B-SETS AND FINE-CYCLIC ELEMENTS(1)

BY

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Introduction. The theory of A-sets and proper cyclic elements of a Peano space proved extremely fruitful not only as an additional insight in the structure of Peano spaces, but also—to cite an example—in applications to surface area theory (see [2; 5; 6]). Nevertheless, as L. Cesari has observed in his papers [2; 3] concerning surfaces that are defined as mappings from closed finitely connected Jordan regions, it is desirable to have a finer decomposition of proper cyclic elements. For, if one considers the middle space M associated with such a continuous mapping (see [6; 9]), the proper cyclic elements of M may have the form of several "leaves" linked together. Each of these constituent "leaves" ought to be considered by itself and should constitute, in the terminology of L. Cesari [2; 3], a *fine-cyclic element.*

Several attempts have been made in the literature to decompose proper cyclic elements further (see [4; 7; 8]). Recently, L. Cesari [2; 3] has succeeded in obtaining such a decomposition for surfaces defined as mappings from closed finitely connected Jordan regions which proved very useful in the theory of surface area. One of the purposes of this paper is to extend Cesari's concepts to Peano spaces.

In the above mentioned papers [4; 7] a decomposition of a proper cyclic element has been obtained by generalizing the concept of *conjugacy* (see [6; 9]). However, the elements so obtained lack many properties that are possessed by proper cyclic elements and which are very desirable. In this paper a new attempt is made to decompose a proper cyclic element by generalizing the concept of a set being *cyclic* (§13 of this paper). The writer's intention has been to obtain as complete a theory for the generalized elements as the theory of proper cyclic elements of a Peano space.

It is well-known that a proper cyclic element of a Peano space is a cyclic A-set. Indeed, a proper cyclic element can be defined to be a cyclic A-set. Accordingly, the greatest part of this paper deals with a generalization of an A-set. In this connection it should be recalled that an A-set of a Peano space P can be defined as a closed nondegenerate subset A of P with the property that each component of $P-A$ has only a single frontier point. An immediate and obvious generalization of an A-set would be a nondegenerate continuum B of P such that the frontier of each component of $P-B$ decomposes into a finite number of points. Such sets will be termed *B-sets.* A *fine-cyclic element* will then be defined to be a B-set which remains connected after removing

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any finite set of points. Such a definition is shown to be equivalent to the one of L. Cesari. Moreover, the fine-cyclic elements of a unicoherent Peano space P are precisely the proper cyclic elements of P.

The theory of B-sets of a general Peano space is, however, not as complete as might be expected. Indeed, a B-set need not be a Peano space. There are many other shortcomings to be pointed out in the course of the paper. The aim of this paper is to obtain a theory of B-sets as complete as possible as the theory of A-sets. Accordingly, the writer was forced to restrict the Peano spaces and to consider only Peano spaces whose degree of *multicoherence* is finite. As the paper attempts to show, the properties of B-sets in such Peano spaces are suitable extensions of the corresponding properties of A-sets.

At this point the writer wishes to express his gratitude to Professor L. Cesari for the privilege he has given him to study his papers [2; 3].

Notation. The following notation will be employed in this paper. The closure of a set $E \subset X$ will be denoted by $c(E)$, and the frontier of E by $\mathrm{Fr}(E)$. These concepts depend upon the containing space. Consequently, if the space relative to which these operations are considered, is not the original space, it will appear as a subscript, i.e., the notation $\mathrm{Fr}_A(E)$ is the frontier of E relative to the space A.

1. **Definition of a B-set and some properties.** Let P be a Peano space, i.e., P is a Hausdorff space which is a continuous image of the closed unit interval $0 \leqq t \leqq 1$. All Peano spaces considered will be nondegenerate (more than one point).

DEFINITION. A nondegenerate continuum B of P will be termed a *B-set* of P provided either $B = P$ or else every component of $P - B$ has only a finite number of frontier points.

The proof of the following three lemmas offers no difficulty.

(i) LEMMA. *Let B be a B-set of a Peano space P and assume that $P - B \neq \emptyset$. If G is a component of $P - B$, then $c(G)$ is a B-set of P.*

(ii) LEMMA. *Under the same conditions as in* (i), *$P - G$ is a B-set of P.*

(iii) LEMMA. *If $B_1, \cdots, B_n$ is a finite collection of B-sets of a Peano space P and if $B = B_1 \cup \cdots \cup B_n$ is connected, then B is a B-set of P.*

2. **Condition $r(P)$.** In the sequel, unless otherwise stated, we will have to restrict ourselves to Peano spaces whose *degree of multicoherence* is finite (Whyburn [9, p. 83]). Let P be a Peano space, and let for any two continua F_1, F_2 of P whose union is P, $r(F_1, F_2)$ be the number of components of $F_1 \cap F_2$ less one. The degree of multicoherence $r(P)$ of P is defined by $r(P) = \text{l.u.b. } r(F_1, F_2)$, where the least upper bound is taken over all decompositions $P = F_1 \cup F_2$ of two continua F_1, F_2.

(i) LEMMA. *Let P, P^* be Peano spaces with $r(P) = n < \infty$, and let m be a continuous and monotone mapping from P onto P^*. Then $r(P^*) \leqq n$.*

Proof. Let F_1^*, F_2^* be any two continua of P^* with $F_1^*\cup F_2^*=P^*$, and set $F_1=m^{-1}(F_1^*)$, $F_2=m^{-1}(F_2^*)$. Since m is monotone, F_1, F_2 are continua of P such that $F_1\cup F_2=P$. Therefore, $r(F_1, F_2)\leqq n$. Let now F be a component of $F_1\cap F_2$. Since $m(F_1\cap F_2)=F_1^*\cap F_2^*$, there is a component F^* of $F_1^*\cap F_2^*$ such that $m(F)\subset F^*$. Therefore, the number of components of $F_1^*\cap F_2^*$ is not greater than the number of components of $F_1\cap F_2$. Consequently, $r(F_1^*, F_2^*)\leqq n$, and the lemma follows.

For a slightly different proof of (i) see Whyburn [9, p. 154].

3. **Preliminary results.** In this section we will state some lemmas which will be needed in the sequel. Most of the results are well-known and therefore some of the proofs are omitted.

(i) Lemma. *Let H, K be disjoint closed sets of a metric space M, and assume that $M-(H\cup K)\neq\varnothing$. Then there exists a closed set E in M separating H, K in M.*

(ii) Lemma. *Let A be a finite union of disjoint continua of a Peano space P. Then there are two continua H, K of P such that* (1) $H\cap K=\varnothing$, (2) $A\subset H\cup K$, (3) $A\cap H\neq\varnothing$, $A\cap K\neq\varnothing$.

