ON THE LOCATION OF THE ZEROS OF THE DERIVATIVE OF RATIONAL FUNCTIONS OF DISTANCE POLYNOMIALS

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1. **Introduction.** Nagy [6](2) studied the class of polynomials in E_m consisting of all expressions of the type (1.1).

$$(1.1)\qquad \begin{aligned} &F(x_1, x_2, \cdots, x_m) \\ &= c\prod_{k=1}^{n}\left[(x_1 - x_{1,k})^2 + (x_2 - x_{2,k})^2 + \cdots + (x_m - x_{m,k})^2\right], \qquad c > 0. \end{aligned}$$

The function F, to which the name of "distance polynomial" was given, is a non-negative, real function of the form $c(\sum_{i=1}^{n} x_i^{2n}) + \Phi(x_1, x_2, \cdots, x_m)$ where Φ is a real polynomial of degree at most $2n-1$. The "derivative" of F was defined to be

$$(1.2)\qquad F'(x_1, x_2, \cdots, x_m) = \sum_{h=1}^{n} F_{x_h}^2/4F, \qquad F_{x_h} = \frac{\partial F}{\partial x_h},$$

Nagy extended some theorems of Gauss, Lucas, Jensen, and Laguerre, concerning the location of the zeros of the derivative of a polynomial in two real variables, to the class of distance polynomials in E_m.

In this paper several other results concerning the geometry of the zeros of a polynomial in a single complex variable are extended to E_n. It is found convenient to introduce vector methods. $1/A = A/A\cdot A = A/A^2$ denotes a particular reciprocal with respect to scalar multiplication; $\|A\| = [A\cdot A]^{1/2}$ the norm or length of the vector A; $Q{:}\,w$ the point Q with position vector w; e_i $i = 1, 2, \cdots, n$ a basis for E_n.

(1.3) is a distance polynomial of degree r in E_n.

$$(1.3)\qquad \begin{gathered} F(x_1, x_2, \cdots, x_n) = c\prod_{j=1}^{p}\|v - v_j\|^{2m_j} = c\prod_{j=1}^{p} d_j^{m_j}(x_1, x_2, \cdots, x_n), \; c > 0, \\ \sum_{j=1}^{p} m_j = r. \qquad v = \sum_{i=1}^{n} x_i e_i. \qquad v_j = \sum_{i=1}^{n} x_i^{(j)} e_i. \\ d_j = \|v - v_j\|^2. \end{gathered}$$

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(1) These results were obtained in a doctoral dissertation completed under the direction of Professor Morris Marden of the University of Wisconsin.

(2) [6] indicates item 6 in the attached bibliography.

Its derivative is (1.4).

$$F'(x_1, x_2, \cdots, x_n) = \frac{\|\nabla F\|^2}{4F} = \frac{F}{4}\|\nabla \log F\|^2 \tag{1.4}$$

where $\nabla F = \sum_{i=1}^n F_{x_i}\mathbf{e}_i$ and $F_{x_i} = \partial F/\partial x_i$.

$$R(x_1, x_2, \cdots x_n) = \prod_{j=0}^{q} F_j(x_1, x_2, \cdots, x_n) \Big/ \prod_{j=q+1}^{p} F_j(x_1\ x_2, \cdots, x_n) \qquad 0 \leqq q \leqq p \tag{1.5}$$

and $F_0(x_1, x_2, \cdots, x_n) = 1$.

(1.5) is a rational function of the distance polynomials F_j. It will be assumed that R has been "reduced to lowest terms," i.e. that $\prod_{j=0}^q F_j$ and $\prod_{j=q+1}^p F_j$ have no zeros in common. The "derivative" of (1.5) is

$$R' = \frac{\|\nabla R\|^2}{4R} = \frac{R}{4}\|\nabla \log R\|^2 \tag{1.6}$$

where $\nabla \boldsymbol{R} = \sum_{i=1}^n R_{x_i}\mathbf{e}_i$ and $R_{x_i} = \partial R/\partial x_i$.

Theorem I, the central result of this paper, is an extension to the class of distance polynomials in E_n of a theorem due to Marden [4, Theorem I].

THEOREM I. *Let $F_j(x_1, x_2, \cdots, x_n)$ be a distance polynomial of degree n_j all of whose zeros lie in the spherical region $\sigma_j S_j(v) \equiv \sigma_j[(v-c_j)^2 - r_j^2] \leqq 0$, $r_j > 0$ and $\sigma_j = \pm 1$* for $j = 1, 2, \cdots, p$. *Then every finite zero $P:V$ of R'* (1.6) *is such that V satisfies at least one of the inequalities*

$$\sigma_j S_j(v) \leqq 0, \qquad j = 1, 2, \cdots, p, \tag{1.7}$$

or

$$E(v) \Big/ \prod_{j=1}^{p} S_j(v) = \sum_{j=1}^{p} \frac{NN_j}{S_j(v)} - \sum_{j=1,k=j+1}^{p} \frac{N_j N_k T_{jk}}{S_j(v) S_k(v)} \leqq 0 \tag{1.8}$$

where $N_j = \nu_j n_j$, $\nu_j = 1$ for $j \leqq q$ and $\nu_j = -1$ for $j > q$, i.e., N_j and n_j are the signed and unsigned degrees respectively of F_j. $N = \sum_{k=1}^p N_j$ is the total degree of R. $\tau_{jk} = \|c_j - c_k\|^2 - (\lambda_j r_j - \lambda_k r_k)^2$ where $\lambda_j = \nu_j \sigma_j$.

