ON SOME CLASSES OF NONCONTINUABLE ANALYTIC FUNCTIONS

BY

F. W. CARROLL[1]

1. **Introduction.** This paper is concerned with analytic functions of the type

$$F(z) = \sum_{n=0}^{\infty} |a_n| e^{2\pi i f(n)} z^n.$$

Here, $f(n)$ is a real valued function, $\{|a_n|\}$ is a bounded sequence, such that $\inf_k \sum_{k+1}^{k+N} |a_n| \to \infty$ if $N \to \infty$. Using a lemma due to Riesz and certain results on uniform distributions (mod 1), simple conditions are obtained sufficient in order that $F(z)$ have $|z| = 1$ as its natural boundary. Some of the more applicable such conditions are (cf. §4): (i) From some positive integer r, $\Delta^r f(n) \to \infty$ in a monotone fashion, $\Delta^{r+1} f(n) \to 0$; (ii) $|a_n| = 1$ and $f(n)$ is a finite sum of terms $Cn^\alpha (\log n)^\beta (\log \log n)^\gamma$, at least one of these terms being of higher order than n and not of the form Cn^α with C rational and α integral; (iii) $|a_n| = 1$ and $f(n) = An^\lambda \sin Bn^\alpha$, where $A \neq 0$, $B \neq 0$, $0 < \alpha < 1$, $\alpha + \lambda > 1$.

As is well-known, such classes of noncontinuable functions can be enlarged considerably by using Hadamard's multiplication theorem. For, let $b_n \neq 0$ $(n = 0, 1, \cdots)$ be such that $L(z) = \sum_0^\infty b_n z^n$ has a radius of convergence 1 with only one singularity z_1 on its circle of convergence. Then, assuming that $F_1(z) = \sum_0^\infty b_n^{-1} |a_n| e^{2\pi i f(n)} z^n$ has also a radius of convergence 1, each singularity z_0 of $F(z)$ with $|z_0| = 1$ is equal to $z_1 z_2$, z_2 denoting a singularity of $F_1(z)$ with $|z_2| = 1$. Consequently, if $F(z)$ has $|z| = 1$ as its natural boundary then so has $F_1(z)$. In §5, cf. Theorem 6, this principle is applied in showing that certain generalized hypergeometric series represent a function analytic in a circle $|z| < R$, and having $|z| = R$ as a natural boundary.

2. **Principal results.** In this paper, $\{a_0, a_1, a_2, \cdots\}$ denotes a *bounded* sequence of complex numbers satisfying

$$\limsup |a_n|^{1/n} = 1. \tag{2.1}$$

Further, $f(n)$ $(n = 0, 1, 2, \cdots)$ denotes a real valued function which, when $a_n \neq 0$, is (mod 1) uniquely determined by

---

Presented to the Society, April 18, 1958, under the title *On a class of noncontinuable analytic functions*; received by the editors December 16, 1958.

[1] This paper, which was prepared while the author was an International Business Machine Fellow at Purdue University, constitutes a portion of a doctoral dissertation written under the direction of Professor J. H. B. Kemperman.

$$a_n = |a_n| e^{2\pi i f(n)}. \tag{2.2}$$

We shall derive a number of sufficient conditions on $|a_n|$ and $f(n)$ in order that

$$F(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} |a_n| e^{2\pi i f(n)} z^n \tag{2.3}$$

(which is analytic for $|z| < 1$) shall have $|z| = 1$ as its natural boundary. The proofs are all based on a result due to Riesz [8], stating that (for a_n bounded) the partial sums of (2.3) are uniformly bounded on each arc $z = e^{2\pi i\phi}$, $\alpha \leqq \phi \leqq \beta$, on which $F(z)$ is regular. Consequently,

Lemma 2.1. *Let $\alpha \leqq \beta < \alpha + 1$. If to each constant $M > 0$ there corresponds a positive integer N and a real number $\phi \in [\alpha, \beta]$, such that*

$$\limsup_{k\to\infty} \left| \sum_{n=k+1}^{k+N} |a_n| e^{2\pi i(f(n)+n\phi)} \right| \geqq M, \tag{2.4}$$

then $F(z)$ has at least one singularity on the arc $z = e^{2\pi i\phi}$, $\alpha \leqq \phi \leqq \beta$.