(iii) Lemma. *Let H, K be two disjoint continua of a Peano space P and let A be a closed subset of P consisting of n distinct components. If A separates H, K in P, then there exists a number ν, $1\leqq\nu\leqq n$, such that* (1) *no union of less than ν components of A separates H, K in P*; (2) *there exists a union of ν components separating H, K in P.*

By a change in notation we may assume that $A_1, \cdots, A_\nu$, $1\leqq\nu\leqq n$, is such a minimal collection of continua of A. If we set $A^*=A_1\cup\cdots\cup A_\nu$, the set A^* separates H, K in P. Let G be the union of all components of $P-A^*$ not containing H.

(iv) Lemma. *$A^*\cup G$ is a continuum of P.*

Proof. Let γ be the component of $P-A^*$ containing H. Then $A^*\cup G=P-\gamma$, which shows that $A^*\cup G$ is closed. To verify that $A^*\cup G$ is connected, let us assume that

$$A^*\cup G = E\cup F, \tag{1}$$

where E, F are nonempty disjoint closed sets. We may assume that $K\subset F$. If $F\cap A^*=\varnothing$, then $P=(\gamma\cup E)\cup F$, where $\gamma\cup E$, F are nonempty disjoint closed sets of P. Since P is connected, this is impossible, and hence

$$F\cap A^*\neq\varnothing, \qquad E\cap A^*\neq\varnothing. \tag{2}$$

From (1) and (2) it follows that $F\cap A^*$ decomposes into less than ν components of A.

Consider the set equality $P-(F\cap A^*)=(\gamma\cup E)\cup(F-A^*)$. Since $\mathrm{Fr}(\gamma)$

$\subset A^*$, we have $c(\gamma\cup E)\cap(F-A^*)=[E\cup c(\gamma)]\cap(F-A^*)=\varnothing$. Moreover, $(\gamma\cup E)\cap c(F-A^*)\subset(\gamma\cup E)\cap F=\varnothing$, and $H\subset\gamma\cup E$, $K\subset F-A^*$. Thus $F\cap A^*$ separates H, K in P. Since, as noted above, $F\cap A^*$ decomposes into less than ν components of A, we arrived at a contradiction to the definition of A^*.

(v) Lemma. *Let K be continuum of a Peano space P and let G be a component of $P-K$. If $r(P)=n<\infty$, then* $\mathrm{Fr}(G)$ *decomposes into at most $n+1$ distinct components.*

(vi) Lemma. *Let K_1, K_2 be two disjoint continua of a Peano space P with $r(P)=n<\infty$. Let A be a closed set of P separating K_1, K_2 in P. Then there exist ν components $A_1, \cdots, A_\nu$, $\nu\leqq n+1$, of A such that $A^*=A_1\cup\cdots\cup A_\nu$ separates K_1, K_2 in P.*

Proof. Let G_1 be the component of $P-A$ containing K_1. Then $\mathrm{Fr}(G_1)\subset A$, whence $K_2\subset P-c(G_1)$. Let G_2 be the component of $P-c(G_1)$ containing K_2. Since $c(G_1)$ is a continuum, we have from (v) that $\mathrm{Fr}(G_2)$ decomposes into at most $n+1$ distinct components each of which is in A. Let then $k_1, \cdots, k_i$, $i\leqq n+1$, be the components of $\mathrm{Fr}(G_2)$, and let $A_1, \cdots, A_\nu$, $\nu\leqq i$, be all those distinct components of A such that $k_1\cup\cdots\cup k_i\subset A_1\cup\cdots\cup A_\nu$. Since $k_1\cup\cdots\cup k_i$ separates K_1, K_2 in P so does $A^*=A_1\cup\cdots\cup A_\nu$. This completes the proof.

4. **Further properties of B-sets.** Let P be a Peano space with $r(P)=n<\infty$ and let R be a continuum of P. If G is a component of $P-R$, then by 3(v), $\mathrm{Fr}(G)$ decomposes into at most $n+1$ distinct components. The next theorem gives some information concerning the number of components of $P-R$.

(i) Theorem. *Let R be a continuum of a Peano space P with $r(P)=n<\infty$. Then there can be at most n distinct components of $P-R$ the frontier of each of which decomposes into more than one component.*

Proof. If we deny the assertion we have at least $n+1$ distinct components $G_1, \cdots, G_{n+1}$ of $P-R$ such that for each i, $\mathrm{Fr}(G_i)$ reduces to more than one component. By 3(ii) we have for each i two disjoint continua H_i, K_i of P such that $\mathrm{Fr}(G_i)\subset H_i\cup K_i$, $\mathrm{Fr}(G_i)\cap K_i\neq\varnothing$, $\mathrm{Fr}(G_i)\cap H_i\neq\varnothing$. By 3(i) in conjunction with 3(vi) we have a closed set $A_i\subset P$ consisting of at most $n+1$ components separating H_i, K_i in P. We may also assume that the number of components of A is minimal in the sense of 3(iii). Note also that A_i separates $K_i\cap c(G_i)$, $H_i\cap c(G_i)$ in $c(G_i)$.

For each i, let S_i be the union of all components of $P-A_i$ not containing K_i and let δ_i be the component of $P-A_i$ containing K_i. Then $\delta_i\cap G_i\neq\varnothing$ and $S_i\cap G_i\neq\varnothing$. By 3(iv)

$$A_i\cup S_i,\qquad i=1,\cdots,n+1 \tag{1}$$

is a continuum of P.

