) - (f^- \vee \alpha)$. We shall warn the reader when there is danger of confusing a truncate of f with a power of f. The symbol Σ^* will denote a field of subsets of a given set S, and Σ will be an ideal (i.e. $e \cap E \in \Sigma$ if $e \in \Sigma$ and $E \in \Sigma^*$) of Σ^*. The elements of Σ will be denoted by small Roman letters and called *integrable sets*; the elements of Σ^* will be denoted by capital Roman letters and called *measurable sets*. The Greek letter μ will denote a positive ($\mu e \geqq 0$) additive ($\mu(e_1 \cup e_2) = \mu e_1 + \mu e_2$, $e_1 \cap e_2 = \varnothing$) function from Σ into $\mathfrak{X}$. The terms *subadditive*, *superadditive*, *increasing*, and *decreasing* have definitions analogous to those of numerical set functions. A positive superadditive function $\phi: \Sigma \rightarrow \mathfrak{X}$ is increasing and

$\phi\varnothing = 0$. If D is an arbitrary subset of S which is contained in some $e \in \Sigma$, then $\phi^* D = \bigwedge (e \supseteq D) \phi e$ and $\phi_* D = \bigvee (e \subseteq D) \phi e$ exist in $\mathfrak{X}$. The function μ^* is subadditive and μ_* is superadditive.

1. **Set functions.**

1.1. DEFINITIONS. An increasing function $\phi: \Sigma^* \to \mathfrak{X}$ is *Σ-decreasing* at a set $E_0 \in \Sigma^*$ if $\phi E_0 = \lim (E \supseteq E_0 \cdot E \in \Sigma) \phi E$. A class $\{\phi_b\}$ of increasing functions from Σ^* into $\mathfrak{X}$ is *uniformly Σ-decreasing* at a set E_0 if

$$\phi_b E_0 = \lim (E \supseteq E_0 \cdot E \in \Sigma) \phi_b E$$

uniformly in b.

1.2. Viewed as a set function, the Lebesgue integral $\int_E f(s) ds$ of a positive function is a Σ-decreasing function for every Lebesgue measurable set.

1.3. We shall assume that $E \in \Sigma$ whenever $E \in \Sigma^*$ and $\{\mu e \mid e \subseteq E \cdot e \in \Sigma\}$ is bounded in $\mathfrak{X}$.

2. **Convergence in measure.**

2.1. DEFINITIONS. A subset $D \subseteq S$ is a *μ-null set* if $\mu^* D = 0$. Any statement which is true for all points of S except possibly those in a μ-null set is said to hold *μ-almost everywhere.* A real-valued function f defined on S is a *μ-null function* if $\{s \mid |f(s)| \geqq \epsilon\}$ is a μ-null set for each $\epsilon > 0$. The function f is *μ-essentially bounded* if it is the sum of a bounded function and a null function. The *μ-essential supremum* of an essentially bounded function g is ess sup $g = \bigwedge_h \bigvee_s h(s)$, where $\{h\}$ is the class of all bounded functions for which $g - h$ is a null function. Similarly, the *μ-essential infimum* of g is ess inf $g = \bigvee_h \bigwedge_s h(s)$. A real-valued function f is a *simple function* if it assumes only a finite number of distinct values $\alpha_1, \cdots, \alpha_n$, and $f^{-1}(\{\alpha_i\}) = E_i \in \Sigma^*$ for $i = 1, \cdots, n$. If all E_i belong to Σ, then f is a *μ-integrable* simple function. If the values $-\infty$ and ∞ are admitted among the α_i, the function f is an *extended simple function.*

2.2. Every simple function f can be expressed uniquely in the form $f = \sum_{i=1}^n \alpha_i \chi_{E_i}$, where the α_i are distinct real numbers and the E_i are disjoint elements of Σ^* whose union is S. The class of all simple functions and the class of all integrable simple functions are commutative lattice ordered algebras.

2.3. DEFINITION. Let $\{f_a\}$ be a generalized sequence of arbitrary extended real-valued functions defined on S. If f is an arbitrary real-valued function, then $\{f_a\}$ *converges to f in μ-measure* if, for each $\epsilon > 0$,

(a) $E(a, \epsilon) = \{s \mid |f_a(s) - f(s)| \geqq \epsilon\}$ is eventually contained in an integrable set $e(a, \epsilon)$;

(b) $\lim \mu^* E(a, \epsilon) = 0$.

Classic techniques can be used to prove the following:

2.4. LEMMA. (a) *The limit in measure of a generalized sequence is uniquely determined up to a null function;*

(b) *the limit in measure of a generalized sequence of null functions is a null function.*

2.5. THEOREM. *Suppose that* $\{f_a\}$ *converges to* f *in measure and* $\{g_b\}$ *converges to* g *in measure. Let* $\alpha>0$ *and* $D\subseteq S$ *be arbitrary, and let* β *be an arbitrary real number. Then*

(a) $\{\chi_D f_a\}$ *converges to* $\chi_D f$ *in measure;*
(b) $\{\beta f_a\}$ *converges to* βf *in measure;*
(c) $\{f_a+g_b\}$ *converges to* $f+g$ *in measure;*
(d) $\{|f_a|\}$ *converges to* $|f|$ *in measure;*
(e) $\{f_a^\alpha\}$ *converges to* f^α *in measure;*
(f) $\{f_a^\alpha g_b^\alpha\}$ *converges to* $f^\alpha g^\alpha$ *in measure.*

3. μ-measurable functions.

3.1. DEFINITIONS. A real valued function f is *totally μ-measurable* if and only if there exists a generalized sequence $\{f_a\}$ of simple functions which converges to f in measure. The function f is *μ-measurable* if $\chi_e f$ is totally μ-measurable for every $e\in\Sigma$.

3.2. Although no confusion will result from dropping the "μ-" which prefixes "μ-null set" and "μ-null function", it is important to retain this prefix when referring to μ-measurable functions. In the sequel we shall define *measurable* functions. Although every measurable function is μ-measurable, the converse does not always hold.

3.3. LEMMA. (a) *Every simple function is totally μ-measurable;*
(b) *every null function is totally μ-measurable:*
(c) *the class of all null functions is a vector lattice;*
(d) *the class of all essentially bounded functions is a vector lattice;*
(e) *the product of two null functions is a null function;*
(f) *the product of two essentially bounded functions is essentially bounded;*
(g) *if f and g are (totally) μ-measurable, if $E\in\Sigma^*$, and if δ is a real number, then $f+g$, δf, $\chi_E f$, f^+, f^-, $|f|$, $f\wedge g$, $f\vee g$, and each α-truncate f^α are (totally) μ-measurable.*

Proof. Parts (a) and (b) are clear. Part (c) follows from (b) and §2.5. Part (d) follows from (c). Part (e) follows from the inclusion relation

$$\{s\mid |(n_1n_2)(s)|\geqq\epsilon^2\}\subseteq\{s\mid |n_1(s)|\geqq\epsilon\}\cup\{s\mid |n_2(s)|\geqq\epsilon\}.$$

Part (f) is then clear from the preceding parts and the definition of essential boundedness. Finally, (g) follows from §2.5.

3.4. DEFINITION. A function g is an *extended (totally) μ-measurable function* if and only if it is of the form $g=f+(\infty)\chi_{E_1}+(-\infty)\chi_{E_2}$, where E_1 and E_2 are disjoint elements of Σ^* and f is (totally) μ-measurable.