If $g(n)$ is defined for $n = 0, 1, \cdots$, we denote

$$g(n+1) - g(n) = \Delta g(n), \qquad \Delta^{j+1} g(n) = \Delta(\Delta^j g(n)).$$

A first application of Lemma 2.1 is:

Theorem 1. *Let ψ be a given real number. Suppose that to each constant $M > 0$ there corresponds a positive integer N and an increasing sequence $\{k_\nu\}$ of positive integers such that*

$$\sum_{n=k_\nu+1}^{k_\nu+N} |a_n| \geqq M \qquad (\nu = 1, 2, \cdots), \tag{2.5}$$

and

$$\begin{cases} \lim_{\nu\to\infty} (\Delta f(k_\nu) + \psi) = 0 \pmod 1, \\ \lim_{\nu\to\infty} \Delta^j f(k_\nu) = 0 \pmod 1, \qquad (j = 2, 3, \cdots, N). \end{cases} \tag{2.6}$$

Then $F(z)$ is singular at $z_0 = e^{2\pi i\psi}$.

Proof. From Lemma 2.1, it suffices to prove that

$$\limsup_{\nu\to\infty} \left| \sum_{n=k_\nu+1}^{k_\nu+N} |a_n| e^{2\pi i[g(n)-g(k_\nu)]} \right| \geqq M, \tag{2.7}$$

where $g(n) = f(n) + n\psi$. But

$$g(k_\nu + m) - g(k_\nu) = m(\Delta f(k_\nu) + \psi) + \sum_{j=2}^{m} \binom{m}{j} \Delta^j f(k_\nu),$$

thus, from (2.6), ($\binom{m}{j}$ being an integer),

$$\lim_{\nu \to \infty} (g(k_\nu + m) - g(k_\nu)) = 0 \pmod 1, \quad (m = 1, \cdots, N).$$

Remembering that $\{a_n\}$ is bounded, (2.5) now implies (2.7).

Corollary. *The function*

$$F(z) = \sum_{n=0}^{\infty} | a_n | e^{2\pi i f(n)} z^n,$$

where

$$\lim_N \inf_k \sum_{k+1}^{k+N} | a_n | = + \infty,$$

has $|z| = 1$ *as its natural boundary if* $\lim_{n\to\infty} \Delta^2 f(n) = 0$ *and the sequence* $\{\Delta f(n)\}$ *is dense* (mod 1).

The same is true, therefore, if

$$(2.8) \qquad \lim_{n\to\infty} \Delta f(n) = \infty, \qquad \lim_{n\to\infty} \Delta^2 f(n) = 0.$$

Here, (2.8) is satisfied for $f(n) = Cn^\alpha (\log n)^\beta$ with $1 < \alpha < 2$ or $\alpha = 1, \beta > 0$, or $\alpha = 2, \beta < 0, (C > 0)$. (2.8) is also implied by

$$(2.9) \qquad f(n) = n\phi(n), \qquad \lim_{n\to\infty} \phi(n) = \infty, \qquad \lim_{n\to\infty} n\Delta\phi(n) = 0.$$

It was shown by Fabry [3], that if (2.9) holds then (2.1) alone implies that $F(z)$ has $|z| = 1$ as its natural boundary. On the other hand, for instance, $f(n) = n \log n$ does not satisfy (2.9).

In §3, less trivial applications of Theorem 1 will be given. It will be shown, for instance, that (2.6) can be satisfied for each number ψ if $f(n) = Cn^\alpha$, $C \neq 0$, $\alpha > 1$, $\alpha \not\equiv 0$ (mod 1). Unfortunately the simple-looking case

$$(2.10) \qquad \Phi(z) = \sum_{n=0}^{\infty} e^{2\pi i \theta n^2} z^n, \qquad (\theta \text{ irrational}),$$

cannot be handled by use of Theorem 1 (since $\Delta^2(\theta n^2) = 2\theta \neq 0$), though Cooper [1] already proved that $\Phi(z)$ has $|z| = 1$ as its natural boundary(2).