Let now $N_i = c(G_i) \cap (A_i \cup S_i)$, $M_i = c(\delta_i) \cap c(G_i)$. Then N_i, M_i are closed and nonempty. Since $c(G_i) = N_i \cup M_i$ and $c(G_i)$ is connected, we infer that $N_i \cap M_i \neq \emptyset$. Moreover, $N_i \cap M_i \subset G_i$. Since, for $i \neq j$, $G_i \cap G_j = \emptyset$, we obtain

(2) $$(N_i \cap M_i) \cap (N_j \cap M_j) = \emptyset, \qquad i \neq j,\ i, j = 1, \cdots, n+1.$$

Let now G^* be the union of all components G of $P - R$ not among $G_1, \cdots, G_{n+1}$. Define

(3) $$F_1 = R \cup G^* \cup \bigcup_{i=1}^{n+1} N_i, \qquad F_2 = R \cup \bigcup_{i=1}^{n+1} M_i.$$

Since $R \cup G^* = P - \bigcup_{i=1}^{n+1} G_i$, the set F_1 is closed. Clearly, F_2 is also closed. Next we will show that F_1 is connected. Let us deny this and assume that $F_1 = H \cup K$, where H, K are nonempty disjoint closed sets. Since $R \cup G^*$ is connected, let

(4) $$R \cup G^* \subset H.$$

From (3) and (4) we infer that $K \subset \bigcup_{i=1}^{n+1} N_i$, and since $G_i \cap N_j = \emptyset$, $i \neq j$, we have

(5) $$K \cap G_i \subset N_i.$$

Let $K_i' = K \cap N_i$. Then K_i' is not empty for some i. The remainder of the proof will be in terms of such an index i. Since $K_i' \subset A_i \cup S_i$, consider the set equality

(6) $$A_i \cup S_i = K_i' \cup [(A_i \cup S_i) - K_i'].$$

Since from (4), $K_i' \subset P - R$, we have that $K_i' \subset G_i$, and hence

(7) $$K_i' \neq \emptyset, \qquad (A_i \cup S_i) - K_i' \neq \emptyset.$$

Moreover,

(8) $$c(K_i') \cap [(A_i \cup S_i) - K_i'] = K_i' \cap [(A_i \cup S_i) - K_i'] = \emptyset.$$

We will show now that $[(A_i \cup S_i) - K_i'] \cap G_i \subset H$. First of all the left-hand set is contained in F_1. Assume now that there is a point

$$x \in [(A_i \cup S_i) - K_i'] \cap G_i$$

which is also in K. Then $x \in K \cap G_i$, and hence by (5), $x \in N_i$. Thus $x \in K \cap N_i = K_i'$, which is impossible. Therefore

(9) $$\begin{aligned} c[(A_i \cup S_i) - K_i'] \cap K_i' &= c[(A_i \cup S_i) - K_i'] \cap G_i \cap K_i' \\ &\subset c\{[(A_i \cup S_i) - K_i'] \cap G_i\} \cap K_i' \subset H \cap K_i' \subset H \cap K = \emptyset. \end{aligned}$$

Since (6), (7), (8) and (9) contradict (1), we have shown that F_1 is a continuum.

By an entirely analogous argument it can be shown that F_2 is also a continuum. It follows from (3) that $F_1 \cup F_2 = P$, and from (2) we infer that $F_1 \cap F_2$ decomposes into at least $n+2$ distinct components. This, however, is contrary to the hypothesis that $r(P)=n$. The proof is now complete.

(ii) COROLLARY. *Let B be a B-set of a Peano space P with $r(P)=n<\infty$. Then there can be at most n distinct components of $P-B$ with more than one frontier point.*

5. A**-sets.** For the purpose of simplifying proofs in the subsequent discussion of B-sets, we will state in this section some properties of A-sets of a Peano space P. Let us recall that a nondegenerate closed subset A of P is termed an A-set of P provided every component of $P-A$ has a single frontier point. For the proof of (i) the reader is referred to Whyburn [9], and the proof of (ii) is immediate.

(i) LEMMA. *Let $\mathfrak{A}$ be a collection of A-sets of a Peano space P. If $A^* = \bigcap A$, $A \in \mathfrak{A}$, is nondegenerate, then A^* is an A-set of P.*

(ii) LEMMA. *Let B be a B-set of a Peano space P, and let A be an A-set of P containing B. Then B is also a B-set of A.*

Let K be a nondegenerate subset of a Peano space P. Then the intersection of all A-sets of P containing K is by (i) an A-set of P and will be referred to as the *smallest A-set* of P containing K.

(iii) LEMMA. *Under the above conditions, if A is the smallest A-set of P containing K, then for every component G of $A-K$, $Fr_A(G)$ decomposes into at least two distinct points.*

Proof. Assume there is a component G of $A-K$ such that $\mathrm{Fr}(G)$ reduces to a single point. Then $A-G$ is an A-set A^* of A containing K. Since A is an A-set of P, A^* is also an A-set of P. Since A^* is properly contained in A, we have a contradiction.

6. **More properties of** B**-sets.** Let A be an A-set of a Peano space P and let K be a connected subset of P. Then by Whyburn [9], $A \cap K$ is connected (possibly empty). The following two theorems assert an analogous property for B-sets.

(i) THEOREM. *Let B be a B-set of a Peano space P with $r(P)=n<\infty$, and let K be a continuum of P. Then $B \cap K$ decomposes into at most $n+1$ distinct components.*

Proof. Let A be the smallest A-set of P containing B. Since A is a monotone retract of P, we have by 2(i) that $r(A) \leqq n$. From 5(ii), B is a B-set of A and as a consequence of 5(iii) every component of $A-B$ has more than one frontier point (relative to A). Since $r(A) \leqq n$, we infer from 4(i) that there is only a finite number of components of $A-B$.

Since A is an A-set of P, $A\cap K$ is connected. We may exclude the trivial cases $B\cap K=K$, $B\cap K=\varnothing$, and $A=B$. Let S be the union of the closures of all components G of $A-B$ for which $c(G)\cap K\neq\varnothing$, and let S' be the union of all other components. Since there is only a finite number of components of $A-B$, we conclude that $F_1=B\cup S'$, $F_2=K'\cup S$, where $K'=A\cap K$, are two continua whose union is A. Therefore, $F_1\cap F_2$ decomposes into at most $n+1$ distinct components. Now $F_1\cap F_2=(B\cap K')\cup(S'\cap K')\cup(B\cap S)\cup(S'\cap S)$ $=(B\cap K')\cup(B\cap S)$. Since $B\cap S$ is a finite set of points, $B\cap K'=B\cap A\cap K$ $=B\cap K$ decomposes into at most $n+1$ distinct components.

Let us denote by $\rho(K)$ the diameter of a set K.

(ii) THEOREM. *Let B be a B-set of a Peano space P with $r(P)=n<\infty$. Then there exists a $\delta=\delta(B)>0$ such that for every connected subset K of P with $\rho(K)<\delta$ the set $B\cap K$ is connected.*

Proof. As in (i) let A be the smallest A-set of P containing B. Then $C=\bigcup \mathrm{Fr}_A(G)$, where the union is taken over all components G of $A-B$, is a finite set and we may write $C=\{x_1, \cdots, x_k\}$, $x_i\neq x_j$ for $i\neq j$. Define

$$(1)\qquad \delta=\delta(B)=\frac{1}{2}\min\,[\rho(x_i, x_j),\, i\neq j,\, i, j=1, \cdots, k].$$

Let now K be any connected subset of P with diameter less than δ. Then $K'=K\cap A$ is connected. We may assume that $K'\cap B\neq\varnothing$, $(A-B)\cap K'\neq\varnothing$. Since $\rho(K')<\delta$, $K'\cap B$ contains at most one point of C. From $(A-B)\cap K'$ $\neq\varnothing$ we infer that $K'\cap B$ has precisely one point x_0 in common with C.