This theorem determines a closed region of space, bounded by the surface $E(v) = 0$, which contains all the zeros of R'/R, the "logarithmic derivative" of (1.5), as soon as the spherical regions over which the zeros of the F_j are distributed are known. The class of spherical regions consists of the closed interiors and exteriors of spheres as well as of closed half spaces. This Marden-type theorem, applied to finite products and quotients, yields sharper results than those arising from the application of the extended Gauss-Lucas type theorem of Nagy [6, Theorem I].

2. **Some lemmas.** In the subsequent work one finds the following lemmas useful.

LEMMA I. *Let* (1) $P:V$, $Q_1:v_1$, *and* $Q_1':w_1=1/(V-v_1)\neq 0$ *be points in* E_n.

(2) $S(v)\equiv(v-c)^2-r^2=0$, $r>0$, *and* $S'(v)\equiv(v-c')^2-r'^2=0$, $r'>0$, *be the equations of two spheres in* E_n *with center and radius*

$$C:c,\ r \quad and \quad C':c'=V-c/S(V), \quad r'=r/|S(V)|$$

respectively. Then the point Q_1' *lies* (1) *inside or outside the sphere* S' *according as the sphere* S *does or does not separate the two points* P *and* Q_1, *or*

(2) *on the sphere* S' *if* S *passes through* Q_1 *and not through* P, *and conversely.*

Proof.

$$\begin{aligned} S'(w_1) &= (w_1-c')^2-r'^2 \\ &= \left[\frac{1}{V-v_1}-\frac{V-c}{S(V)}\right]^2-\frac{r^2}{|S(V)|^2} \\ &= \frac{1}{(V-v_1)^2}-2\left[\frac{V-c}{S(V)}\right]\cdot\left[\frac{V-v_1}{(V-v_1)^2}\right]+\frac{S(V)}{S^2(V)} \\ &= \frac{S(V)-2(V-c)\cdot(V-v_1)}{S(V)(V-v_1)^2}+\frac{1}{S(V)} \\ &= \frac{S(v_1)}{S(V)}\,w_1^2. \end{aligned}$$

$w_1^2\geqq 0$. $w_1^2=0$ if, and only if, $w_1=0$. Since $w_1\neq 0$, the sign of $S'(w_1)$ is the same as that of $S(v_1)/S(V)$. It follows that if

(1) $S(v)=0$ separates P and Q_1, $S(V)$ and $S(v_1)$ are opposite in sign and $S'(w_1)<0$. Q_1' is in the sphere $S'(v)=0$.

(2) the points P and Q_1 are both exterior to or interior to $S(v)=0$, $S(V)$ and $S(v_1)$ are similarly signed and $S'(w_1)>0$. Q_1' is outside of the sphere $S'(v)=0$.

(3) S passes through Q_1, $S(v_1)=0$. If S does not pass through P, $S(V)\neq 0$. If P is understood to be distinct from Q_1, it follows that $S'(w_1)=0$, i.e. S' passes through Q_1'.

Starting with any one of the statements $S'(w_1)<0$, $S'(w_1)>0$, or $S'(w_1)=0$, one may without difficulty retrace his steps and establish the converse.

LEMMA II. *If for* $j=1, 2, \cdots, p$ *the points* $Q_j:w_j$ *vary independently over the closed interiors of the spheres* $S_j(v)\equiv(v-c_j)^2-r_j^2=0$, $r_j>0$, *the locus of the point* $Q:w=\sum_{j=1}^{p} m_jw_j$ *where the* m_j *are real will be the closed interior of the sphere* $S(v)=0$ *of radius* $r=\sum_{j=1}^{p}|m_j|r_j$ *and center* $C:c=\sum_{j=1}^{p} m_jc_j$.

Proof.

$$\|\mathbf{w}-\mathbf{c}\| = \left\| \sum_{j=1}^{p} m_j(\mathbf{w}_j - \mathbf{c}_j)\right\| \leqq \sum_{j=1}^{p} |m_j| \, \|\mathbf{w}_j - \mathbf{c}_j\|.$$

By hypothesis Q_j is in the closed interior of S_j, i.e., $\|\mathbf{w}_j-\mathbf{c}_j\| \leqq r_j$, for $j=1, 2, \cdots, p$. It follows that $\|\mathbf{w}-\mathbf{c}\| \leqq \sum_{j=1}^{p} |m_j| r_j = r$ and $(\mathbf{w}-\mathbf{c})^2 - r^2 \leqq 0$. Consequently Q lies in the closed interior of the sphere $S(\mathbf{v})=0$.

Conversely, it can be established that if $Q:\mathbf{w}$ lies in the closed interior of $S(\mathbf{v}) = 0$, there exists a point $Q_j:\mathbf{w}_j$ in each of the spheres $S_j(\mathbf{v}) = 0$, $j=1, 2, \cdots, p$, such that $\mathbf{w} = \sum_{j=1}^{p} m_j \mathbf{w}_j$.

If $Q:\mathbf{w} = \sum_{i=1}^{n} w_i \mathbf{e}_i$ lies in the closed interior of $S(\mathbf{v})=0$ and $\mathbf{c} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$, $\mathbf{w}-\mathbf{c} = \sigma r \sum_{i=1}^{n} \lambda_i \mathbf{e}_i$ where $0 \leqq \sigma \leqq 1$ and $\lambda_i = (w_i - \alpha_i)\left[\sum_{i=1}^{n} (w_i-\alpha_i)^2\right]^{-1/2}$.

