A FORMULA FOR THE MULTIPLICITY OF A WEIGHT[1]

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1. Introduction. 1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let π_λ be an irreducible representation of $\mathfrak{g}$, with highest weight λ, on a finite dimensional vector space V_λ. A well known theorem of E. Cartan asserts that the highest weight, λ, of π_λ occurs with multiplicity one. It has been a question of long standing to determine, more generally, the multiplicity of an arbitrary weight of π_λ. Weyl's formula (1.12) for the character of π_λ is an expression for the function $\chi_\lambda(x) = \mathrm{tr}\exp \pi_\lambda(x)$, $x \in \mathfrak{h}$, on $\mathfrak{h}$ in terms of λ and quantities independent of the representation. In the same spirit the author has always understood the multiplicity question to mean the following: Let I be the set of all integral linear forms on $\mathfrak{h}$. Let m_λ be the function in I which assigns to each integral linear form $\nu \in I$ the multiplicity $m_\lambda(\nu)$ of its occurrence as a weight of π_λ. Find a formula for the multiplicity function m_λ in terms of λ and quantities independent of the representation. It is the purpose of this paper to give such a formula (1.1.5).

Obviously a knowledge of the multiplicity function m_λ determines $\chi_\lambda(x)$, $x \in \mathfrak{h}$. That is,

$$\chi_\lambda(x) = \sum_{\nu \in I} m_\lambda(\nu) \exp(\nu, x). \tag{1.1.1}$$

On the other hand Weyl's formula asserts that

$$\chi_\lambda(x) = \frac{\sum_{\sigma \in W} sg(\sigma) \exp(\sigma(g + \lambda), x)}{\sum_{\sigma \in W} sg(\sigma) \exp(\sigma(g), x)} \tag{1.1.2}$$

where g is one half the sum of the positive roots and W is the Weyl group. Finding a formula for the multiplicity function m_λ in a sense then "accomplishes" the division indicated by the formula of Weyl. We hasten to add—this does not in any way detract from Weyl's formula since it still retains its overriding and quite remarkable feature of expressing what is in general a very complicated trigonometric polynomial on $\mathfrak{h}$ as a quotient of two relatively simple trigonometric polynomials.

A direct interest in the multiplicity function arises from sources other than those mentioned above. Included are the following:

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(1) Let U be a compact connected Lie group and $T\subseteq U$ a maximal toroidal subgroup. By virtue of the Frobenius reciprocity theorem the induced representation of U by a character of T is determined as soon as one knows $m_\lambda(\nu)$ for all dominant λ and a fixed suitable $\nu\in I$. On the other hand such an induced representation is of special interest in algebraic geometry since as one knows it is equivalent to the representation of U defined by the natural action of U on the cross sections of the complex line bundle corresponding to ν over the algebraic manifold (flag manifold) U/T. The knowledge of $m_\lambda(\nu)$ for $\lambda\neq\nu$ supplements the Borel-Weil theorem with the information that π_λ occurs only on nonholomorphic cross-sections of the line bundle and does so with multiplicity $m_\lambda(\nu)$. (See [1] for details.)

(2) Concerning infinite dimensional representations a theorem of Gelfand-Neumark asserts that the restriction, to a maximal compact subgroup U of a complex semi-simple group G, of an irreducible unitary representation of G belonging to a nondegenerate series is given, as in (1), as soon as one knows $m_\lambda(\nu)$ for all dominant λ and a fixed suitable ν. (See [5].)

A means of computing $m_\lambda(\nu)$ has been given by Freudenthal in [4]. The computation is based upon a recursive relation satisfied by the values $m_\lambda(\nu)$, $\nu\in I$, for a fixed λ. This relation is an immediate consequence of what Freudenthal calls the Hauptformel. It is given as (see [3, 2.1 and 3.1])

$$\sum m_\lambda(\nu+k\phi)\cdot(\nu+k\phi,\phi)=m_\lambda(\nu)((\lambda+g,\lambda+g)-(\nu+g,\nu+g)) \tag{1.1.3}$$

where the summation is over all positive integers k and all positive roots ϕ. For the purposes of finding a formula for $m_\lambda(\nu)$, use of the relation (1.1.3) carries the repeated disadvantage of having always to divide by terms of the form $((\lambda+g,\lambda+g)-(\nu+g,\nu+g))$ even in the case when ν is not even a weight of π_λ. We could find no way in which (1.13) leads to a closed expression for m_λ.

Let $P(\mu)$, $\mu\in I$, be the integer valued function on I defined by

$$P(\mu) = \text{no. of ways } \mu \text{ may be partitioned into a sum of positive roots.}$$

It follows from elementary considerations in representation theory that the inequality

$$m_\lambda(\lambda-\mu)\leqq P(\mu) \tag{1.1.4}$$

holds for all dominant λ and all $\mu\in I$. Now one can show (and we exploit this fact) that, fixing μ, for λ sufficiently "far out" in the fundamental chamber and sufficiently far from the "walls" of the chamber the equality sign in (1.1.4) will always hold. It seems clear then that a formula for m_λ must necessarily involve the function P. It is the main result of this paper to establish the formula

$$m_\lambda(\nu)=\sum_{\sigma\in W} sg(\sigma)P(\sigma(g+\lambda)-(g+\nu)). \tag{1.1.5}$$

Putting $\lambda=0$ yields the following recursive relation for the partition function P.

$$P(\mu) = - \sum_{\sigma\in W;\sigma\neq e} sg(\sigma)P(\mu - (g - \sigma g)) \tag{1.1.6}$$

for $\mu\neq 0$, $\mu\in I$. (The recursive nature of (1.1.6) is further clarified when it is recalled that P vanishes outside the cone generated by the positive roots and $g-\sigma g$ lies in that cone. Also $P(0)=1$.)

1.2. An auxiliary result is Theorem 5.1. Theorem 5.1 bears the same relation to the relation "totally subordinate" among representations, introduced in §4.4, as does a theorem of Dynkin (Theorem 4.3) to the relation "subordinate." Theorem 5.1 may be regarded as a weak generalization of the Clebsch-Gordan theorem.

2. **Preliminaries.** 1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra of dimension n. Let l be the rank of $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$(dim $\mathfrak{h}=l$). Let B be the Cartan-Killing bilinear form on $\mathfrak{g}$. The value B assigns to vectors $x, y\in\mathfrak{g}$ will be denoted by (x, y). One knows that the restriction of B to $\mathfrak{h}$ is nonsingular and hence one may identify $\mathfrak{h}$ with its dual space. In particular Δ, the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, is then a subset of $\mathfrak{h}$. Let $\mathfrak{h}_0$ be the real subspace of $\mathfrak{h}$ generated by Δ. Then one knows that $\mathfrak{h}_0$ has real dimension l, the restriction B_0 of B to $\mathfrak{h}_0$ is positive definite, and

$$\mathfrak{h} = \mathfrak{h}_0 + i\mathfrak{h}_0$$

is a real direct sum.

Let W be the Weyl group of $\mathfrak{g}$ regarded as operating on $\mathfrak{h}$. Elements $x, y\in\mathfrak{h}$ are said to conjugate under W if $\sigma x=y$ for some $\sigma\in W$. One knows that $\mathfrak{h}_0$ is invariant under W. In fact it is only the action of W on $\mathfrak{h}_0$ which is of interest to us. For each root $\phi\in\Delta\subseteq\mathfrak{h}_0$ let $\mathfrak{h}_\phi\subseteq\mathfrak{h}_0$ designate the hyperplane orthogonal to ϕ and therefore given by

$$\mathfrak{h}_\phi = \{x\in\mathfrak{h}_0 \mid (x, \phi) = 0\}.$$

Also let $R_\phi\in W$ designate the reflection of $\mathfrak{h}_0$ through $\mathfrak{h}_\phi$. This is given algebraically by

$$R_\phi x = x - \frac{2(\phi, x)}{(\phi, \phi)}\phi$$

for $x\in\mathfrak{h}_0$.