Let us assume that $B\cap K'$ separates into N and M. Then we may take $x_0\in N$. We assert that $M\subset B^0$, where B^0 denotes the interior of B relative to A. Let x be an arbitrary point of M. Then $x\neq x_0$, and hence $0<\rho(x, x_0)<\delta$. Since A is a Peano space, let $\mathcal{O}$ be a connected open set in A such that $x\in\mathcal{O}$ and $\rho(\mathcal{O})<\rho(x, x_0)$. We will show now that $(A-B)\cap\mathcal{O}=\varnothing$. If this is not the case, there is a component G of $A-B$ intersecting $\mathcal{O}$ and since $\mathcal{O}$ is connected, $\mathrm{Fr}_A(G)\cap\mathcal{O}\neq\varnothing$. Consequently, there is a point $x'\in C$ which is also in $\mathcal{O}$. Since $\rho(\mathcal{O})<\rho(x, x_0)$, we have that $x'\neq x_0$, and thus $\rho(x', x_0)\leqq\rho(x, x_0)+\rho(x, x')<2\delta$. This contradicts (1) and therefore $x\in\mathcal{O}\subset B^0$. Thus M is a subset of B^0.

It follows now that $K'=(K'-M)\cup M$, $K'-M\neq\varnothing$, $M\neq\varnothing$. Taking closures relative to A, we have $c(M)\cap(K'-M)=c(M)\cap B\cap(K'-M)$ $=c(M)\cap[(B\cap K')-M]=c(M)\cap N=\varnothing$, and $c(K'-M)\cap M$ $=c(K'-M)\cap B^0\cap M\subset c[(K'-M)\cap B]\cap M=c[(B\cap K')-M]\cap M=c(N)\cap M$ $=\varnothing$. This contradicts the connectedness of K', and hence $B\cap K'=B\cap K\cap A$ $=B\cap K$ is connected.

(iii) REMARK. If $r(P)=\infty$, then (ii) is generally not true as the example in §8 will show. The following more general result can be proved by an entirely analogous argument. Let B be a B-set of a Peano space P and suppose that there is a Peano subspace Q of P such that $B \subset Q$ and $Q-B$ decomposes into a finite number of components. Then there exists a $\delta = \delta(B) > 0$ such that for every connected subset K of Q with $\rho(K) < \delta$ the set $B \cap K$ is connected. Note that B is also a B-set of Q.

7. **Peano subspaces and B-sets.** Let B be a B-set of a Peano space P.

(i) THEOREM. *Suppose that there is a Peano subspace Q of P such that $B \subset Q$ and $Q-B$ decomposes into a finite number of components. Then B is a Peano space*(2).

Proof. We only have to show that B is locally connected, and this will be accomplished if B can be written as a finite union of connected sets with arbitrarily small diameter. Let $\epsilon > 0$ be given and let $0 < \eta = \min [\delta(B), \epsilon]$. Since Q is *locally* connected, there is a finite number of connected sets $K_1, \cdots, K_t$ such that $Q = K_1 \cup \cdots \cup K_t$ and $\rho(K_i) < \eta$. By 6(iii), $B \cap K_i$ is connected, and hence B can be written as a finite union of connected sets with diameter less than ϵ.

(ii) COROLLARY. *If G is a component of $P-B$, then $c(G)$ and $P-G$ are Peano spaces. Moreover, if $r(P) < \infty$, then a B-set B of P is a Peano space.*

Proof. The smallest A-set of P containing $c(G)$, or $P-G$, or else B can be taken as the Peano space Q in (i).

8. **Example.** Let $Q \equiv [0 \leqq u, v \leqq 1]$ be the closed unit square in the Euclidean (u, v)-plane, and let Q^* be the boundary of Q. Moreover, let $L_n \equiv [v = 2^{-n}, 0 \leqq u \leqq 1]$, $n = 0, 1, \cdots$, and let $K_n \equiv [u = i2^{-(n+1)}, 0 \leqq v \leqq 2^{-n}, i = 0, 1, \cdots, 2^{n+1}]$, $n = 0, 1, \cdots$. Finally let

$$P = Q^* \cup \left(\bigcup_{n \geqq 0} L_n \right) \cup \left(\bigcup_{n \geqq 0} K_n \right).$$

It is easily verified that P is a Peano space with $r(P) = \infty$. Let now

$$B = Q^* \cup \left(\bigcup_{n \geqq 0} L_n \right).$$

Then each component of $P-B$ has two distinct frontier points, and hence B is a B-set of P. Since B is not locally connected, B is not a Peano subspace of P. It is also seen that there is no $\delta = \delta(B)$ satisfying 6(ii). Moreover, there are continua $K \subset P$ such that $K \cap B$ decomposes into an infinite number of components (see 6(i)).

9. **Further properties of B-sets.**

(i) LEMMA. *Let B be a B-set of a Peano space P with $r(P) = n < \infty$. Then $r(B) \leqq n$.*

(2) The author is indebted to the referee for suggesting this theorem.

Proof. Let A be the smallest A-set of P containing B. Since B is also a B-set of A, we may without loss of generality assume that $P=A$. In view of 4(ii) and 5(iii), $P-B$ decomposes into a finite number of components each of which has more than one frontier point.

Let now F_1, F_2 be any two continua whose union is B. Let S_1 be the union of the closures of all components G of $P-B$ for which $\mathrm{Fr}(G)\cap F_1\neq\emptyset$, and let S_2 be the union of all other components of $P-B$. Then the sets $F_1^* = F_1\cup S_1$, $F_2^*=F_2\cup S_2$ are two continua of P whose union is P. Therefore, $F_1^*\cap F_2^*$ reduces to at most $n+1$ distinct components. Now $F_1^*\cap F_2^* = (F_1\cap F_2)\cup(F_2\cap S_1)\cup(F_1\cap S_2)\cup(S_1\cap S_2) = (F_1\cap F_2)\cup(F_2\cap S_1)$. Since $F_2\cap S_1$ is either empty or else finite, we conclude that $F_1\cap F_2$ decomposes into at most $n+1$ components. The proof is complete.

