ON A SUBRECURSIVE HIERARCHY AND PRIMITIVE RECURSIVE DEGREES[1]

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1. Prior to the Herbrand-Gödel-Kleene definition of general recursive function, certain classes of functions defined more restrictively by particular "recursions" had been studied. Subsequently subclasses of the general recursive functions have continued to be of interest, not only because of the simplicity and naturalness of certain types of recursive definition, but because of the insight such classes might provide into recursiveness and effectiveness and because of the need for some measurement of the level or complexity of recursiveness of a function or predicate.

Pursuit of this interest led therefore to the devising of various hierarchies of recursive functions. It would seem that such a hierarchy ought to satisfy at least the following conditions: (1) that it be generated on the basis of some general principle, (2) that the order of a class should correspond to the complexity of the functions it contains in the sense that functions which enumerate or majorize a given class should appear in a higher class, (3) that the union of its classes should contain all the recursive functions or, if this fails, should be sufficiently large relative to one's purpose for the hierarchy.

The hierarchies of recursive functions which have been studied fall short of fully satisfying these conditions (cf. [4, §1]).

In [4] Kleene makes a new attempt at a classification of general recursive functions, by using the notion of relative primitive recursiveness and of the uniform effective enumerability of the functions primitive recursive in an assumed function. A general recursive function h_y, and a class C_y of the functions primitive recursive in h_y, is associated with each element y of a system O of ordinal notations. If $y <_O z$, h_y is primitive recursive in h_z but h_z is not primitive recursive in h_y, so $C_y \subset C_z$ and $C_y \neq C_z$. In addition the relation of primitive recursive equivalence is used to divide the number-theoretic functions into equivalence classes called primitive recursive degrees which may be studied after the manner of [5].

We shall investigate some problems connected with these classifications. First a uniqueness property of classes C_y associated with notations for the same ordinal is described and shown to hold at ordinals less than ω^2 and to fail at ω^2. The k-recursive functions of Péter [6] are located in the hierarchy below the ω^ω level. Although it is not yet settled whether all recursive functions are obtained, it is clear that $U_{y \in O} C_y$ is a large and interesting class.

Finally primitive recursive degrees are studied, and certain similarities to and differences from the theory of general recursive degrees of [5] are obtained.

2. For the hierarchy of classes C_y, the "uniqueness property" will be said to hold at an ordinal α if, whenever y, $z \in O$, $|y| = |z| = \alpha$, then $C_y = C_z$ (i.e. h_y and h_z are of the same primitive recursive degree). As is remarked in [4, §7], the use of the O of S_3, involving as it does general recursive fundamental sequences, would be out of keeping with the purpose of building a hierarchy based on primitive recursiveness. We now show that if the O of S_3 is used, the uniqueness property fails at the first possible place in the hierarchy, namely at the ω level. In fact, the nonuniqueness occurs in such a way that a function of arbitrary primitive recursive degree for a general recursive predicate is definable at the ω level.

In this section O refers to the O of S_3.

If ϕ is a function in C_y, $y \in O$, we shall refer to an index of ϕ from h_y [4, §3] as a "y-index" of ϕ. Let e_1 be an index of the primitive recursive function $\lambda ba \langle b, a \rangle$ (i.e. an index under [4, §3] for $l = 0$), and hence also a y-index of $\lambda ba \langle b, a \rangle$ for all $y \in O$. For y, $z \in O$, if p is a y-index of h_{2^z}, then

$$\langle 4, 2, p, \langle 2, 2, \langle 0, 2, 1 \rangle \rangle, e_1 \rangle$$

is a y-index of h_z.

Define the function $\zeta(n)$ primitive recursively as follows:

$$\zeta(0) = \langle 0, 2, 1 \rangle,$$
$$\zeta(n') = \langle 4, 2, \zeta(n), \langle 2, 2, \langle 0, 2, 1 \rangle \rangle, e_1 \rangle.$$

Now we can show by induction on n that, for all $z \in O$, $\zeta(n)$ is a $(z +_O n_0)$-index of h_z. In particular, taking $z = 0_O$, $\zeta(n)$ is an n_0-index of $h_{0_O}(b, a)$, which is identically 0. Thence it follows by induction on n, using the second recursion equation for ζ that, for $n \geq m$, $\zeta(n)$ is an m_0-index of the constant function $\lambda ba\ 0$. Define the function fin (y) by course-of-values recursion as follows:

$$\text{fin } (y) = \begin{cases} \text{fin } ((y)_0) + 1 & \text{if } y = 2^{(y)_0} \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

So if y, $z \in O$, $z = 1$ or $z = 3 \cdot 5^{(z)_2}$, and $y = z +_O n_0$, then fin $(y) = n$.

Define $\eta(n)$ primitive recursively:

$$\eta(n) = \langle 4, 1, \zeta(n), \langle 2, 1, \langle 2, 1, 1 \rangle \rangle, \langle 3, 1, 1 \rangle \rangle.$$

The value $\eta(n)$ is a y-index of the one-place constant function 1 if $n < $ fin (y), and (if $|y| < \omega$) of the constant function 0 otherwise.

If for each n, u_n, $v_n \in O$ and $v_n <_O v_{n+1}$, and $\lim_n |v_n| = \alpha$, we say the values of u_n "run through" the values of v_n if (1) for each n there is an m such that $u_n = v_m$, (2) for each n $u_n <_O u_{n+1}$, and (3) $\lim_n |u_n| = \alpha$.

Let $Q(n)$ be any recursive predicate. Define:

$$y_n = (2n + 1)_o$$

and

$$z_n = \begin{cases} y_n +_o 1_o & \text{if } \overline{Q}(n), \\ y_n & \text{if } Q(n). \end{cases}$$

Clearly the values of y_n and of z_n run through the finite notations, n_o. In addition $\lambda n\ y_n$ is primitive recursive and $\lambda n\ z_n$ is recursive. So for $u = 3 \cdot 5^v$ and $v = 3 \cdot 5^z$ where $\lambda n\ y_n = \lambda n \{y\} (n_o)$ and $\lambda n\ z_n = \lambda n \{z\} (n_o)$ (notation of [1, §65]), h_u and h_v are initial functions at the ω level. For every $n \geqq 0$, $\eta(\text{fin } (y_n))$ is a z_n-index ($\eta(\text{fin } (y_n) \div 1)$ is a $(z_n)_0$-index) of the constant function 1 if fin (y_n) $<$ fin (z_n) and of the constant function 0 otherwise. Therefore

$$h_{z_n}(\eta(\text{fin } (y_n) \div 1), a) = \begin{cases} 1 & \text{if fin } (y_n) < \text{fin } (z_n), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore $[\text{fin } (y_n) < \text{fin } (z_n)] \equiv [y_n <_o z_n] \equiv \overline{Q}(n)$. So $\lambda n\ h_{z_n}(\eta(\text{fin } (y_n) \div 1),1)$ is the representing function of $Q(n)$. And $\lambda n\ h_{z_n}(\eta(\text{fin } (y_n) \div 1), 1)$ is primitive recursive in $\lambda nba\ h_{z_n}(b,a)$. Therefore $Q(n)$ is primitive recursive in $\lambda nba\ h_{z_n}(b,a)$, and hence in the initial function h_v. So if, in particular, a recursive Q is chosen which is not primitive recursive in h_u, then h_v is not primitive recursive in h_u, where u and v are the numbers defined above.

3. We now restrict ourselves to the system O of ordinal notations (called O' in §§10 and 11 of [4]) which differs from the O of S_3 by the exclusive use of primitive recursive fundamental sequences of notations in the formation of limit notations and the use of indices instead of Gödel numbers. With this restriction on O we prove the following theorem:

THEOREM 1. *If $\alpha < \omega^2$, then the uniqueness property holds for α.*

Proof, *by course-of-values induction up to ω^2.*

CASE 1. $\alpha = |2^v|$. If $\alpha < \omega$, the uniqueness property holds because the ordinal notation is unique. Otherwise the argument of [4, §6] applies.

CASE 2. $\alpha = |3 \cdot 5^v|$. By a "complete sequence" of ordinal notations between $l \cdot \omega$ and $(l+1) \cdot \omega$ is meant the sequence of values of a function $\lambda n\ z_n$ such that $z_0 \in O$, $|z_0| = l \cdot \omega$, and $z_{n'} = 2^{z_n}$. If $3 \cdot 5^z \in O$ and $|3 \cdot 5^z| = (l+1) \cdot \omega$, then the values of the function $\lambda n\ z_n = \lambda n\ \text{pr}\ (z, \langle n_o \rangle)$ must form either a complete sequence or (except possibly for a finite initial sequence of function values) a subsequence which runs through a complete sequence. Now let $v = 3 \cdot 5^z$ and $u = 3 \cdot 5^v$, where $y_n = \text{pr}\ (y, \langle n_o \rangle)$ and $\alpha = |u| = |v|$. Then to prove Case 2 we shall compare C_u and C_v, first where y_n is such a subsequence of a complete sequence z_n, second where y_n except for a finite initial sequence is such, and third where y_n and z_n are complete sequences.

To make the first of these comparisons, note that for all n, $z_n \leqq_o y_n$ and

$C_{z_n} \subset C_{y_n}$ (cf. $[4 \ (13b)]$). Recalling the properties of the functions ζ and fin of §2, observe that $\zeta(\text{fin } (y_n) \doteq n')$ is a $(y_n)_0$-index of h_{z_n} provided fin $(y_n) > n$, and that $\lambda n \zeta(\text{fin } (y_n) \doteq n')$ is primitive recursive. Then we may write:

$$h_v(b, a) = \begin{cases} h_u(\langle \zeta(\text{fin } (y_{(b)_1}) \doteq (b)_1'), (b)_1 \rangle, \langle (b)_0, a \rangle) \\ \qquad \text{if } z_{(b)_1} \neq y_{(b)_1}, \\ h_u(b, a) \quad \text{if } z_{(b)_1} = y_{(b)_1}. \end{cases}$$

Thus h_v is primitive recursive in h_u.

