CONTINUA AND VARIOUS TYPES OF HOMOGENEITY(1)

BY

C. E. BURGESS

1. **Definitions.** A point set M is said to be *n-homogeneous* if for any n points $x_1, x_2, \cdots, x_n$ of M and any n points $y_1, y_2, \cdots, y_n$ of M there is a homeomorphism of M onto itself that carries $x_1+x_2+\cdots+x_n$ onto $y_1+y_2+\cdots+y_n$. If there is such a homeomorphism which carries x_i into y_i $(i\leqq n)$, then M is said to be *strongly n-homogeneous*. A point set M is said to be *nearly n-homogeneous* if for any n points $x_1, x_2, \cdots, x_n$ of M and any n open subsets(2) $D_1, D_2, \cdots, D_n$ of M there exist n points $y_1, y_2, \cdots, y_n$ of $D_1, D_2, \cdots, D_n$, respectively, and a homeomorphism of M onto itself that carries $x_1+x_2+\cdots+x_n$ onto $y_1+y_2+\cdots+y_n$. For convenience throughout this paper, the terms "n-homogeneous" and "nearly n-homogeneous" are used only where $n>1$. Where $n=1$, the terms "homogeneous" and "nearly homogeneous" are used. Continua possessing these types of homogeneity have been investigated previously in [4; 6].

A continuum M is said to be *aposyndetic at the point x of M* if for any point y of $M-x$ there is a subcontinuum K of M and an open subset U of M such that $M-y\supset K\supset U\supset x$. The continuum M is said to be *aposyndetic* if it is aposyndetic at each of its points.

A subset H of the connected point set M is said to *separate* M if $M-H$ is not connected.

A point x of a continuum M is said to be a *cut point* of M if there exist two points y and z in $M-x$ such that every subcontinuum of M that contains $y+z$ also contains x.

2. **Homogeneous plane continua.** Let K denote a nondegenerate homogeneous bounded plane continuum. F. B. Jones [9] has shown that K is a simple closed curve provided it either is aposyndetic or contains no cut point, and H. J. Cohen [7] has shown that K is a simple closed curve if it either contains a simple closed curve or is arc-wise connected. These results generalized Mazurkiewicz's theorem that K is a simple closed curve if it is locally connected [13]. Similar results are obtained here for a bounded plane continuum which is nearly homogeneous and separates the plane into a finite number of connected domains. It can easily be seen that it is necessary in

Presented to the Society, November 27, 1954; received by the editors June 25, 1956 and, in revised form, October 18, 1956 and December 21, 1956.

(1) Some of this work was supported by the National Science Foundation under G-2574.

(2) A subset of a continuum M that is open relative to M is called an *open subset* of M.

Theorem 1 to require that M separate the plane as there exists a dendron which is nearly homogeneous. Also, a locally connected bounded plane continuum described by Sierpinski [17] is nearly homogeneous and has infinitely many complementary domains. It was shown in [4] that a bounded plane continuum M is the boundary of each of its complementary domains provided M is nearly homogeneous and does not have infinitely many complementary domains. Sierpinski's example cited above shows that it is necessary to require that M should not have infinitely many complementary domains, and it is shown here (Theorem 2) that this requirement is not necessary if M is homogeneous. Results more complete than those in Theorems 3, 4, and 5 have been obtained by F. B. Jones [11] for homogeneous continua. A more complete bibliography and a history of results on homogeneous plane continua will appear in a paper by Bing and Jones [2].

THEOREM 1. *If the bounded plane continuum M is nearly homogeneous and separates the plane into a finite number of connected domains, then M is a simple closed curve provided any one of the following four requirements is fulfilled*:

(a) *M contains a simple closed curve.*
(b) *M is aposyndetic.*
(c) *M contains no cut point.*
(d) *M is arc-wise connected.*

Proof of (a). It follows from [4, Theorem 7] that no proper subcontinuum of M separates the plane. Hence M is a simple closed curve.