3.5. LEMMA. *If f is the limit in measure of a generalized sequence $\{f_a\}$ of (totally) μ-measurable functions, then there exist directed sets $\{g_b\}^{\geqq}$ and $\{h_c\}^{\leqq}$ of extended (totally) μ-measurable functions, converging to f in measure, such that $g_b\leqq f\leqq h_c$. If $\{f_a\}$ consists of simple functions, then $\{g_b\}$ and $\{h_c\}$ may be* *chosen to consist of extended simple functions. Further, the functions g_b may be chosen to be (totally) μ-measurable if f is bounded below, and the functions h_c may be chosen to be (totally) μ-measurable if f is bounded above.*

Proof. For each $\epsilon>0$, let $E(a, \epsilon)=\{s \mid |f_a(s)-f(s)| \geqq \epsilon\}$. If e is an integrable set containing $E(a, \epsilon)$ let

$$g(a, \epsilon, e) = [f_a - \epsilon]\chi_e + (-\infty)\chi_{e'} \text{ and}$$
$$h(a, \epsilon, e) = [f_a + \epsilon]\chi_e + (\infty)\chi_{e'}.$$

Then by §§3.3 and 3.4, $g(a, \epsilon, e)$ and $h(a, \epsilon, e)$ are extended (totally) μ-measurable functions; and they are extended simple functions if every f_a is simple.

Let $\{g_b\}$ consist of all finite suprema of functions of the form $g(a, \epsilon, e)$, where $\epsilon>0$ and $E(a, \epsilon)\leqq e\in\Sigma$ are allowed to vary, and let $\{h_c\}$ consist of all finite infima of functions of the form $h(a, \epsilon, e)$. Since, for each $\epsilon>0$, $E(a, \epsilon)$ is eventually contained in an integrable set, neither $\{g_b\}$ nor $\{h_c\}$ is vacuous. Also, $g_b\leqq f\leqq h_c$, and both $\{g_b\}^{\geqq}$ and $\{h_c\}^{\leqq}$ are directed sets. Further, $\{g_b\}$ and $\{h_c\}$ consist of extended simple functions if $\{f_a\}$ consists of simple functions.

Let e^* be an integrable set containing $E(a, \epsilon)$. If $s\notin e$, then

$$-\epsilon < f(s) - f_a(s) < \epsilon, \quad \text{so} \quad 0 < f(s) - [f_a(s) - \epsilon] < 2\epsilon.$$

Hence

$$\{s \mid f(s) - g(a, \epsilon, e)(s) \geqq 2\epsilon\} \subseteq e.$$

Similarly,

$$\{s \mid h(a, \epsilon, e)(s) - f(s) \geqq 2\epsilon\} \subseteq e.$$

Since $\{g_b\}$ and $\{h_c\}$ are directed sets, it follows that

$$L_1 = \bigwedge_b \mu^*\{s \mid f(s) - g_b(s) \geqq 2\epsilon\}$$

and

$$L_2 = \bigwedge_c \mu^*\{s \mid h_c(s) - f(s) \geqq 2\epsilon\}$$

exist. Moreover, $L_i\leqq \mu e$, $i=1, 2$. Hence $L_i\leqq\mu^*E(a, \epsilon)$. Since $\lim \mu^*E(a, \epsilon)=0$, this implies that $L_i=0$. Hence $\{g_b\}$ and $\{h_c\}$ converge in measure to f. This proves the first part of the lemma.

If f is bounded below, let $g_b'=g_b\vee[\bigwedge_s f(s)]$; if f is bounded above, let $h_c'=h_c\wedge[\bigvee_s f(s)]$. Then $\{g_b'\}^{\geqq}$ and $\{h_c'\}^{\leqq}$ are directed sets of (totally) μ-measurable functions satisfying (a), (b), and (c). This finishes the proof.

3.6. Lemma. *If f is totally μ-measurable, then* $\lim \mu^*\{s \mid |f(s)| \geqq \alpha\}=0$; *if f is μ-measurable, then* $\lim \mu^*\{s \mid |(\chi_e f)(s)| \geqq\alpha\}=0$ *for each* $e\in\Sigma$.

Proof. It is sufficient to prove only the first part of the lemma, the second part being an immediate corollary. First, f is the limit in measure of a generalized sequence $\{f_a\}$ of simple functions. For each a, then, there exists a real number α such that $|f_a| \leqq \alpha$. Hence

$$\begin{aligned} \{s \mid |f(s)| \geqq \alpha + 1\} &\subseteq \{s \mid |f(s)| \geqq |f_a(s)| + 1\} \\ &\subseteq \{s \mid \big| |f(s)| - |f_a(s)| \big| \geqq 1\} \\ &\subseteq \{s \mid |f(s) - f_a(s)| \geqq 1\}. \end{aligned}$$

Hence $\bigwedge(\alpha > 0)\mu^*\{s \mid |f(s)| \geqq \alpha\} \leqq \mu^*\{s \mid |f(s) - f_a(s)| \geqq 1\}$ for all f_a. This implies that $\lim \mu^*\{s \mid |f(s)| \geqq \alpha\} = 0$.

3.7. LEMMA. *The product of two (totally) μ-measurable functions is (totally) μ-measurable.*

Proof. We first show that f^2 is totally μ-measurable whenever f is totally μ-measurable. By §3.3(g) we may assume without loss of generality that $f \geqq 0$. Since f is totally μ-measurable, there exists a generalized sequence $\{f_a\}$ of simple functions converging to f in measure. By §3.5 the f_a can be chosen such that $0 \leqq f_a \leqq f$ and such that $\{f_a\}^{\geqq}$ is a directed set. Let $\alpha > 0$ and $\epsilon > 0$ be arbitrary, let $E_a = \{s \mid f^2(s) - f_a^2(s) \geqq \epsilon\}$, let $E(a, \alpha) = \{s \mid (f^\alpha)^2(s) - (f_a^\alpha)^2(s) \geqq \epsilon\}$, and let $E_\alpha = \{s \mid f(s) \geqq \alpha\}$. Here the superscript α indicates truncation. From §2.5 it is clear that $\{\mu^*E(a, \alpha)\}^{\leqq}$ is a directed set with limit zero for each $\alpha > 0$. By the preceding lemma $\{\mu^*E_\alpha \mid \alpha > 0\}^{\leqq}$ is a directed set with limit zero. Now $E_a = (E_a \cap \{s \mid f(s) \leqq \alpha\}) \cup (E_a \cap E_\alpha) \subseteq E(a, \alpha) \cup E_\alpha$. Hence $\mu^*E_a \leqq \mu^*E(a, \alpha) + \mu^*E_\alpha$. By Theorem I.2.8 it follows that $\lim \mu^*E_a = 0$. Therefore f^2 is totally μ-measurable for each totally μ-measurable function f. The lemma now follows from §3.3(g) and the identity $4fg = (f+g)^2 - (f-g)^2$.

3.8. LEMMA. *If $\{f_a\}$ is a generalized sequence of (totally) μ-measurable functions which converges to a function f in measure, then f is (totally) μ-measurable.*

Proof. We prove the lemma only for the case where all f_a are totally μ-measurable. In view of §§2.5, 3.3(g), and 3.5, it suffices to prove the lemma under the assumptions that $f_a \geqq 0$, that $f \geqq 0$, that $\{f_a\}$ is directed by $\geqq$, and that $f_a \leqq f$. Each f_a is the limit in measure of a generalized sequence of simple functions. Hence by §3.5 there exists for each f_a a directed set $\{f_a^b\}$ of simple functions such that $0 \leqq f_a^b \leqq f_a$ and $\{f_a^b\}$ converges to f_a in measure. Let $\{g_c\}^{\geqq}$ be the directed set of all finite suprema of functions f_a^b. We shall show that $\{g_c\}$ converges to f in measure, thus showing that f is totally μ-measurable.