It remains to be shown that h_u is primitive recursive in h_v. Denote z_{y_n} by r_n and z_{y_n-1} by s_n. Since $n < z_n$, $y_n <_0 r_n$ and $C_{y_n} \subset C_{r_n}$. So h_{y_n} is primitive recursive in h_{s_n} with index $\zeta(y_n \doteq (\text{fin } (y_n))')$ and we can write

$$h_{y_n}(b, a) = h_{r_n}(\zeta(y_n \doteq (\text{fin } (y_n))'), \langle b, a \rangle).$$

Thus $\lambda n b a \, h_{y_n}(b, a)$ is primitive recursive in $\lambda n b a \, h_{r_n}(b, a)$. Further $\lambda n b a \, h_{r_n}(b, a)$ is primitive recursive in $\lambda n b a \, h_{z_n}(b, a)$ since $\lambda n \, y_n$ is primitive recursive. And hence $\lambda n b a \, h_{y_n}(b, a)$ is primitive recursive in $\lambda n b a \, h_{z_n}(b, a)$, so h_u is primitive recursive in h_v. Therefore h_u and h_v are of the same primitive recursive degree.

In case $y_i <_0 z_0$, $i = 0, \cdots, M-1$, but $z_0 \leq_0 y_M$, let $x_n = y_{n+M}$. Then $\lambda n \, x_n$ is a primitive recursive function running through the values of z_n. Write $x_n = \text{pr } (x, \langle n_0 \rangle)$ and $w = 3 \cdot 5^x$. By the argument just completed h_w and h_v are of the same (primitive recursive) degree. The following equations show that h_w and h_u are also of the same degree.

$$h_w(b, a) = h_u(\langle (b)_0, (b)_1 + M \rangle, a),$$

$$h_u(b, a) = \begin{cases} h_w(\langle t_i, 1 \rangle, \langle (b)_0, a \rangle) & \text{if } (b)_1 = i \quad (i = 0, 1, \cdots, M-1), \\ h_w(\langle (b)_0, (b)_1 \doteq M \rangle, a) & \text{if } (b)_1 \geq M, \end{cases}$$

where t_i, $i = 0, \cdots, M-1$, is an $(x_1)_0$-index of h_{y_i}.

Finally we compare classes C_u for which the values of the function $\lambda n \text{ pr } ((u)_2, \langle n_0 \rangle)$ form a complete sequence.

Denote by z_n and y_n two complete sequences of notations between $l \cdot \omega$ and $(l+1) \cdot \omega = \alpha$. For each $n \geq 0$, $|z_n| = |y_n|$ and hence by the hypothesis of the induction h_{z_n} and h_{y_n} are of the same primitive recursive degree. Let t be an index of the primitive recursive function $\text{tr}^2 (b, c)$ of $[4, §4]$. Then using $[4, §6]$, if r is an index of h_{z_n} from h_{y_n},

$$\langle 4, 2, \langle 0, 2, 1 \rangle, \langle 4, 2, t, \langle 3, 2, 1 \rangle, \langle 2, 2, r \rangle \rangle, \langle 3, 2, 2 \rangle \rangle$$

will be an index of $h_{z_{n'}}$ from $h_{y_{n'}}$. Let c be an index of h_{z_0} from h_{y_0}. Define $\rho(n)$ primitive recursively as follows:

$$\rho(0) = c,$$

$$\rho(n') = \langle 4, 2, \langle 0, 2, 1 \rangle, \langle 4, 2, t, \langle 3, 2, 1 \rangle, \langle 2, 2, \rho(n) \rangle \rangle, \langle 3, 2, 2 \rangle \rangle.$$

Now by induction on n, $\rho(n)$ is an index of h_{z_n} from h_{y_n}. Thus $h_{z_n}(b, a)$ $= h_{y_{n+1}}(\rho(n), \langle b, a \rangle)$, so h_v is primitive recursive in h_u. Similarly h_u is primitive recursive in h_v.

4. The uniqueness property fails at ω^2, however; there exist u, v such that $u, v \in O$, $|u| = |v| = \omega^2$, and $C_u \neq C_v$. To show this we shall first establish two lemmas.

4.1. Suppose $\phi(x_1, \cdots, x_n)$ is primitive recursive in $\theta(z, y_1, \cdots, y_m)$ with index e. The index e corresponds to a particular primitive recursive derivation of ϕ from θ (cf. [4, §3]). Carrying out the computation of the value $\phi((a)_0, \cdots, (a)_{n-1})$ for a given a by the method corresponding to the index e will require function values $\theta(z, y_1, \cdots, y_m)$ for a specific finite set S of $(m+1)$-tuples $z, y_1, \cdots, y_m$. S depends on e and a and on the function θ. For the following lemma we define $\delta_e(a)$ to be the largest value of z which appears in the $(m+1)$-tuples of S, or 0 if S is empty. (Kleene has examples with $\lambda a \, \delta_e(a)$ not primitive recursive.)

LEMMA 1. *If $\phi(x_1, \cdots, x_n)$ is primitive recursive in $\theta(z, y_1, \cdots, y_m)$ with index e, then $\lambda a \, \delta_e(a)$ is primitive recursive in $\theta(z, y_1, \cdots, y_m)$.*

Proof, *by a course-of-values induction on the index e.* Suppose true for all indices less than e.

CASE 0. $(e)_0 = 0$.

$$\phi \text{ is } \theta. \quad \delta_e(a) = (a)_0.$$

CASES 1, 2, 3. $(e)_0 = 1, 2, 3$. ϕ is primitive recursive and its definition based on e is independent of θ. $\delta_e(a) = 0$.

CASE 4. $(e)_0 = 4$.

$$\phi(x_1, \cdots, x_{(e)_1}) = \psi(\chi_1(x_1, \cdots, x_{(e)_1}), \cdots, \chi_k(x_1, \cdots, x_{(e)_1})),$$

where $k = (e)_{2,1}$.

$$\delta_e(a) = \max \big[\delta_{(e)_2}(\langle \chi_1((a)_0, \cdots, (a)_{(e)_1-1}), \cdots, \chi_k((a)_0, \cdots, (a)_{(e)_1-1}) \rangle),$$
$$\delta_{(e)_3}(\langle (a)_0, \cdots, (a)_{(e)_1-1} \rangle), \cdots, \delta_{(e)_{k+2}}(\langle (a)_0, \cdots, (a)_{(e)_1-1} \rangle) \big].$$

Since $\chi_1, \cdots, \chi_k$ are primitive recursive in θ by the hypothesis of the lemma, and $\delta_i(a)$, for indices $i < e$, are primitive recursive in θ by the hypothesis of induction, and max is a primitive recursive function, $\delta_e(a)$ is primitive recursive in θ.

CASE 5. $(e)_0 = 5$.

$$\phi(0, x_2, \cdots, x_n) = \psi(x_2, \cdots, x_n),$$
$$\phi(x_1', x_2, \cdots, x_n) = \chi(x_1, \phi(x_1, \cdots, x_n), x_2, \cdots, x_n).$$

Define $\delta_e(a)$ by course-of-values recursion on a as follows (noting that when $(a)_0 > 0$, then $\langle (a)_0 \dot{-} 1, (a)_1, \cdots, (a)_{(e)_1-1} \rangle < a$):

$$\delta_e(a) = \begin{cases} \delta_{(e)_2}(\langle (a)_1, \cdots, (a)_{(e)_1-1} \rangle) \text{ if } (a)_0 = 0, \\ \max \left[\delta_e(\langle (a)_0 \dot- 1, (a)_1, \cdots, (a)_{(e)_1-1} \rangle), \right. \\ \qquad \delta_{(e)_3}(\langle (a)_0 \dot- 1, \phi((a_0) \dot- 1, (a)_1, \cdots, (a)_{(e)_1-1}), \\ \qquad\qquad (a)_1, \cdots, (a)_{(e)_1-1}) \rangle) \right] \text{ if } (a)_0 > 0. \end{cases}$$

The functions $\delta_{(e)_2}(a)$ and $\delta_{(e)_3}(a)$ are primitive recursive in θ by the hypothesis of induction, ϕ is primitive recursive in θ, and max is primitive recursive. Hence $\delta_e(a)$ is primitive recursive in θ. This completes the proof.

From the definition of $\delta_e(a)$ it follows that if an index e defines $\phi_1(x_1, \cdots, x_n)$ from $\theta_1(z, y_1, \cdots, y_m)$ and $\phi_2(x_1, \cdots, x_n)$ from $\theta_2(z, y_1, \cdots, y_m)$ respectively, and given any a, if, for all $z < \delta_e(a)$ and all $y_1, \cdots, y_m$,

$$\theta_1(z, y_1, \cdots, y_m) = \theta_2(z, y_1, \cdots, y_m),$$

then $\phi_1((a)_0, \cdots, (a)_{n-1}) = \phi_2((a)_0, \cdots, (a)_{n-1})$.

LEMMA 2. *Given any general recursive function $\phi(a)$, there is a general recursive function $\nu(a)$ of the form $\mu y R(a, y)$, R primitive recursive, such that $\nu(a) > \phi(a)$ for all a and $\nu(a)$ is monotone increasing.*

Proof. Let e be a Gödel number of ϕ. Now by the proof of the normal form theorem [1, §§56–58], $T_1(e, a, x) \to U(x) < x$. Hence $\mu x T_1(e, a, x) > \phi(a)$. So we may take $\nu(a) = \mu y(i)_{i \le a} T_1(e, i, (y)_i)$.

4.2. We define for arbitrary $v \in O$ a monotone increasing function $\xi_v(a) \in C_{v^*}$ (where $v^* = v + _o 1_o$, notation of [2, footnote 23]) which majorizes the one-place functions in C_v (and of course those in C_w for all $w <_o v$):

$$\xi_v(a) = \prod_{i \le a} p_i^{h_{v^*}(i, \langle a \rangle)+1}$$

where p_i is the $(i+1)$-st prime number. To see that this function has the desired majorizing property, let $\alpha(a)$ be an arbitrary one-place function in C_v with v-index e. Then for each $a \ge e$ the function value $\xi_v(a)$ has as a factor $p_e^{h_v(e, \langle a \rangle)+1} = p_e^{\alpha(a)+1}$. Hence $\xi_v(a) > \alpha(a)$ for all $a \ge e$.

4.3. The set of ordinal notations $<_o u$ for some $u \in O$, ordered by $<_o$, we call the "segment" of O determined by u. Extending the definition of §2, we say the values of y_n "run through" the segment determined by $u \in O$ if, for each n, $y_n <_o y_{n'} <_o u$ and $\lim_n |y_n| = |u|$.