Proof of (b). No proper subcontinuum of M separates the plane [4, Theorem 7], and hence it follows from a theorem proved by R. L. Moore [14, Theorem 2] that no subcontinuum of M separates M. Bing [1, Theorem 2] has shown that any such aposyndetic continuum is locally connected. That M is a simple closed curve follows from (a) and the fact that every locally connected bounded plane continuum which separates the plane contains a simple closed curve.

Proof of (c). As in the proof of (b), no proper subcontinuum of M separates M. Bing [1, Theorem 10] has shown that such a continuum is a simple closed curve if it is not cut by any one of its points.

Proof of (d). It follows from [4, Theorem 7] that no proper subcontinuum of M separates the plane. Since M is decomposable, it follows from a theorem proved by Kuratowski [12, Theorem 5] that M is the sum of two continua M_1 and M_2 irreducible between the same pair of points. Hence $M_1 \cdot M_2$ is the sum of two mutually separated sets H_1 and H_2, and there is an arc K in M irreducible from H_1 to H_2 and lying in one of the sets M_1 and M_2. Consider the case in which K is a subset of M_1. Since $K \cdot M_2$ is not connected, it follows from a well known theorem (Janiszewski) that $K+M_2$ separates the plane. Hence $K+M_2=M$ and $M_1-M_1 \cdot M_2=K-K \cdot M_2$. This implies that M is aposyndetic at each point of the set $M_1-M_1 \cdot M_2$, and since this set is open relative to M, it follows from the near-homogeneity of M that M is aposyndetic. Hence it follows from (b) that M is a simple closed curve.

THEOREM 2. *Every homogeneous bounded plane continuum is the boundary of each of its complementary domains.*

Proof. Suppose that the boundary K of some complementary domain of a homogeneous bounded plane continuum M is different from M. It is well known that K is a proper subcontinuum of M and it is easy to see that K separates the plane. From a classification of homogeneous decomposable bounded plane continua given by F. B. Jones [11, Theorem 2], it follows that M is indecomposable. Since any homeomorphic image of K separates the plane (Brouwer), it follows from the homogeneity of M that every composant(3) of M contains a continuum that separates the plane. But no indecomposable continuum is separated by one of its subcontinua. Hence each composant of M contains a continuum that is the boundary of some complementary domain of M. Since M has uncountably many mutually exclusive composants, this leads to the contradiction that M has uncountably many complementary domains.

THEOREM 3. *If the compact metric continuum M is nearly homogeneous and H is an indecomposable proper subcontinuum of M, then* $\mathrm{cl}(M-H)$ *intersects every composant of H.*

Proof. Suppose some composant L of H does not intersect $\mathrm{cl}(M-H)$. Let R be an open subset of M intersecting L but not $\mathrm{cl}(M-H)$. For each point y of M there is a point x of R and a homeomorphism of M onto itself that carries x into y. Hence for each point y of M, there is an indecomposable subcontinuum K of M and an open subset D of M such that $K \supset D \supset y$ and some composant of K does not intersect $\mathrm{cl}(M-K)$. Since M can be covered by a finite number of open sets such as D, it follows that there is a least integer n such that M is the sum of n indecomposable continua $M_1, M_2, \cdots, M_n$ such that no one of them is a subset of the sum of the others and for each i $(i \leqq n)$, some composant of M_i does not intersect $\mathrm{cl}(M-M_i)$.