Let $\epsilon > 0$ be arbitrary. Then

$$\{s \mid f(s) - f_a^b(s) \geqq 2\epsilon\} \subseteq \{s \mid f(s) - f_a(s) \geqq \epsilon\} \cup \{s \mid f_a(s) - f_a^b(s) \geqq \epsilon\}.$$

Hence $\mu^*\{s|f(s)-g_c(s)\geqq 2\epsilon\}$ exists for some g_c. Since $\{g_c\}$ is directed, $L=\bigwedge_c \mu^*\{s|f(s)-g_c(s)\geqq 2\epsilon\}$ exists. From the above, eventually

$$0 \leqq L \leqq \mu^*\{s\,|\,f(s) - f_a(s) \geqq \epsilon\} + \mu^*\{s\,|\,f(s) - \overset{b}{f_a}(s) \geqq \epsilon\}.$$

Hence eventually $0\leqq L\leqq\mu^*\{s|f(s)-f_a(s)\geqq\epsilon\}$, so $L=0$. This proves that $\{g_c\}$ converges in measure to f.

3.9. Many of the foregoing lemmas are summarized in the following

THEOREM. *The class of all (totally) μ-measurable functions is a commutative lattice ordered algebra. This algebra contains as subspaces the class of all simple functions and the class of all null functions, and is closed relative to the operation of taking limits in measure.*

4. **μ-integrable functions.**

4.1. DEFINITIONS. Suppose that $f=\sum_{i=1}^k \alpha_i\chi_{e_i}$ is an integrable simple function, where the real numbers α_i and the sets $e_i\in\Sigma$ are distinct. The *μ-integral of f over the set $E\in\Sigma^*$* is the quantity $\int_E f(s)\mu(ds)=\sum_{i=1}^k \alpha_i\mu(e_i\cap E)$.

4.2. For each $E\in\Sigma^*$ the mapping $f\to\int_E f(s)\mu(ds)$ is a positive linear transformation of the vector lattice of all integrable simple functions into $\mathfrak{X}$. For each integrable simple function f, the mapping $E\to\int_E f(s)\mu(ds)$ is an additive function from Σ^* into $\mathfrak{X}$. If $f\geqq 0$, this function is positive.

4.3. Let f be an arbitrary bounded totally μ-measurable function. Then by §3.5 there exist directed sets $\{g_b\}^{\geqq}$ and $\{h_c\}^{\leqq}$ of simple functions such that $g_b\leqq f\leqq h_c$, and $\{g_b\}$ and $\{h_c\}$ converge to f in measure. If $e_1\in\Sigma$, let $g^*=\bigvee_b\int_S\chi_{e_1}g_b$ and $h^*=\bigwedge_c\int_S\chi_{e_1}h_c$. Then $0\leqq h^*-g^*=\bigwedge(b,\,c)\int_S\chi_{e_1}(h_c-g_b)$. Let $\epsilon>0$ be arbitrary, and let $E(b,\,c,\,\epsilon)=\{s|\,h_c(s)-g_b(s)\geqq 2\epsilon\}$. Then $E(b,\,c,\,\epsilon)\subseteq\{s|\,h_c(s)-f(s)\geqq\epsilon\}\cup\{s|f(s)-g_b(s)\geqq\epsilon\}$, so eventually $\mu^*E(b,\,c,\,\epsilon)$ exists, and $\bigwedge(b,\,c)\mu^*E(b,\,c,\,\epsilon)=0$. If $e_1\supseteq e\supseteq E(b,\,c,\,\epsilon)\cap e_1$, then

$$\int_S \chi_{e_1}(h_c - g_b) = \int_e + \int_{e_1-e} \leqq \bigvee_s\, [h_c(s) - g_b(s)]\mu e + \epsilon\mu e_1.$$

Since $\lim\mu^*E(b,\,c,\,\epsilon)=0$ and $\epsilon>0$ is arbitrary, it follows that $h^*-g^*=0$. Now if $\{f_a\}^{\geqq}$ is any directed set of simple functions such that $f_a\leqq f$ and $\{f_a\}$ converges to f in measure, then the preceding argument yields $h^*=\bigvee_a\int_S\chi_{e_1}f_a$. Similarly, if $\{f_a\}^{\leqq}$ is any directed set of simple functions such that $f_a>f$ and $\{f_a\}$ converges to f in measure, then $g^*=\bigwedge_a\int_S\chi_{e_1}f_a$.

4.4. DEFINITION. An arbitrary positive totally μ-measurable function f is *μ-integrable over the set $E\in\Sigma^*$* if and only if the set $\{\int_E f_a\}$ is bounded above, where $\{f_a\}^{\geqq}$ is the directed set of all integrable simple functions $f_a\leqq f$.

4.5. DEFINITION. If $f\geqq 0$ is integrable over E, then the *μ-integral of f over E* is the quantity $\bigvee_a\int_E f_a$.

4.6. Note that a positive integrable simple function is integrable over any $E\in\Sigma^*$, and that the integrals given for such a function in §§4.1 and 4.5 are equal. Further, any positive simple function which is integrable over any

$E \in \Sigma^*$ is an integrable simple function. This removes any ambiguity attached to the concept of positive integrable simple function.

4.7. LEMMA. (a) *Suppose that $f > 0$ is totally μ-measurable and $\int_e f^\alpha$ exists for every $\alpha > 0$ and every integrable set $e \subseteq E \in \Sigma^*$. If $\{\int_e f^\alpha\}$ is bounded, then f is integrable over E.*

If $f \geqq 0$ is integrable over E, then:

(b) $\int_E f = \bigvee_a \bigvee (e \in \Sigma \cdot e \subseteq E) \bigvee_\alpha \int_e f_a^\alpha$, *and the suprema may be taken in any order*;

(c) $\chi_E f$ *is the limit in measure of a directed set of integrable simple functions*;

(d) $\chi_E g$ *is integrable over E and $\int_E \chi_E g \leqq \int_E f$ if g is a μ-measurable function such that $0 \leqq g \leqq f$*;

(e) *if $\{g_b\}^{\geqq}$ is any directed set of simple functions such that $0 \leqq g_b \leqq \chi_E f$ and $\{g_b\}$ converges in measure to $\chi_E f$, then each g_b is integrable over E and $\int_E f = \bigvee_b \int_E g_b$.*

4.8. LEMMA. *Let $f \geqq 0$ and $g \geqq 0$ be integrable over E, and let $\beta \geqq 0$ be arbitrary. Then $f+g$ and βf are integrable over E, and $\int_E (f+g) = \int_E f + \int_E g$ and $\int_E \beta f = \beta \int_E f$.*

Proof. Let $\{f_a\}$ be the directed set of all simple functions such that $0 \leqq f_a \leqq f$ and let $\{g_b\}$ be the directed set of all simple functions such that $0 \leqq g_b \leqq g$. If $e \in \Sigma$, $e \subseteq E$, and $\alpha \geqq 0$, then $\{\chi_e (f_a + g_b)^\alpha\}$ converges in measure to $\chi_e (f+g)^\alpha$. Hence $\int_e (f+g)^\alpha \leqq \bigvee_a \bigvee_b \int_e (f_a + g_b) \leqq \int_E f + \int_E g$, by §§4.7(e), 4.2, and 4.3. Hence $\int_E (f+g) \leqq \int_E f + \int_E g$. Now $\int_E (f_a + g_b) \leqq \int_E (f+g)$, so the additivity follows. Similarly for the positive-homogeneity.

4.9. DEFINITION. An arbitrary totally μ-measurable function f is *integrable over the set $E \in \Sigma^*$* if and only if f^+ and f^- are integrable over E. The integral of f over E is the quantity

$$\int_E f = \int_E f^+ - \int_E f^-.$$

4.10. We remark that by §4.8 the functions f and $|f|$ are integrable over the same sets. Also, by §3.3, every null function is integrable, and the integral of a null function is zero over any measurable set.

4.11. LEMMA. *If f is integrable over $E \in \Sigma^*$, then $\int_E |f| = 0$ if and only if $\chi_E f$ is a null function.*

Proof. We have remarked that $\int_E |f| = 0$ if $\chi_E f$ is a null function. Conversely, if $\int_E |f| = 0$, then $|\chi_E f|$ is the limit in measure of a directed set $\{f_a\}$ of positive integrable simple functions, such that $0 \leqq f_a \leqq |\chi_E f|$. Therefore $\int_E f_a = 0$. If $f_a = \sum_i \alpha_i \chi_{e_i}$ and $\alpha_i \neq 0$, then $\mu e_i = 0$. Hence f_a is a null function. Since the limit in measure of a generalized sequence of null functions is a null function, this implies that $|\chi_E f|$, and hence $\chi_E f$, is a null function.