(ii) LEMMA. *Let A be an A-set of a Peano space P and let B be a B-set of A. Then B is also a B-set of P.*

Proof. Let G be a component of $P-B$. If $G\subset P-A$, then G is also a component of $P-A$, and hence $\mathrm{Fr}(G)$ reduces to a single point. We may therefore assume that $G\cap A\neq\emptyset$. Since $G'=G\cap A$ is connected, G' is a component of $A-B$. Since B is a B-set of A, $\mathrm{Fr}_A(G')$ is finite.

We now assert that

$$\mathrm{Fr}_A\,(G') = \mathrm{Fr}\,(G). \tag{1}$$

Since $\mathrm{Fr}_A(G')\subset\mathrm{Fr}(G)$ is obvious, let $x\in\mathrm{Fr}(G)$. Then there exists a sequence of points $\{x_n\}$ in G such that $x_n\to x$. If infinitely many x_n are in A, then $x_n\in G'$ and hence $x\in\mathrm{Fr}_A(G')$. Thus we may proceed with the proof under the assumption that $x_n\notin A$ for each n. Let γ_n be the component of $P-A$ containing x_n. Let us observe that only a finite number of the x_n lie in a given γ_k. Otherwise, $x\in\mathrm{Fr}(\gamma_k)$, and since $G-\gamma_k\neq\emptyset$, $x\in G$ which is impossible.

Hence we have a sequence $\{n_i\}$ with the property that $x_{n_i}\in\gamma_{n_i}$, $\gamma_{n_i}\cap\gamma_{n_j} =\emptyset$, $i\neq j$, $x_{n_i}\to x$. Since γ_{n_i} is a component of $P-A$, $\mathrm{Fr}(\gamma_{n_i})$ reduces to a single point p_{n_i} in A. By Whyburn [9], $\rho[c(\gamma_{n_i})]\to 0$ as $i\to\infty$, and therefore $\rho(p_{n_i}, x)\leqq\rho(p_{n_i}, x_{n_i})+\rho(x_{n_i}, x)\leqq\rho[c(\gamma_{n_i})]+\rho(x_{n_i}, x)\to 0$ as $i\to\infty$. Thus $p_{n_i}\to x$, and since $p_{n_i}\in G'$, we infer that $x\in\mathrm{Fr}_A(G')$. Thus (1) follows and the proof of (ii) is complete.

(iii) LEMMA. *Let B be a B-set of a Peano space P with $r(P)<\infty$. If B' is a B-set of B, then B' is also a B-set of P.*

Proof. Let A be the smallest A-set of P containing B. Then B is also a B-set of A and $A-B$ decomposes into a finite number of components each of which has more than one frontier point. We will show that B' is a B-set of A, which by (ii) concludes the proof.

Let now $L=\bigcup\mathrm{Fr}_A(G)$, where the union is taken over all components G of $A-B$. Then $L\subset B$ and L is finite. Consider a component G' of $A-B'$.

If $G'\cap L=\emptyset$, then either G' is a component of $A-B$, or else G' is a component of $B-B'$. To prove this assume that G' is not a component of $A-B$. Then $G'\cap B\neq\emptyset$. If now $G'-B\neq\emptyset$, then there is a component γ of $A-B$ contained in G'. Since $G'-\gamma\neq\emptyset$, there follows that $G'\cap \mathrm{Fr}_A(\gamma)\neq\emptyset$, and thus $G'\cap L\neq\emptyset$, a contradiction. In either case $\mathrm{Fr}_A(G')$ is finite.

Hence we assume that $G'\cap L\neq\emptyset$. Since L is a finite set of points in B, let $\gamma_1, \cdots, \gamma_k$ be the components of $B-B'$ such that $\gamma_i\cap L\neq\emptyset$, $i=1, \cdots, k$. We assert that

$$(1) \qquad \gamma_1 \cup \cdots \cup \gamma_k \supset G' \cap B.$$

Let $x\in G'\cap B$ and let γ be the component of $B-B'$ containing x. We will prove that $\gamma\cap L\neq\emptyset$. If we deny this, then it follows readily that γ is open in A and that $\mathrm{Fr}_A(\gamma)\subset B'$. From this we deduce that

$$(2) \qquad G' = (G' - \gamma) \cup \gamma,\ G' - \gamma \neq \emptyset,\ \gamma \neq \emptyset,$$

$$(3) \qquad c(\gamma) \cap (G' - \gamma) = \gamma \cap (G' - \gamma) = \emptyset, \text{ since } \mathrm{Fr}_A(\gamma) \subset B',$$

$$(4) \qquad c(G' - \gamma) \cap \gamma \subset c[(G' - \gamma) \cap \gamma] = \emptyset,$$

where the above closures are relative to A. However, (2), (3), and (4) contradict the connectedness of G'. Thus $\gamma\cap L\neq\emptyset$, and (1) follows.

We will prove now that

$$(5) \qquad \mathrm{Fr}_A(G') \subset \bigcup_{i=1}^{k} \mathrm{Fr}_B(\gamma_i) \cup L.$$

If $x\in \mathrm{Fr}_A(G')$, then there is a sequence $\{x_n\}$ in G' such that $x_n\to x$. If infinitely many x_n are in $G'\cap B$, then from (1), $x\in \mathrm{Fr}_B(\gamma_i)$ for some i. Otherwise, x is in L, thus proving (5).

Since L is a finite set of points and since B' is a B-set of B, $\mathrm{Fr}_B(\gamma_1)\cup\cdots\cup\mathrm{Fr}_B(\gamma_k)$ is a finite set of points. Thus (5) implies that B' is a B-set of A. Thereby (iii) is proved.

Remark. Without the restriction $r(P)<\infty$, (iii) is in general not true. As an example consider

$$I \equiv [0 \leqq x \leqq 1,\ y = 0], \qquad I_1 \equiv [0 \leqq x \leqq 1/2,\ y = x],$$
$$I_2 \equiv [1/2 \leqq x \leqq 1,\ y + x = 1],$$
$$I_n' \equiv [x = n^{-1},\ 0 \leqq y \leqq n^{-1}], \qquad I_n'' \equiv [x = 1 - n^{-1},\ 0 \leqq y \leqq n^{-1}].$$

Then $P=I\cup I_1\cup I_2\cup\bigcup_{n\geqq 2}(I_n'\cup I_n'')$ is a Peano space. The set $B=I\cup\bigcup_{n\geqq 2}(I_n'\cup I_n'')$ is a B-set of P, and $B'=I$ is a B-set, even A-set, of B. However, B' is not a B-set of P.