Now suppose that some composant K_1 of M_1 intersects M_2 and does not intersect $\mathrm{cl}(M-M_1)$. Since M_2 intersects $\mathrm{cl}(M-M_1)$, it follows that some subcontinuum Z of M_2 is irreducible from $\mathrm{cl}(M-M_1)$ to some point of K_1. From the fact that $Z=\mathrm{cl}(Z-\mathrm{cl}(M-M_1))$ [15, Theorem 37, p. 22], it follows that Z is a proper subcontinuum of M_1 intersecting both K_1 and $\mathrm{cl}(M-M_1)$. This involves the contradiction that K_1 intersects $\mathrm{cl}(M-M_1)$. Hence it follows that some composant of M_1 does not intersect M_2, and similarly, for each j $(2 \leqq j \leqq n)$, some composant of M_1 does not intersect M_j. By [5, Theorem 2], some composant of M_1 does not intersect $M_2+M_3+\cdots+M_n$. In a

(3) If x is a point of a continuum M, the sum of all of the proper subcontinua of M that contain x is called a *composant* of M.

similar manner it can be shown that for each i ($i \leqq n$), some composant of M_i does not intersect $M_1+M_2+ \cdots +M_{i-1}+M_{i+1}+ \cdots +M_n$. By [3, Theorem 1], M is n-indecomposable(4), and from the requirement that H is a proper subcontinuum of M it readily follows that $n>1$. But this is impossible for a compact metric continuum which is nearly homogeneous [4, Theorem 2]. That cl$(M-H)$ intersects every composant of H follows from this contradiction.

THEOREM 4. *If the bounded plane continuum M is nearly homogeneous and H is an indecomposable proper subcontinuum of M such that $M-H$ has only a finite number of components, then* cl$(M-H)=M$.

Proof. Suppose that cl$(M-H)$ is a proper subset of M. It has been shown by Rutt [16, Lemma 1] that no set which is the closure of a component of $M-H$ intersects every composant of M. Hence by [5, Theorem 2], some composant of H does not intersect cl$(M-H)$. This is contrary to Theorem 3.

THEOREM 5. *If the bounded plane continuum M is nearly homogeneous and separates the plane into a finite number of connected domains, then every indecomposable proper subcontinuum of M is a continuum of condensation of M.*

Proof. It has been shown that M is the boundary of each of its complementary domains [4, Theorem 7]. Since such a continuum is not separated by any one of its subcontinua [14, Theorem 2], the conclusion of this theorem then follows from Theorem 4.

3. **Separation properties and homogeneity.** It has been shown previously that a decomposable plane continuum is locally connected if it is nearly n-homogeneous [4; 6]. By imposing additional conditions in the hypotheses, some similar results are obtained here for compact metric continua.

THEOREM 6. *If the decomposable metric continuum M is nearly 2-homogeneous, then M is aposyndetic.*

Proof. Let M_1 and M_2 be two proper subcontinua of M such that $M_1+M_2=M$, and let x_1 and x_2 be any two points of M. There exist two points y_1 and y_2 in $M-M_1$ and $M-M_2$ respectively and a homeomorphism f of M onto itself such that $f(y_1+y_2)=x_1+x_2$. Then $f(M_1)+f(M_2)=M$ and neither of the continua $f(M_1)$ and $f(M_2)$ contains both x_1 and x_2. Hence M is aposyndetic at x_1 with respect to x_2, and from this it follows that M is aposyndetic.

THEOREM 7. *Every 2-homogeneous compact metric continuum is aposyndetic.*

---

(4) A continuum M is said to be *n-indecomposable* if M is the sum of n continua such that no one of them is a subset of the sum of the others and M is not the sum of $n+1$ such continua. In some of the references such continua are said to be *indecomposable under index n.*

Proof. This theorem follows from Theorem 6 and the fact that every 2-homogeneous compact metric continuum is decomposable. (See proof of [4, Theorem 10].)

REMARK. Using the fact that a decomposable bounded plane continuum is locally connected if it is nearly 2-homogeneous [4, Theorem 9], I proved that every 2-homogeneous bounded plane continuum is a simple closed curve [4, Theorem 10]. I wish to observe here that this result is a direct consequence of Theorem 7 and F. B. Jones' theorem that every homogeneous aposyndetic bounded plane continuum is a simple closed curve [9]. In fact, it follows that a decomposable bounded plane continuum is a simple closed curve if it is homogeneous and nearly 2-homogeneous. This result is also included in a more recent result by F. B. Jones [11, Theorem 2].