The open set $\mathfrak{R}\subseteq\mathfrak{h}_0$ defined by

$$\mathfrak{R} = \mathfrak{h}_0 - \bigcup_{\phi\in\Delta}\mathfrak{h}_\phi$$

is called the set of regular elements in $\mathfrak{h}_0$. The connected components D_i^0 of $\mathfrak{R}$ are called open Weyl chambers. One knows that there are w of them where

w is the order of W and that in fact they may be indexed by W in such a way that if

$$\mathfrak{R} = \bigcup_{\sigma \in W} D_\sigma^0$$

is the decomposition of $\mathfrak{R}$ into its connected components $D_\sigma^0 = \sigma(D^0)$ for any $\sigma \in W$ and $D^0 = D_e^0$, where e is the identity element of W. The closure D_σ of D_σ^0 will be called a closed Weyl chamber or simply a Weyl chamber. Obviously one has

$$\mathfrak{h}_0 = \bigcup_{\sigma \in W} D_\sigma$$

and $D_\sigma = \sigma(D)$. Having fixed D—now called the fundamental chamber—among all the equally suitable Weyl chambers, we will say that an element $x \in \mathfrak{h}_0$ is dominant if $x \in D$. We will say x is strongly dominant if $x \in D^0$. That is, x is strongly dominant if it is both dominant and regular.

Each chamber D_σ, $\sigma \in W$, decomposes Δ into a union of two disjoint subsets Δ_σ^+ and Δ_σ^- where $\Delta_\sigma^- = (-1)\Delta_\sigma^+$ and $\phi \in \Delta_\sigma^+$ if and only if $(\phi, x) \geqq 0$ for all $x \in D_\sigma$. Conversely, one knows that $x \in D_\sigma$ if and only if $(\phi, x) \geqq 0$ for all $\phi \in \Delta_\sigma^+$. Of course the inequality $\geqq$ becomes a strict inequality $>$ when $x \in D_\sigma^0$. Write Δ^+ and Δ^- for Δ_e^+ and Δ_e^-. The elements of Δ^+ are called positive roots and they are in fact just the positive elements of Δ with respect to a suitable lexicographical ordering in $\mathfrak{h}_0$. We shall assume from now on that such an ordering is given in $\mathfrak{h}_0$.

2.2. Consider the lattice I (also, a discrete subgroup of $\mathfrak{h}_0$) of integral elements in $\mathfrak{h}_0$. By definition $\mu \in I$ if and only if $2(\mu, \phi)/(\phi, \phi)$ is an integer for all $\phi \in \Delta$. The set I is the set of all weights of all representations of $\mathfrak{g}$.

Let $\Pi \subseteq \Delta^+$, $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$, be the set of simple positive roots. The elements α_i, $i = 1, 2, \cdots, l$ form a basis of $\mathfrak{h}_0$. Let f_j, $j = 1, 2, \cdots, l$ be the dual basis to the basal elements $2\alpha_i/(\alpha_i, a_i)$, $i = 1, 2, \cdots, l$. That is

$$\frac{2(f_j, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{ij}. \tag{2.2.1}$$

Then $f_j \in I$, $j = 1, 2, \cdots, l$ and in fact these elements form a basis of I. That is, if $\mu \in \mathfrak{h}_0$ then upon writing

$$\mu = \sum_{i=1}^{l} n_i f_i$$

$\mu \in I$ if and only if the n_i are integers. On the other hand $f_j \in D$ for $j = 1, 2, \cdots, l$ and in fact if $x \in \mathfrak{h}_0$ then upon writing

$$x = \sum_{i=1}^{l} c_i f_i \tag{2.2.2}$$

$x \in D$ if and only if $c_i \geqq 0$, $i=1, 2, \cdots, l$ and $x \in D^0$ if and only if $c_i > 0$, $i=1, 2, \cdots, l$. Important in representation theory is the intersection $I_D = I \cap D$, the set of dominant integral elements in $\mathfrak{h}_0$ and $I_{D^0} = I \cap D^0$, the set of strongly dominant integral elements in $\mathfrak{h}_0$.

2.3. Now let π be a representation of $\mathfrak{g}$ on the finite dimensional complex vector space V_π. We shall always assume that the representation is complex linear.

In such a case one knows that there is a unique decomposition of V_π as a direct sum of weight spaces $V_\pi(\mu)$, $\mu \in I$. That is

$$V_\pi = \sum_{\mu \in I} V_\pi(\mu)$$

where $V_\pi(\mu)$ is defined by

$$V_\pi(\mu) = \left\{ v \in V_\pi \mid \pi(x)v = (\mu, x)v \text{ for all } x \in \mathfrak{h} \right\}.$$

Of course $V_\pi(\mu) \neq 0$ for only a finite number of μ. An element $\mu \in I$ such that $V_\pi(\mu) \neq 0$ is called a weight of π. We will let $\Delta(\pi) \subseteq I$ designate the set of weights of π.

Now for any $\mu \in I$ let $m_\pi(\mu) = \dim V_\pi(\mu)$. One always has

$$m_\pi(\mu) = m_\pi(\sigma\mu)$$

for any $\mu \in I$, $\sigma \in W$.

Now in case π is irreducible the convex set in $\mathfrak{h}_0$ generated by all the weights of π has as its extremal points a unique dominant weight λ and all its conjugates $\{\sigma\lambda\}$, $\sigma \in W$. Any one of these extremal points will be called an extremal weight. The weight λ is the highest weight of π relative to any lexicographical ordering in $\mathfrak{h}_0$ making Δ^+ the set of positive elements in Δ. It is an already classical theorem, due to E. Cartan, that $m_\pi(\lambda) = 1$ and that π is characterized by its highest weight. Furthermore, since any element $\nu \in I_D$ is the highest weight of some irreducible representation of $\mathfrak{g}$, we may use I_D as the index set for the set of equivalence classes of all irreducible representations of $\mathfrak{g}$. In fact, for simplicity, for each $\lambda \in I_D$ we choose a fixed irreducible representation of $\mathfrak{g}$ with highest weight λ and designate it by π_λ. The vector space for this representation will be designated by V_λ and, for simplicity, we will write $V_\lambda(\mu)$ for $V_{\pi_\lambda}(\mu)$, $m_\lambda(\mu)$ for $m_{\pi_\lambda}(\mu)$ and $\Delta(\lambda)$ for $\Delta(\pi_\lambda)$.

3. **The partition function** P. 1. Now $\mathfrak{g}$ admits the direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\phi \in \Delta} (e_\phi)$$

where e_ϕ, $\phi \in \Delta$, is a root vector associated with ϕ. Thus

$$[x, e_\phi] = (\phi, x)e_\phi \tag{3.1.1}$$

for any $x \in \mathfrak{h}$. Now let $\mathcal{E}(\mathfrak{g})$ be the universal enveloping algebra (Birkhoff-

Witt algebra) of $\mathfrak{g}$. Let z_i, $i=1, 2, \cdots, l$ be a basis of $\mathfrak{h}_0$. Then (see [7, Théorème 1′, p. 1–07]) the elements

$$(3.1.2)\quad (e_{\phi_1})^{\xi_1}(e_{\phi_2})^{\xi_2}\cdots(e_{\phi_r})^{\xi_r}(z_1)^{\zeta_1}(z_2)^{\zeta_2}\cdots(z_l)^{\zeta_l}(e_{-\phi_1})^{\eta_1}(e_{-\phi_2})^{\eta_2}\cdots(e_{-\phi_r})^{\eta_r}$$

form a basis of $\mathcal{E}(\mathfrak{g})$. Here r is the number of positive roots. ϕ_i, $i=1, 2, \cdots, r$, are the positive roots indexed so that $\phi_i<\phi_{i+1}$ and ξ_i, ζ_j, η_k, $1\leqq i\leqq r$, $1\leqq j\leqq l$, $1\leqq k\leqq r$ are non-negative integers, designating, of course, the powers of the corresponding basal elements. We shall need a simplified notation for this basal element. Towards this end let θ designate the n-tuple (ξ_i, ζ_j, η_k) and write e^θ for the basal element (3.1.2). Let Λ designate the index set of all n-tuples θ with non-negative integer coefficients. Thus the most general element $p\in\mathcal{E}(\mathfrak{g})$ may be uniquely written

$$p = \sum_{\theta\in\Lambda} a_\theta e^\theta$$

where a_θ are complex numbers, only a finite number of which are distinct from zero.