4.12. LEMMA. *If f is a null function and g is integrable over E, then $fg\chi_E$ is a null function.*

Proof. The function $\alpha|f|$ is a null function for each $\alpha \geqq 0$. Since $|g^\alpha \chi_E| \leqq \alpha$, it is clear that $fg^\alpha\chi_E$ is a null function. Now $\{s \mid |(fg\chi_E)(s) - (fg^\alpha\chi_E)(s)| \geqq \epsilon\}$ is contained in $\{s \mid |(\chi_E g)(s)| \geqq \alpha\}$ for each $\epsilon > 0$. By §3.6 it follows that $\{fg^\alpha\chi_E\}$ converges to $fg\chi_E$ in measure. Hence $fg\chi_E$ is a null function.

4.13. THEOREM. *For each $E \in \Sigma^*$, the class of all functions which are integrable over E forms a vector lattice, and the mapping $T_E: f \rightarrow \int_E f$ is a positive linear transformation from this vector lattice into $\mathfrak{X}$.*

Proof. Since $|f+g| \leqq |f| + |g|$ it is clear from §4.7 that $f+g$ is integrable over E. Now $(f+g)^+ - (f+g)^- = f+g = f^+ - f^- + g^+ - g^-$. Hence $\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+$, whence $\int_E (f+g) = \int_E f + \int_E g$. This now implies that $f \wedge g$ and $f \vee g$ are integrable. The rest is clear.

4.14. THEOREM. (a) $-\int_E |f| \leqq \int_E f \leqq \int_E |f|$;

(b) $\int_E |f+g| \leqq \int_E |f| + \int_E |g|$;

(c) *if f is a μ-measurable function such that $|f|$ is dominated μ-almost everywhere by a function g which is integrable over E, then $\chi_E f$ is integrable over E and $\int_E \chi_E f \leqq \int_E g$;*

(d) *if f is an essentially bounded μ-measurable function, then $\chi_e f$ is integrable over S for every integrable set e, and* (ess inf f) $\mu e \leqq \int_S \chi_e f \leqq$ (ess sup f)μe.

(e) *If f is integrable over E and g is essentially bounded, then fg is integrable over E and*

$$\int_E |fg| \leqq (\text{ess sup } |g|) \int_E |f|.$$

4.15. LEMMA. *If f is integrable over E, then for each $\epsilon > 0$ the set $\{s \mid |(\chi_E f)(s)| \geqq \epsilon\}$ is contained in an integrable set.*

4.16. THEOREM. *If f is integrable over E, then $\chi_E f$ is the limit in measure of a generalized sequence $\{f_a\}$ of integrable simple functions; this generalized sequence may be decomposed into the difference of two directed sets of integrable simple functions, one converging to f^+ and the other converging to f^-, in measure. Hence there is a generalized sequence $\{g_b\}$ of integrable simple functions which 1-converges in measure to $\chi_E f$.*

4.17. THEOREM. *If f is totally μ-measurable and the set $\{\int_e |f|^\alpha \mid e \subseteq E \cdot \alpha \geqq 0\}$ is bounded, then f is integrable over E.*

4.18. THEOREM. *For each totally μ-measurable function f, the class Π of all sets $E \in \Sigma^*$ over which f is integrable is an ideal of Σ^*, and the mapping ϕ_f: $E \rightarrow \int_E f$ is an additive function from Π into $\mathfrak{X}$. If $f \geqq 0$ and f is integrable over S, then ϕ_f is positive and Σ-decreasing at each $E \in \Sigma^*$.*

4.19. COROLLARY. *If $\int_{E_1} f$ exists, then $\int_E f$ and $\int_{E_1}\chi_E f$ exist for any measurable set $E \subseteq E_1$, and $\int_E f = \int_{E_1}\chi_E f$.*

4.20. The proofs of the foregoing statements are fairly easy consequences of what has already been done. As shown in §4.6, any positive simple function which is integrable over S is a positive integrable simple function. Since every simple function which is integrable over S is the difference of two positive integrable simple functions, it is clear that a simple function which is integrable over every $E \in \Sigma^*$ (or over S) is an integrable simple function. It now follows that a set $E \in \Sigma^*$ is integrable if and only if χ_E is integrable.

4.21. DEFINITION. A positive function $\phi: \Sigma \to \mathfrak{X}$ is *μ-continuous* if and only if $\lim \phi^* D_a = 0$ whenever $\lim \mu^* D_a = 0$. A class $\{\phi_b\}$ of positive functions from Σ into $\mathfrak{X}$ is *uniformly μ-continuous* if and only if $\lim_a \phi_b^* D_a = 0$ uniformly in b whenever $\lim \mu^* D_a = 0$.

4.22. THEOREM. *If f is integrable over S, then $\phi: e \to \int_e |f|$ is a μ-continuous set function. Specifically, if $\{\mu^* D_a\}$ is k-convergent to zero, then $\{\phi^* D_a\}$ is $(k+1)$-convergent to zero.*

Proof. For notational convenience we assume that $f \geqq 0$. If $e \supseteq D_a$, then $\phi e = \int_e (f - f^\alpha) + \int_e f^\alpha \leqq \int_S (f - f^\alpha) + \alpha \mu e$. Hence $\phi^* D_a \leqq \int_S (f - f^\alpha) + \alpha \mu^* D_a$. The result now follows from §I.2.8.

5. Convergence theorems.

5.1. DEFINITION. A generalized sequence $\{f_a\}$ of integrable functions *converges in the mean* to an integrable function f if

$$\lim \int_S |f_a - f| = 0.$$

5.2. LEMMA. *If $\{f_a\}$ is a generalized sequence of integrable functions which converges in the mean to the integrable function f, then $\{f_a\}$ converges to f in measure.*

Proof. Let $h_a = |f_a - f|$ and let $\epsilon > 0$ be arbitrary. As usual, define $E(a, \epsilon) = \{s \mid h_a(s) \geqq \epsilon\}$. We shall show that $\lim \mu^* E(a, \epsilon) = 0$. Since $\int_S h_a^\epsilon$ converges to zero and $E(a, \epsilon) = \{s \mid h_a^\epsilon(s) \geqq \epsilon\}$, we may assume without loss of generality (for the purpose of proving the lemma) that the h_a are uniformly bounded by ϵ.

Let a be fixed. Then by §4.15 $E(a, \epsilon)$ is contained in an integrable set e. By §4.3, $\int_e h_a = \bigwedge_b \int_e g(a, b)$, where $\{g(a, b)\}$ is the class of all simple functions such that $h_a \leqq g(a, b) \leqq \epsilon$. Now

$$\chi_e g(a, b) = \alpha_1 \chi_{e_1} + \cdots + \alpha_m \chi_{e_m}, \text{ where } g(a, b)(s) = \alpha_i \geqq 0$$

for $s \in e_i$, and $\bigcup_{i=1}^m e_i = e$. Hence

$$\int_e g(a, b) = \alpha_1\mu(e_1) + \cdots + \alpha_m\mu(e_m)$$
$$\geqq \alpha_1\mu^*(e_1 \cap E(a, \epsilon)) + \cdots + \alpha_m\mu^*(e_m \cap E(a, \epsilon)).$$

If $\alpha_i < \epsilon$, then $h_a(s) \leqq g(a, b)(s) < \epsilon$ for $s \in e_i$. Hence $e_i \cap E(a, \epsilon) = \varnothing$ in this case. This implies that $\mu^*(e_i \cap E(a, \epsilon)) = 0$ whenever $\alpha_i < \epsilon$. Hence

$$\epsilon\mu^*(e_1 \cap E(a, \epsilon)) + \cdots + \epsilon\mu^*(e_m \cap E(a, \epsilon)) \leqq \int_e g(a, b).$$

Now $E(a, \epsilon) = \bigcup_{i=1}^m (e_i \cap E(a, \epsilon))$. Since μ^* is subadditive, this fact, together with the preceding statement, yields $\epsilon\mu^*E(a, \epsilon) \leqq \int_e g(a, b)$ for all b. Hence $\epsilon\mu^*E(a, \epsilon) \leqq \bigwedge_b \int_e g(a, b) = \int_e h_a \leqq \int_S h_a$. It follows that $\lim \mu^*E(a, \epsilon) = 0$, thus proving the lemma.