Let $\lambda n\, y_n = \lambda n\, \mathrm{pr}\, (y, \langle n_o \rangle)$ be a primitive recursive function the values of which run through the segment determined by u and have u as limit notation $3 \cdot 5^y$. We shall define a monotone increasing function $\xi(a)$ which majorizes the one-place functions of C_w for all $w <_o u$, and which under the assumption that the uniqueness property holds at $|u|$ will be contained in C_u.

Applying 4.2, there is an index d which, for all $v \in O$, defines $\lambda a\, \xi_v(a)$ from h_{v^*}; so $\xi_v(a) = h_{v^{**}}(d, \langle a \rangle)$. Let

$$\xi(a) = \prod_{i \leq a} p_i^{h_{y^{**}_n}(d,\langle a \rangle)+1}$$

It is clear that $\xi(a)$ is primitive recursive in $\lambda nba\ h_{y^{**}_n}(b, a)$. Supposing that the uniqueness property holds at $|u|$, $\lambda nba\ h_{y^{**}_n}(b, a)$ is primitive recursive in h_u, since putting $y^{**}_n = \mathrm{pr}\ (s, \langle n_0 \rangle)$, $|u| = |3 \cdot 5^s|$ and $h_{y^{**}_n}(b, a) = h_{3.5^s}(\langle b, n \rangle, a)$. Thus (under the assumption of the uniqueness property at $|u|$), $\xi(a) \in C_u$.

Also $\xi(a)$ is monotone increasing. Furthermore $\xi(a)$ majorizes the one-place functions in classes C_w, for $w <_O u$, in the following way. If $\alpha(a) \in C_w$, then (using [4, (13)]) $\alpha(a)$ must appear in C_{y_n} for some value of n; let its y_n-index be e. Then for all $a \geq n$, the function value $\xi(a)$ has as a factor

$$p_n^{h_{y^{**}_n}(d,\langle a \rangle)+1} = p_n^{\xi_{y_n}(a)+1};$$

and for all $a \geq e$, $\xi_{y_n}(a) > \alpha(a)$. Hence for all $a \geq \max\ (e, n)$, $\xi(a) > \alpha(a)$.

4.4. THEOREM 2. *The uniqueness property does not hold at ω^2.*

Proof. We shall contradict the assumption that the uniqueness property holds at ω^2 by constructing two primitive recursive sequences y_n and z_n ($y_n = \mathrm{pr}\ (y, \langle n_0 \rangle)$, $z_n = \mathrm{pr}\ (z, \langle n_0 \rangle)$, $u = 3 \cdot 5^y$, $v = 3 \cdot 5^z$) such that each sequence runs through a segment of the notations of O for the ordinals less than ω^2, but z_n advances through the segment so slowly in comparison with y_n that C_u can be shown (under the assumption that the uniqueness property holds) to contain functions not in C_v.

First, consider some $w \in O$, $|w| = \omega^2$. Then by 4.2 with $v = w$, $\xi_w(a)$ majorizes all the one-place functions of C_w. So by the assumption that the uniqueness property holds at ω^2, ξ_w majorizes all the one-place functions of C_u and of C_v. By Lemma 2 with [4, §9] there is a primitive recursive predicate $R(a, y)$ such that $\nu(a) = \mu y R(a, y)$ is monotone increasing and for all a, $\nu(a) > \xi_w(a)$. Thus $\nu(a)$ majorizes the functions of C_u and C_v.

Now use the predicate R just described to define a function ψ primitive recursively as follows:

$$\psi(b, k, n) = n \text{ if } k = 0,$$
$$= \psi(b, k \dotminus 1, n) \text{ if } k > 0\ \&\ (i)_{i \leq \mathrm{fin}(n)} \overline{R}(k \dotminus 1, i),$$
$$= 3 \cdot 5^{\mathrm{sb}^1_1(b, k \dotminus 1)} +_O (\mathrm{fin}\ (n) \dotminus \mu i_{i \leq \mathrm{fin}(n)} R(k \dotminus 1, i))_O \text{ otherwise,}$$

where sb^1_1 is the function defined in [4, §10]. By the recursion theorem for primitive recursive functions [4, §10] there is a number f which is an index of $\psi(k, n) = \psi(f, k, n)$. Let $\zeta(k, n) = \psi(k, n_0)$.

Observe that, as we can prove by induction on k (using the monotonicity property of $\nu(a)$), the first $\nu(k)$ values $\zeta(k, 0), \cdots, \zeta(k, \nu(k) - 1)$ of the function $\lambda n\ \zeta(k, n)$ are

$0_o, 1_o, \cdots, (\nu(0)-1)_o,$

$\zeta(1, \nu(0)), \zeta(1, \nu(0))+_o 1_o, \cdots, \zeta(1, \nu(0))+_o(\nu(1)-\nu(0)-1)_o,$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots,$

$\zeta(k, \nu(k-1)), \zeta(k, \nu(k-1))+_o 1_o, \cdots, \zeta(k, \nu(k-1))+_o(\nu(k)-\nu(k-1)-1)_o,$

and the subsequent values of $\zeta(k, n)$ are obtained by successive additions $(+_o)$ of 1_o, where, for $l=1, \cdots, k$, $\zeta(l, \nu(l-1)) = 3 \cdot 5^{\text{sb}^1_1(J, l-1)}$ is a notation for $l \cdot \omega$ given by the sequence of values of the function $\zeta(l-1, n)$. Thus for $l=0, \cdots, k-1$, $\zeta(k, \nu(l)) = 3 \cdot 5^{\text{sb}^1_1(J, l)}$ and $|\zeta(k, \nu(l))| = (l+1) \cdot \omega$.

Let $z_n = \zeta(n, n)$. Clearly $z_n \in O$. For each n, $\zeta(k, n) \leqq_o \zeta(k+1, n)$; and for each k, $\zeta(k, n) <_o \zeta(k, n+1)$. Hence $z_n <_o z_{n+1}$. Furthermore for every n and l, if $n \geqq \nu(l)$ then $|z_n| \geqq (l+1) \cdot \omega$. Hence $\lim_n |z_n| = \omega^2$. Thus the values of the primitive recursive function $\lambda n \, z_n$ run through a segment of notations for ordinals less than ω^2.

Define y_n as follows:

$$y_n = \begin{cases} 0_o & \text{if } n = 0, \\ 3 \cdot 5^{\text{sb}^1_1(J, n-1)} & \text{otherwise.} \end{cases}$$

Then $\lambda n \, y_n$ is also primitive recursive, and y_n and z_n run through the same segment of notations for ordinals less than ω^2. By induction on n, using the properties of $\nu(n)$ and $\zeta(k, n)$, $z_n <_o y_n$ for every $n > 0$.

We now show that C_u and C_v are not identical by observing that the function $\xi(a)$, defined as in 4.3 using the sequence of notations y_n just defined (and by 4.3, under the assumption that the uniqueness property holds at ω^2, a function of class C_u), majorizes all the one-place functions of C_v.

Let $\alpha(a)$ be an arbitrary one-place function in C_v, and let e be an index of α from $\lambda nba \, h_{z_n}(b, a)$. Then $\delta_e(\langle a \rangle)$ (as defined in 4.1) is the largest value of n, for which values of the function $\lambda nba \, h_{z_n}(b, a)$ are used in the computation based on e of the function value $\alpha(a)$. By Lemma 1, $\lambda a \, \delta_e(\langle a \rangle)$ is primitive recursive in $\lambda nba \, h_{z_n}(b, a)$. Hence the $\nu(a)$ we have defined majorizes $\delta_e(\langle a+1 \rangle)$. Let d_1 be an index of $\lambda nba \, \theta(\langle b, n \rangle, a)$ from θ, and let c be an arbitrary number such that

$$c > \max \, (\mu z_{z>0}(l)_{l \geqq z}(\nu(l - 1) > \delta_e(\langle l \rangle)), \, \text{tr}^3(e, d_1)).$$

Denote by $\lambda a \, \alpha_c(a)$ the function primitive recursive in

$$\lambda nba \, h_{\zeta(c-1, n)}(b, a) \quad (=\lambda nba \, h_{y_c}(\langle b, n \rangle, a))$$

with index e. Then α_c is primitive recursive in h_{y_c} with index $\text{tr}^3(e, d_1)$.

We shall now show that $\alpha_c(c) = \alpha(c)$. First note that the computation based on e of $\alpha(c)$ requires values of $\lambda nba \, h_{z_n}(b, a)$ only for values of $n \leqq \delta_e(\langle c \rangle)$ $< \nu(c-1)$. It follows from the properties of $\zeta(k, n)$ that $(t)(i)_{i<\nu(t)}[\zeta(t, i) = z_i]$

and hence $(i)_{i<\nu(c-1)}[h_{\zeta(c-1,i)}(b, a) = h_{z_i}(b, a)]$. Thus by the remark at the conclusion of Lemma 1, $\alpha_c(c) = \alpha(c)$.

Furthermore $\alpha_c \in C_{y_c}$. Now by 4.3, $\xi(a)$ majorizes the functions of C_{y_c} in such a way that $(x)_{x \geq \max(\mathrm{tr}^3(e, vd_1), c)}[\xi(x) > \alpha_c(x)]$. For all c under consideration, $c > \mathrm{tr}^3(e, d_1)$; hence $\xi(c) > \alpha_c(c) = \alpha(c)$. Thus ξ majorizes α. Now ξ is independent of the choice of α and α is an arbitrary one-place function in C_v, so ξ majorizes all one-place functions of C_v.

§5. We next investigate the problem of locating Péter's k-recursive functions [6, §10] in the hierarchy of [4].

5.1. First we extend the indexing of functions primitive recursive in assumed functions [4, §3] to functions k-recursive in assumed functions for $k > 1$.