3.2. Now since $\Delta\subseteq I$ we can define a mapping of Λ into I as follows: the image of $\theta\in\Lambda$ is denoted by $\langle\theta\rangle$ and $\langle\theta\rangle$ is defined by

$$(3.2.1)\qquad \langle\theta\rangle = \sum_{i=1}^{r} (\xi_i - \eta_i)\phi_i$$

where θ is the integral n-tuple (ξ_i, ζ_j, η_k). The significance of $\langle\theta\rangle$ will be ap parent from the following: consider the infinite dimensional (purely algebraic) representation ρ of $\mathfrak{g}$ on $\mathcal{E}(\mathfrak{g})$ defined by $\rho(x)p=[x, p]$ where $x\in\mathfrak{g}$, $p\in\mathcal{E}(\mathfrak{g})$. Then we observe that $\mathcal{E}(\mathfrak{g})$ admits the direct sum decomposition

$$(3.2.2)\qquad \mathcal{E}(\mathfrak{g}) = \sum_{\mu\in I} \mathcal{E}_\mu(\mathfrak{g})$$

where $\mathcal{E}_\mu(\mathfrak{g})$, the "weight space for the weight μ" is defined by

$$\mathcal{E}_\mu(\mathfrak{g}) = \{p \in \mathcal{E}(\mathfrak{g}) \mid [x, p] = (\mu, x)p, x \in \mathfrak{h}\}.$$

Indeed it is clear from (3.1.1) and (3.2.1) that $e^\theta\in\mathcal{E}_{\langle\theta\rangle}(\mathfrak{g})$ and that in fact the set of all e^θ such that $\langle\theta\rangle=\mu$ forms a basis of $\mathcal{E}_\mu(\mathfrak{g})$. Since the set $\{e^\theta\}$, $\theta\in\Delta$, forms a basis of $\mathcal{E}(\mathfrak{g})$ we have (3.2.2).

Now any representation π of $\mathfrak{g}$ on a vector space V_π admits a unique extension to $\mathcal{E}(\mathfrak{g})$ as a homomorphism of $\mathcal{E}(\mathfrak{g})$ into the algebra of operators on V_π. We shall always regard π as so extended. The significance of the decomposition as far as representation theory is concerned is that $\pi(p)$ maps $V_\pi(\nu)\to V_\pi(\nu+\mu)$ for every $p\in\mathcal{E}_\mu(\mathfrak{g})$. That is

$$(3.2.3)\qquad \tilde{\pi}\colon \mathcal{E}_\mu(\mathfrak{g}) \times V_\pi(\nu) \to V_\pi(\mu + \nu)$$

where $\pi(p, v)=\pi(p)v$ for $p\in\mathcal{E}_\mu(\mathfrak{g})$, $v\in V_\pi(\nu)$. This is, of course, clear from the definition of $\mathcal{E}_\mu(\mathfrak{g})$ and $V_\pi(\nu)$.

3.3. The space $\mathcal{E}_\mu(\mathfrak{g})$ is clearly infinite-dimensional. For many purposes it suffices to consider a particular finite dimensional subspace of $\mathcal{E}_\mu(\mathfrak{g})$, the subspace generated by e_ϕ where ϕ is positive. Let $\mathfrak{n}^+$, a maximal nilpotent subalgebra of $\mathfrak{g}$, be the Lie subalgebra spanned linearly by the root vectors e_ϕ where $\phi \in \Delta^+$. Then as usual one may regard $\mathcal{E}(\mathfrak{n}^+)$, the enveloping algebra of $\mathfrak{n}^+$, as a subalgebra of $\mathcal{E}(\mathfrak{g})$.

Let Λ^+ designate the subset of Λ consisting of all n-tuples ξ_i, ζ_j, η_k such that $\zeta_j = \eta_k = 0$, $j = 1, 2, \cdots, l$, $k = 1, 2, \cdots, r$. We will use the letter ξ to designate elements of Λ^+.

Now we observe (applying [7, Théorème 1′, p. 1–07] again) that the elements e^ξ, $\xi \in \Lambda^+$ form a basis of $\mathcal{E}(\mathfrak{n}^+)$. Let

$$\mathcal{E}_\mu(\mathfrak{n}^+) = \mathcal{E}_\mu(\mathfrak{g}) \cap \mathcal{E}(\mathfrak{n}^+).$$

Then it is clear that if $\Lambda^+(\mu)$, $\mu \in I$, is defined by

$$\Lambda^+(\mu) = \{\xi \in \Lambda^+ \mid \langle \xi \rangle = \mu\}$$

the elements e^ξ, $\xi \in \Lambda^+(\mu)$ form a basis of $\mathcal{E}_\mu(\mathfrak{n}^+)$. Furthermore it is also clear that $\mathcal{E}_\mu(\mathfrak{n}^+)$ is finite dimensional. The function on I which assigns to each $\mu \in I$ the dimension of $\mathcal{E}_\mu(\mathfrak{n}^+)$ plays a central role in this paper. Thus for any $\mu \in I$ let

$$\begin{aligned} P(\mu) &= \dim \mathcal{E}_\mu(\mathfrak{n}^+) \\ &= \text{number of elements in } \Lambda^+(\mu). \end{aligned}$$

We wish to make the following observations about $P(\mu)$. First of all each $\xi \in \Lambda^+(\mu)$ may be regarded as a "way" of writing μ as a sum of positive roots. Since repetitions of roots are permitted and the order in which the roots occur does not enter, ξ may be regarded as a "partition" of μ into a sum of positive roots. Thus $P(\mu)$ is, in effect, a partition function counting the number of partitions of μ as a sum of positive roots.

Upon writing

$$\mu = \sum_{i=1}^{l} n_i \alpha_i \tag{3.3.1}$$

it is obvious that $P(\mu) = 0$, if for some i, n_i is not a non-negative integer.

Also note that $P(0) = 1$ (2). These facts are used in the recurrence formula given in §6 for P. We note more generally that if n_i is a non-negative integer for all i then $P(\mu) \geqq 1$. Indeed the formula (3.3.1) provides a way of writing μ as a sum of simple positive roots. Since the simple positive roots are linearly independent there is only one such way of writing μ.

Now what was defined above for $\mathfrak{n}^+$ we define similarly for $\mathfrak{n}^-$, the Lie subalgebra generated linearly by all the root vectors $e_{-\phi}$, $\phi \in \Delta^+$. Let Λ^- be

(2) Recall that $\mathcal{E}(n^+)$ contains the scalars and that $\xi_i = \zeta_j = \eta_k = 0$ defines an element $\xi \in \Lambda^+$ such that $e^\xi = 1$.

all n-tuples ξ_i, ζ_j, η_k such that $\xi_i = \zeta_j = 0$, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, l$. We will use the letter η to designate an element in Λ^-. Analogously $\mathcal{E}_\mu(\mathfrak{n}^-)$ has as basis the elements e^η where $\eta \in \Lambda^-(\mu)$. Obviously

$$\dim \mathcal{E}_\mu(\mathfrak{n}^+) = \dim \mathcal{E}_{-\mu}(\mathfrak{n}^-)$$

for any $\mu \in I$.

3.4. Returning to representation theory, for any $\lambda \in I_D$ let $v_\lambda \in V_\lambda$ be a weight vector belonging to the highest weight λ. That is, $(v_\lambda) = V_\lambda(\lambda)$. It is a well known and simple fact that every vector $v \in V_\lambda$ may be put in the form $v = \pi_\lambda(p)v_\lambda$ where $p \in \mathcal{E}(\mathfrak{n}^-)$. Indeed to prove this it suffices to know, (1), that the root vectors e_{α_i} and $e_{-\alpha_j}$, where α_i and α_j run through the simple positive roots, generate $\mathfrak{g}$ and, (2), that $[e_{\alpha_i}, e_{\alpha_j}] = \delta_{ij}\alpha_i$. Furthermore it follows from (3.2.3) that for any $\mu \in I$ every vector $v \in V_\lambda(\lambda - \mu)$ may be written $v = \pi_\lambda(p)v_\lambda$ where $p \in \mathcal{E}_{-\mu}(\mathfrak{n}^-)$. That is,

$$\pi_\lambda(\mathcal{E}_{-\mu}(\mathfrak{n}^-))v_\lambda = V_\lambda(\lambda - \mu).$$

It follows immediately then that

$$m_\lambda(\lambda - \mu) \leqq P(\mu) \tag{3.4.1}$$

for any $\lambda \in I_D$. We will show that given any $\mu \in I$, λ can be chosen so that the equality holds in (3.4.1). This and more will be needed in §6.2.