5.3. Lemma. *If $\{\int_S |f_a|\}$ is eventually bounded above (say by M) and $\{f_a\}$ converges to f in measure, then f is integrable over S and $\int_S |f| \leqq M$.*

Proof. Let $\alpha \geqq 0$ be arbitrary. For $\epsilon > 0$, let $E(a, \epsilon) = \{s \mid |f^\alpha(s) - f_a^\alpha(s)| \geqq \epsilon\}$, and let e_1 be any integrable set. By §3.8, f is totally μ-measurable, and by §§3.3(g) and 4.14, f^α is integrable over e_1. If $e \supseteq e_1 \cap E(a, \epsilon)$, then

$$\int_{e_1} |f^\alpha - f_a^\alpha| = \int_{e_1 \cap e} + \int_{e_1 \cap \tilde{e}} \leqq \int_e + \epsilon\mu e_1 \leqq 2\alpha\mu e + \epsilon\mu e_1.$$

Hence

$$\int_{e_1} |f^\alpha - f_a^\alpha| \leqq 2\alpha u^*(e_1 \cap E(a, \epsilon)) + \epsilon\mu e_1 \leqq 2\alpha\mu^*E(a, \epsilon) + \epsilon\mu e_1.$$

Now

$$\int_{e_1} |f^\alpha| \leqq \int_{e_1} |f^\alpha - f_a^\alpha| + \int_S |f_a^\alpha| \leqq 2\alpha\mu^*E(a, \epsilon) + \epsilon\mu e_1 + M.$$

Since $\lim \mu^*E(a, \epsilon) = 0$, we have $\int_{e_1} |f^\alpha| \leqq \epsilon\mu e_1 + M$. But $\epsilon > 0$ is arbitrary, so $\int_{e_1} |f^\alpha| \leqq M$. By §4.17, this implies that f is integrable over S and $\int_S |f| \leqq M$.

5.4. Theorem. *Let $\{f_n\}$ be a sequence of functions which are integrable over S. A function f is integrable over S and the limit in the mean of $\{f_n\}$ if and only if*
(a) *the sequence $\{\int_S f_n\}$ is eventually bounded;*
(b) *$\{f_n\}$ converges to f in measure;*
(c) *the sequence $\{\int_e |f_n|\}$ of set functions is uniformly μ-continuous;*
(d) *the sequence $\{\int_E |f_n|\}$ is uniformly Σ-decreasing at $\varnothing$.*

Proof. We suppose first that f is integrable and is the limit in the mean of $\{f_n\}$. Then (a) is immediate from the inequality $\int_S |f_n| \leqq \int_S |f_n - f| + \int_S |f|$. Next, (b) follows from §5.2. Now suppose that $\{\mu^*D_a\}$ is a generalized sequence which k-converges to zero. Let $F(n, a)=\bigwedge(e\supseteq D_a)\int_e f_n$. We shall show that $F(n,a)$ converges to zero uniformly in n. Let $G(n, a)=\bigwedge(e\supseteq D_a)\int_e|f_n-f|$ and $H(a)=\bigwedge(e\supseteq D_a)\int_e|f|$. Then $0\leqq F(n, a)\leqq G(n, a)+H(a)$. Now $\lim_a G(n, a)=0$ for each n and $\lim_n G(n, a)=0$ uniformly in a. By §I.2.10 it follows that $\lim_a G(n, a)=0$ uniformly in n. This implies that $\lim_a F(n, a)=0$ uniformly in n, or that $\{\int_e|f_n|\}$ is uniformly μ-continuous.

We turn to the proof of (d). By §4.18 it is clear that $\bigwedge_{e\in\Sigma}\int_{\tilde{e}}g=0$ for every positive function g which is integrable over S. Let $\mathfrak{K}=\{\int_{\tilde{e}}|f|\,|\,e\in\Sigma\}$, and let $\omega^{\leqq}=\{\{\mathfrak{K}\}\}$. Then ω is a 1-tower with base zero associated with $\{\int_{\tilde{e}}|f|\}$. Similarly, let $\omega_n^{\leqq}=\{\{\{\int_{\tilde{e}}|f_n|\}\}\}$ be a 1-tower with base zero associated with $\{\int_{\tilde{e}}|f_n|\}$. By hypothesis, there exists a p-tower $\sigma^{\leqq}=\{\psi_1, \cdots, \psi_p\}$ with base zero such that $\int_S|f_n-f|\leqq P\in\mathfrak{P}\in\psi_p$ for $n\geqq n(P)$. For each $P\in\mathfrak{P}\in\psi_p$, let $\sigma_P^{\leqq}$ be the elementary 1-tower with base P, and let $\rho_P=\sigma_P+\omega+\omega_1+\cdots+\omega_{n(P)}$. Then ρ_P is a 1-tower with base P. Let $\sigma'=\{\Phi_1, \cdots, \Phi_{P+1}\}$ be the extension of σ by means of the 1-towers ρ_P, and let $M=P+\int_{\tilde{e}}|f|+\int_{\tilde{e}_1}|f_1|+\cdots+\int_{\tilde{e}_{n(P)}}|f_n|$, $M\in\mathfrak{M}\in\Phi_{p+1}$. Let $e_M=e\cup e_1\cup\cdots\cup e_{(n(P))}$. Then e_M is an integrable set. If $n=1, \cdots, n(P)$, then $\int_{\tilde{e}_M}|f_n|\leqq\int_{\tilde{e}_n}|f_n|\leqq M$; if $n>n(P)$, then

$$\int_{\tilde{e}_M}|f_n|\leqq\int_{\tilde{e}}|f_n|\leqq\int_{\tilde{e}}|f_n-f|+\int_{\tilde{e}}|f|\leqq P+\int_{\tilde{e}}|f|\leqq M.$$

Hence $\int_{\tilde{e}_M}|f_n|\leqq M$ uniformly in n. It follows that $\{\int_E f_n\}$ is uniformly Σ-decreasing at $\varnothing$. This proves (d).

Now assume (a), (b), (c), and (d). By §5.3, f is integrable over S. We shall show that $\{\int_S|f_n-f|\}$ converges to zero.