A function ϕ is said to be "k-recursive" in a fixed list of assumed functions $\theta_1(a_1, \cdots, a_{m_1}), \cdots, \theta_l(a_1, \cdots, a_{m_l})$ if it is introduced by one of the schemata ( ), $\cdots$, (IV) of [4, §3], or one of the following schemata (V_j), $j = 1, \cdots, k$, where $n \geq 0$ and $\psi_1, \cdots, \psi_j, \chi, \gamma_1^{(1)}, \cdots, \gamma_{j-1}^{(1)}, \gamma_1^{(2)}, \cdots,$ $\gamma_{j-2}^{(2)}, \cdots, \gamma_1^{(j-2)}, \gamma_2^{(j-2)}, \gamma_1^{(j-1)}$ are functions previously introduced by applications of the schemata:

$$(V_j) \begin{cases} \phi(0, x_2, \cdots, x_j, y_1, \cdots, y_n) = \psi_1(x_2, \cdots, x_j, y_1, \cdots, y_n), \\ \phi(x_1', 0, x_3, \cdots, x_j, y_1, \cdots, y_n) = \psi_2(x_1, x_3, \cdots, x_j, y_1, \cdots, y_n), \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ \phi(x_1', \cdots, x_{j-1}', 0, y_1, \cdots, y_n) = \psi_j(x_1, \cdots, x_{j-1}, y_1, \cdots, y_n), \\ \phi(x_1', \cdots, x_j, y_1, \cdots, y_n) = \chi(x_1, \cdots, x_j, \phi_1, \cdots, \phi_j, y_1, \cdots, y_n), \end{cases}$$

where

$$\phi_i = \phi(x_1', \cdots, x_{i-1}', x_i,$$
$$\overset{(i)}{\gamma_1}(x_1, \cdots, x_j, \phi(x_1', \cdots, x_{j-1}', x_j, y_1, \cdots, y_n), y_1, \cdots, y_n), \cdots,$$
$$\overset{(i)}{\gamma_{j-i}}(x_1, \cdots, x_j, \phi(x_1', \cdots, x_{j-1}', x_j, y_1, \cdots, y_n), y_1, \cdots, y_n), y_1, \cdots, y_n).$$

For a given k, (V_k) is the schema of [6, middle p. 77] (but read "$\phi(n_1+1, \cdots,$ $n_k+1)$" for "$\phi(n_1, \cdots, n_{k+1})$"), except without the normalization to 0 of the value of $n_1 \cdot n_2 \cdot \cdots \cdot n_k = 0$, and except that we allow parameters. The indices corresponding to schemata ( ), $\cdots$, (IV) are those given in [4, §3]. When $s_1, \cdots, s_j, h, g_1^{(1)}, g_2^{(1)}, \cdots, g_1^{(j-1)}$ are respective indices of $\psi_1, \cdots, \psi_j, \chi,$ $\gamma_1^{(1)}, \gamma_2^{(1)}, \cdots, \gamma_1^{(j-1)}$ as previously introduced, we assign $\langle 4+j, n+j, s_1, \cdots,$ $s_j, h, g_1^{(1)}, g_2^{(1)}, \cdots g_1^{(j-1)} \rangle$ as index to the function ϕ introduced by (V_j). In particular, this gives to (V) ($=(V_1)$) the same index as in [4, §3].

For any fixed l, $m_1, \cdots, m_l \geq 0$, a primitive recursive predicate $\mathrm{In}_k^{m_1, \cdots, m_l}(b)$ may be defined to express that b is an index of ϕ from assumed functions $\theta_1, \cdots, \theta_l$ k-recursively. For each $k \geq 1$,

$$
\mathrm{In}_k^{m_1,\cdots,m_l}(b) \equiv \left\{ [\mathrm{In}^{m_1,\cdots,m_l}(b) \ \& \ (b)_0 = 0, 1, 2, 3]\right.
$$

$$
\vee \left[b = \prod_{i < (b)_{2,1+3}} p_i^{(b)_i} \ \& \ (b)_0 = 4 \ \& \ \mathrm{In}_k^{m_1,\cdots,m_l}((b)_2) \right.
$$

$$
\left. \& \ (i)_{2<i<(b)_{2,1+3}}(\mathrm{In}_k^{m_1,\cdots,m_l}((b)_i) \ \& \ (b)_{i,1} = (b)_1) \right]
$$

$$
\vee \left[b = \prod_{i \leq (j^2+j+4)/2} p_i^{(b)_i} \ \& \ (b)_0 = 4 + j \right.
$$

$$
\& \ (i)_{1<i\leq(j^2+j+4)/2}\mathrm{In}_k^{m_1,\cdots,m_l}((b)_i)
$$

$$
\&(i)_{1<i<j+2}((b)_{i,1} = (b)_1 \dot- 1) \ \& \ (b)_{j+2,1} = (b)_1 + j
$$

$$
\left.\left. \& \ (i)_{j+2<i\leq(j^2+j+4)/2}((b)_{i,1} = (b)_1 + 1) \ \& \ 0 < j < k+1 \right] \right\},
$$

where $\mathrm{In}^{m_1,\cdots,m_l}(b)$ is the predicate of $[4, §3]$. $\mathrm{In}_1^{m_1,\cdots,m_l}(b) \equiv \mathrm{In}^{m_1,\cdots,m_l}(b)$.

5.2. Let c_1, c_2 be indices from β of the respective functions

$$
\alpha_1(x) = \beta(\text{fin }(x)), \qquad \begin{cases} \alpha_2(y, 0) = 1, \\ \alpha_2(y, x') = y +_0 \beta(x), \end{cases}
$$

where fin (x) (§2) has the property fin $(n_0) = n$. Let

$$
\eta(w, 0, n) = n_0,
$$

$$
\eta(w, m', 0) = 0_0 = 1,
$$

$$
\eta(w, m', 1) = \langle 0, 1, \overset{1}{\mathrm{tr}}(c_1, \overset{1}{\mathrm{sb}}_1(w, m))\rangle,
$$

$$
\eta(w, m', n'') = \langle 0, 1, \overset{1}{\mathrm{tr}}(c_1, \overset{1}{\mathrm{sb}}_1(\overset{1}{\mathrm{tr}}(c_2, \overset{1}{\mathrm{sb}}_1(w, m)), \eta(w, m', n')))\rangle.
$$

Thus $\eta(w, m, n)$ is primitive recursive, and by the recursion theorem for primitive recursive functions there is an index e of $\eta(e, m, n)$. Let $\eta(m, n) = \eta(e, m, n)$.

Observe that $\eta(m', 1)$, $\eta(m', n'')$ (shown below at the right) are each of the form $3 \cdot 5^z$, where $z_x(= \mathrm{pr}\ (z, \langle x_0 \rangle))$ for $x = 0, 1, 2, \cdots$ are shown at the left:

$$
\eta(m, 0), \eta(m, 1), \eta(m, 2), \cdots \qquad\qquad\qquad \eta(m', 1),
$$

$$
1, \eta(m', n') +_0 \eta(m, 0), \eta(m', n') +_0 \eta(m, 1), \cdots \qquad \eta(m', n'').
$$

It follows by induction on m, and on n for each m, that, for each m, n: $\eta(m, n) \in O$ and $\eta(m, n) <_0 \eta(m, n')$ and

$$
|\eta(m, n)| = \begin{cases} n \ \text{if}\ m \cdot n = 0, \\ n \cdot \omega^m \ \text{if}\ m, n > 0. \end{cases}
$$

5.3. Since 1 is the first notation in the "fundamental sequence" of nota-

tions with $\eta(m', n'')$ as "limit notation" (as shown above), $h_{y+_O\eta(m',n)}(b, a)$ $= h_{y+_O\eta(m',n')}(\langle b, 0\rangle, a)$. So, for c_3 an index of $\lambda ba\,\beta(\langle b, 0\rangle, a)$ from β, $h_{y+_O\eta(m',n)}$ is primitive recursive in $h_{y+_O\eta(m',n')}$ with index c_3. It follows by induction on p that $h_{y+_O\eta(m',n)}$ is primitive recursive in $h_{y+_O\eta(m',n+p)}$ with index $v(p)$, where

$$v(0) = \langle 0, 2, 1\rangle,$$
$$v(p') = \text{tr}^2(v(p), c_3).$$

Since $\eta(m', 1)$ is a limit notation with the fundamental sequence $\eta(m, 0)$, $\eta(m, 1)$, $\eta(m, 2)$, $\cdots$, we have $h_{y+_O\eta(m,x)}(b, a) = h_{y+_O\eta(m',1)}(\langle b, x\rangle, a)$. So if c_4 is an index of $\lambda xba\,\beta(\langle b, x\rangle, a)$ from β, then $\mu(x) = \text{sb}_2^1(x, c_4)$ is an index of $h_{y+_O\eta(m,x)}$ from $h_{y+_O\eta(m',1)}$.

If ϕ has $y+_O\eta(m, 1)$-index b, it has $y+_O\eta(m', 1)$-index $\text{tr}^2(b, \mu(1))$. Thus ϕ has $y+_O\eta(m+p, 1)$-index $\iota(b, p)$, where

$$\iota(b, 0) = b,$$
$$\iota(b, p') = \text{tr}^2(\iota(b, p), \mu(1)),$$

defines ι primitive recursively.

Homology of notation is defined for the present system O in the same manner as for the former system [2, pp. 327–328]. Although the associative law "$(a+_Ob)+_Oc = a+_O(b+_Oc)$" does not hold in general for ordinal notations [3, footnote 29], it does for the corresponding h-types, i.e. $(a+_Ob)+_Oc$ is always homologous to $a+_O(b+_Oc)$. The proof is by induction on c over O. And (just as was the case for the predicates H_y, [2, p. 329]) the functions h_y depend only on the h-type of y, i.e. $h_y(b, a) = h_z(b, a)$ where y and z are homologous.

Now $(y+_O\eta(0, n))+_O\eta(0, 1) = y+_O\eta(0, n')$, and $(y+_O\eta(m', 0))+_O\eta(m', 1)$ $= y+_O\eta(m', 1)$. Using the remark on the associative law for h-types under $+_O$, $(y+_O\eta(m', n'))+_O\eta(m', 1)$ and $y+_O\eta(m', n'')$ are limit notations for fundamental sequences of notations thus:

$$q_0, q_1, q_2, \cdots \qquad (y +_O \eta(m', n')) +_O \eta(m', 1),$$
$$1, r_0, r_1, \cdots \qquad\qquad y +_O \eta(m', n''),$$

where q_i and r_i are homologous, $i = 0, 1, 2, \cdots$. Hence

$$h_{(y+_O\eta(m',n'))+_O\eta(m',1)}(b, a) = h_{y+_O\eta(m',n'')}(\langle (b)_0, (b)_1 + 1\rangle, a).$$

To combine the cases $m \cdot n = 0$ and $m \cdot n \neq 0$, let c_5 be an index of

$$\beta(\langle (b)_0, (b)_1 + 1\rangle, a)$$

from β, and put

$$\lambda(m, n) = \begin{cases} \langle 0, 2, 1\rangle & \text{if } m \cdot n = 0, \\ c_5 & \text{if } m \cdot n \neq 0. \end{cases}$$

Then $h_{(y+_0\eta(m,n))+_0\eta(m,1)}$ is primitive recursive in $h_{y+_0\eta(m,n')}$ with index $\lambda(m, n)$.