4. **Theorems of Dynkin and Brauer.** 1. For any $\lambda \in I_D$ let $\lambda^* \in I_D$ be the highest weight of the contragredient representation to π_λ. We may always choose V_{λ^*} so that V_{λ^*} is the dual space to V_λ and $\pi_{\lambda^*}(x)$ is the negative transpose of $\pi_\lambda(x)$ for any $x \in \mathfrak{g}$.

Now for any $\lambda \in I_D$ we recall that $\Delta(\lambda) = -\Delta(\lambda^*)$ and in fact $m_\lambda(\mu) = m_{\lambda^*}(-\mu)$. Note that this implies $-\lambda^*$ is the extremal weight of π_λ which lies in the chamber $-D$. It follows then that the one dimensional space $V_\lambda(-\lambda^*)$ may be characterized by

$$V_\lambda(-\lambda^*) = \{v \in V_\lambda \mid \pi_\lambda(x)v = 0 \text{ for all } x \in \mathfrak{n}^-\}.$$

Now let π be any representation of $\mathfrak{g}$ on a vector space V_π. Define the subspace $Z_\pi \subseteq V_\pi$ as follows:

$$Z_\pi = \{v \in V_\pi \mid \pi(x)v = 0 \text{ for all } x \in \mathfrak{n}^-\}.$$

It follows immediately that

$$\dim Z_\pi = C(\pi) \tag{4.1.0}$$

where $C(\pi)$ is the number of irreducible representations appearing in the decomposition of V_π into irreducible components.

Now let $\lambda_1, \lambda_2 \in I_D$ and consider the case when $\pi = \pi_{\lambda_2} \otimes \pi_{\lambda_1^*}$, the tensor product of the representations π_{λ_2} and $\pi_{\lambda_1^*}$. It is well known that we may identify the vector space $V_\pi = V_{\lambda_2} \otimes V_{\lambda_1}^*$ with the space $L(V_{\lambda_1}, V_{\lambda_2})$ of all linear transformations A mapping V_{λ_1} into V_{λ_2} and that with respect to this identification

$$\pi(x)(A) = \pi_{\lambda_2}(x)A - A\pi_{\lambda_1}(x).$$

But then we find that

$$Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*) = \{A \in L(V_{\lambda_1}, V_{\lambda_2}) \mid \pi_{\lambda_2}(x)A = A\pi_{\lambda_1}(x) \text{ for all } x \in \mathfrak{n}^-\}.$$

That is, $Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*)$ is the set of all intertwining operators for the pair of restriction representations $\pi_{\lambda_1}|\mathfrak{n}^-$ and $\pi_{\lambda_2}|\mathfrak{n}^-$(3). But then we note that if $A \in Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*)$

$$\pi_{\lambda_2}(p)A = A\pi_{\lambda_1}(p) \tag{4.1.1}$$

holds for all $p \in \mathcal{E}(\mathfrak{n}^-)$. On the other hand since every vector $v \in V_{\lambda_1}$ may be put in the form $v = \pi_{\lambda_1}(p)v_{\lambda_1}$ where $p \in \mathcal{E}(\mathfrak{n}^-)$ and $(v_\lambda) = V_\lambda(\lambda)$, it follows that every $A \in Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*)$ is uniquely determined by what it does to the single vector v_{λ_1}.

Define the subspace $W_{\lambda_2}(\lambda_1) \subseteq V_{\lambda_2}$ by

$$W_{\lambda_2}(\lambda_1) = \{v \in V_{\lambda_2} \mid v = Av_{\lambda_1} \text{ for some } A \in Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*)\}.$$

Then, as we have just noted, the mapping

$$\sigma: Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*) \to W_{\lambda_2}(\lambda_1) \tag{4.1.2}$$

defined by

$$\sigma(A) = Av_{\lambda_1}$$

for $A \in Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*)$ is an isomorphism onto. Recalling (4.1.0) we note in passing that we have proved

LEMMA 4.1. *Let $\lambda_1, \lambda_2 \in I_D$. For any finite dimensional representation π of $\mathfrak{g}$ let $C(\pi)$ denote the number of irreducible representations occurring in the complete reduction of π into irreducible components. Then*

$$C(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*) = \dim W_{\lambda_2}(\lambda_1) \tag{4.1.3}$$

$$\leqq \dim V_{\lambda_2}. \tag{4.1.4}$$

We will be interested in the case when equality holds in (4.1.4). That is, when $W_{\lambda_2}(\lambda_1) = V_{\lambda_2}$. Towards this end we wish to characterize the space $W_{\lambda_2}(\lambda_1)$.

4.2. For any $\lambda \in I_D$ and any $v \in V_\lambda$ let the left ideal $\mathcal{E}(v, \lambda)$ in $\mathcal{E}(\mathfrak{n}^-)$ be defined by

$$\mathcal{E}(v, \lambda) = \{p \in \mathcal{E}(\mathfrak{n}^-) \mid \pi_\lambda(p)v = 0\}.$$

(3) If $\mathfrak{k} \subseteq \mathfrak{g}$ is a subalgebra and π is a representation of $\mathfrak{g}$ denote by $\pi|\mathfrak{k}$ the restriction of π to $\mathfrak{k}$.

The following lemma is then an elementary fact in general ring theory:

LEMMA 4.2. *Let* $\lambda_1, \lambda_2 \in I_D$; *then when* $v \in V_{\lambda_2}$, $v \in W_{\lambda_2}(\lambda_1)$ *if and only if* $\mathcal{E}(v_{\lambda_1}, \lambda_1) \subseteq \mathcal{E}(v, \lambda_2)$. *That is, if and only if*

$$\pi_{\lambda_1}(p)v_{\lambda_1} = 0 \textit{ implies } \pi_{\lambda_2}(p)v = 0$$

for any $p \in \mathcal{E}(\mathfrak{n}^-)$.

Proof. If $v \in W_{\lambda_2}(\lambda_1)$ it is obvious from (4.1.1) that the condition of Lemma 4.2 is satisfied. Conversely if the condition is satisfied then setting

$$A(\pi_{\lambda_1}(p)v_{\lambda_1}) = \pi_{\lambda_2}(p)v$$

for all $p \in \mathcal{E}(\mathfrak{n}^-)$ defines (in a well defined way) an element $A \in Z(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*)$ such that $Av_{\lambda_1} = v$.

4.3. In [3], Dynkin introduces (Definition 3.2, p. 283) the notion of one representation being subordinate to another. In the case of irreducible representations, say π_{λ_1} and π_{λ_2}, Dynkin's definition is as follows: π_{λ_2} is said to be subordinate to π_{λ_1} (or simply λ_2 is subordinate to λ_1) in case $v_{\lambda_2} \in W_{\lambda_1}(\lambda_2)$ (4). Recall $(v_{\lambda_2}) = V_{\lambda_2}(\lambda_2)$.

Thus according to Lemma 4.2, λ_2 is subordinate to λ_1 if and only if $\mathcal{E}(v_{\lambda_1}, \lambda_1) \subseteq \mathcal{E}(v_{\lambda_2}, \lambda_2)$. That is, if and only if for all $p \in \mathcal{E}(\mathfrak{n}^-)$

(4.3.1) $$\pi_{\lambda_1}(p)v_{\lambda_1} = 0 \text{ implies } \pi_{\lambda_2}(p)v_{\lambda_2} = 0.$$

Dynkin then goes on to prove the following theorem (Theorem 4.3 below, Theorem 3.15 in [3, p. 285]), which asserts in effect that λ_2 is subordinate to λ_1 if and only if (4.3.1) holds for a much smaller class of elements p. First however, we recall the following well known facts in representation theory. Let π be a representation of $\mathfrak{g}$ on V_π. Let $\phi \in \Delta$, $\mu \in \Delta(\pi)$, v be any vector in $V_\pi(\mu)$ which is also an eigenvector of $\pi(e_{-\phi}e_\phi)$. (It is known that one may find a basis in $V_\pi(\mu)$ which has this property.) Then

(4.3.2) $$\frac{2(\phi, \mu)}{(\phi, \phi)} = p - q$$

where p is the smallest value of j such that $\pi((e_{-\phi})^{j+1})v = 0$ and q is the smallest value of j such that $\pi(e_\phi^{j+1})v = 0$. For use later on we note the following easy consequence of (4.3.2). Let $M_\phi(\lambda)$ be defined by

$$M_\phi(\lambda) = \max_{\mu \in \Delta(\lambda)} \frac{2(\phi, \mu)}{(\phi, \phi)};$$

then $M_\phi(\lambda)$ is the smallest value of j such that

(4.3.3) $$\pi_\lambda(e_\phi^{j+1}) = 0.$$

(4) Note that in [3] extremal vector means weight vector for the highest weight, not any extremal weight.