Let $\omega^{\leqq}=\{\Phi_1, \cdots, \Phi_k\}$ be a k-tower associated with the class $\{\int_E|f_n|\}$ which is uniformly Σ-decreasing at $\varnothing$. For each $M\in\mathfrak{M}\in\Phi_k$ there exists an integrable set e_M such that $\int_{\tilde{e}_M}|f_n|\leqq M$ uniformly in n. Since the set function $\int_E|f|$ is Σ-decreasing at $\varnothing$, we may without loss of generality assume that $\int_{\tilde{e}_M}|f|\leqq M$ also. For arbitrary $\epsilon>0$, let $E(n, \epsilon)=\{s\,|\,|f_n(s)-f(s)|\geqq\epsilon\}$. Since $\{f_n\}$ converges to f in measure, eventually $E(n, \epsilon)$ is contained in some $e\in\Sigma$. Hence

$$\int_{e_M}|f_n-f|=\int_{e_M-e}+\int_{e_M\cap e}\leqq\epsilon\mu e_M+\int_{e_M\cap e}|f_n-f|.$$

Now $\int_{e_M\cap e}|f_n-f|\leqq\int_e|f_n|+\int_e|f|$. Since $\lim\mu^*E(n, \epsilon)=0$, it follows from (c) and §4.22 that $\lim_n\bigwedge(e\supseteq E(n,\epsilon))\int_{e_M\cap e}|f_n-f|=0$. Let $\sigma_M^{\leqq}=\{\psi_1, \cdots, \psi_p\}$ be associated with the sequence $\{\bigwedge(e\supseteq E(n, \epsilon))\int_{e_M\cap e}|f_n-f|\}$. For each $N\in\mathfrak{N}\in\psi_p$, $\{\{\{\epsilon\mu e_M+N\,|\,\epsilon>0\}\}\}$ is a 1-tower with base N. Let σ_M' be the extension of σ_M by means of these 1-towers. From the above work σ_M' is a $(p+1)$-tower with base zero associated with the sequence $\{\int_{e_M}|f_n-f|\}$. For each $M\in\mathfrak{M}\in\Phi_k$, $\{\{\{2M\}\}\}+\sigma_M'$ is a $(p+1)$-tower with base $2M$. Let ω' be the extension of 2ω by means of these $(p+1)$-towers. Then, since

$$\int_S |f_n - f| \leqq \int_{e_M} + \int_{\tilde{e}_M} \leqq \int_{e_M} |f_n - f| + \int_{\tilde{e}_M} |f_n| + \int_{\tilde{e}_M} |f|$$
$$\leqq \int_{e_M} |f_n - f| + 2M,$$

it follows that ω' is a tower with base zero associated with the sequence $\{\int_S |f_n - f|\}$. This finishes the proof of the theorem.

5.5. THEOREM. *Let $\{f_n\}$ be a sequence of μ-measurable functions which converges in measure to a function f. Suppose that every $|f_n|$ is dominated μ-almost everywhere by the function g which is integrable over S. Then f and f_n are integrable over S and $\{f_n\}$ converges in the mean to f.*

Proof. By §4.14 all f_n are integrable over S and $\int_E |f_n| \leqq \int_E g$ for any measurable set E. This implies that $\{\int_e f_n\}$ is uniformly μ-continuous and $\{\int_E f_n\}$ is uniformly Σ-decreasing at $\varnothing$. The result now follows from the preceding theorem.

6. **Countable additivity.**

6.1. DEFINITIONS. A *σ-field* Σ^* of subsets of a set S is a field of subsets of S which is closed under the formation of countable disjoint unions. If Σ is an ideal of a field Σ then an additive function $\phi: \Sigma \to \mathfrak{X}$ (where $\mathfrak{X}$ is a Dedekind complete partially ordered vector space) is *countably additive* if $\phi(\bigcup_{i=1}^{\infty} (E_i \cap e)) = \lim_n \sum_{i=1}^{n} \phi(E_i \cap e)$ for every $e \in \Sigma$ and every countable disjoint collection $\{E_i\}$ of elements of Σ^*.

6.2. DEFINITION. A real-valued function f defined on S is *measurable* if $f^{-1}(B) \in \Sigma^*$ whenever B is a Borel set of real numbers. Here Σ^* is a σ-field.

6.3. In contrast with that of μ-measurability, the concept of measurability is independent of any function μ, depending only on the underlying σ-field Σ^*. The theory of measurable functions is well known.

6.4. LEMMA. *If μ is countably additive, then every measurable function f is μ-measurable.*

Proof. Let $e \in \Sigma$ and $\epsilon > 0$ be arbitrary. It will be sufficient to prove the lemma for the case where $f \geqq 0$. Let $e_n = \{s \mid (n-1)\epsilon \leqq f(s) < n\epsilon\} \cap e$, and let $f_k = \sum_{n=1}^{k} (n-1)\epsilon \chi_{e_n}$. Then $e_n \in \Sigma$, $e = \bigcup_{n=1}^{\infty} e_n$, each f_n is a simple function, and $\{s \mid |(\chi_e f)(s) - f_k(s)| \geqq \epsilon\} = \bigcup_{k+1}^{\infty} e_n \in \Sigma$. Hence $\lim_a \mu^* \{s \mid |(\chi_e f)(s) - f_a(s)| \geqq \epsilon\} = 0$, where $\{f_a\}^{\geqq}$ is the directed set of all simple functions such that $0 \leqq f_a \leqq \chi_e f$. This implies that f is μ-measurable.

6.5. THEOREM. *Let $\{f_n\}$ be a sequence of measurable functions which converges μ-almost everywhere to a measurable function f. If there exists an integrable (over S) function g such that $|f_n(s)| \leqq g(s)$ μ-almost everywhere for $n = 1, 2, \cdots$, then f and f_n are integrable over S and $\{f_n\}$ converges in the mean to f. This convergence is 1-convergence.*

Proof. We first show that $\{f_n\}$ converges to f in measure. Let $E(n, \epsilon) = \{s \mid |f_n(s) - f(s)| \geqq 2\epsilon\}$. Then $E(n, \epsilon) \in \Sigma^*$. If $E_k = \bigcup_{n=k}^{\infty} E(n, \epsilon)$, then $E_k \in \Sigma^*$, $E_k \supseteq E_{k+1}$, and $\bigcap_k E_k \in \Sigma^*$. Further, $\bigcap_k E_k \subseteq N = \{s \mid f_n(s) \nrightarrow f(s)\}$, so $\mu^*(\bigcap_k E_k) = 0$. Now

$$E_1 = \left(\bigcap_k E_k\right) \cup (E_1 - E_2) \cup (E_2 - E_3) \cup \cdots = (E_1 - E_2) \cup \cdots \cup (E_{n-1} - E_n) \cup E_n.$$

If $M = \{s \mid g(s) \geqq \epsilon\}$ then $M \in \Sigma$ by §4.15. Hence

$$\lim \sum_1^{n-1} \mu[(E_k - E_{k+1}) \cap M] = \lim \left(\sum_1^{n-1} \mu[(E_k - E_{k+1}) \cap M] + \mu(E_n \cap M) \right).$$

Hence $\lim \mu(E_n \cap M) = 0$, so $\lim \mu[(E(n, \epsilon) \cap M] = 0$. But $E(n, \epsilon) \subseteq E(n, \epsilon) \cap (M \cup N \cup K_n) \subseteq [E(n,\epsilon) \cap M] \cup N \cup K_n$, where $K_n = \{s \mid |f_n(s)| > g(s)\}$. Since $\mu^* N = \mu^* K = 0$, it follows that $\lim \mu^* E(n, \epsilon) = 0$. Hence $\{f_n\}$ converges to f in measure. In view of §§5.5 and 6.5, it remains to show that the mean convergence is 1-convergence.

For each $s \in S$, let $g_n'(s) = \bigwedge_{k \geqq n} f_k(s)$ and $g_n''(s) = \bigvee_{k \geqq n} f_k(s)$. Then from the above work all g_n' and g_n'' are integrable over S and $\int_S |f_n - f| = \int_e |f_n - f| + \int_{\tilde{e}} |f_n - f| \leqq \int_e |f_n - f| + \int_S (g_n'' - g_n')$ where $e \supseteq N$ and $e \in \Sigma$. Hence by §4.22 $\int_S |f_n - f| \leqq \int_S (g_n'' - g_n')$. Now $\{g_n'' - g_n'\}$ is a decreasing sequence so $\{\int_S (g_n'' - g_n')\}$ is a decreasing sequence. The preceding work shows that $\lim \int_S (g_n'' - g_n') = 0$, so the convergence of $\{\int_S |f_n - f|\}$ is 1-convergence.