THEOREM 8. *If the compact metric continuum M is aposyndetic and hereditarily unicoherent*(5)*, then M is locally connected.*

Proof. Let x be a point of M and let K be a subcontinuum of M not containing x. Jones [8, Theorem 7] has shown that every aposyndetic continuum is freely decomposable(6). Hence for each point y of K, M is the sum of two continua X and Y such that x and y lie in $M-Y$ and $M-X$ respectively. Consider the collection of all such pairs of continua. Then K is covered by the collection of all open sets such as $M-X$, and hence K is covered by a finite number of such open sets. There exists a positive integer m and continua $X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_m$ such that $K \subset Y_1 + Y_2 + \cdots + Y_m$ and for each i $(i \leqq m)$, $X_i + Y_i = M$ and $x \subset M - Y_i$. Since M is hereditarily unicoherent, it follows that the common part X' of the continua $X_1, X_2, \cdots, X_m$ is a continuum that does not intersect K. Clearly $M = X' + Y_1 + Y_2 + \cdots + Y_m$. Hence it has been shown that for any point x of M and any subcontinuum K of M not containing x, the continuum M is the sum of a finite number of continua no one of which both intersects K and contains x. This condition is sufficient for a compact metric continuum to be locally connected [15, Theorem 51, p. 134]. (Also, see [8, Theorem 8].)

COROLLARY 8.1. *If the decomposable compact metric continuum M is hereditarily unicoherent and nearly 2-homogeneous, then M is locally connected*(7).

THEOREM 9. *If the decomposable compact metric continuum M is unicoherent and nearly homogeneous, then for every positive integer n some subcontinuum of M separates M into more than n components.*

(5) A continuum M is said to be *unicoherent* if every two continua whose sum is M have a connected intersection. A continuum is said to be *hereditarily unicoherent* if each of its subcontinua is unicoherent.

(6) A continuum M is said to be *freely decomposable* if for any two points x and y of M there exist two continua whose sum is M such that neither of them contains both x and y.

(7) F. B. Jones [10] has shown that there does not exist a homogeneous continuum which satisfies the hypothesis of this corollary.

Proof. Suppose the contrary. Since M is decomposable and unicoherent, it follows that some subcontinuum of M separates M. Hence it follows from the above supposition that for some integer k greater than one some subcontinuum of M separates M into k components and no subcontinuum of M separates M into more than k components. Sorgenfry [18, Theorem 3.6] has shown that such a continuum is irreducible about some k of its points. This involves a contradiction since no decomposable compact metric continuum is both nearly homogeneous and irreducible about some finite set [4, Theorem 4].

COROLLARY 9.1. *If the nondegenerate compact metric continuum M is unicoherent and n-homogeneous, then for every positive integer k some subcontinuum of M separates M into more than k components.*

COROLLARY 9.2. *If the unicoherent compact metric continuum M is n-homogeneous and H is a subset of M consisting of n points, then there exists a subcontinuum of M which separates each two points of H from each other in M.*

4. **Some characterizations of a simple closed curve.** Simple closed curves in the plane have been characterized with some type of homogeneity requirement by Mazurkiewicz [13], Jones [9], Cohen [7], and the author [4]. (Also, see Theorem 1 in this paper.) A decomposable compact metric continuum that is continuum-wise homogeneous has been characterized as a simple closed curve [6]. The proofs of the characterizations given here rely heavily upon Bing's characterization of a simple closed curve as a compact metric continuum that is neither cut by any one of its points nor separated by any one of its subcontinua [1] and upon Whyburn's theorem that all except a countable number of local separating points of a compact metric continuum are points of order two in that continuum [19].