The equivalence of the two statements in the following theorem arises from the fact that if $\lambda \in I_D$ and $\lambda = \sum_{i=1}^{l} c_i f_i$ then by (2.2.1) and (4.3.2) c_i is the smallest value of j such that

$$\pi_\lambda(e_{-\alpha_i}^{j+1})v_\lambda = 0,$$

$i = 1, 2, \cdots, l$.

THEOREM (DYNKIN) 4.3. *Let $\lambda_1, \lambda_2 \in I_D$. Then π_{λ_2} is subordinate to π_{λ_1} if and only if $\lambda_1 - \lambda_2 \in I_D$. That is, writing*

$$\lambda_k = \sum_{i=1}^{l} \overset{k}{c}_i f_i, \qquad k = 1, 2$$

π_{λ_2} is subordinate to π_{λ_1} if and only if

$$\overset{1}{c}_i \geqq \overset{2}{c}_i$$

for $i = 1, 2, \cdots, l$. Equivalently π_{λ_2} is subordinate to π_{λ_1} if and only if for all j, and $i = 1, 2, \cdots, l$

$$\pi_{\lambda_1}(e_{-\alpha_i}^{j})v_{\lambda_1} = 0 \text{ implies } \pi_{\lambda_2}(e_{-\alpha_i}^{j})v_{\lambda_2} = 0$$

where $(v_{\lambda_k}) = V_{\lambda_k}(\lambda_k)$, $k = 1, 2$.

4.4. For any $\mu \in I$, $\lambda \in I_D$, $v \in V_\lambda$ let

$$\mathcal{E}_{-\mu}(v, \lambda) = \mathcal{E}_{-\mu}(\mathfrak{n}^-) \cap \mathcal{E}(v, \lambda).$$

Then the inequality (3.4.1) in fact becomes

$$P(\mu) - m_\lambda(\lambda - \mu) = \dim \mathcal{E}_{-\mu}(v_\lambda, \lambda). \tag{4.4.1}$$

Dynkin's theorem asserts that as the coefficients of λ go up the right side of (4.4.1) goes down. We shall need the fact (proved later) that it can be made zero.

Let $\lambda_1, \lambda_2 \in I_D$; we will now say that λ_2 is totally subordinate to λ_1 in case $V_{\lambda_2} = W_{\lambda_2}(\lambda_1)$. Obviously totally subordinate implies subordinate.

From the point of view of general ring theory the notions of subordinate and totally subordinate are very easy to describe. Let $k = 1, 2$, then V_{λ_k} may be regarded as a cyclic module over the ring $\mathcal{E}(\mathfrak{n}^-)$, with cyclic vector v_{λ_k}. Now associated with every cyclic module over a ring are two prominent subrings (1) the left ideal (here $\mathcal{E}(v_{\lambda_k}, \lambda_k)$) of all ring elements which annihilate the cyclic vector and (2) the two sided ideal—now written as J_k—of all ring elements which annihilate the entire module. Obviously $J_k \subseteq \mathcal{E}(v_{\lambda_k}, \lambda_k)$. Now we observe that λ_2 is subordinate to λ_1 when $\mathcal{E}(v_{\lambda_1}, \lambda_1) \subseteq \mathcal{E}(v_{\lambda_2}, \lambda_2)$ and totally subordinate to λ_1 when $\mathcal{E}(v_{\lambda_1}, \lambda_1) \subseteq J_2$.

Continuing from §4.1, obviously Lemma 4.1 implies

LEMMA 4.4. *Let $\lambda_1, \lambda_2 \in I_D$; then λ_2 is totally subordinate to λ_1 if and only if*

$$C(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*) = \dim V_{\lambda_2}.$$

4.5. If λ_2 is totally subordinate to λ_1 it follows immediately from Lemma 4.2 and (4.3.2) that

$$\frac{2(\alpha_i, \lambda_1)}{(\alpha_i, \alpha_i)} \geqq \frac{2(\alpha_i, \mu)}{(\alpha_i, \alpha_i)}$$

for all $\mu \in \Delta(\lambda_2)$. Thus by §2.1, (4.3.2) and (4.3.3) we have

LEMMA 4.5. *Let $\lambda_1, \lambda_2 \in I_D$. If λ_2 is totally subordinate to λ_1 then $\lambda_1 - \mu \in I_D$ for all $\mu \in \Delta(\lambda_2)$ or equivalently*

$$\pi_{\lambda_1}(e^j_{-\alpha_i})v_{\lambda_1} = 0 \textit{ implies } \pi_{\lambda_2}(e^j_{-\alpha_i}) = 0$$

for all j and all $i = 1, 2, \cdots, l$.

4.6. We will prove an analogue of Dynkin's theorem (Theorem 5.1) for the notion of totally subordinate (instead of subordinate). Theorem 5.1 asserts in effect that the condition of Lemma 4.5 is also a sufficient condition for totally subordinate. For this we need a theorem of Brauer. First, however, we wish to observe,

LEMMA 4.6. *Let $\lambda_1, \lambda_2 \in I_D$ then λ_2 is totally subordinate to λ_1 if and only if λ_2^* is totally subordinate to λ_1^*.*

Proof. It is obvious that $\pi_{\lambda_2^*} \otimes \pi_{\lambda_1}$ is the contragredient representation to $\pi_{\lambda_2} \otimes \pi_{\lambda_1^*}$. Hence $C(\pi_{\lambda_2^*} \otimes \pi_{\lambda_1}) = C(\pi_{\lambda_2} \otimes \pi_{\lambda_1^*})$. Since, of course, $\dim V_{\lambda^*} = \dim V_{\lambda_2}$ the result follows from Lemma 4.4.

4.7. Let $\mathfrak{A}(I)$ designate the group algebra over I. We admit into $\mathfrak{A}(I)$ only functions h on I with finite support. It is also convenient to regard elements of $\mathfrak{A}(I)$ as finite formal combinations of the elements of I. However, since the group operation in I is written additively we will designate the function (Dirac measure at ν) which is 1 at ν and zero at μ for all $\mu \neq \nu$ by δ_ν. Thus when h is regarded as a formal combination of elements of I, h is written uniquely as

$$h = \sum_{\nu \in I} a_\nu \delta_\nu$$

where only a finite number of the a_ν are distinct from zero. When regarded as a function, $h(\nu) = a_\nu$. The function m_λ, defined in §2.3, which assigns to each $\nu \in I$ its multiplicity in π_λ is an element of $\mathfrak{A}(I)$. Now for each $\nu \in I$ let $F_\nu \in \mathfrak{A}(I)$ be defined by

$$F_\nu = \sum_{\sigma \in W} sg(\sigma \delta)_{\sigma\nu}.$$

Note that $F_\nu = sg(\sigma)F_{\sigma\nu}$ and as a function $F_\nu(\mu) = sg(\sigma)F_\nu(\sigma\mu)$. Next we observe that $F_\nu \neq 0$ if and only if $\nu \in \mathfrak{R}$; that is, if and only if ν is regular. Indeed if $\nu\in\mathcal{R}$ then (see §2.1) all the elements $\sigma\nu$, $\sigma\in W$ are distinct and hence certainly $F_\nu\neq 0$. If, however, $\nu\in\mathfrak{h}_\phi$ for some $\phi\in\Delta$ then $R_\phi\nu=\nu$, and since $sg(R_\phi)=-1$, $F_\nu=F_{R_\phi\nu}=-F_\nu$ and hence $F_\nu=0$. Thus it suffices to consider F_ν for regular ν. In fact it suffices to consider only F_ω where $\omega\in I_D{}^0$, that is, where ω is strongly dominant. Indeed if $\nu\in\mathcal{R}$ there exists a unique $\omega\in I_D{}^0$ such that $F_\nu=\pm F_\omega$. This is obvious since there exists a unique σ such that $\sigma\nu\in I_D{}^0$. Let $\omega=\sigma\nu$; then $F_\nu=sg(\sigma)F_\omega$.