Consider the set of points $Q_j:\mathbf{w}_j = \mathbf{c}_j + (\sigma|m_j|/m_j) r_j \sum_{i=1}^{n} \lambda_i \mathbf{e}_i$, $0 \leqq \sigma \leqq 1$, $j=1, 2, \cdots, p$. Since $\sum_{i=1}^{n} \lambda_i^2 = 1$ and $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, $(\mathbf{w}_j - \mathbf{c}_j)^2 = \sigma^2 r_j^2 (\sum_{i=1}^{n} \lambda_i^2) \leqq r_j^2$. Consequently $Q_j:\mathbf{w}_j$ lies in the closed interior of the sphere $S_j(\mathbf{v})=0$. Moreover, since

$$\sum_{j=1}^{p} m_j \mathbf{w}_j = \sum_{j=1}^{p} m_j \mathbf{c}_j + \sigma \left(\sum_{j=1}^{p} |m_j| \, r_j \right)\left(\sum_{i=1}^{n} \lambda_i \mathbf{e}_i \right) = \mathbf{c} + \sigma r \left(\sum_{i=1}^{n} \lambda_i \mathbf{e}_i \right)$$
$$= \mathbf{c} + (\mathbf{w} - \mathbf{c}) = \mathbf{w},$$

the desired locus is the closed interior of the sphere $S(\mathbf{v})=0$.

Lemma II is the n space analogue of Lemma I in Marden [4].

LEMMA III. *If $P:\mathbf{V}$ is a finite zero of R'/R, the "logarithmic derivative" of the rational function* (1.5) *of the distance polynomials F_j, $F_j(x_1, x_2, \cdots, x_n) = c_j \prod_{\delta_{j-1}+1}^{\delta_j} d_k^{m_k}(x_1, x_2, \cdots, x_n)$, $\delta_0=0$, $c_j>0$, $d_k = \|\mathbf{v}-\mathbf{v}_k\|^2$, $m_k>0$ and real, $j=1, 2, \cdots, p$, $\mathbf{V}$* must satisfy (2.1).

$$\sum_{k=1}^{\delta_p} \frac{M_k}{\mathbf{V} - \mathbf{v}_k} = \mathbf{0}, \tag{2.1}$$

$M_k = \mu_k m_k$. $\mu_k = 1$ for $1 \leqq k \leqq q$. $\mu_k = -1$ for $q+1 \leqq k \leqq p$.

Proof. A necessary and sufficient condition that $P:\mathbf{V}$ be a finite zero of R'/R is that $\nabla \log R$ vanish at P. $\nabla \log R = \sum_{j=1}^{q} \nabla \log F_j - \sum_{j=q+1}^{p} \nabla \log F_j = \sum_{j=1}^{p} \mu_j \nabla \log F_j$ where $\mu_j = 1$ for $0 \leqq j \leqq q$ and $\mu_j = -1$ for $q+1 \leqq j \leqq p$. If $\mathbf{v} = \sum_{i=1}^{n} x_i \mathbf{e}_i$ and $\mathbf{v}_k = \sum_{i=1}^{n} x_i^{(k)} \mathbf{e}_i$, $d_k = \|\mathbf{v}-\mathbf{v}_k\|^2 = \sum_{i=1}^{n} (x_i - x_i^{(k)})^2$.

$$\log F_j = \log c_j + \sum_{k=\delta_{j-1}+1}^{\delta_j} m_k \log d_k$$

and therefore

$$\frac{\partial \log F_j}{\partial x_i} = \sum_{k=\delta_{j-1}+1}^{\delta_j} m_k \left(\frac{\partial \log d_k}{\partial x_i} \right) = \sum_{k=\delta_{j-1}+1}^{\delta_j} \frac{2m(x_i - x_i^{(k)})}{d_k}.$$

Since

$$\nabla \log F_j = \sum_{i=1}^{n} \frac{\partial \log F_j}{\partial x_i} e_i = \sum_{k=\delta_{j-1}+1}^{\delta_j} \frac{2m_k\left[\sum_{i=1}^{n}(x_i - x_i^{(k)})e_i\right]}{d_k}$$

$$= \sum_{k=\delta_{j-1}+1}^{\delta_j} \frac{2m_k}{V - v_k}, \qquad \nabla \log R = \sum_{j=1}^{p} \mu_j \left(\sum_{k=\delta_{j-1}+1}^{\delta_j} \frac{2m_k}{V - v_k} \right)$$

and (2.2) holds.

$$\nabla \log R = \sum_{k=1}^{\delta_p} \frac{2M_k}{V - v_k}, \qquad M_k = \mu_k m_k. \tag{2.2}$$

Clearly V must satisfy (2.1).

LEMMA IV. If (1) $Q_j: V_j$ *lies in the spherical region* $\sigma_j S_j(v) \equiv \sigma_j[(v - c_j)^2 - r_j^2] \leqq 0$,

$$r_j > 0, \sigma = \pm 1, \qquad j = 1, 2, \cdots, p.$$

(2) $P: V$ *is exterior to all of the spherical regions* $\sigma_j S_j(v) \leqq 0, j = 1, 2, \cdots, p$ *and*

(3) $\sum_{j=1}^{p} m_j/(V - v_j) = 0$, m_j *real*,
V *must satisfy inequality* (2.3).

$$I(v) \equiv \left[\sum_{j=1}^{p} \frac{m_j(v - c_j)}{S_j(v)} \right]^2 - \left[\sum_{j=1}^{p} \frac{|m_j| r_j}{|S_j(v)|} \right]^2 \leqq 0. \tag{2.3}$$

Proof. By hypothesis $S_j(v) = 0$ separates Q_j and P. It follows from Lemma I that the point $Q_j': w_j = 1(V - v_j)$ is in the sphere

$$S_j'(v) \equiv \left[v - \frac{(V - c_j)}{S_j(V)} \right]^2 - \left[\frac{r_j}{|S_j(V)|} \right]^2 = 0.$$