Let $c_6 = \langle 4, 2, \langle 0, 2, 1 \rangle, \langle 2, 2, \langle 0, 2, 1 \rangle, e_1 \rangle$ be an index of $h_{y+_0\eta(0,n)}$ from $h_{y+_0\eta(0,n')}$, where e_1 is an index of $\lambda ba \langle b, a \rangle$. Using the c_3 given above, define ρ primitive recursively as follows:

$$\rho(m, 0, b) = b,$$

$$\rho(0, n', b) = \mathrm{tr}^2(\rho(0, n, b), c_6),$$

$$\rho(m', n', b) = \mathrm{tr}^2(b, c_3).$$

By induction on m, and on n for each m, if a function α has y-index b, it has $y+_0\eta(m, n)$-index $\rho(m, n, b)$.

5.4. THEOREM 3. *For each $k \geq 1$, there is a primitive recursive function $\nu_k(b)$ and, for each $l, m_1, \cdots, m_l \geq 0$, a primitive recursive function $\sigma_k(b, c_1, \cdots, c_l)$ such that, if ϕ is k-recursive in $\theta_1(a_1, \cdots, a_{m_1}), \cdots, \theta_l(a_1, \cdots, a_{m_l})$ with index b, and if, for some $y \in O$, $c_1, \cdots, c_l$ are y-indices of $\theta_1, \cdots, \theta_l$ respectively, then $\sigma_k(b, c_1, \cdots, c_l)$ is a $y+_0\eta(k \dotdiv 2, \nu_k(b))$-index of ϕ.*

Proof. For any $k \geq 1$, we define ν_k by the course-of-values recursion

$\nu_k(b) = \max (\nu_k((b)_2), \cdots, \nu_k((b)_{(b)_{2,1}+2}))$ if $(b)_0 = 4$,

$\qquad = \mathrm{sg}\,(k-1) + \max (\nu_k((b)_2), \cdots, \nu_k((b)_{(k^2+k+4)/2}))$

$\qquad\qquad\qquad\qquad\qquad$ if $(b)_0 = 4+j$ for $j = 1, \cdots, k$,

$\qquad = 0 \quad$ otherwise.

The proof of the theorem and the definition of σ_k now proceed by course-of-values induction on k.

First we treat the trivial case, $k = 1$. By induction on b, $\nu_1(b) = 0$ for all b, so $\eta(1 \dotdiv 2, \nu_1(b)) = \eta(0, 0) = 1$, and $y +_0 \eta(1 \dotdiv 2, \nu_1(b)) = y$, and it suffices to take

$$\sigma_1(b, c_1, \cdots, c_l) = \mathrm{tr}^{m_1, \cdots, m_l}(b, c_1, \cdots, c_l).$$

For the induction step $(k > 1)$, we assume the theorem to be already proved for preceding values of k which are ≥ 1, and infer it for the given k by an induction on b, which we present under $5+k$ cases.

CASE 0. $(b)_0 = 0$. Then $\phi = \theta_i$, $i = (b)_2$, $\nu_k(b) = 0$. Let $\sigma_k(b, c_1, \cdots, c_l) = c_i$.

CASES 1, 2, 3. $(b)_0 = 1, 2, 3$. Then ϕ is primitive recursive, $\nu_k(b) = 0$. Let $\sigma_k(b, c_1, \cdots, c_l) = b$.

CASE 4. $(b)_0 = 4$. $\phi(x_1, \cdots, x_n) = \psi(\chi_1(x_1, \cdots, x_n), \cdots, \chi_m(x_1, \cdots, x_n))$, where $n = (b)_1$, $m = (b)_{2,1}$. By the hypothesis of the secondary induction on b, ψ has $y+_0\eta(k-2, \nu_k((b)_2))$-index $\sigma_k((b)_2, c_1, \cdots, c_l)$, and χ_i has

$$y + _0\eta(k - 2, \nu_k((b)_{2+i}))\text{-index}$$

$\sigma_k((b)_{i+2}, c_1, \cdots, c_l)$, $i = 1, \cdots, m$. But

$$\nu_k(b) = \max (\nu_k((b)_2), \cdots, \nu_k((b)_{m+2})).$$

So using the function v defined in 5.3, ψ has $y + o\eta(k-2,\ \nu_k(b))$-index $\mathrm{tr}^2\ (\sigma_k((b)_2,\ c_1,\ \cdots,\ c_l),\ v(\nu_k(b) - \nu_k((b)_2)))$, and similarly with each χ_i. Hence it suffices to take

$$\sigma_k(b,\ c_1,\ \cdots,\ c_l)$$
$$= 2^4 \cdot 3^{(b)_1} \cdot \prod_{1 < i < (b)_2,\ 1+3} p_i \exp \mathrm{tr}^2\ (\sigma_k((b)_i,\ c_1,\ \cdots,\ c_l),\ v(\nu_k(b) - \nu_k((b)_i))).$$

CASE 5. $(b)_0 = 5$. Similarly we take

$$\sigma_k(b, c_1, \cdots, c_l) = 2^5 \cdot 3^{(b)_1} \cdot \big[5 \exp \mathrm{tr}^2\ (\sigma_k((b)_2,\ c_1,\ \cdots,\ c_l),\ v(\nu_k(b) - \nu_k((b)_2)))\big]$$
$$\cdot \big[7 \exp \mathrm{tr}^2\ (\sigma_k((b)_3,\ c_1,\ \cdots,\ c_l),\ v(\nu_k(b) - \nu_k((b)_3)))\big].$$

There remain $k-1$ cases for the given k. The treatment will be sufficiently illustrated (and the notation kept simple) by treating the two remaining cases for $k=3$ without parameters. We begin with Case 7.

CASE 7. $(b)_0 = 7$. Then

$$\phi(0,\ x_2,\ x_3) = \psi_1(x_2,\ x_3),$$
$$\phi(x_1',\ 0,\ x_3) = \psi_2(x_1,\ x_3),$$
$$\phi(x_1',\ x_2',0) = \psi_3(x_1,\ x_2),$$

$$\phi(x_1',\ x_2',\ x_3') = \chi(x_1,\ x_2,\ x_3,\ \phi(x_1,\ \gamma_1^{(1)}(x_1,\ x_2,\ x_3,\ \phi(x_1',\ x_2',\ x_3)),$$
$$\gamma_2^{(1)}(x_1,\ x_2,\ x_3,\ \phi(x_1',\ x_2',\ x_3))),$$
$$\phi(x_1',\ x_2,\ \gamma_1^{(2)}(x_1,\ x_2,\ x_3,\ \phi(x_1',\ x_2',\ x_3))),$$
$$\phi(x_1',\ x_2',\ x_3)).$$

Using the hypothesis of the induction on b, and a construction illustrated in Case 4, the functions ψ_1, ψ_2, ψ_3, χ, $\gamma_1^{(1)}$, $\gamma_2^{(1)}$, $\gamma_1^{(2)}$ are primitive recursive in h_w (where $w = y +_0 \eta(1,\ r)$ for $r = \max\ (\nu_3((b)_2),\ \cdots,\ \nu_3((b)_8)))$ with indices s_1, s_2, s_3, h, $g_1^{(1)}$, $g_2^{(1)}$, $g_1^{(2)}$, respectively, where $s_i = \mathrm{tr}^2\ (\sigma_k((b)_{i+1},\ c_1,\ \cdots,\ c_l)$, $v(r - \nu_3((b)_i)))$, $i = 1, 2, 3$, and h, $g_1^{(1)}$, $g_2^{(1)}$, $g_1^{(2)}$ are constructed similarly from $(b)_5$, $(b)_6$, $(b)_7$, $(b)_8$.

Now observe that for each x_1 the equations of (V_3) define $\lambda x_2 x_3 \phi(x_1',\ x_2\ x_3,)$ 2-recursively from ψ_2, ψ_3, χ, $\gamma_1^{(1)}$, $\gamma_2^{(1)}$, $\gamma_1^{(2)}$, and $\lambda x_2 x_3 \phi(x_1,\ x_2,\ x_3)$. In order to exhibit the 2-recursion let

$$\chi_{x_1}(a,\ b,\ c,\ d) = \chi(x_1,\ a,\ b,\ \phi(x_1,\ \gamma_1^{(1)}(x_1,\ a,\ b,\ d),\ \gamma_2^{(1)}(x_1,\ a,\ b,\ d)),\ c,\ d).$$

For each x_1, χ_{x_1} is primitive recursive in χ, $\lambda x_2 x_3 \phi(x_1,\ x_2,\ x_3)$, $\gamma_1^{(1)}$, and $\gamma_2^{(1)}$, say with index g (independent of x_1). And let

$$\gamma_{1,x_1}^{(2)}(a,\ b,\ c) = \gamma_1^{(2)}(x_1,\ a,\ b,\ c).$$

Then $\lambda x_2 x_3 \, \phi(x_1', x_2, x_3)$ is introduced by the 2-recursion

$$\phi(x_1', 0, x_3) = \psi_2(x_1, x_3),$$

$$\phi(x_1', x_2', 0) = \psi_3(x_1, x_2),$$

$$\phi(x_1', x_2', x_3') = \chi_{x_1}(x_2, x_3, \phi(x_1', x_2, \gamma_{1,x_1}^{(2)}(x_2, x_3, \phi(x_1', x_2', x_3)))), \phi(x_1', x_2', x_3)).$$

And $f = \langle 6, 2, \langle 0, 1, 1\rangle, \langle 0, 1, 2\rangle, \langle 0, 4, 3\rangle, \langle 0, 3, 4\rangle \rangle$ defines $\lambda x_2 x_3 \, \phi(x_1', x_2, x_3)$ from $\lambda x_3 \, \psi_2(x_1, x_3)$, $\lambda x_2 \, \psi_3(x_1, x_2)$, χ_{x_1} and $\gamma_{1,x_1}^{(2)}$ 2-recursively.