THEOREM 10. *If the decomposable compact metric continuum M is nearly 2-homogeneous and there exist two points of M such that no subcontinuum of M separates them in M, then M is a simple closed curve.*

Proof. Let x_1 and x_2 denote two points of M such that no subcontinuum of M separates them in M. Suppose that some subcontinuum K of M separates M. Then $M-K$ is the sum of two mutually separated sets D_1 and D_2 that are open relative to M. There exist a point y_1 in D_1 and a point y_2 in D_2 such that there is a homeomorphism of M onto itself that carries x_1+x_2 onto y_1+y_2. This involves the contradiction that the inverse of this homeomorphism carries the continuum K onto a continuum that separates x_1 from x_2. Hence no subcontinuum of M separates M.

By Theorem 6, M is aposyndetic. Jones [8, Theorems 3 and 4] has shown that the class of aposyndetic compact metric continua is equivalent to the class of semi-locally-connected compact metric continua, and Whyburn [21, 6.21] has shown that every nonseparating point of a semi-locally-connected compact metric continuum is also a noncut point of that continuum. Since no point separates M, it follows that no point cuts M. Bing [1, Theorem 10] has shown that a compact metric continuum is a simple closed curve if it is neither cut by any point nor separated by any one of its subcontinua. Hence M is a simple closed curve.

COROLLARY 10.1. *If the nondegenerate compact metric continuum M is 2-homogeneous and is not separated by any one of its subcontinua, then M is a simple closed curve.*

COROLLARY 10.2. *If the nondegenerate compact metric continuum M is strongly 4-homogeneous, then some subcontinuum of M separates M.*

Proof. This corollary follows from Corollary 10.1 and the fact that no simple closed curve is strongly 4-homogeneous.

THEOREM 11. *If the nondegenerate compact metric continuum M is multicoherent*(8) *and 2-homogeneous, then M is a simple closed curve.*

Proof. This theorem follows from Corollary 10.1 and the fact that no multicoherent compact metric continuum is separated by one of its subcontinua.

THEOREM 12. *If M is a compact metric continuum such that for any proper subcontinuum H of M and any two sets K_1 and K_2 in $M-H$ each consisting of two points there is a homeomorphism of M onto itself that carries H onto itself and K_1 onto K_2, then M is a simple closed curve.*

Proof. Suppose that M is not a simple closed curve. By Corollary 10.1, some subcontinuum H of M separates M. Let x_1 and x_2 be two points lying in the same component of $M-H$ and let y_1 and y_2 be two points lying in different components of $M-H$. A contradiction results since there is no homeomorphism of M onto itself that carries H onto itself and x_1+x_2 onto y_1+y_2.

THEOREM 13. *If the homogeneous compact metric continuum M is separated by some countable set, then M is a simple closed curve.*

Proof. Any countable set that separates M contains a local separating point of M [19, Corollary 5a]. Since M is homogeneous, every point of M is a local separating point of M. Hence M contains a point of order 2 [19, Theorem 9]. The homogeneity of M thus requires that every point of M is of order 2, and hence M is a simple closed curve (Menger).

THEOREM 14. *If the nondegenerate compact metric continuum M is homogeneous and hereditarily locally connected, then M is a simple closed curve.*

Proof. Every hereditarily locally connected compact metric continuum is

(8) A continuum M is said to be *multicoherent* if there do not exist two proper subcontinua M_1 and M_2 of M such that $M_1+M_2=M$ and $M_1\cdot M_2$ is connected.

a rational curve [20], and hence some countable set separates M. That M is a simple closed curve follows from Theorem 13.

5. **Homogeneous triodic continua.** A continuum is said to be a triod if it is the sum of three continua such that their intersection is a proper subcontinuum of each of them and is the intersection of each two of them.

THEOREM 15. *If the decomposable compact metric continuum M is nearly 2-homogeneous and is not a triod, then M is locally connected.*

Proof. The near 2-homogeneity and the decomposability of M imply that M is freely decomposable, and Sorgenfry [18, Theorem 3.1] has shown that a freely decomposable compact metric continuum is locally connected if it is not a triod.