If we let $w = \sum_{j=1}^{n} m_j w_j$ where the m_j are real, by Lemma II, the locus of $Q: w$ is the closed interior of the sphere $T(v) = 0$, i.e. (2.4) holds.

$$T(w) \equiv \left(w - \sum_{j=1}^{p} \frac{m_j(V - c_j)}{S_j(V)} \right)^2 - \left(\sum_{j=1}^{p} \frac{|m_j| r_j}{|S_j(V)|} \right)^2 \leqq 0. \tag{2.4}$$

By hypothesis $0 = \sum_{j=1}^{p} m_j/(V - v_j) = \sum_{j=1}^{p} m_j w_j = w$. Therefore $w = 0$ is a value corresponding to a set of suitably chosen points Q_j in the given regions $\sigma_j S_j(v) \leqq 0$. $w = 0$ must satisfy (2.4) and it follows that, under the hypotheses of this lemma, (2.3) holds.

3. **A Marden-type theorem.** Theorem I can now be established.

Proof. Let $P: V$ be a finite zero of R'. If P is also a zero of R, P must coincide with at least one zero of some F_j for $1 \leqq j \leqq q$. Therefore it must lie in at least one of the given spherical regions and (1.7) holds. If P is not a zero of R, either P lies in at least one of the given spherical regions, in which event V satisfies at least one of the inequalities (1.7), or P is exterior to all of the given regions. It will be shown that in the latter case V satisfies (1.8).

Since P is a zero of R' and not of R, it follows from Lemma III that $\sum_{k=1}^{\delta p} M_k/(V-v_k)=0$ where $M_k=\mu_k m_k$, $M_k\neq 0$ and real, $m_k>0$ and real, $\mu_k=1$ for $k\leqq q$ and $\mu_k=-1$ for $k>q$.

It is convenient to first consider distance polynomials F_j each of which has a single zero of multiplicity n_j which lies in the spherical region $\sigma_j S_j(v)\leqq 0$ for $j=1, 2, \cdots, p$, i.e.

$$F_j(x_1, x_2, \cdots, x_n) = c_j d_j^{n_j}(x_1, x_2, \cdots, x_n), \qquad c_j > 0. \tag{3.1}$$

Since P is exterior to all p of the given spherical regions, Lemma IV applies and it follows that V must satisfy inequality (2.3), $I(v)\leqq 0$, with $N_j=v_j n_j$ replacing m_j. $v_j=1$ for $j\leqq q$ and $v_j=-1$ for $j>q$. When $I(v)$ is expanded and simplified it becomes

$$I(v) \equiv \sum_{j=1}^{p} \frac{N_j^2}{S_j(v)} + \sum_{j=1,k=j+1}^{p} \frac{2N_jN_k[(v-c_j)\cdot(v-c_k)-\lambda_j\lambda_k r_j r_k]}{S_j(v)S_k(v)} \tag{3.2}$$

where $\lambda_j=\mu_j\sigma_j$.

Since $(c_j-c_k)^2=(v-c_j)^2+(v-c_k)^2-2(v-c_j)\cdot(v-c_k)$ and $(\lambda_j r_j-\lambda_k r_k)^2=r_j^2+r_k^2-2\lambda_j\lambda_k r_j r_k$, $2[(v-c_j)\cdot(v-c_k)-\lambda_j\lambda_k r_j r_k]=S_j(v)+S_k(v)-\tau_{jk}$ where $\tau_{jk}=[(c_j-c_k)^2-(\lambda_j r_j-\lambda_k r_k)^2]$.

If $\sum_{j=1}^{p} N_j$ is replaced by N, (3.2) reduces to (3.3)

$$I(v) = \sum_{j=1}^{p} \frac{NN_j}{S_j(v)} - \sum_{j=1,k=j+1}^{p} \frac{N_jN_k\tau_{jk}}{S_j(v)S_k(v)}. \tag{3.3}$$

It follows that V must satisfy condition (1.8).

Now consider the distance polynomials (3.4)

$$\begin{aligned} &F_j(x_1, x_2, \cdots, x_n)\\ &= c_j \prod_{k=1}^{p} d_{jk}^{m_{jk}}(x_1, x_2, \cdots, x_n), j = 1, 2, \cdots, p, \qquad c_j > 0 \end{aligned} \tag{3.4}$$

each of which has all of its zeros in the spherical region $\sigma_j S_j^*(v)\leqq 0$ where $S_i^*(v)\equiv(v-c_j^*)^2-r_j^{*2}$, $r_j^*>0$. Let Q_{mk} be the zero corresponding to the first degree distance polynomial d_{mk}. The zeros Q_{mk}, $k=1, 2, \cdots, p_m$, of F_m all lie in $\sigma_m S_m^*(v)\leqq 0$. This is a specialization of the preceding case in which the p_m spheres S_j coalesce in the sphere S_m^*. The derivation of the desired vector inequalities follows as above.

The boundary surface $E(v)=0$ of the region of space described by the inequality (1.8) can be written in the form $(\sum_{i=1}^{n} x_i^2)^p+\Phi(x_1, x_2, \cdots, x_n)=0$ where Φ is a real polynomial of degree at most $2p-1$. In E_3 this surface contains the circle at infinity as a p-fold curve and it will be referred to as a p spherical $2p-ic$ surface. When $p=2$ the surface is a cyclide. These surfaces were studied in detail by Darboux [2].

It is not difficult to show that in certain cases when the spheres S_j are all symmetrically located with respect to a fixed point 0 and $\sigma_j=1$ for all j, the surface $E(v)=0$ degenerates into a set of spheres centered at 0, and the desired region $E(v)\leqq 0$ is the closed interior of the largest sphere in that set.

4. **The zeros of** $R-\lambda^2R'$. An interesting application of Theorem I is displayed in the next result.