Using 5.3, $\tau_1(x_1) = \mathrm{sb}_1^1(\rho(0, x_1, s_2), x_1)$ is a $w +_o \eta(0, x_1)$-index of $\lambda x_3 \psi_2(x_1, x_3)$, $\tau_2(x_1) = \mathrm{sb}_1^1(\rho(0, x_1, s_3), x_1)$ of $\lambda x_2 \, \psi_3(x_1, x_2)$, $\tau_3(x_1) = \mathrm{sb}_3^1(\rho(0, x_1, g_1^{(2)}), x_1)$ of $\gamma_{1,x_1}^{(2)}$, $\tau_4(x_1) = \rho(0, x_1, h)$ of χ, $\tau_5(x_1) = \rho(0, x_1, g_1^{(1)})$ of $\gamma_1^{(1)}$, and $\tau_6(x_1) = \rho(0, x_1, g_2^{(1)})$ of $\gamma_2^{(1)}$.

Now we shall work out the definition of a function $\kappa(x_1)$ such that, for each x_1, $\kappa(x_1)$ will be a $w +_o \eta(0, x_1)$-index of $\lambda x_2 x_3 \phi(x_1, x_2, x_3)$. Clearly $\kappa(0) = s_1$. If $\kappa(x_1)$ is a $w +_o \eta(0, x_1)$-index of $\lambda x_2 x_3 \, \phi(x_1, x_2, x_3)$, then $\tau_7(x_1, \kappa(x_1)) = \mathrm{tr}^{6,2,4,4}(g, \tau_4(x_1), \kappa(x_1), \tau_5(x_1), \tau_6(x_1))$ is such an index of χ_{x_1}, and hence $\tau_8(x_1, \kappa(x_1)) = \sigma_2(f, \tau_1(x_1), \tau_2(x_1), \tau_7(x_1, \kappa(x_1)), \tau_3(x_1))$ is a $(w +_o \eta(0, x_1)) +_o \eta(0, \nu_2(f))$-index of $\lambda x_2 x_3 \, \phi(x_1', x_2, x_3)$. But under the above definition of ν_k, $\nu_2(f) = 1$; $\tau_8(x_1, \kappa(x_1))$ is thus a $(w +_o \eta(0, x_1)) +_o \eta(0, 1)$-index of $\lambda x_2 x_3 \, \phi(x_1', x_2, x_3)$. Using the function λ of 5.3,

$$\tau_9(x_1, \kappa(x_1)) = \mathrm{tr}^2(\tau_8(x_1, \kappa(x_1)), \quad \lambda(0, x_1))$$

is a $w +_o \eta(0, x_1')$-index of $\lambda x_2 x_3 \, \phi(x_1', x_2, x_3)$. So take $\kappa(x_1') = \tau_9(x_1, \kappa(x_1))$. The functions $\tau_1, \cdots, \tau_9$ are primitive recursive and

$$\kappa(0) = s_1,$$

$$\kappa(x_1') = \tau_9(x_1, \kappa(x_1)),$$

so κ is primitive recursive.

Since $\kappa(x_1)$ is a $w +_o \eta(0, x_1)$-index of $\lambda x_2 x_3 \, \phi(x_1, x_2, x_3)$, $\phi(x_1, x_2, x_3) = h_{w +_o \eta(1,1)}(\langle \kappa(x_1), x_1\rangle, \langle x_2, x_3\rangle)$ and (letting q be an index of κ and e_1 an index of $\lambda ba \, \langle b, a\rangle$)

$$i = \langle 4, 3, \langle 0, 2, 1\rangle, \langle 4, 3, e_1, \langle 4, 3, q, \langle 3, 3, 1\rangle\rangle, \langle 3, 3, 1\rangle\rangle, \langle 4, 3, e_1, \langle 3, 3, 2\rangle, \langle 3, 3, 3\rangle\rangle \rangle$$

is a $w +_o \eta(1, 1)$ index of ϕ. But $w = y +_o \eta(1, r)$. So $\mathrm{tr}^2(i, \lambda(1, r))$ is a $y +_o \eta(1, r')$-index of ϕ. Since $r' = \nu_3(b)$, $\mathrm{tr}^2(i, \lambda(1, r))$ is a $y +_o \eta(1, \nu_3(b))$-index of ϕ. Hence it suffices to take $\sigma_3(b, c_1, \cdots, c_l) = \mathrm{tr}^2(i, \lambda(1, \nu_3(b) - 1))$.

For any $k > 1$, the proof of Case $4 + k$ proceeds similarly, and we take $\sigma_k(b, c_1, \cdots, c_l) = \mathrm{tr}^2(i_k, \lambda(k - 2, \nu_k(b) - 1))$, where i_k is a $w +_o \eta(k - 2, 1)$-index of ϕ $(i_3 = i)$. (If $k = 2$, the proof is a bit simpler.) If parameters are allowed, slight alterations in the proof give us i_k as a primitive recursive function of $(b)_1$ (the number of variables of ϕ), hence as a primitive recursive function of b.

The treatment of Case 6 can be described briefly in terms of the work on Case 7.

CASE 6. $(b)_0 = 6$. $\phi(x_1, x_2)$ is introduced by an application of (V_2). The auxiliary functions $\psi_1, \psi_2, \chi, \gamma_1^{(1)}$ for (V_2) are primitive recursive in h_w (for a w defined similarly to that of Case 7). By the proof of Case 6 for $k = 2$ (under the hypothesis of induction on k) ϕ has a $w +_0 \eta(0, 1)$-index i_2, and hence a $w +_0 \eta(1, 1)$-index $\iota(i_2, 1)$, using the ι of 5.3. As in the last step of Case 7, we may take $\sigma_3(b, c_1, \cdots, c_l) = \text{tr}^2(\iota(i_2, 1), \lambda(1, \nu_3(b) - 1))$.

For any $k > 2$, the proof of Case $4 + j$, $1 < j < k$, proceeds similarly. Here $\iota(i_j, k - j)$ is a $w +_0 \eta(k - 2, 1)$ index of ϕ. Take

$$\sigma_k(b, c_1, \cdots, c_l) = \text{tr}^2(\iota(i_j, k - j), \lambda(k - 2, \nu_k(b) - 1)).$$

We exhibit the primitive recursive definition of σ_k separately.

$\sigma_k(b, c_1, \cdots, c_l) = c_i$ if $(b)_0 = 0$ & $(b)_2 = i$ $(i = 1, \cdots, l)$,

$\quad = b$ if $(b)_0 = 1, 2, 3$,

$\quad = 2^4 \cdot 3^{(b)_1} \cdot \prod_{1 < i < (b)_2, 1 + 3} p_i \exp \text{tr}^2(\sigma_k((b)_i, c_1, \cdots, c_l), v(\nu_k(b) - \nu_k((b)_i)))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ if $(b_0) = 4$,

$\quad = 2^5 \cdot 3^{(b)_1} \cdot [5 \exp \text{tr}^2(\sigma_k((b)_2, c_1, \cdots, c_l), v(\nu_k(b) - \nu_k((b)_2)))]$

$\qquad\qquad \cdot [7 \exp \text{tr}^2(\sigma_k((b)_3, c_1, \cdots, c_l), v(\nu_k(b) - \nu_k((b)_3)))]$ if $(b)_0 = 5$,

$\quad = \text{tr}^2(\iota(i_j, k - j), \lambda(k - 2, \nu_k(b) - 1))$ if $(b_0) = 4 + j$, $(j = 2, \cdots, k - 1)$,

$\quad = \text{tr}^2(i_k, \lambda(k - 2, \nu_k(b) - 1))$ if $(b)_0 = 4 + k$, $(k > 2)$,

$\quad = 0$ otherwise.

Note that for $k = 1$, this definition agrees with p. 96 (cf. [4, §4]).

COROLLARY 1. *If ϕ is k-recursive, then there is an n ($n = 0$, if $k = 1$) such that ϕ is primitive recursive in $h_{\eta(k-2,n)}$; thus for all $k > 1$, the k-recursive functions appear in the hierarchy below the ω^{k-1} level.*

Proof. If $k > 1$, apply the theorem with $l = 0$ and $y = 0_0$.

6. In [4, §6] the notion of primitive recursive degree is introduced in analogy to that of degree of recursive unsolvability (general recursive degree) of Kleene and Post [5]. Operations $\cup$ and $'$ are given under which formulas (1)–(12) of [5] hold.

We shall use the term degree, unless otherwise qualified, to refer to primitive recursive degree.

This system of degrees is a refinement of the system of general recursive degrees of Kleene and Post, since clearly no functions of the same degree will be of different general recursive degrees. The degrees of objects of general recursive degree **0** might be called "degrees of solvability."

For any two functions α and β, either α is primitive recursive in β or not, so that for each two degrees $\mathbf{a}$ and $\mathbf{b}$, either $\mathbf{a} < \mathbf{b}$, $\mathbf{a} = \mathbf{b}$, $\mathbf{a} > \mathbf{b}$, or $\mathbf{a} \mid \mathbf{b}$. That the fourth possibility occurs follows of course from the existence of incomparable general recursive degrees. But we shall later exhibit incomparable degrees within a single general recursive degree, i.e. incomparable degrees associated with objects having the same general recursive degree.

That there is no highest degree is an immediate consequence of the absence of a highest general recursive degree. The question whether there is a highest degree within a given general recursive degree can easily be answered in the negative. Suppose $\alpha(x)$ is of degree $\mathbf{a}$ within a given general recursive degree $\mathbf{d}$. Then $\mathrm{pr}^\alpha (b, a)$, of general recursive degree $\mathbf{d}$, is of degree $\mathbf{a}' > \mathbf{a}$.

There are $\aleph_0$ functions of each degree, and $2^{\aleph_0}$ degrees. There are $\aleph_0$ degrees within each general recursive degree since ($\leqq$) there are only $\aleph_0$ functions of each general recursive degree (by [5, 1.2]) and since ($\geqq$) there is no highest degree within any general recursive degree.

The definition of independence of degrees and the proof that for $n > 2$ independence implies pairwise incomparability but not conversely translate directly from [5, 1.3], replacing recursive by primitive recursive. In 8.2 we will show the existence for any $n \geqq 1$, of n independent degrees within any given general recursive degree, so the definition and proof apply to degrees within a given general recursive degree.

As was true for general recursive degrees, the degree of an infinite join of a set of functions is not determined by the degrees of the functions. It depends on the functions themselves and on the order in which they are joined.