6.6. The preceding theorem improves a special case of one of McShane's theorems [8, pp. 59 and 81].

6.7. COROLLARY. *For each integrable function f, $\int_E f$ is a countably additive set function. The convergence in the definition of countable additivity may be taken to be* 1-*convergence in this case.*

III

1. A representation theorem.

1.1. In this section S will denote a normal topological space and Σ the field generated by the closed subsets of S. Also, $C(S)$ will denote the space of all real-valued bounded continuous functions on S, and $\mathfrak{X}$ a Dedekind complete partially ordered vector space.

1.2. DEFINITION. If $\mu: \Sigma \to \mathfrak{X}$ is a positive additive function and $e \in \Sigma$, then μ is *regular* if $\mu e = \bigvee \mu F$, where $F \subseteq e$ is closed; the class of all regular functions will be denoted by $r(\Sigma)$.

1.3. If $\mu \in r(\Sigma)$ then $\mu e = \bigwedge \{\mu G \mid G \supseteq e \cdot G \text{ open}\}$. If $f \in C(S)$ then $f^{-1}[\alpha, \delta) \in \Sigma$ for all real numbers α and δ. Since f is bounded,

$$f_\epsilon = \sum_{n=-\infty}^{\infty} (n\epsilon) f^{-1}[n\epsilon, n\epsilon + \epsilon)$$

is a simple function for any $\epsilon>0$. Since $0\leqq f-f_\epsilon<\epsilon$, clearly $\{f_\epsilon\}$ converges to f in μ-measure. Hence f is μ-measurable. Since μS is defined and f is bounded, f is μ-integrable over S.

1.4. LEMMA. *Let ϕ be a function defined from the class of all subsets of S into $\mathfrak{X}$ such that $\phi\varnothing=0$. If*

$$\mathfrak{C} = \left\{K \subseteq S \,\middle|\, \phi E = \phi(E \cap K) + \phi(E \cap \tilde{K}) \text{ for all } E \subseteq S\right\},$$

then $\mathfrak{C}$ is a field and ϕ is additive on this field.

Proof. Formally identical with that on p. 45 of [6].

1.5. THEOREM. *If T: $C(S)\to\mathfrak{X}$ is a positive linear transformation, then there exists $\mu_T\in r(\Sigma)$ such that $Tf=\int_S f(s)\mu_T(ds)$ for all $f\in C(S)$. Conversely, if $\mu\in r(\Sigma)$, then $T_\mu f=\int_S f(s)\mu(ds)$ defines a positive linear transformation from $C(S)$ into $\mathfrak{X}$, and the correspondence $T\leftrightarrow\mu$ is reciprocal.*

Proof. The proof in [1, p. 577] may be adapted to the new situation.

1.6. Note that §1.5 is valid for the space of all complex valued bounded continuous functions defined on S, as well as for the real space $C(S)$: Since $T(f_1+if_2)=Tf_1+iTf_2$, a linear transformation on the complex space is determined by its action on the real functions. The integral of a complex function $f=f_1+if_2$ (where the f_i are real functions) is then, by definition, $\int_S f(s)\mu(ds) =\int_S f_1+i\int_S f_2$. Naturally $\mathfrak{X}$ must be a vector space over the complex number system (or a direct sum $\mathfrak{X}\oplus\mathfrak{X}$ of real spaces) but this causes no trouble.

1.7. If $\mathfrak{X}$ is the real or complex number system, ordered by $\alpha\geqq\beta$ if $\alpha-\beta\geqq 0$, and S is a compact Hausdorff space, then a general form of the celebrated Riesz theorem may be obtained from §1.5 with only a little more argument. The key lemmas follow.

1.8. LEMMA [1, p. 590]. *If S is a compact Hausdorff space then every bounded regular finitely additive real valued function μ defined on the field Σ generated by the closed sets of S is countably additive.*

1.9. LEMMA [5, p. 76]. *Every positive countably additive μ defined on Σ can be extended to a positive regular measure μ defined on the σ-field $\mathfrak{B}$ of Borel subsets of S.*

1.10. THEOREM. *Let S be a compact Hausdorff space, let $\mathfrak{X}$ be a Dedekind complete partially ordered vector space, and let T be a positive linear transformation from $C(S)$ into $\mathfrak{X}$. If there exist enough continuous positive linear functionals in $\mathfrak{X}$ to distinguish between points of $\mathfrak{X}$, then there exists a positive countably additive regular mapping μ_* from the σ-field $\mathfrak{B}$ of Borel subsets of S into $\mathfrak{X}$ such that $Tf=\int_S f(s)\mu_*(ds)$. Conversely, if μ_* is such a mapping, then the integral $\int_S f(s)\mu_*(ds)$ defines a positive linear transformation from $C(S)$ into $\mathfrak{X}$.*

Proof. Let $\mu:\Sigma\to\mathfrak{X}$ be the mapping mentioned in §1.5. For each $B\in\mathfrak{B}$, let $\mu_*B=\bigvee\{\mu E \mid E\subseteq B\cdot E\in\Sigma\}$. Then the mapping μ_*: $\mathfrak{B}\to\mathfrak{X}$ is superadditive and increasing. If $\{B_k\}$ is a sequence of pairwise disjoint elements of B and $B=\bigcup_{k=1}^{\infty} B_k$, then $\mu_*B\geqq\mu_*(\bigcup_{k=1}^{n} B_k)\geqq\sum_{k=1}^{n}\mu_*B_k$. Hence $\mu_*B\geqq\bigvee_n\sum_{k=1}^{n}\mu_*B_k \geqq\sum_{k=1}^{n}\mu_*B_k$. Let X^* be an arbitrary positive continuous linear functional defined on $\mathfrak{X}$. Then by §§1.8 and 1.9 $X^*\mu_*$ is countably additive. Hence $(X^*\mu_*)B=\bigvee_n\sum_{k=1}^{n}X^*\mu_*B_k$. But $X^*\mu_*B\geqq X^*\bigvee_n\sum_{k=1}^{n}\mu_*B_k\geqq X^*\sum_{k=1}^{n}\mu_*B_k =\sum_{k=1}^{n}X^*\mu_*B_k$. It follows that $X^*\mu_*B=X^*\bigvee_n\sum_{k=1}^{n}\mu_*B_k$. Therefore $\mu_*B =\bigvee_n\sum_{k=1}^{n}\mu_*B_k$. From its definition μ_* is regular. Since μ_* is an extension of μ, any generalized sequence $\{f_a\}$ of μ-simple functions converging to f in μ-measure must be a generalized sequence of μ_*-simple functions converging to f in μ_*-measure. If $\{f_a\}^{\geqq}$ is the directed set of all μ-simple functions such that $0\leqq f_a\leqq f^+$ and $\{g_b\}$ is the directed set of all μ-simple functions such that $0\leqq g_b\leqq f^-$, then

$$\int_S f(s)\mu(ds)=\int_S f^+-\int_S f^-=\bigvee_a\int_S f_a-\bigvee_b\int_S g_b$$

$$=\int_S f^+(s)\mu_*(ds)-\int_S f^-(s)\mu_*(ds)=\int_S f(s)\mu_*(ds).$$

This completes the proof.

2. **When T is multiplicative.**

2.1. Definition. A positive linear transformation T from a partially ordered algebra $\mathfrak{Y}$ into a partially ordered algebra $\mathfrak{X}$ is *multiplicative* if $T(Y_1Y_2)=(TY_1)(TY_2)$ for all $Y_1,\ Y_2\in\mathfrak{Y}$.