THEOREM II. *Under the hypotheses of Theorem* I *every finite zero of* $R-\lambda^2R'$, λ *real, satisfies at least one of the* $p+1$ *inequalities*

$$\sigma_jS_j(v) \leqq 0, \qquad j = 1, 2, \cdots, p. \tag{4.1}$$

$$\frac{M(v)}{\prod_{j=1}^{p}S_j(v)} = \sum_{j=1}^{p} \frac{N_j\lambda}{NS_j(v)}T_j(v) - \sum_{j=1,k=j+1}^{p} \frac{N_jN_k\tau_{jk}}{S_j(v)S_k(v)} \leqq 0 \tag{4.2}$$

where $T_j(v)=[v-(\mathbf{c}_j+N\mathbf{w})]^2-r_j^2$, $\mathbf{w}=\lambda\mathbf{e}$, *and* $\mathbf{e}$ *is the unit vector in the* ∇R *direction.*

Proof. Let $P:\boldsymbol{V}$ be a zero of $R-\lambda^2R'$. If P is a zero of R, it is also a zero of R' and $\boldsymbol{V}$ corresponds to a point which lies in at least one of the spherical regions described by (4.1). If P is not a zero of R, either P lies in one of the given spherical regions, in which case $\boldsymbol{V}$ satisfies at least one of the vector inequalities (4.1), or P is exterior to all of these regions. In the latter event it will be shown that $\boldsymbol{V}$ must satisfy (4.2).

Since P is a zero of $R-\lambda^2R'$ and not of R,

$$\left.\frac{R'}{R}\right|_P = \left.\frac{\|\nabla R\|^2}{4R^2}\right|_P = \frac{1}{4}\|\nabla \log R\|_P^2 = \lambda^{-2}$$

and from (2.2) one obtains

$$\left.\frac{1}{2}\frac{\nabla R}{R}\right|_P = \left.\frac{\nabla \log R}{2}\right|_P = \sum_{j=1}^{p}\frac{N_j}{\boldsymbol{V}-\boldsymbol{v}_j} = \frac{1}{\lambda}\mathbf{e}$$

where $\mathbf{e}$ is the unit vector in the ∇R direction.

Since P is exterior to all of the given spherical regions one can proceed as in the proof of Lemma II and it follows that $\sum_{j=1}^{p} N_j/\boldsymbol{V}-\boldsymbol{v}_j=\boldsymbol{w}$, a value corresponding to a suitably chosen set of points $Q_j:\boldsymbol{w}_j$ $j=1, 2, \cdots, p$, each of which lies in the corresponding spherical region $\sigma_jS_j(v)\leqq 0$, must satisfy (2.4). That is

$$\left(\boldsymbol{w} - \sum_{j=1}^{p} N_j(\boldsymbol{V}-\mathbf{c}_j)/S_j(\boldsymbol{V})\right)^2 - \left(\sum_{j=1}^{p} |N_j|\,\sigma_jr_j/S_j(\boldsymbol{V})\right)^2 \leqq 0.$$

When the left hand side is expanded this inequality reduces to (4.2). $T_j(v)=0$ is the sphere obtained by translating $S_j(v)=0$ in the direction of $N\lambda\mathbf{e}$ by an amount equal to the magnitude of that vector.

COROLLARY. *If all the zeros of a distance polynomial $F(x_1, x_2, \cdots, x_n)$ of degree m lie in a sphere $S_1(v)=0$, any zero of $F-\lambda^2F'$, λ real, will lie either in S_1 or in the sphere obtained by translating S_1 in the direction of $m\lambda\mathbf{e}$ by an amount equal to the magnitude of that vector. $\mathbf{e}$ is the unit vector in the ∇F direction.*

Proof. If $p=q=1$, $\sigma_1=1$, i.e. $R=F$, a distance polynomial of degree m with all of its zeros in $S_1(v)=0$, it follows from Theorem II that the zeros of $F-\lambda^2F'$ either lie in S_1 or are such that their position vectors satisfy (4.2). In this case (4.2) reduces to $mw^2T(V)/mS_1(V)\leqq 0$. Since P is exterior to S_1, $S_1(V)>0$ and consequently $T_1(V)=[V-\{c_1+m\lambda\mathbf{e}\}]^2-r_1^2\leqq 0$.

The results of this section are generalizations of Theorem II in Marden [4].

5. **The critical points of finite products of distance polynomials in E_3.** Nagy [6] developed the following Gauss-Lucas type theorem for the class of distance polynomials in E_3: "Let $F(x_1, x_2, x_3)$ be a distance polynomial with zeros Q_j, $j=1, 2, \cdots, n$, and let K be the smallest convex region of space which contains those zeros. Then all the zeros of F' also lie in K. No zero of F' is on the boundary of K unless it is a multiple zero of F or unless all the Q_j are coplanar."

As soon as the location of all the zeros of F is known Nagy's theorem singles out a portion of space which must contain the zeros of the logarithmic derivative of F. If F is a finite product of distance polynomials, Theorem I will restrict even further the region of space within which the critical points may lie.

Let $F(x_1, x_2, x_3)=\prod_{j=1}^{p} F_j(x_1, x_2, x_3)$ be a finite product of the distance F_j where each F_j is of degree n_j and has all of its zeros in the closed interior of the sphere S_j. In this section the region containing the critical points of F is determined for $p=2$ and $p=3$ when the spheres in question have collinear centers and a common external center of similitude. In each case the desired region consists not only of the closed interiors of the given spheres but also of the closed interiors of another set of spheres having the same external center of similitude as the given set. The centers of the second set are located at the zeros of the logarithmic derivative of the distance polynomial $G(x_1, x_2, x_3)$ obtained from F by coalescing all the zeros of each F_j at the center of the corresponding sphere S_j.