7.7.1. Lemma 1. *For each $l, m_1, \cdots, m_l \geqq 0$, there is a primitive recursive function $\rho(b)$ such that, if $\theta_i(x_1, \cdots, x_{m_i})$ is bounded by $\beta_i(x_1, \cdots, x_{m_i})$, $i = 1, \cdots, l$, then $\mathrm{pr}^{\theta_1, \cdots, \theta_l}(b, a) \leqq \mathrm{pr}^{\beta_1, \cdots, \beta_l}(\rho(b), a)$ and, when $\mathrm{In}^{m_1, \cdots, m_l}(b)$, the number $\rho(b)$ is an index from $\beta_1, \cdots, \beta_l$ of a function of $(b)_1$ variables which is monotone nondecreasing in each variable.*

Proof. A function which fails to be monotone nondecreasing can be introduced under Schemata ( ), $\cdots$, (V) only by ( ) or (V) or by application of (IV) to previously introduced functions which fail to be monotone nondecreasing. Let d_i, $i = 1, \cdots, l$, be an index of

$$\lambda x_1, \cdots, x_{m_i} \sum_{y_1 \leqq x_1, \cdots, y_{m_i} \leqq x_{m_i}} \beta_i(y_1, \cdots, y_{m_i}) \text{ from } \beta_i.$$

Let c be an index of $\lambda xy\, x + y$. If an index b defines ϕ from $\theta_1, \cdots, \theta_l$ and if $(b)_0 = 5$, then Schema (V), altered by replacing $\chi(x_1, y, x_2, \cdots, x_n)$ by $y + \chi(x_1, y, x_2, \cdots, x_n)$, introduces a function defined from $\theta_1, \cdots, \theta_l$ by

$$\langle 5, (b)_1, (b)_2, \langle 4, (b)_{3,1}, c, \langle 3, (b)_{3,1}, 2 \rangle, (b)_3 \rangle \rangle.$$

Now define by course-of-values recursion

$$\rho(b) = d_i \text{ if } (b)_0 = 0 \ \& \ (b)_2 = i \qquad\qquad (i = 1, \cdots, l),$$
$$= b \text{ if } (b)_0 = 1, 2, 3,$$
$$= 2^4 \cdot 3^{(b)_1} \cdot \prod_{1 < i < (b)_{2,1}+3} p_i^{\rho((b)_i)} \qquad\qquad \text{if } (b)_0 = 4,$$
$$= \langle 5, (b)_1, \rho((b)_2), \langle 4, (b)_{3,1}, c, \langle 3, (b)_{3,1}, 2 \rangle, \rho((b)_3) \rangle \rangle \text{ if } (b)_0 = 5,$$
$$= 0 \qquad \text{otherwise.}$$

The function ρ is primitive recursive. An induction on b shows that, if $\mathrm{In}^{m_1,\cdots,m_l}(b)$, the function defined by $\rho(b)$ from $\beta_1, \cdots, \beta_l$ is monotone nondecreasing and simultaneously (using the monotonicity) that it bounds the function defined by b from $\theta_1, \cdots, \theta_l$. If $\bar{\mathrm{I}}\mathrm{n}^{m_1,\cdots,m_l}(b)$, then $\mathrm{pr}^{\theta_1,\cdots,\theta_l}(b, a) = 0$.

7.2. To any function, predicate or set there is an object of each of the other two kinds having the same general recursive degree [5, 1.2]. This is not the case for primitive recursive degrees.

THEOREM 4. *If $\alpha(x)$, of degree $\mathbf{a}$, is a function which majorizes the one-place primitive recursive functions, then there is no predicate or set of degree $\mathbf{a}$.*

Proof. The degree of a predicate or set is the degree of its representing function θ. To apply Lemma 1 (7.1) with $l=1$, let $\beta(x)=1$ since for all $x, \theta(x) \leq 1$. Then to each function $\phi(x)$ primitive recursive in a predicate or set (i.e. primitive recursive, say with index e, in θ) there is a primitive recursive function $\mathrm{pr}^\beta(\rho(e), \langle x \rangle)$ which bounds $\phi(x)$. Thus a function $\alpha(x)$ which majorizes all the one-place primitive recursive functions cannot be primitive recursive in any such θ (hence not in any predicate or set) and so cannot be of the same degree. This completes the proof.

Observe that if $\mathbf{d} > \mathbf{a}$, $\mathbf{d}$ is also the degree of a function which majorizes the one-place primitive recursive functions since, if δ is of degree $\mathbf{d}$, then $\langle \alpha(x), \delta(x) \rangle$ is of degree $\mathbf{d}$ [5, (7)] and for all x, $\langle \alpha(x), \delta(x) \rangle \geq \alpha(x)$.

It can be noted, in addition, that $\mathbf{a}$ is actually incomparable with the degree of some predicates. For example, define $v(x) = \bar{\mathrm{s}}\mathrm{g}(\mathrm{pr}^\alpha(x, \langle x \rangle))$; then $v(x)$ takes on only values 0 and 1 (and is thus the representing function of a predicate $U(x)$), and differs from every function primitive recursive in α. The degree of U is incomparable with that of α. And if α is recursive (e.g. $\alpha(x) = \sum_{a,b \leq x} h_2(b, a)$, which is of degree $\mathbf{0}'$) then U is recursive. The degrees of α and of U may thus both be within general recursive degree $\mathbf{0}$.

8. 8.1. For the purposes of Theorem 5 we show that for a given function $\alpha(x)$ a function $\pi(b, a)$, primitive recursive in pr^α (b, a), can be defined which, when $\mathrm{In}^1(b)$, gives for all functions $\theta(x)$ which take on only values 0 and 1, an upper bound for the values of x for which values of $\langle \alpha(x), \theta(x) \rangle$ are required for the computation (based on index b) of the function value $\mathrm{pr}^{\lambda x \langle \alpha(x), \theta(x) \rangle}(b, a)$. In other words, if

$$(x)_{x < \pi(b,a)}[\phi_1(x) = \phi_2(x) \ \& \ (\phi_i(x))_0 = \alpha(x) \ \& \ (\phi_i(x))_1 \leq 1 \ \& \ \phi_i(x)$$
$$= \langle (\phi_i(x))_0, (\phi_i(x))_1 \rangle], \qquad i = 1, 2,$$

then $\mathrm{pr}^{\phi_1}(b, a) = \mathrm{pr}^{\phi_2}(b, a)$.

To use Lemma 1 (7.1) with $l = m_1 = 1$, choose $\beta(x) = \langle \alpha(x), 1 \rangle$. Thus, for every function $\theta(x)$ which takes on only values 0 and 1, for all x, $\beta(x) \geq \langle \alpha(x), \theta(x) \rangle$, and $\psi(b, a) = \mathrm{pr}^\beta \, (\rho(b), a) \geq \mathrm{pr}^{\lambda x \langle \alpha(x), \theta(x) \rangle}(b, a)$.

Then define π as follows:

$$\pi(b, a) = a \text{ if } (b)_0 = 0,$$

$$= \max_{i,j} \left[\pi((b)_2, i), \pi((b)_j, a) \right] \text{ where } i \leq \prod_{k < (b)_{2,1}} p_k^{\psi((b)_{k+3}, a)}$$

$$\text{and } 2 < j < (b)_{2,1} + 3 \text{ if } (b)_{2,1} > 0 \text{ and } (b)_0 = 4,$$

$$= \pi \left((b)_2, \prod_{k < (b)_1 \dot- 1} p_k^{(a)_{k+1}} \right) \text{ if } (b)_0 = 5 \ \& \ (b)_1 > 1 \ \& \ (a)_0 = 0,$$

$$= \max_j \left[\pi(b, [a/2]), \pi \left((b)_3, 2^{(a)_0 \dot- 1} \cdot 3^j \cdot \prod_{1 < k \leq (b)_1} p_k^{(a)_k \dot- 1} \right) \right]$$

$$\text{where } j \leq \psi(b, [a/2]) \text{ if } (b)_0 = 5 \ \& \ (a)_0 > 0,$$

$$= 0 \text{ otherwise (including when } (b)_0 = 1, 2, 3).$$

A course-of-values induction on index b shows that $\pi(b, a)$ has the required property. If $(b)_0 = 0, 1, 2, 3$, the property clearly is obtained. If $(b)_0 = 4$ we need only observe that it suffices to choose $\pi(b, a)$ no smaller than

$$\max \left[\pi \left((b)_2, \prod_{k < m} p_k \exp \mathrm{pr}^{\lambda x \langle \alpha(x), \theta(x) \rangle}((b)_{k+3}, a) \right), \pi((b)_3, a), \cdots, \pi((b)_{m+2}, a) \right],$$

where $m = (b)_{2,1}$. Since $\psi(b, a) \geq \mathrm{pr}^{\lambda x \langle \alpha(x), \theta(x) \rangle}(b, a)$, the number

$$\prod_{k < m} p_k \exp \mathrm{pr}^{\lambda x \langle \alpha(x), \theta(x) \rangle}((b)_{k+3}, a)$$

lies among the values of i used under the fourth case in the definition of $\pi(b, a)$. Similarly if $(b)_0 = 5$.

Now $\beta(x) = \langle \alpha(x), 1 \rangle$ is primitive recursive in $\alpha(x)$. So by (10) of [4, §6], $\mathrm{pr}^\beta \, (b, a)$ is primitive recursive in $\mathrm{pr}^\alpha \, (b, a)$. Hence $\psi(b, a)$ is primitive recursive in $\mathrm{pr}^\alpha \, (b, a)$. Observe that the definition of $\pi(b, a)$ from $\psi(b, a)$, and thence from $\mathrm{pr}^\alpha \, (b, a)$ is an unnested course-of-values double recursion: for given b and a, the value $\pi(b, a)$ is obtained primitive recursively from values $\pi(b, y)$, where $y < a$, and values $\pi(x, z)$, where $x < b$ and z is given primitive recursively from $\mathrm{pr}^\alpha \, (b, a)$. Hence by combining the methods of [6, §§3 and 6], $\pi(b, a)$ is primitive recursive in $\mathrm{pr}^\alpha \, (b, a)$.

8.2. THEOREM 5. *For any $n \geq 1$, given any function $\alpha(x)$, of degree $\mathbf{a}$, which majorizes the primitive recursive functions (e.g. $\mathbf{a} = \mathbf{0}'$), there exist functions $\theta_1(x), \cdots, \theta_n(x)$ such that:*

(1) for $i = 1, \cdots, n$, θ_i is primitive recursive in $\mathrm{pr}^\alpha \, (b, a)$,

(2) for $i = 1, \cdots, n$, θ_i is not primitive recursive in $\alpha, \theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_n$, and

(3) α *is not primitive recursive in* $\theta_1, \cdots, \theta_n$.