An element $h\in\mathfrak{A}(I)$ will be called alternating in case

$$h(\mu) = sg(\sigma)h(\sigma\mu)$$

for every $\mu\in I$. Let $\mathfrak{A}_s(I)\subseteq\mathfrak{A}(I)$ be the space of all alternating elements in $\mathfrak{A}(I)$. The elements F_ν belong to $\mathfrak{A}_s(I)$ and are called elementary alternating sums. If $h\in\mathfrak{A}_s(I)$ then since

$$h(\mu) = \frac{1}{w}\sum_{\sigma\in W} sg(\sigma)h(\sigma\mu)$$

where $w=$order (W) it follows that every element in $\mathfrak{A}_s(I)$ is spanned by elementary alternating sums. Hence from above every $h\in\mathfrak{A}_s(I)$ may be uniquely written as $h=\sum_{\omega\in I_D{}^0} b_\omega F_\omega$ where $b_\omega=0$ except for a finite number of ω. That is, the elements F_ω, $\omega\in I_D{}^0$ form a basis of $\mathfrak{A}_s(I)$.

Now let

$$g = \frac{1}{2}\sum_{\phi\in\Delta^+}\phi.$$

Then it is a well known fact that $g\in I$ and in fact, $2(g_1\alpha_i)/(\alpha_i,\alpha_i)=1$ for all i. That is

$$g = f_1 + f_2 + \cdots + f_l. \tag{4.7.1}$$

Thus from (2.2.2), $g\in I_D{}^0$. In fact g is a very special element of $I_D{}^0$. It is the unique minimal element of $I_D{}^0$ in the sense that $\omega-g\in I_D$ for every $\omega\in I_D{}^0$. Conversely it is also clear that $\lambda+g\in I_D{}^0$ for every $\lambda\in I_D$. Thus if τ_g designates the operation of translation by g in I then

$$\tau_g\colon I_D\to I_D{}^0 \tag{4.7.2}$$

where the isomorphism is onto. This property characterizes g as an element of I. Thus we see that any $h\in\mathfrak{A}_s(I)$ may be uniquely written as

$$h = \sum_{\lambda\in I_D} a_\lambda F_{\lambda+g} \tag{4.7.3}$$

where $a_\lambda=0$ except for a finite number of λ.

4.8. Let $\lambda_1,\lambda_2\in I_D$. In [2] R. Brauer has given a formula for determining the irreducible representations which occur in the tensor product of π_{λ_1} and π_{λ_2}. This formula is given as Theorem 4.8 below. (In Theorem 4.8, however, we consider $\pi_{\lambda_2^*}\otimes\pi_{\lambda_1}$ instead of $\pi_{\lambda_2}\otimes\pi_{\lambda_1}$. We do this because the formula for the former is more directly applicable for us.) Since $\pi_{\lambda_2^*}\otimes\pi_{\lambda_1}$ is symmetric in λ_2^* and λ_1 it is somewhat surprising at first to note that the formula is decidedly unsymmetric in λ_1 and λ_2^*. However, it is this feature which suggested the use to which this formula is put here.

Recalling that $m_{\lambda^*}(\mu)=m_\lambda(-\mu)$, (see §4.1), we have

THEOREM (BRAUER) 4.8. *Let $\lambda_1, \lambda_2\in I_D$. Let n_λ, $\lambda\in I_D$, be the multiplicity of the irreducible representation π_λ occurring in the tensor product of $\pi_{\lambda_2^*}$ and π_{λ_1} (of course $n_\lambda=0$ except for a finite number of λ) so that*

$$\pi_{\lambda_2^*}\otimes\pi_{\lambda_1}=\sum_{\lambda\in I_D} n_\lambda\pi_\lambda$$

where the equality sign actually stands for equivalence.

Now according to (4.7.3) *let the coefficients a_λ, $\lambda\in I_D$, be defined by*

$$\sum_{\mu\in I} m_{\lambda_2}(\mu)F_{\varrho+\lambda_1-\mu}=\sum_{\lambda\in I_D} a_\lambda F_{\varrho+\lambda}. \tag{4.8.1}$$

Then $a_\lambda=n_\lambda$.

5. **A weak generalization of the Clebsch-Gordan theorem.** 1. In the general case the formula (4.8.1) for the coefficients n_λ has one serious drawback. That is because in general $\lambda_1-\mu\notin I_D$ for $\mu\in\Delta(\lambda_2)$ and hence (see §4.7) there will be considerable cancellation in the contribution towards the coefficients on the right side of (4.8.1). But now if we assume λ_2 is totally subordinate to λ_1 then Lemma 4.5 asserts that this drawback is eliminated and hence in fact we can read off the coefficients n_λ directly from the left side of (4.8.1). On the other hand we observe again that we need only assume $\lambda_1-\mu\in I_D$ for all $\mu\in\Delta(\lambda_2)$ for this to be true.

But under this weaker assumption we get the relation

$$\sum_{\lambda\in I} n_\lambda=\sum_{\mu\in I_D} m_{\lambda_2}(\mu).$$

That is,

$$C(\pi_{\lambda_2^*}\otimes\pi_{\lambda_1})=\dim V_\lambda.$$

$$=\dim V_{\lambda_2^*}.$$

But by Lemma 4.6 this implies λ_2 is totally subordinate to λ_1. We have thus proved the following theorem. The tensor product aspects of Theorem 5.1 may be regarded as a weak generalization of the Clebsch-Gordan theorem. A generalization, since if $\mathfrak{g}$ is the Lie algebra of all 2×2 complex matrices of trace zero then for any pair $\lambda_1, \lambda_2\in I_D$, either λ_1 is totally subordinate to λ_2 or vice versa. Weak, since for general $\mathfrak{g}$ not every pair $\lambda_1, \lambda_2\in I_D$ are so related. Applying Lemmas 4.4 and 4.5 we have

THEOREM 5.1. *The following statements are all equivalent. Let* $\lambda_1, \lambda_2 \in I_D$

(1) λ_2 *is totally subordinate to* λ_1,

(2) $\lambda_1 - \mu \in I_D$ *for all* $\mu \in \Delta(\lambda_2)$,

(3) *If* $\lambda_1 = \sum_{i=1}^{l} c_i f_i$ *then* $c_i \geqq M_{\alpha_i}(\lambda_2)$, $i = 1, 2, \cdots, l$ (*see* §4.3),

(4) $\pi_{\lambda_1}(e_{-\alpha_i}^j)v_{\lambda_1} = 0$ *implies* $\pi_{\lambda_2}(e_{-\alpha_i}^j) = 0$ *for* $i = 1, 2, \cdots, l$; $j = 1, 2, \cdots$.

(5) $C(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*) = \dim V_{\lambda_2}$ *where* $C(\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*)$ *is the number of irreducible representations appearing in the decomposition of the tensor product* $\pi_{\lambda_2} \otimes \pi_{\lambda_1}^*$ *into irreducible representations.*

(6) *For any* $\lambda \in I_D$ *the representation* π_λ *appears* $m_{\lambda_2}(\mu)$ *times, where* $\mu = \lambda_1 - \lambda$, *in the decomposition of the tensor product* $\pi_{\lambda_2}^* \otimes \pi_{\lambda_1}$ *into irreducible representations.*

5.2. Let g be as in §4.7. Consider now the "one-parameter" family of representations π_{kg}, $k = 0, 1, 2, \cdots$. But now for any $\lambda \in I_D$ where $\lambda = \sum_{i=1}^{l} c_i f_i$ it follows immediately from (4.7.1) and Dynkin's theorem, Theorem 4.3, that π_λ is subordinate to π_{kg} if and only if $k \geqq \max_i c_i$. In particular we note that π_{kg} is subordinate to $\pi_{(k+1)g}$ for all k. Thus for all $\mu \in I$

$$\mathcal{E}_\mu(v_{(k+1)g}, (k+1)g) \subseteq \mathcal{E}_\mu(v_{kg}, kg). \tag{5.2.1}$$

Since as we recall

$$P(\mu) - m_{kg}(kg - \mu) = \dim \mathcal{E}_\mu(v_{kg}, kg) \tag{5.2.2}$$

it follows that the left side of (5.2.2) is monotone decreasing with increasing k. Next as an immediate consequence of (1) and (3) in Theorem 5.1 we have