THEOREM III. *If $F(x_1, x_2, x_3)$ is an nth degree distance polynomial n_1 of whose zeros lie in the closed interior of the sphere S_1 and the remainder of whose zeros, $n_2=n-n_1$ in number, lie in the closed interior of the sphere S_2, all the zeros of $F'(x_1, x_2, x_3)$ lie either in the closed interior of S_1 or S_2, or in the closed interior of a third sphere $S(v)=0$ where*

$$S(v) \equiv \left(v - \frac{n_1 c_2 + n_2 c_1}{n_1 + n_2}\right)^2 - \left(\frac{n_1 r_2 + n_2 r_1}{n_1 + n_2}\right)^2.$$

S has a common center of similitude with S_1 and S_2 and its center is located at the centroid of a system of two particles, one of mass n_2 located at the center of the sphere S_1 and the other of mass n_1 located at the center of the sphere S_2.

Proof. This is a special case of Theorem I for which $q=2$; $N_j=n_j\ j=1, 2$; $N=n_1+n_2$; $\nu_i=1$, $\sigma_i=1$, $\lambda_i=1$ for $i=1, 2$.

$F(x_1, x_2, x_3)=F_1(x_1, x_2, x_3)F_2(x_1, x_2, x_3)$. Each F_j, a distance polynomial of degree n_j, has all of its zeros in the closed interior of the sphere S_j. As a consequence of Theorem I $P:V$, a zero of F', lies either in the closed interior of S_1 or S_2 or, if P is exterior to both of these spheres, V satisfies (1.8) which in this case reduces to

$$(n_1 + n_2)\left[\frac{n_1}{S_1(V)} + \frac{n_2}{S_2(V)}\right] - \frac{n_1 n_2 \tau_{12}}{S_1(V)S_2(V)} \leqq 0.$$

If both sides of the inequality are multiplied by $S_1(V)S_2(V)>0$ and the S_j and τ_{12} are replaced by their equivalents, this condition reduces to

$$[(n_1+n_2)V-(n_1c_2+n_2c_1)]^2-(n_1r_2+n_2r_1)^2=(n_1+n_2)^2S(V)\leqq 0.$$

Consequently if P is exterior to both S_1 and S_2, it must lie in the closed interior of S.

This theorem is an extension of a result due to Walsh [7].

THEOREM IV. *If all the zeros of the distance polynomial $F_j(x_1, x_2, x_3) = \prod_{k=1}^{p_j} \|v-v_k\|^{2m_k}$ of degree n_j, $j=1, 2, 3$, lie in the closed interior of the sphere S_j and if all the S_j have a common external center of similitude 0, then each zero of the derivative of $F=\prod_{j=1}^{3} F_j(x_1, x_2, x_3)$ lies either in the closed interior of one of the S_j or in the closed interior of one of the spheres S_1' or S_2'. The spheres S_1' and S_2' also have the external center of similitude 0 and their centers correspond to the zeros of the logarithmic derivative of the distance polynomial $G(v) = \prod_{j=1}^{3} G_j(v) = \prod_{j=1}^{3} \|v-c_j\|^{2n_j}$ where the c_j are the position vectors of the centers C_j of the spheres S_j.*

Proof. Without any loss of generality the line of centers may be taken as the x_1 axis and the center of similitiude 0 as the origin of a rectangular coordinate system. C_j:c_j where $c_j=c_je_1$ is the center of S_j. Since 0 is a center of similitude $r_jc_j^{-1}=\lambda$ for $j=1, 2, 3$. It follows that S_j has the form $S_j\equiv(\sum_{i=1}^{3} x_i^2) -2c_jx_1+\mu x_1^2$ and $\tau_{jk}=\|c_j-c_k\|^2-(r_j-r_k)^2=\mu(c_j-c_k)^2$ where $\mu=1-\lambda^2$.

As a consequence of Theorem I all the zeros $P:V$ of F' lie either in one of the given spheres S_j or in the closed region of space described by (1.8). Under these hypotheses $p=3$, $\nu_j=1$, $\sigma_j=1$, $\lambda_j=1$, and consequently $N_j=n_j$ for $j=1, 2, 3$. If P is exterior to all of the S_j, $\prod_{j=1}^{3} S_j(V)>0$ and (1.8) reduces to

$$\sum_{i,j,k,=1;i\neq j\neq k}^{3} [NN_iS_j(V)S_k(V) - \mu N_jN_k(c_j - c_k)^2S_i(V)] \leqq 0$$

where $N = \sum_{i=1}^{3} N_i$. The boundary surface of the desired region of space is clearly of order 4.

After substituting the appropriate expressions for the S_j, the equation of the boundary surface, $E(v) = 0$, reduces to

$$(5.1) \qquad n^2\left(\sum_{i=1}^{3} x_i^2\right)^2 - (2Ax_1 - B)\left(\sum_{i=1}^{3} x_i^2\right) + 4Cx_1^2 - 2Dx_1 + E = 0$$

where

$$A = n\sum_{i=1}^{3} (n - n_i)c_i;$$

$$B = \mu\left[\left\{\sum_{i=1}^{3} (n - n_i)^2c_i^2\right\} + 2(n_1c_1 + n_2c_2)n_3c_3\right]$$

$$C = n\left\{\sum_{i,j,k=1;i\neq j\neq k;j<k}^{3} n_ic_jc_k\right\};$$

$$D = \mu\left[\sum_{i,j,k=1;i\neq j\neq k,j<k}^{3} n_ic_jc_k\{(n - n_j)c_j + (n - n_k)c_k\} + 2c_1c_2c_3 \sum_{i,j=1;i\neq j,i<j}^{3} n_in_j\right];$$

and

$$E = \mu^2\left[c_1^2\left(\sum_{i,j=1;i\neq j,i<j}^{3} n_i^2c_j^2\right) + 2c_1c_2c_3\left(\sum_{i,j,k=1;i\neq j\neq k;j<k}^{3} c_in_jn_k\right)\right].$$