Proof. For simplicity, we illustrate the proof with $n=2$. For larger n, replace $\langle \alpha(x), \theta(x) \rangle$ in the definition of $\pi(b, a)$ by

$$\langle \alpha(x), \theta_1(x), \theta_2(x), \cdots, \theta_{n-1}(x) \rangle$$

and let the $\beta(x)$ for the application of Lemma 1 (7.1) instead be $2^{\alpha(x)} \cdot \prod_{0 < i < n} p_i^1$.

Let $\pi(b, a)$ be defined as in 8.1 for the $\alpha(x)$ of our theorem. Let

$$\nu(0) = 0,$$

$$\nu(g') = \tilde{\pi}([g/2]', \nu(g)') + g'.$$

The function $\tilde{\pi}(b, a)$ $[1, \text{p. } 291]$ is primitive recursive in $\pi(b, a)$ and is monotone nondecreasing in both variables. Thus $\nu(g)$ is primitive recursive in $\pi(b, a)$, hence in pr^α (b, a), and is monotone increasing. Furthermore, for all b, $a \,\tilde{} (b', a') > \pi(b, a)$; so for any g and for any θ which takes on only values 0 and 1, the computation based on index b of $\text{pr}^{\lambda x \langle \alpha(x), \theta(x) \rangle}(b, \nu(g))$, $b = [g/2]$, requires only values of $\alpha(x)$ and of $\theta(x)$ for $x < \nu(g')$.

Suppose a function $\phi(x)$ is primitive recursive in $\langle \alpha(x), \theta(x) \rangle$ with index b and that $\theta(x)$ is a primitive recursive function with index t. Then ϕ is primitive recursive in α with index $\text{tr}^1 (b, \langle 4, 1, e_1, \langle 0, 1, 1 \rangle, t \rangle)$, where e_1 is an index of $\lambda yz \langle y, z \rangle$. In particular, if $\theta(x) = \lambda x (c)_x \dot- 1$, then $t = \text{sb}_1^1(e, c)$ for e an index of $\lambda cx (c)_x \dot- 1$ (cf. $[4, \S 10]$). So letting

$$\epsilon(b, c) = \text{tr}^1 (b, \langle 4, 1, e_1, \langle 0, 1, 1 \rangle, \text{sb}_1^1(e, c) \rangle),$$

ϵ is primitive recursive and

$$\text{pr}^{\lambda x \langle \alpha(x), (c)_x \dot- 1 \rangle}(b, a) = \text{pr}^\alpha (\epsilon(b, c), a).$$

We now proceed to define the functions θ_1 and θ_2. First let

$$\zeta(0) = 2 \exp \prod_{i < \nu(1)} p_i,$$

$$\zeta(g') = \begin{cases} \zeta(g) \cdot p_{g'}^{\gamma_1(g)} & \text{if } g = 2b + 1, \\ \zeta(g) \cdot p_{g'}^{\gamma_2(g)} & \text{if } g = 2b, \end{cases}$$

where

$$\gamma_1(g) = 2^{1 + \overline{\text{sg}}(\text{pr}^\alpha(\epsilon(b, (\zeta(g))_1 * (\zeta(g))_3 * \cdots * (\zeta(g))_g), \nu(g)))} \cdot \prod_{0 < i < m} p_i^1,$$

and

$$\gamma_2(g) = 2^{1 + \overline{\text{sg}}(\text{pr}^\alpha(\epsilon(b, (\zeta(g))_0 * (\zeta(g))_2 * \cdots * (\zeta(g))_g), \nu(g)))} \cdot \prod_{0 < i < m} p_i^1,$$

and $m = \nu(g+2) - \nu(g)$. Observe that, for $h \geq i$, $(\zeta(h))_i = (\zeta(i))_i$; and furthermore that, for g odd $[g$ even$]$,

$$(\zeta(g))_1 * (\zeta(g))_3 * \cdots * (\zeta(g))_g \ [(\zeta(g))_0 * (\zeta(g))_2 * \cdots * (\zeta(g))_g]$$

is a product of the first $\nu(g')$ primes to nonvanishing exponents, where for even $y < g$ [odd $y < g$] the exponent of $p_{\nu(y)}$ is

$$1 + \bar{\mathrm{sg}}(\mathrm{pr}^\alpha(\epsilon([y/2], (\zeta(y))_0 * (\zeta(y))_2 * \cdots * (\zeta(y))_y), \nu(y)))$$

$$[1 + \bar{\mathrm{sg}}(\mathrm{pr}^\alpha(\epsilon([y/2], (\zeta(y))_1 * (\zeta(y))_3 * \cdots * (\zeta(y))_y), \nu(y)))], \text{ all other}$$

exponents being 1. The function ζ is primitive recursive in pr^α, since the expression $(\zeta(g))_1 * (\zeta(g))_3 * \cdots * (\zeta(g))_g$ in $\gamma_1(g)$ can be written

$$\prod_{i \leq [g/2]}^{*} (\zeta(g))_{2i+1}$$

where $\prod^*$ is defined from $*$ in the same way as $\prod$ from $\cdot$ [1, p. 224], and similarly in $\gamma_2(g)$.

Now let

$$\theta_1(x) = ((\zeta(2x + 1))_1 * (\zeta(2x + 1))_3 * \cdots * (\zeta(2x + 1))_{2x+1})_x \dot{-} 1,$$

$$\theta_2(x) = ((\zeta(2x))_0 * (\zeta(2x))_2 * \cdots * (\zeta(2x))_{2x})_x \dot{-} 1.$$

Since θ_1 and θ_2 are primitive recursive in ζ, they are primitive recursive in pr^α (b, a). Also $\theta_1(x)$ and $\theta_2(x)$ take on only values 0 and 1. Furthermore, if $g = 2b+1$, then, using the above observations and the monotonicity of $\nu(g)$, and writing $\bar{\theta}_1(x) = \prod_{i < x} p_i^{\theta_1(t)+1}$ (cf. [2, footnote 2]),

$$\theta_2(\nu(g)) = \bar{\mathrm{sg}}(\mathrm{pr}^\alpha (\epsilon(b, (\zeta(g))_1 * (\zeta(g))_3 * \cdots * (\zeta(g))_g), \nu(g)))$$

$$= \bar{\mathrm{sg}}(\mathrm{pr}^\alpha (\epsilon(b, \bar{\theta}_1(\nu(g'))), \nu(g)))$$

$$= \bar{\mathrm{sg}}(\mathrm{pr}^{\lambda x \langle \sigma(x), (\bar{\theta}_1(\nu(g')))_x \dot{-} 1 \rangle}(b, \nu(g))).$$

Now by a property of $\nu(g)$, since $\theta_1(x)$ takes on only values 0 and 1, the computation based on index $b = [g/2]$ of $\mathrm{pr}^{\lambda x \langle \alpha(x), \theta_1(x) \rangle}(b, \nu(g))$ requires only values of $\alpha(x)$ and of $\theta_1(x)$ for $x < \nu(g')$. Hence $\theta_2(\nu(g)) = \bar{\mathrm{sg}}(\mathrm{pr}^{\lambda x \langle \alpha(x), \theta_1(x) \rangle}(b, \nu(g)))$. Thus $\theta_2(x)$ differs from every one-place function primitive recursive in $\langle \alpha(x), \theta_1(x) \rangle$ for at least one value of x and so is not primitive recursive in $\langle \alpha(x), \theta_1(x) \rangle$. Similarly $\theta_1(x)$ is not primitive recursive in $\langle \alpha(x), \theta_2(x) \rangle$. Thus conditions (1) and (2) are satisfied.

To show that condition (3) is satisfied we apply Lemma 1 (7.1) as in the proof of Theorem 4. Recall that for all x, $\theta_1(x) < 2$ and $\theta_2(x) < 2$. Since the primitive recursive (constant) function $\langle 1, 1 \rangle$ bounds $\langle \theta_1(x), \theta_2(x) \rangle$, each function primitive recursive in $\langle \theta_1(x), \theta_2(x) \rangle$ is bounded by a primitive recursive function by Lemma 1. By hypothesis, $\alpha(x)$ majorizes the one-place primitive recursive functions, and hence $\alpha(x)$ is not primitive recursive in $\langle \theta_1(x), \theta_2(x) \rangle$. This completes the proof of the theorem.

The arguments which yield Corollaries 1–3, [5, 2.2], may now be used to give three analogous corollaries in which degree is understood to mean primitive recursive degree and the degree $\mathbf{a}$ satisfies the condition of the theorem.

No analog of Post's problem [5, p. 391] arises for this system of degrees.

Theorem 5 and its proof may be altered slightly to eliminate the function α and to produce, for any $n \geq 1$, functions θ_i with independent degrees t_i, where $t_i < 0'$, $i = 1, \cdots . n$.

THEOREM 5 (CONCLUDED). *For any $n \geq 1$, there exist functions $\theta_1(x), \cdots, \theta_n(x)$ such that: (1) for $i = 1, \cdots, n$, θ_i is primitive recursive in* pr (b, a), *(2) for $i = 1, \cdots, n$, θ_i is not primitive recursive in $\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_n$.*

Proof. The proof is essentially that just given with the last paragraph eliminated and the following additional alterations. Let $\alpha(x) = 1$. In redefining $\pi(b, a)$ choose $\beta(x) = \langle 1, 1 \rangle$. Then $\pi(b, a)$ is primitive recursive in pr (b, a). The primitive recursive function $\epsilon(b, c)$ is redefined replacing

$$\langle 4, 1, e_1, \langle 0, 1, 1 \rangle, t \rangle \text{ by } \langle 4, 1, e_1, \langle 2, 1, 1 \rangle, t \rangle.$$

Finally, replace pr^α in the exponent of 2 in both $\gamma_1(g)$ and $\gamma_2(g)$ by pr.

Corollaries 2 and 3 may then be amended to allow a to be any degree.

8.3. The upper semi-lattice of general recursive degrees is shown not to be a lattice, according to [5, Theorem 3]. The proof establishes that the upper semi-lattice of (primitive recursive) degrees is also not a lattice (leaving the question unanswered concerning degrees within a given general recursive degree). For the recursiveness in B_1 and B_2 mentioned in (B) and (C) of [5, Theorem 3] is actually shown to be primitive [5, p. 405 and p. 406 lines 14–15], since, for $j \leq k$, L_j is primitive recursive in L_k (using [5, p. 384] lines 1–3 $k - j$ times).

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