(2) A much simpler proof than Cooper's can be given. Writing in (2.10) $n = k + m$, the identity $\Phi(z) = P_{k-1}(z) + C_k z^k \Phi(z e^{4\pi i k\theta})$ is obtained; here, $P_{k-1}(z)$ denotes a polynomial, $C_k \neq 0$ a constant. Since $\Phi(z)$ has at least one singularity z_0 on $|z| = 1$, and since the set of points $\{z_0 e^{4\pi i k\theta}\}$ is dense on $|z| = 1$ Cooper's theorem follows.

In order to handle functions of the type $f(n) = \theta n^k$, θ irrational, $k=2, 3, 4, \cdots$, we introduce:

THEOREM 2. *Let $f(n)$ be a real valued function such that to each positive integer N there corresponds a sequence $\{k_\nu\}$ of positive integers such that*

$$\Delta f(k_\nu) \text{ is dense } (\text{mod } 1),$$

$$\lim_{\nu\to\infty} \Delta^j f(k_\nu) = c_j \ (\text{mod } 1) \text{ exists}, \qquad j = 2, 3, \cdots, N. \tag{2.11}$$

Then

$$G(z) = \sum_{n=1}^{\infty} e^{2\pi i f(n)} z^n \tag{2.12}$$

has $|z| = 1$ as its natural boundary.

Proof. Let $\alpha < \beta \leqq \alpha + 1$ and

$$S_N(k, \phi) = \sum_{m=k}^{k+N} \exp\{2\pi i(f(n) + n\phi)\}. \tag{2.13}$$

Let N be a large positive integer. From

$$f(k+m) = f(k) + \sum_{j=1}^{m} \binom{m}{j} \Delta^j f(k)$$

and (2.11),

$$|S_N(k_\nu, \phi)| = \left| \sum_{m=1}^{N} \exp\left\{2\pi i\left[m(\Delta f(k_\nu) + \phi) + \sum_{j=2}^{m} \binom{m}{j} c_j\right]\right\}\right| + \theta, \tag{2.14}$$

where $|\theta| < 1$ for $\nu > \nu_0(N)$, say. Let μ be fixed, $\mu > \nu_0(N)$. From (2.13),

$$\int_0^1 |S_N(k_\mu, \phi)|^2 d\phi = N,$$

hence, there exists a number ϕ_μ such that $|S_N(k_\mu, \phi_\mu)| \geqq N^{1/2}$. Because $\Delta f(k_\nu)$ is dense (mod 1) there exists an index $\nu > \nu_0(N)$ such that $\psi = -\Delta f(k_\nu) + \Delta f(k_\mu) + \phi_\mu$ is contained in (α, β). From (2.14)

$$|S_N(k_\nu, \psi)| \geqq |S_N(k_\mu, \phi_\mu)| - 2 \geqq N^{1/2} - 2.$$

Applying Lemma 2.1, we obtain the stated assertion.

In the next section, using certain results on uniform distributions (mod 1), auxiliary theorems are presented making it possible to obtain nontrivial applications of the Theorems 1 and 2, cf. §4.

3. **Auxiliary results.** In this section $\{(a_\mu, b_\mu)\}$ will always denote a sequence of intervals (a_μ, b_μ), $\mu = 1, 2, \cdots$, where the a_μ, b_μ are non-negative

numbers, and $b_\mu - a_\mu \to \infty$. The system of real-valued functions $g_i(n)$ ($i=0, 1, \cdots, r-1$), defined for $n=0, 1, \cdots$, is said to be *uniformly distributed* (mod 1), on $\{(a_\mu, b_\mu)\}$ if for each choice of the real numbers $\gamma_0, \cdots, \gamma_{r-1}$, $0 \leqq \gamma_i \leqq 1$,

$$\lim_{\mu\to\infty} N_\mu/(b_\mu - a_\mu) = \gamma_0 \cdots \gamma_{r-1},$$

where N_μ denotes the number of integers n in (a_μ, b_μ) satisfying

$$0 \leqq g_i(n) \leqq \gamma_i \pmod 1, \qquad (i = 0, \cdots, r-1).$$

It is known [2] that this is the case *if and only if* each linear combination $\sum h_i g_i(n)$ (with integer coefficients h_i not all zero) is uniformly distributed (mod 1) on the sequence $\{(a_\mu, b_\mu)\}$.