Consider the function $G(v) = \prod_{j=1}^{3} \|v - c_j\|^{2n_j}$. This function can be obtained by coalescing all the zeros of each F_j at the center of the sphere S_j. The zeros of the logarithmic derivative of G must satisfy the relation $\sum_{j=1}^{3} (n_j/(v - c_j)) = 0$. Nagy's Lucas-type theorem, cited at the beginning of this section, shows that the zeros of $G'(v)$ must lie on the x_1 axis. If V corresponds to a zero of G', $V = Xe_1$. The condition which must be satisfied by the zeros of the logarithmic derivative of G may now be rewritten as $\sum_{j=1}^{3} (n_j/(X - c_j)e_1) = 0$ which is equivalent to (5.2).

$$(5.2) \qquad nX^2 - \left[\sum_{i=1}^{3} (n - n_i)c_i\right]X + \sum_{i,j,k=1;i\neq j\neq k,j<k}^{3} n_ic_jc_k = 0.$$

If γ_1 and γ_2 are the zeros of (5.2), $n(\gamma_1 + \gamma_2) = \sum_{i=1}^{3} (n - n_i)c_i$ and $n\gamma_1\gamma_2 = (\sum_{i,j,k=1;i\neq j\neq k,j<k}^{33} n_ic_jc_k)$. The spheres S_j' with centers $C_j' : \gamma_je_1$ and with 0 as their external center of similitude are described by the equation

$$S_j'(x_1, x_2, x_3) \equiv \left(\sum_{i=1}^{3} x_i^2\right) - 2\gamma_jx_1 + \mu\gamma_j^2 = 0, \qquad j = 1, 2.$$

After forming the expression $n^2S_1'S_2'$ it is not difficult to show that it is identical to (5.1). In short, $E(v)=0$, the boundary surface in question, degenerates into two spheres S_1' and S_2' with centers whose position vectors correspond to the zeros of the logarithmic derivative of $G(v)$. The region of space described by $E(v)\equiv n^2S_1'(v)S_2'(v)\leqq 0$ consists of the closed interiors of the spheres S_1' and S_2' and contains all the zeros of F' which do not lie in any of the S_j.

By introducing a coordinate system with its origin at the center of S_1 and proceeding as in Theorem IV, the following result may be obtained.

THEOREM V. *Let $F(x_1, x_2, x_3)=\prod_{j=1}^{3} F_j(x_1, x_2, x_3)$ where each F_j is a distance polynomial of degree n_j all of whose zeros lie in the sphere S_j of radius r and center C_j:$\mathbf{c}_j$. Then the zeros of F' lie either in the closed interiors of the given spheres S_j or in the closed interiors of S_1' and S_2'. S_1' and S_2' are spheres of radius r whose centers correspond to the zeros of the logarithmic derivative of $G(v)=\prod_{j=1}^{3}\|v-\mathbf{c}_j\|^{2n_j}$.*

The general case of both of these theorems was developed for polynomials in a single complex variable by Walsh [8b].

6. **The critical points of finite quotients of distance polynomials in E_3.** Proceeding as in Theorem III the following two theorems may easily be verified.

THEOREM VI. *If the distance polynomial $F_j(x_1, x_2, x_3)$ of degree n_j has all of its zeros in or on the sphere $S_j(v)\equiv(v-\mathbf{c}_j)^2-r_j^2=0$, $r_j>0$, for $j=1, 2$, $n_1\neq n_2$, all of the finite zeros of the derivative of the quotient $F=F_1/F_2$ lie in the closed interiors of S_1, S_2, and a third sphere*

$$S(v)\equiv\left(v-\frac{n_2\mathbf{c}_1-n_1\mathbf{c}_2}{n_2-n_1}\right)^2-\left(\frac{n_2r_1+n_1r_2}{|n_2-n_1|}\right)^2=0.$$

This is an extension of a result obtained by Walsh [8a].

THEOREM VII. *If the distance polynomial $F_j(x_1, x_2, x_3)$ of degree n has all of its zeros in the spherical region $\sigma_jS_j(v)\leqq 0$ where $\sigma_j=\pm 1$, $j=1, 2$, and the two spherical regions have no points in common, all the finite zeros of $F=F_1/F_2$ lie in the two given spherical regions.*

This result was obtained by Bôcher [1] for a polynomial in a single complex variable.

The special case of Theorem I for which $N_1=n_1$, $N_2=n_2$, $N_3=-n_3$, and $N=\sum_{i=1}^{3} N_i=n_1+n_2-n_3=0$ leads to some interesting results which are summarized in the final two theorems.

THEOREM VIII. *If the points Q_j:$\mathfrak{y}_j$ vary independently over the spherical regions $\sigma_jS_j(v)\leqq 0$ for $j=1, 2, 3$, any point P:V whose position vector forms a constant cross ratio with the $\mathfrak{y}_j$ lies in a fourth spherical region.*

THEOREM IX. *If* (1) $F_j(x_1, x_2, x_3)$ *is a distance polynomial of degree* n_j *all of whose zeros lie in the spherical region* $\sigma_j S_j(v) \leqq 0$ *for* $j=1, 2, 3$, (2) $n_3 = n_1 + n_2$ *and* (3) *the given spherical regions have no point in common, every finite zero of the derivative of* $F = F_1F_2/F_3$ *is such that it lies in at least one of the given spherical regions or in a fourth spherical region. This fourth region is described by a point* P *whose position vector* V *forms a constant cross ratio,* $-n_2/n_1$, *with the position vectors* ψ_j *of the points* Q_j *which describe the given spherical regions* $\sigma_j S_j(v) \leqq 0$, $j=1, 2, 3$.