LEMMA 5.2. *Let* $\lambda \in I_D$. *Then* π_λ *is totally subordinate to* π_{kg} *if and only if*

$$k \geqq \max M_{\alpha_i}(\lambda).$$

5.3. But now we can prove easily that the right side of (5.2.2) vanishes for k sufficiently high. From (5.2.1) it obviously suffices to show that for any $p \in \mathcal{E}_\mu(\mathfrak{n}^-)$ there exists k such that $\pi_{kg}(p)v_{kg} \neq 0$. [This insures that we can always strictly drop the dimension of $\mathcal{E}_\mu(v_{kg}, kg)$ by choosing k large enough.] But now by Theorem 1 in [6] for any $p \in \mathcal{E}(\mathfrak{g})$ there exists $\lambda \in I_D$ such that $\pi_\lambda(p) \neq 0$. In particular if $p \in \mathcal{E}_\mu(\mathfrak{n}^-)$ there exists $v \in V_\lambda$ such that $\pi_\lambda(p)v \neq 0$. On the other hand by Lemma 5.2 for any $k \geqq \max_i M_{\alpha_i}(\lambda)$, π_λ is totally subordinate to π_{kg}. Thus from the definition of totally subordinate $\pi_{kg}(p)v_{kg} \neq 0$. Thus we have proved

LEMMA 5.3. *Let* $\mu \in I$ *be arbitrary. Then there exists a positive integer* N *such that*

$$P(\mu) = m_{kg}(kg - \mu)$$

for all $k \geqq N$.

Now for every $\sigma \in W$ let the subset $T(\sigma) \subseteq \Delta^+$ be defined by

$$T(\sigma) = \{\phi \in \Delta^+ \mid \sigma^{-1}(\phi) \in \Delta^-\}$$

and let $s(\sigma) \in I$ be defined by

$$s(\sigma) = \sum_{\phi \in T(\sigma)} \phi.$$

It is clear that if we write

$$s(\sigma) = \sum_{i=1}^{l} b_i(\sigma)\alpha_i \tag{5.3.1}$$

then all the integers $b_i(\sigma)$ are non-negative. If $\sigma \neq e$ then $b_i(\sigma) \geqq 1$ for at least one i.

Now observe that

$$\sigma(g) = \frac{1}{2}\left(\sum_{\psi \in (\Delta^+ - T(\sigma))} \psi - \sum_{\phi \in T(\sigma)} \phi\right).$$

Hence we see that

$$s(\sigma) = g - \sigma(g) \tag{5.3.2}$$

for all $\sigma \in W$. In particular then it follows immediately from §2.1 and §4.7 that the elements $s(\sigma)$, $\sigma \in W$ are all distinct. These elements of I will play a fundamental role in the remaining portions of this paper. In the definition of the function P on I we were concerned with the ways of expressing an element $\mu \in I$ in terms of the positive roots. Now we shall be concerned with expressing μ in terms of the elements $s(\sigma)$, $\sigma \in W$. However, although in general there are far more elements $s(\sigma)$ than there are roots the matter is nevertheless simplified since now we will be concerned with the number of *ordered* ways of doing this—in fact ordered ways with a signature. Hence a recursion formula can immediately be given in this case. In the case of $P(\mu)$, until we establish the equality given in Lemma 6.2 no recursive formula is apparent to the author.

5.4. For convenience write the elements $s(\sigma), \sigma \in W$, as s_i, $i = 1, 2, \cdots, w$ where $w = $ order W. We may choose the ordering so that $s_1 = e$ and if $s_i = s(\sigma)$ then $-sg(\sigma) = (-1)^i$. Now let Γ be the set of all finite sequences γ,

$$\gamma = (i_1, i_2, \cdots, i_q)$$

such that i_j is a positive integer and $2 \leqq i_j \leqq w$.

For any $\mu \in I$ let the subset $\Gamma_\mu \subseteq \Gamma$ be defined by

$$\Gamma_\mu = \left\{\gamma \in \Gamma \,\middle|\, \sum_{j=1}^{q} s_{i_j} = \mu\right\}.$$

Observe that since $\gamma_i \geqq 2$ for all $\gamma \in \Gamma$ it follows that Γ_μ is a finite set. Now for any $\gamma \in \Gamma$ define

$$sg(\gamma) = (-1)^{i_1+\cdots+i_q}$$

Observe that if γ is the empty sequence, that is, $q=0$, then $sg(\gamma)=1$.

We can now define the function Q on I. For any $\mu \in I$ let

$$Q(\mu) = \sum_{\gamma \in \Gamma_\mu} sg(\gamma).$$

If we write

$$\mu = \sum_{i=1}^{l} b_i \alpha_i$$

we observe the following properties: (1) $Q(\mu)=0$ in case b_i is not a non-negative integer for some i, (2) $Q(0)=1$, by a remark made above. Finally, (3), we note that if $\mu \neq 0$, Q satisfies the following recursive relation. (Here we revert to the original notation, $s(\sigma)$.)

$$(5.4.1) \qquad Q(\mu) = - \sum_{\sigma \in W, \sigma \neq e} sg(\sigma) Q(\mu - s(\sigma)).$$

From (1) and (2) above, and from (5.3.1) we note that (5.4.1) provides an effective computation for Q.

It is especially interesting from the geometric point of view to write (5.4.1) in the form

$$\sum_{\sigma \in W} sg(\sigma) Q(\mu - s(\sigma)) = 0$$

for all $\mu \in I$, $\mu \neq 0$.

6. **The multiplicity formula.** 1. Let $\lambda \in I_D$. We now recall the formula of Weyl for the character of the representation π_λ. For any $x \in \mathfrak{h}$ Weyl's theorem (see [8] or [7, p. 19–07])

$$(6.1.1) \qquad \operatorname{tr} \exp \pi_\lambda(x) = \frac{\sum_{\omega \in I} F_{g+\lambda}(\omega) \exp(\omega, x)}{\sum_{\omega \in I} F_g(\omega) \exp(\omega, x)}$$

where the (known) functions $F_\nu \in \mathfrak{A}_s(I)$, $\nu \in I$, are given in §4.7. On the other hand in terms of the (unknown) function m_λ, obviously

$$(6.1.2) \qquad \operatorname{tr} \exp \pi_\lambda(x) = \sum_{\omega \in I} m_\lambda(\omega) \exp(\omega, x).$$

Setting (6.1.1) equal to (6.1.2) and clearing the denominator in (6.1.1) it follows that

$$F_{g+\lambda} = F_g * m_\lambda$$

where $*$ designates multiplication (convolution) in the group algebra $\mathfrak{A}(I)$. Thus recalling the definition of F_g and the usual expression for convolution

(6.1.3) $$F_{g+\lambda}(\omega) = \sum_{\sigma\in W} sg(\sigma)m_\lambda(\omega - \sigma(g)).$$

This immediate but none the less important consequence of Weyl's formula was first pointed out to us by Raoul Bott.

Now let $\nu=\omega-g$ and define

$$\begin{aligned} G(\nu) &= F_{g+\lambda}(\omega) \\ &= F_{g+\lambda}(\nu + g). \end{aligned}$$

Then $G\in\mathfrak{A}(I)$ has the following properties

(6.1.4) $$G(\nu) = \begin{cases} 0 \text{ for } \nu \neq \sigma(\lambda + g) - g, \\ sg(\sigma) \text{ for } \nu = \sigma(\lambda + g) - g. \end{cases}$$

Substituting $\nu+g$ for ω in the right side of (6.1.3), solving for the case when $\sigma=e$, and recalling that $g-\sigma g=s(\sigma)$, we get

(6.1.5) $$m_\lambda(\nu) = G(\nu) - \sum_{\sigma\in W;\sigma\neq e} sg(\sigma)m_\lambda(\nu + s(\sigma)).$$

Like (1.1.3) this sets up a recursive formula for $m_\lambda(\nu)$. On the other hand (6.1.5) has one significant advantage over (1.1.3)—it does not necessitate the division of a term which might vanish. Now as in §5.4 write $s(\sigma)=s_i$ where $-sg(\sigma)=(-1)^i$ and $\sigma\neq e$ implies $i\geqq 2$. Then applying (6.1.5) a second time we get

$$m_\lambda(\nu) = G(\nu) + \sum_{i_1=2}^{w}(-1)^{i_1}G(\nu + s_i) + \sum_{i_1,i_2=2}^{w}(-1)^{i_1+i_2}m_\lambda(\nu + s_{i_1} + s_{i_2}).$$