Consider the following two conditions, where $g(n)$ denotes a function defined for $n=0, 1, \cdots$ and $\{(a_\mu, b_\mu)\}$ is a sequence of intervals.

(A) In each of the intervals (a_μ, b_μ), $g(n)$ is monotone and of one sign, while

$$\lim_{\mu\to\infty} \max(|g(a_\mu)|, |g(b_\mu)|) = 0,$$

$$\lim_{\mu\to\infty} (b_\mu - a_\mu) \min(|g(a_\mu)|, |g(b_\mu)|) = \infty.$$

(B) $\lim_{n\to\infty} g(n) = \theta$, θ irrational, where n is restricted to the union of the intervals (a_μ, b_μ).

LEMMA 3.1. *Let r be a positive integer, and let $f(n)$ be a real-valued function defined for $n=0, 1, \cdots$. Suppose that, for some $\{(a_\mu, b_\mu)\}$, $\Delta^r f(n)$ satisfies either condition* (A) *or condition* (B). *Then for each positive integer q the system $\Delta^i f(qn)$ $(i=0, 1, \cdots, r-1)$ is uniformly distributed* (mod 1) *on the sequence $\{(a_\mu/q, b_\mu/q)\}$.*

Proof. As was shown by van der Corput [2], a function $h(n)$ is uniformly distributed (mod 1) on $\{(c_\mu, d_\mu)\}$ as soon as $g(n) = \Delta h(n)$ satisfies either (A) or (B) on $\{(c_\mu, d_\mu)\}$. Now $h(n) = \Delta^{r-1} f(qn)$ is such that $h(n+1) - h(n) = \sum_{j=0}^{q-1} \Delta^r f(qn+j)$ satisfies (A) or (B) on $\{(a_\mu/q, b_\mu/q - 1)\}$, hence, $\Delta^{r-1} f(qn)$ is uniformly distributed on $\{(a_\mu/q, b_\mu/q)\}$.

Let j be a fixed integer, $1 \leqq j \leqq r-1$ and assume that the system $\Delta^{r-i} f(qn)$ ($i=1, 2, \cdots, j$) is uniformly distributed (mod 1) on $\{(a_\mu/q, b_\mu/q)\}$, (a correct assumption if $j=1$); it suffices to show that the system $\Delta^{r-i} f(qn)$ ($i=1, 2, \cdots, j+1$) is likewise. Let $h_1, \cdots, h_{j+1}$ be integers not all zero. It must be shown that

$$H(n) = \sum_{i=1}^{j+1} h_i \Delta^{r-i} f(qn)$$

is uniformly distributed (mod 1) on $\{(a_\mu/q, b_\mu/q)\}$. From the induction assumption, we may assume $h_{j+1}\neq 0$. From a result due to van der Corput [2], it suffices to show that, for each integer $\nu\geqq 1$, $H_1(n)=H(n+\nu)-H(n)$ is uniformly distributed (mod 1) on $\{(a_\mu/q,\ b_\mu/q)\}$. Now

$$\begin{aligned} H_1(n) &= \sum_{i=1}^{j+1} h_i[\Delta^{r-i}f(qn+q\nu)-\Delta^{r-i}f(qn)] \\ &= q\nu h_{j+1}\Delta^{r-j}f(qn)+\sum_{i=-q\nu+1}^{j-1} c_i\Delta^{r-i}f(qn), \end{aligned}$$

the c_i's denoting fixed integers. Since $\Delta^r f(n)$ satisfies (A) or (B) on $\{(a_\mu,\ b_\mu)\}$, $\Delta^{r-i}f(n)$ converges to a constant (mod 1) for each $i\leqq 0$, at least when n is restricted to the interval $(a_\mu/q,\ b_\mu/q-\nu)$. Thus $H_1(n)$ is uniformly distributed (mod 1) on $\{(a_\mu/q,\ b_\mu/q)\}$, provided that

$$q\nu h_{j+1}\Delta^{r-j}f(qn)+\sum_{i=1}^{j-1} c_i\Delta^{r-i}f(qn)$$

is. But this is certainly the case, since by the induction assumption the system $\Delta^{r-i}f(qn)$ $(i=1,\cdots,j)$ is uniformly distributed (mod 1) on $\{(a_\mu/q,\ b_\mu/q)\}$.