2.2. Theorem. *Let S be a normal topological space and suppose that T is a positive multiplicative linear transformation from $C(S)$ into a Dedekind completely partially ordered algebra $\mathfrak{X}$. If E_1 and E_2 are elements of the field generated by the closed subsets of S, then*

$$\mu(E_1\cap E_2)=\mu E_1\mu E_2,$$

where μ is the function whose existence was stated in §1.5. *Conversely, if μ is such a function, then the mapping*

$$T_\mu f=\int_S f(s)\mu(ds)$$

defined in §1.5 *is multiplicative.*

Proof. Assume that T is multiplicative. We shall first show that $\mu(F\cap K) =\mu F\mu K$ for arbitrary closed sets F and K. Let $f\in C(S)$ be greater than χ_F, and let $g\in C(S)$ be greater than χ_K. Then $fg\geqq\chi_{F\cap K}$. Hence

$$\mu F\mu K = \left(\bigwedge (f \geqq \chi_F)Tf\right)\left(\bigwedge (g \geqq \chi_K)Tg\right) = \bigwedge_g \left(\left(\bigwedge_f Tf\right)Tg\right)$$
$$= \bigwedge_g \bigwedge_f TfTg = \bigwedge_{f,g} T(fg) \geqq \bigwedge (h \geqq \chi_{F\cap K})Th = \mu(F \cap K).$$

Given $h\in C(S)$ such that $h\geqq\chi_{F\cap K}$, we shall now construct decreasing sequences $\{f_n\}$ and $\{g_n\}$ of elements of $C(S)$ such that $f_n\geqq\chi_F$, $g_n\geqq\chi_K$, and $\bigwedge_n (Tf_nTg_n)\leqq Th$. From this construction we shall then have $\mu F\mu K \leqq\bigwedge_n (Tf_nTg_n)\leqq Th$. Since h is an arbitrary continuous function greater than $\chi_{F\cap K}$, it will then follow that $\mu F\mu K\leqq\mu(F\cap K)$. Combined with the result of the preceding paragraph, this will prove the first part of the theorem for closed sets F and K.

Without loss of generality we may assume that $h(s)=1$ for each $s\in K\cap F$. Let $h_n=h\vee(1/n)$. If we define $f_n'(s)$ as 1 for $s\in F$ and $f_n'(s)=h_n(s)$ for $s\in K$, it follows from elementary topology that f_n' is a continuous function on $K\cup F$; by Tietze's extension theorem f_n' can be extended to a bounded continuous function f_n'' defined on all of S. Let $f_n^*=f_n''\vee h_n$. If $s\in F$, then $f_n^*(s)\geqq f_n''(s)=1$. Since $f_n^*\geqq h\geqq 0$, it follows that $f_n^*\geqq\chi_F$. Also, f_n^* is a continuous function which never takes on a value less than $(1/n)$. Hence $g_n^*=h_n/f_n^*\in C(S)$.

If $s\in K$, then $h_n(s)=f_n^*(s)g_n^*(s)=h_n(s)g_n^*(s)$, so $g_n^*(s)=1$. Since both h_n and f^* are positive, g_n must be positive also. Hence $g_n\geqq\chi_K$.

Since $0\leqq h_n-h\leqq(1/n)$ and T is positive, we have $0\leqq Th_n-Th\leqq(1/n)Te$, where e is the function identically 1 on S. Since $\{h_n\}$ is a decreasing sequence, it follows that $\bigwedge Th_n$ exists and equals Th. Now $Th_n=T(f_n^*g_n^*)$. Let $f_n =f_1^*\wedge\cdots\wedge f_n^*$ and $g_n=g_1^*\wedge\cdots\wedge g_n^*$. Then $\{f_n\}$ and $\{g_n\}$ are decreasing sequences of functions, $f_n^*\geqq f_n\geqq\chi_F$, and $g_n^*\geqq g_n\geqq\chi_K$. Hence $Th=\bigwedge_n Th_n =\bigwedge_n T(f_n^*g_n^*)=\bigwedge_n (Tf_n^*Tg_n^*)\geqq\bigwedge_n(Tf_nTg_n)$. As we observed before, this yields $\mu(F\cap K)=\mu F\mu K$ for closed sets F and K.

Now let E_1 and E_2 be arbitrary elements of Σ. Then

$$\mu(E_1 \cap E_2) = \bigvee\{\mu P \mid P \subseteq E_1 \cap E_2\cdot P \text{ closed}\}.$$

But every closed $P\subseteq E_1\cap E_2$ is the intersection $F\cap K$ of two closed sets $F\subseteq E_1$ and $K\subseteq E_2$ (we may take $F=K=P$), and every intersection $K\cap F$ is a closed set contained in $E_1\cap E_2$. Hence

$$\mu(E_1 \cap E_2) = \bigvee_{F,K} \mu(K \cap F) = \bigvee_{K,F} (\mu K\mu F)$$
$$= \bigvee_F\left[\bigvee_K (\mu F\mu K)\right] = \bigvee_F\left[\mu F \bigvee_K \mu K\right]$$
$$= \bigvee_F \mu F \bigvee_K \mu K = \mu E_1\mu E_2.$$

Conversely, suppose that μ is a regular positive additive function from Σ into $\mathfrak{X}$ such that $\mu(E_1E_2)=\mu E_1\mu E_2$. If $f\geqq 0$ and $g\geqq 0$, then f is the limit in measure of a directed set $\{f_a\}^{\geqq}$ of simple functions such that $0 \leqq f_a \leqq f$, and g is the limit in measure of a directed set $\{g_b\}$ of simple functions such that $0 \leqq g_b \leqq g$. Hence $\{f_a g_b\}^{\geqq}$ is a directed set of simple functions converging to fg in measure. Now

$$\int_S f_a \cdot \int_S g_b = \left(\sum_{i=1}^{k} \alpha_i \mu e_i \right) \left(\sum_{j=1}^{n} \beta_j \mu e_j' \right)$$
$$= \sum_{i,j} \alpha_i \beta_j \mu(e_i \cap e_j') = \int_S (\alpha_i \beta_j) \chi_{e_i \cap e_i'} = \int_S f_a g_b.$$

Hence $Tf\,Tg = \int_S f \cdot \int_S g = \int_S fg = T(fg)$ for positive functions f and g. The extension to arbitrary elements f and g of $C(S)$ is easy:

$$\begin{aligned} T(fg) &= T(f^+g^+ - f^+g^- - f^-g^+ + f^-g^-) \\ &= (Tf^+)(Tg^+) - (Tf^+)(Tg^-) - (Tf^-)(Tg^+) + (Tf^-)(Tg^-) \\ &= T(f^+ - f^-)T(g^+ - g^-) = TfTg. \end{aligned}$$

3. **Final remarks.** Let $\mathfrak{A}$ be a commutative C^* algebra containing the identity operator I. Then by the Gelfand-Neumark theorem, $\mathfrak{A}$ is isometrically isomorphic to the algebra $C(S)$ of all continuous complex-valued functions on a compact Hausdorff space S. Let T: $C(S) \rightarrow \mathfrak{A}$ be this isometric isomorphism. Then Tf is a Hermitian operator for every real function f. If $f \geqq 0$, then $((Tf^{1/2})x, (Tf^{1/2})x) \geqq 0$ for every $x \in \mathfrak{H}$. It follows that T is positive. Now the closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ in the strong topology is also a commutative C^* algebra with identity I. By §I.2.15, $\overline{\mathfrak{A}}$ is a Dedekind complete partially ordered algebra. Noting §1.6, it follows that T may be represented by means of §1.5. With §§1.10 and 2.2, and the fact that the functionals $X \rightarrow (Xx, x)$ distinguish between points of $\overline{\mathfrak{A}}$, this furnishes another proof of the spectral theorem for bounded Hermitian operators.

References

1. A. D. Alexandroff, *Additive set functions in abstract spaces*, Rec. Math. (Mat. Sbornik) N.S. 2, vol. 51 (1941) pp. 563–621.

2. R. G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canadian Journal of Mathematics vol. 7 (1955) pp. 289–305.

3. Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, XXV, 1948.

4. O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 569–582.

5. H. Hahn and A. Rosenthal, *Set functions*, Albuquerque, New Mexico, 1948.

6. P. Halmos, *Measure theory*, van Nostrand, New York, 1950.

7. H. MacNeille, *Partially ordered sets*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 416–460.

8. E. J. McShane, *Order-preserving maps and integration processes*, Annals of Mathematics Studies, vol. 31, Princeton, 1953.

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