10. **Intersection of B-sets.** The property mentioned in 5(i) for A-sets will not be true for B-sets, since the intersection of B-sets need not be connected. We have, however, the following results.

(i) LEMMA. *Let $B_1, \cdots, B_k$ be a finite number of B-sets of a Peano space P with $r(P) = n < \infty$. If B is a nondegenerate component of $B_1 \cap \cdots \cap B_k$, then B is a B-set of P.*

Proof. We proceed by induction on k. Let then $H = B_1 \cap B_2$. By 7(ii), B_1 is a Peano space and by 9(i), $r(B_1) \leqq n$. Let A be the smallest A-set of B_1 containing H. Then $r(A) \leqq n$.

If G is a component of $A - H$, we will prove that $\mathrm{Fr}_A(G)$ reduces to a finite number of points. Let us observe that $B_2 \cap G = \varnothing$, and hence there is a component γ of $P - B_2$ such that $\gamma \supset G$. Since $\mathrm{Fr}_A(G) \subset \mathrm{Fr}(\gamma)$ and since B_2 is a B-set of P, $\mathrm{Fr}_A(G)$ reduces to a finite number of points. If H is connected, we infer that H is a B-set of A and hence from §9, H is a B-set of P.

We assume now that H is not connected. By 6(i), H decomposes into a finite number of components. Since A is a Peano space, the number of components G of $A - H$ such that $\mathrm{Fr}_A(G)$ intersects at least two distinct components of H is finite. By 5(iii), each component of $A - H$ has more than one frontier point. Thus, by 4(i), the components G of $A - H$ for which there is a component K of H such that $\mathrm{Fr}_A(G) \subset K$ is also finite (for then G is a component of $A - K$). Let then $G_1, \cdots, G_k$ be the components of $A - H$. If B is a nondegenerate component of H and if G is a component of $A - B$, then $G \subset G_1 \cup \cdots \cup G_k \cup (H - B)$ and $\mathrm{Fr}_A(G) \subset \mathrm{Fr}_A(G_1) \cup \cdots \cup \mathrm{Fr}_A(G_k)$. Thus $\mathrm{Fr}_A(G)$ is finite and hence B is a B-set of A. Consequently, B is a B-set of P.

Assume now that (i) is established for $k-1$ B-sets of P, and consider k B-sets $B_1, \cdots, B_k$ of P. Let B be a nondegenerate component of $B_1 \cap \cdots \cap B_k$ and let B^* be the nondegenerate component of $B_1 \cap \cdots \cap B_{k-1}$ containing B. Then B^* is a B-set of P and B is a nondegenerate component of $B^* \cap B_k$. Thus B is a B-set of P.

REMARK. Without the restriction $r(P) < \infty$, (i) is in general false. As an example consider the Peano space defined in §9. Let

$$B_1 = I \cup \bigcup_{n \geqq 2} (I_n' \cup I_n''),\ n \text{ even},\quad B_2 = I \cup \bigcup_{n > 2} (I_n' \cup I_n''),\ n \text{ odd}.$$

Then B_1, B_2 are B-sets of P and $B_1 \cap B_2 = I$. However, I is not a B-set of P.

(ii) LEMMA. *Let $\mathfrak{B}$ be a collection of B-sets of a Peano space P with $r(P) = n < \infty$, and assume that $H = \bigcap B$, $B \in \mathfrak{B}$, is a nondegenerate continuum. Then H is a B-set of P.*

Proof. Let G be a component of $P - H$ and assume that $\mathrm{Fr}(G)$ reduces to more than $n+1$ distinct points. Let $x_1, \cdots, x_\nu$ be $\nu = n+2$ distinct points of $\mathrm{Fr}(G)$ and let $2\delta = \min [\rho(x_i, x_j),\ i \neq j,\ i, j = 1, \cdots, \nu]$. Since P is a Peano space, let $\mathcal{O}_i$ be connected open sets of P such that $x_i \in \mathcal{O}_i$ and the diameter of $\mathcal{O}_i$ is less than δ. Then $\mathcal{O}_i \cap \mathcal{O}_j = \varnothing$, $i \neq j$, and for each i, $\mathcal{O}_i \cap G \neq \varnothing$.

In view of the arcwise connectedness of G one can construct a continuum $\tau \subset G$ such that $\mathcal{O}_i \cap \tau \neq \emptyset$, $i=1, \cdots, \nu$. Now for each $y \in \tau$, there exists a set $B_y \in \mathfrak{B}$ such that $y \notin B_y$. Since B_y is closed, there is an open set $\mathcal{O}_y$ containing y and $\mathcal{O}_y \cap B_y = \emptyset$. Since τ is compact, we have a finite number of points $y_1, \cdots, y_m$ in τ such that $\mathcal{O}_{y_1} \cup \cdots \cup \mathcal{O}_{y_m} \supset \tau$. Let now C be the component of $B_{y_1} \cap \cdots \cap B_{y_m}$ which contains H. Since τ is disjoint with the set $B_{y_1} \cap \cdots \cap B_{y_m}$, there follows that $C \cap \tau = \emptyset$. In view of (i), C is a B-set of P. It is now easily seen that the component G' of $P-C$ containing τ has the property that $\mathrm{Fr}(G')$ decomposes into at least $n+2$ distinct points. Since C is a B-set of P, this contradicts 3(v). The proof of (ii) is complete.

Let C be a nondegenerate continuum of a Peano space P with $r(P) = n < \infty$, and let $\mathfrak{B}$ be the collection of all B-sets of P containing C.

(iii) LEMMA. *Under the above conditions the set $H = \bigcap B$, $B \in \mathfrak{B}$, is a B-set of P.*

Proof. By (ii) it suffices to show that H is connected. We may assume that $P-H \neq \emptyset$. We will exhibit now a decreasing sequence of B-sets $\{B_j'\}$ of $\mathfrak{B}$ such that $H = \bigcap_j B_j'$.

Since $P-H$ is open, let $\{K_i\}$ be a sequence of compact sets of $P-H$ such that $K_i \subset K_{i+1}$, $i=1, 2, \cdots$, and $\bigcup_i K_i = P-H$. For each i, let B_i be a B-set of $\mathfrak{B}$ with $B_i \subset P-K_i$. For the construction of such a B-set apply the method used in the proof of (ii). Set $B_1' = B_1$, and define inductively B_j' as the component of $B_j \cap B_{j-1}'$ containing C. Then by (i) each B_j' is a B-set of $\mathfrak{B}$, and $B_j' \supset B_{j+1}'$, $j=1, 2, \cdots$. To prove that $H = \bigcap_j B_j'$, observe that $H \subset \bigcap_j B_j'$. In view of the property or the sequence $\{K_i\}$ we also have the complementary inclusion. By a well-known theorem (Whyburn [9, p. 14]) we conclude that H is connected.