Proof, Any zero $P:V$ of $F'(x_1, x_2, x_3)$ which is exterior to all of the given spherical regions must be such that V satisfies the following inequalities: $\sigma_j S_j(V) > 0$ for $j=1, 2, 3$ and

$$\frac{E(V)}{\prod_{j=1}^{3} S_j(V)} = -\left[\frac{N_1N_2\tau_{12}}{S_1(V)S_2(V)}\right] - \left[\frac{N_1N_3\tau_{13}}{S_1(V)S_3(V)}\right] - \left[\frac{N_3N_2\tau_{23}}{S_2(V)S_3(V)}\right] \leqq 0.$$

If the second inequality is multiplied through by $\prod_{j=1}^{3} \sigma_j S_j(V) > 0$, it becomes

$$\sigma_1\sigma_2\sigma_3 E(V) = -\sigma_1\sigma_2\sigma_3[-n_2n_3\tau_{23}S_1(V) - n_1n_3\tau_{13}S_2(V) + n_1n_2\tau_{12}S_3(V)].$$

It is clear that the boundary surface $E(v) = 0$ is again a sphere.

Since the conditions for Lemma III are satisfied (6.1) holds.

$$(6.1) \qquad 0 = \sum_{k=1}^{\delta_3} \frac{M_k}{V - v_k} = \sum_{k=1}^{\delta_1} \frac{m_k}{V - v_k} + \sum_{k=1+\delta_1}^{\delta_2} \frac{m_k}{V - v_k} - \sum_{k=1+\delta_2}^{\delta_3} \frac{m_k}{V - v_k}.$$

Nagy [6] proved the following Laguerre-type theorem: "If the Q_j, $j=1, 2, \cdots, p$, are points in the spherical region $\sigma S(v) \leqq 0$, $\sigma = \pm 1$, $S(v) \equiv (v-c)^2 - r^2$, $r > 0$, and $P:V$ is exterior to $\sigma S(v) \leqq 0$, it follows that

$$\sum_{k=1}^{p} \frac{m_k}{V - v_k} = \sum_{k=1}^{p} \frac{m_k}{V - \gamma} = \frac{n}{V - \gamma},$$

where $m_k > 0$ and real, $n = \sum_{k=1}^{p} m_k$, and γ is the position vector of a point in $\sigma S(v) \leqq 0$.

In short, if all the zeros Q_j of F lie in a spherical region $\mathcal{S}$, they may be coalesced at at least one point Q in that region without altering the value of the logarithmic derivative of F. This theorem indicates that there exists a point $Q_j:\psi_j$ in each of the spherical regions $\sigma_j S_j(v) \leqq 0$, $j=1, 2, 3$, such that (6.2) holds.

$$(6.2) \qquad \frac{n_j}{V - \psi_j} = \sum_{k=\delta_{j-1}+1}^{\delta_j} \frac{m_k}{V - v_k}, \qquad n_j = \sum_{k=\delta_{j-1}+1}^{\delta_j} m_k, \quad \delta_0 = 0, \qquad j = 1, 2, 3.$$

(6.1) becomes $n_1/(V-\psi_1) + n_2/(V-\psi_2) = n_3/(V-\psi_3)$ where ψ_j is such that $\sigma_j S_j(\psi_j) \leqq 0$. Since $n_3 = n_1 + n_2$, this expression may be rewritten in the form

$$\frac{1}{V-\psi_1}+\left(\frac{n_2}{n_1}\right)\frac{1}{V-\psi_2}=\left(1+\frac{n_2}{n_1}\right)\frac{1}{V-\psi_3},$$

which is equivalent to (6.3)

$$\begin{aligned} 0 = &\left\{(V-\psi_1)^2+\lambda(V-\psi_2)^2-(1+\lambda)(V-\psi_3)^2\right\}V+(V-\psi_3)^2\psi_1 \\ &+\lambda(V-\psi_3)^2\psi_2-\left\{(V-\psi_1)^2+\lambda(V-\psi_2)^2\right\}\psi_3. \\ &\lambda=\frac{n_2(V-\psi_1)^2}{n_1(V-\psi_2)^2}>0. \end{aligned} \tag{6.3}$$

(6.3) is a relation of the type $\alpha V+\sum_{i=1}^{3}\beta_i\psi_i=0$ where $\alpha+\sum_{i=1}^{3}\beta_i=0$. This leads one to the conclusion that the points $P: V$, and $Q_i: \psi_i$, $i=1, 2, 3$ lie in the same plane Π. If complex numbers are introduced into Π and z, z_1, z_2, and z_3 represent P, Q_1, Q_2, and Q_3 respectively, (6.3) becomes $-n_2/n_1 =(z-z_2)(z_3-z_1)/(z-z_1)(z_3-z_2)$. That is, the cross ratio formed by the points P and Q_i, $i=1, 2, 3$ is a constant and the region of space bounded by $E(v)=0$ is the spherical region described by a point $P: V$ which moves so as to form a constant cross ratio, $-n_2/n_1$, with the points $Q_j: \psi_j$ as the Q_j describes the spherical region $\sigma_j S_j(v)\leqq 0$ for $j=1, 2, 3$.

The final two cross ratio type theorems are generalizations of those developed by Walsh [7] for polynomials in a single complex variable.

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