In fact repeating the substitution k times we derive the relation

$$\begin{aligned} m_\lambda(\nu) = \sum_{j=0}^{k-1}\ \sum_{i_1,i_2,\cdots,i_j=2}^{w} &(-1)^{i_1+i_2+\cdots+i_j}G(\nu + s_{i_1} + s_{i_2} + \cdots + s_{i_j}) \\ + \sum_{i_1,i_2,\cdots,i_k=2}^{w} &(-1)^{i_1+i_2+\cdots+i_k}m_\lambda(\nu + s_{i_1} + s_{i_2} + \cdots + s_{i_k}). \end{aligned}$$

But now by (5.3.1) there exists an integer M such that for all $k\geqq M$ and any sequence $\gamma=(i_1, i_2, \cdots, i_k)$, $w\geqq i_j\geqq 2$, $m_\lambda(\nu+s_{i_1}+s_{i_2}+\cdots+s_{i_k})=0$. The same is true for G. Thus we can let $k\to\infty$ and obtain from the definition of Q the relation

(6.1.6) $$m_\lambda(\nu) = \sum_{\omega\in I} Q(\omega - \nu)G(\omega).$$

Now recalling (6.1.4) we obtain

LEMMA 6.1. *Let* $\lambda \in I_D$, $\nu \in I$ *then*

$$m_\lambda(\nu) = \sum_{\sigma \in W} sg(\sigma) Q(\sigma(g + \lambda) - (g + \nu)).$$

6.2. We will now proceed to prove the second major point of this paper, namely $P = Q$.

LEMMA 6.2. *The functions* P *and* Q *on* I *are identical.*

Proof. Let $\mu \in I$ be arbitrary. By Lemma 5.3 there exists an integer N such that

$$P(\mu) = m_{k\sigma}(kg - \mu) \tag{6.2.1}$$

for all $k \geqq N$.

Now write

$$\mu = \sum_{i=1}^{\nu} b_i \alpha_i$$

and let $N_1 = \max_i b_i$. Then by (5.3.1) for all integers $k \geqq N_1$ and all $\sigma \in W$, $\sigma \neq e$

$$Q((k + 1)(\sigma(g) - g) + \mu) = 0. \tag{6.2.2}$$

This is clear since $(k+1)(\sigma(g) - g) + \mu = \mu - (k+1)s(\sigma)$ and if we expand $\mu - (k+1)s(\sigma)$ in terms of the α_i, at least one of the coefficients must be negative.

Let $N_2 = \max(N_1, N)$. Apply Lemma 6.1 where $\lambda = kg$, $\nu = kg - \mu$ and $k \geqq N_2$. Then

$$m_{k\sigma}(kg - \mu) = \sum_{\sigma \in W} sg(\sigma) Q(\sigma((k + 1)g) - ((k + 1)g - \mu)). \tag{6.2.3}$$

But

$$\sigma((k + 1)g) - ((k + 1)g - \mu) = (k + 1)(\sigma(g) - g) + \mu.$$

Hence by (6.2.2) all terms but one ($\sigma = e$) drop out of (6.2.3). That is,

$$m_{k\sigma}(kg - \mu) = Q(\mu).$$

But then by (6.2.1) $Q(\mu) = P(\mu)$. Q.E.D.

The following is our main theorem. Summarizing from above we have proved

THEOREM 6.2. *Let* $\mathfrak{g}$ *be a semi-simple Lie algebra with Cartan subalgebra* $\mathfrak{h}$. *Let* $I \subseteq \mathfrak{h}$ *be the discrete group of integral linear forms on* $\mathfrak{h}$ (*see* §2.2). *Let* P *be the function on* I,—*the partition function*— *which assigns to every* $\mu \in I$ *the number of ways* μ *can be partitioned into a sum of positive roots.* (*By a partition is meant multiplicities are permitted and the order is discounted. See* §3.3.)

Let Δ^+ be the set of positive roots and let

$$g = \frac{1}{2} \sum_{\phi \in \Delta^+} \phi.$$

Let W be the Weyl group and let

$$s(\sigma) = g - \sigma(g)$$

for any $\sigma \in W$. Then $s(\sigma)$ lies in the cone generated by the positive roots and for $\mu \in I$, $\mu \neq 0$, $P(\mu)$ satisfies the recursive relation

$$P(\mu) = - \sum_{\sigma \in W; \sigma \neq e} sg(\sigma) P(\mu - s(\sigma)). \tag{6.2.4}$$

[*In fact P can be defined by* (6.2.4) *when given that $P(0)=1$ and $P(\mu)=0$ for any μ having a negative coefficient when expanded in terms of the simple positive roots*(5).]

Let π be an irreducible representation of $\mathfrak{g}$. Write $\pi = \pi_\lambda$ where $\lambda \in I_D$ (see §2.3) *is the highest weight of π. Let $\nu \in I$ be arbitrary. Let $m_\lambda(\nu)=0$ in case ν is not a weight of π_λ and in case ν is a weight of π_λ let $m_\lambda(\nu)$ be the multiplicity of the weight ν. Then the number $m_\lambda(\nu)$ is given by the formula*

$$m_\lambda(\nu) = \sum_{\sigma \in W} sg(\sigma) P(\sigma(g + \lambda) - (g + \nu)). \tag{6.2.5}$$

REMARKS. In the author's opinion one of the most surprising aspects of the result above is the equality of P and Q. This equality, of course, enables us to establish formulas (6.2.4) and (6.2.5). [Note that when written in the symmetric form,

$$\sum_{\sigma \in W} P(\mu - s(\sigma)) = 0$$

for $\mu \neq 0$, the formula (6.2.4) becomes (6.2.5) for the case $\lambda = 0$. As far as multiplicities are concerned the formula then asserts the obvious fact that for the identity representation the only occurring weight is the zero weight.] But now the proof of the equality of P and Q rests heavily on representation theory. Nevertheless the statement that P equals Q has nothing whatsoever to do with representation theory. In fact it is a statement in combinatorial analysis—asserting that the number of partitions of an arbitrary element $\mu \in I$ where the parts are positive roots is equal to the number of ordered ways with signature plus one minus the number of ordered ways with signature minus one of writing μ with the elements $g - \sigma(g)$, $\sigma \in W$, $\sigma \neq e$. We can find no direct proof to establish this equality (even in the case $\mathfrak{g} = A_n$).

(5) This is clear since occurring among the $s(\sigma)$, $\sigma \in W$, are the simple positive roots themselves. That is, $s(R_{\alpha_i}) = \alpha_i$, $i = 1, 2, \cdots, l$, see §2.1.

Bibliography

1. R. Bott, *Homogeneous vector bundles*, Ann. of Math. vol. 66 (1957) pp. 203–255.

2. R. Brauer, *Sur la multiplication des caractéristiques des groupes continus et semi-simples*, C. R. Acad. Sci. Paris vol. 204 (1937) pp. 1784–1786.

3. E. B. Dynkin, *Maximal subgroups of the classical groups*, Amer. Math. Soc. Translations ser. 2 vol. 6 (1957) pp. 245–378.

4. H. Freudenthal, *Zur Berechrung der Charaktere der halbeinfachen Lieschen Gruppen*. I, Nederl. Akad. Wetensch. Indag. Math. vol. 57 (1954) pp. 369–376.

5. I. Gelfand and M. Neumark, *Über der Zusammenhang zwischen der Darstellungen einer komplexen halbeinfachen Lieschen Gruppe und ihrer maximalen kompakten Untergruppe*, Dokl. Akad. Nauk SSSR vol. 63 (1948) pp. 225–228.

6. Harish-Chandra, *On representations of Lie algebras*, Ann. of Math. vol. 50 (1949) pp. 900–915.

7. *Séminaire "Sophus Lie,"* Ecole Normale Supérieure, Paris, 1955.

8. H. Weyl, *Über die Darstellungen der halbeinfachen Gruppen durch lineare Transformationen*, Math. Z. vol. 24 (1926) pp. 328–395.

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