11. **Property λ.** A subset E of a Peano space P is said to satisfy the *property* λ provided (1) E is connected, (2) no finite set of points of E disconnects E. Clearly, if E satisfies the property λ so does $c(E)$. The proof of (i) is left to the reader.

(i) LEMMA. *Let B be a B-set of a Peano space P and let E be a subset of P satisfying the property λ. If $B \cap E$ is infinite, then $E \subset B$.*

(ii) LEMMA. *Let E be a nondegenerate subset of a Peano space P satisfying the property λ and let $\mathfrak{B}$ be the collection of all B-sets of P containing E. If $r(P) = n < \infty$, then $H = \bigcap B$, $B \in \mathfrak{B}$, is a B-set of P satisfying the property λ.*

Proof. In view of 10(iii) it suffices to show that H satisfies the property λ. If we deny this, we have a finite set F of points $x_1, \cdots, x_i$ in H such that $H-F$ is not connected. Since $c(E)$ satisfies the property λ, $c(E)-F$ is connected, and hence $c(E)-F$ lies in a component G of $H-F$. It follows readily that $c(G) \supset c(E)$. If now Q is a component of $H-c(G)$, then $\mathrm{Fr}_H(Q) \subset F$ and hence $\mathrm{Fr}_H(Q)$ reduces to a finite number of points. Thus $c(G)$ is a B-set of H and hence by 9(iii), $c(G)$ is also a B-set of P. Since $c(G)$ is properly contained in H, we have a contradiction.

REMARK. If $r(P)=\infty$, (ii) is in general false. As an example, let $E \equiv [(x, y): 0 \leqq x \leqq 1,\ 0 \geqq y \geqq -1]$ and let I_1, I_2, I_n', I_n'' be defined as in §9. Then $P=E \cup I_1 \cup I_2 \cup \bigcup_{n\geqq 2}(I_n' \cup I_n'')$ is a Peano space, and E satisfies the property λ. The intersection of all B-sets of P containing E is E. However, E is not a B-set of P.

12. **Diameter of B-sets.** In this section we will discuss a property of B-sets which is analogous to the following property of A-sets (T. Radó [6]). If $\mathfrak{A}$ is a collection of A-sets of a Peano space P, and if any two distinct A-sets of $\mathfrak{A}$ are either disjoint or else have a single point in common, then there can only be a finite number of A-sets of $\mathfrak{A}$ with diameter greater than a given positive number. First we will state a lemma whose proof is left to the reader.

(i) LEMMA. *Let M be a connected metric space and let n be a positive integer. Moreover, let x_1, x_{n+2} be two points in M with $\rho(x_1, x_{n+2})=\delta>0$. Then there exist n points $x_2, \cdots, x_{n+1}$ in M such that $\rho(x_j, x_i)\geqq \delta/2^n$, $i\neq j$, $i, j=1, \cdots, n+2$.*

(ii) THEOREM. *Let P be a Peano space with $r(P)=n<\infty$, and let $\mathfrak{B}$ be a collection of B-sets of P with the property that any two distinct B-sets of $\mathfrak{B}$ are either disjoint or else have only a finite number of points in common. Let $\delta>0$ be given. Then there is at most a finite number of B-sets of $\mathfrak{B}$ with diameter $\geqq \delta$.*

Proof. Let B', B'' be any two distinct B-sets of $\mathfrak{B}$. If $B' \cap B'' \neq \emptyset$, then $B' \cap B''$ decomposes into at most $n+1$ distinct points (see 6(i)).

Deny the above theorem, and assume that there is an infinite sequence $B_1, B_2, \cdots, B_n, \cdots$ of B-sets of $\mathfrak{B}$ with diameter $\geqq \delta$. By (i) we can choose in each B_i, $n+2$ distinct points $\overset{i}{x}_1, \cdots \overset{i}{x}_{n+2}$, such that

$$\rho(\overset{i}{x}_l, \overset{i}{x}_k) \geqq \delta/2^n, \qquad l \neq k,\ l, k = 1, \cdots, n+2. \tag{1}$$

Since P is compact, we may assume that

$$\overset{i}{x}_1 \to x_1, \qquad \overset{i}{x}_2 \to x_2, \cdots, \overset{i}{x}_{n+2} \to x_{n+2}. \tag{2}$$

From (1) and (2) we infer that

$$\rho(x_l, x_k) \geqq \delta/2^n, \qquad l \neq k,\ l, k = 1, \cdots, n+2. \tag{3}$$

Since P is a Peano space, let $G_1, \cdots, G_{n+2}$ be connected open sets such that $x_k \in G_k$ and for $l \neq k$,

$$c(G_l) \cap c(G_k) = \emptyset. \tag{4}$$

Since the sets G_k are open, we have in view of (2) an integer $n_0>0$ such that for $i \geqq n_0$, $\overset{i}{x}_k \in G_k$, $k=1, \cdots, n+2$.

Let now i be a fixed integer not less than n_0. Then $B_i \cap G_k \neq \emptyset$, $k = 1, \cdots, n+2$. Consider now the continuum $H = B_{i+1} \cup c(G_1) \cup \cdots \cup c(G_{n+2})$. From 6(i), $H \cap B_i$ decomposes into at most $n+1$ distinct components. In view of (4) and the property that $B_i \cap G_k \neq \emptyset$, $k=1, \cdots, n+2$ and $B_i \cap B_{i+1}$ is finite, this is clearly impossible. The proof is therefore complete.

(iii) COROLLARY. *If $\mathfrak{B}$ is a collection of B-sets satisfying the hypothesis of* (i), *then $\mathfrak{B}$ is denumerable.*

REMARK. Without the restriction $r(P) < \infty$, (ii) is in general false. As an example consider the Peano space in §8, and take as the collection of B-sets the sets $L_n \equiv [v = 2^{-n}, 0 \leqq u \leqq 1]$, $n = 0, 1, 2, \cdots$.

13. **Fine-cyclic elements.**

DEFINITION. A subset Γ of a Peano space P will be termed a *fine-cyclic element* of P if and only if Γ is a B-set of P satisfying the property λ (see §11).

Concerning the existence of fine-cyclic elements of a Peano space P, let us observe that the Peano space considered in §11 has no fine-cyclic elements, even though it has a subset E satisfying the property λ. However, the following theorem is valid.