COROLLARY 15.1. *If the compact metric continuum M is 2-homogeneous and is not a triod, then M is locally connected.*

QUESTION. Is every 2-homogeneous compact metric continuum locally connected?

THEOREM 16. *If the decomposable compact metric continuum M is unicoherent and nearly homogeneous, then M is a triod.*

Proof. Suppose M is not a triod. Sorgenfry [18, Theorem 3.2] has shown that such a unicoherent continuum is irreducible between some two points. This involves a contradiction as no decomposable compact metric continuum is both nearly homogeneous and irreducible about a finite set [4, Theorem 4].

COROLLARY 16.1. *If the nondegenerate compact metric continuum M is n-homogeneous and unicoherent, then M is a triod.*

COROLLARY 16.2. *If the decomposable bounded plane continuum M does not separate the plane and is nearly homogeneous, then M is a triod.*

Added in proof. The theorems in this paper that apply to continua that are 2-homogeneous can be strengthened to include continua that are n-homogeneous, where $n>2$, by using Morton Brown's result that every n-homogeneous continuum is $(n-1)$-homogeneous [*n-Homogeneity implies $(n-1)$-homogeneity*, Bull. Amer. Math. Soc. Abstract 60-4-448].

I have recently noticed that a locally connected compact metric continuum is a simple closed curve provided it is nearly homogeneous and is not a triod. Hence it can be concluded in Theorem 15 and Corollary 15.1 that M is a simple closed curve. A note on this will be submitted in another paper.

BIBLIOGRAPHY

1. R. H. Bing, *Some characterizations of arcs and simple closed curves*, Amer. J. Math. vol. 70 (1948) pp. 497–506.

2. R. H. Bing and F. B. Jones, *Another homogeneous plane continuum* (to appear in this journal).

3. C. E. Burgess, *Continua which are the sum of a finite number of indecomposable continua*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 234–239.

4. ———, *Some theorems on n-homogeneous continua*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 136–143.

5. ———, *Collections and sequences of continua in the plane*, Pacific J. Math. vol. 5 (1955) pp. 325–333.

6. ———, *Certain types of homogeneous continua*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 348–350.

7. Herman J. Cohen, *Some results concerning homogeneous plane continua*, Duke Math. J. vol. 18 (1951) pp. 467–474.

8. F. B. Jones, *Aposyndetic continua and certain boundary problems*, Amer. J. Math. vol. 63 (1941) pp. 545–553.

9. ———, *A note on homogeneous plane continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 113–114.

10. ———, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 855–859.

11. ———, *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 735–740.

12. Casimir Kuratowski, *Sur les coupures irréductibles du plan*, Fund. Math. vol. 6 (1924) pp. 130–145.

13. Stefan Mazurkiewicz, *Sur les continus homogènes*, Fund. Math. vol. 5 (1924) pp. 137–146.

14. R. L. Moore, *Concerning the common boundary of two domains*, Fund. Math. vol. 6 (1924) pp. 203–213.

15. ———, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, vol. 13, 1932.

16. N. E. Rutt, *Some theorems on triodic continua*, Amer. J. Math. vol. 56 (1934) pp. 122–132.

17. W. Sierpinski, *Sur une courbe cantorienne que contient une image biunivoque et continue de toute courbe donnée*, C. R. Acad. Sci. Paris vol. 162 (1916) pp. 629–632.

18. R. H. Sorgenfry, *Concerning triodic continua*, Amer. J. Math. vol. 66 (1944) pp. 439–460.

19. G. T. Whyburn, *Local separating points of continua*, Monatschefte für Math. und Phys. vol. 36 (1929) pp. 305–314.

20. ———, *Concerning hereditarily locally connected continua*, Amer. J. Math. vol. 53 (1931) pp. 374–384.

21. ———, *Semi-locally connected sets*, Amer. J. Math. vol. 61 (1939) pp. 733–749.

University of Utah,
Salt Lake City, Utah