(i) THEOREM. *Let P be a Peano space with $r(P) < \infty$ and let E be a nondegenerate subset of P satisfying the property λ. Then there exists a unique fine-cyclic element Γ of P containing E.*

Proof. From 11(ii) we infer that the intersection Γ of all B-sets of P containing E is a B-set of P satisfying the property λ. The uniqueness of Γ is a consequence of 11(i).

(ii) THEOREM. *Let P be a Peano space with $r(P) < \infty$ and let B be a B-set of P. Then the fine-cyclic elements of B are those of P which are subsets of B.*

Proof. Let Γ be a fine-cyclic element of B. Then by (i) there exists a unique fine-cyclic element Γ' of P such that $\Gamma' \supset \Gamma$. By 11(i), $\Gamma' \subset B$, and hence $\Gamma' = \Gamma$. Conversely, if Γ is a fine-cyclic element of P with $\Gamma \subset B$, then by (i) there is a fine-cyclic element Γ' of B such that $\Gamma \subset \Gamma'$. But then from the first part of the proof, Γ' is also a fine-cyclic element of P, and in view of 11(i), $\Gamma' = \Gamma$.

14. **Properties of fine-cyclic elements.** Since a fine-cyclic element of a Peano space P is a B-set of P, the properties of B-sets apply to fine cyclic elements. For convenient reference a list of those properties is given below. As will be seen from this list, there is a striking similarity with the corresponding properties of proper cyclic elements.

(i) Two distinct fine-cyclic elements of P are either disjoint or else have a finite number of points in common.

The proof follows immediately from 11(i), and no restriction upon P is needed. However, in the sequel we will have to assume that $r(P) = n < \infty$. The Greek letter Γ will be used as a generic notation for a fine-cyclic element.

(ii) There is at most a denumerable number of fine-cyclic elements of P, and if there is an infinite number of fine-cyclic elements of P, they can be arranged in a sequence $\{\Gamma_i\}$ such that the diameter of Γ_i approaches zero as $i\to\infty$.

(iii) If G is a component of $P-\Gamma$, then $\mathrm{Fr}(G)$ consists of at most $n+1$ points.

(iv) There are at most n components of $P-\Gamma$ whose frontier decomposes into more than one point.

(v) If K is a continuum of P, then $\Gamma\cap K$ decomposes into at most $n+1$ distinct components. Moreover, if the diameter of K is sufficiently small, then $\Gamma\cap K$ is connected.

(vi) Γ is a Peano subspace of P, and $r(\Gamma)\leqq n$.

(vii) If P is unicoherent, i.e., if $r(P)=0$, then the fine-cyclic elements of P are the proper cyclic elements of P.

Proof. We only need to prove (vii). Since a fine-cyclic element Γ of P is cyclic, there is a unique proper cyclic element C of P such that $\Gamma\subset C$. Since P is unicoherent, C satisfies the property λ (see [5]). By 13(i) there is a unique fine-cyclic element Γ' of P satisfying $\Gamma\subset C\subset\Gamma'$. Application of (i) yields $\Gamma=C=\Gamma'$. Conversely, a proper cyclic element of P is a B-set satisfying the property λ and hence is a fine-cyclic element of P.

15. **L. Cesari's fine-cyclic elements.** For the content of this section the reader is referred to [2; 3].

Let J be a closed finitely connected Jordan region in the Euclidean plane E_2. If the superscript "^{0}" denotes "interior of", then $J=J_0-(J_1\cup\cdots\cup J_n)^0$ where J_i, $i=0, 1, \cdots, n$ are closed simple Jordan regions and $J_i\subset J_0^0$ for $i=1,\cdots,n$, and $J_i\cap J_j=\emptyset$, $i\neq j$, $i, j=i,\cdots,n$. The integer n is termed the connectivity of J. It can be shown that the degree of multicoherence of J is n.

Let (T, J) be a continuous mapping from J into E_3, the Euclidean three space. Let $G(T, J)$ denote the collection of maximal continua of constancy of (T, J). According to L. Cesari [2; 3], a fine-cyclic element K of (T, J) is defined to be a nonempty continuum of J satisfying the following properties:

(1) K is the union of the continua of constancy in $G(T, J)$ which intersect K, and T is not constant on K;

(2) If γ is a component of $J-K$, then T is constant on each component of $\mathrm{Fr}_J(\gamma)$;

(3) K is minimal with respect to the properties (1) and (2); that is, every proper subcontinuum of K satisfying (1) and (2) is a continuum in $G(T, J)$.

Given (T, J) as above. Then (see [9]), (T, J) admits of a monotone-light factorization $T=lm$, $m: J\Rightarrow M$, $l: M\to E_3$, where m is a monotone mapping from J onto M and l is a light mapping from M into E_3. It follows from [9] that $G(T, J)$ is the collection of continua $\{m^{-1}(x), x\in M\}$. Since $r(J)=n$, we have from 2(i) that $r(M)\leqq n$, and therefore the theory of fine-cyclic elements of M, as developed in this paper, applies. We will prove now the following theorem.

THEOREM. *K is a fine-cyclic element of (T, J) in the sense of Cesari if and only if there is a fine-cyclic element Γ of M in the sense of* §13 *such that $K = m^{-1}(\Gamma)$.*

Proof. Let K be a fine-cyclic element in the sense of Cesari. By 3(v), for each component γ of $J - K$, $\mathrm{Fr}_J(\gamma)$ decomposes into a finite number of components. Let $m(K) = \Gamma$. From the properties (1), (2), and (3) it follows that $K = m^{-1}(\Gamma)$ and that Γ is a B-set of M. We will show now that Γ satisfies the property λ. If this were not the case, we have a finite number of points $x_1, \cdots, x_t$ in Γ such that $R = \Gamma - (x_1 \cup \cdots \cup x_t)$ is not connected. If Q is a component of R, then $c(Q)$ is a B-set of Γ, and hence $c(Q)$ is a B-set of M. Since m is monotone, it follows easily that $m^{-1}[c(Q)]$ is a proper subcontinuum of K satisfying (1) and (2). But then by (3), $c(Q)$ reduces to a single point in M, a contradiction. Therefore, Γ is a B-set of M satisfying the property λ, and hence Γ is a fine-cyclic element of M.

Conversely, if Γ is a fine-cyclic element of M in the sense of §13, then by a similar argument, $m^{-1}(\Gamma)$ is a fine-cyclic element of (T, J) in the sense of Cesari.

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