THEOREM 1′. *Besides the usual conditions on $\{a_n\}$ suppose that*

$$\liminf_{N\quad k}\sum_{n=k+1}^{k+N}|a_n| = \infty \qquad (k=0,1,\cdots). \tag{3.1}$$

Then $F(z)$, as defined by (2.3), *has $|z|=1$ as its natural boundary as soon as, for some integer $r\geqq 2$ and sequence of intervals $\{(a_\mu,\ b_\mu)\}$, the difference $\Delta^r f(n)$ satisfies condition* (A) *on $\{(a_\mu,\ b_\mu)\}$.*

Proof. In view of Theorem 1 and Condition (3.1), it suffices to prove, for each N, the existence of a sequence $\{k_\nu\}$ such that $\Delta f(k_\nu)$ is dense (mod 1) and $\Delta^j f(k_\nu)\to 0$ (mod 1), $(j=2,\cdots,N)$. By hypothesis $\Delta^r f(n)\to 0$ (mod 1) on $\{(a_\mu,\ b_\mu)\}$; it follows that $\Delta^j f(n)\to 0$ (mod 1) on $\{(a_\mu,\ b_\mu-N)\}$, $(j=r,\ r+1,\cdots,N)$. Also, $\Delta^r f(n)$ satisfies condition (A) on $\{(a_\mu,\ b_\mu-N)\}$, so that by Lemma 3.1, the points $\{\Delta f(n),\ \Delta^2 f(n),\cdots,\Delta^{r-1}f(n)\}$ for $n\in(a_\mu,\ b_\mu-N)$ are in an obvious sense uniformly distributed (mod 1) in the $(r-1)$-dimensional unit cube. Thus there exists a sequence $\{k_\nu\}$ of integers chosen from the $\{(a_\mu,\ b_\mu-N)\}$ such that $\Delta f(k_\nu)$ is dense (mod 1) and $\Delta^j f(k_\nu)\to 0$ (mod 1) $(j=2,3,\cdots,r-1)$.

THEOREM 2′. *Let $f(n)=f_1(n)+f_2(n)$, where,* (mod 1), *$f_1(n)$ is periodic: $f_1(n+q)=f(n)$* (mod 1), *while for some integer $r\geqq 2$ and some sequence $\{(a_\mu,\ b_\mu)\}$, $\Delta f_2(n)$ satisfies condition* (B) *on $\{(a_\mu,\ b_\mu)\}$. Then $G(z)$, as defined by* (2.12), *has $|z|=1$ as its natural boundary.*

Proof. From Lemma 3.1, $\Delta^{r-i}f_2(qn)$, $(i=1,\cdots,r-1)$, is uniformly distributed (mod 1) on $\{(a_\mu/q,\ b_\mu/q)\}$, while each difference $\Delta^j f_1(qn)$ is constant.

Hence, as in the previous proof, there exists a sequence of integer multiples $k_\nu = qk'_\nu$ satisfying (2.11), and Theorem 2 implies the stated assertion.

LEMMA 3.2. *A function* $g(n)$ *satisfies condition* (A) *for at least one sequence* $\{(a_\mu, b_\mu)\}$ *if there exists a sequence* $\{(c_\mu, d_\mu)\}$ *such that* (i) $\lim_{\mu\to\infty} c_\mu = \infty$; (ii) $g(n)$ *is monotone and of one sign in each* (c_μ, d_μ) *and* $|g(d_\mu)-g(c_\mu)| \geqq 1$; (iii) $\lim_{n\to\infty} \Delta g(n) = 0$, $n \in (c_\mu, d_\mu)$.

Proof. Replacing $\{(c_\mu, d_\mu)\}$ by a proper subsequence, it may be assumed that for n in the μth interval (c_μ, d_μ), $|\Delta g(n)| < 1/\mu^3$. From $|g(d_\mu)-g(c_\mu)| \geqq 1$, there exists at least one interval (a_μ, b_μ), $c_\mu < a_\mu < b_\mu < d_\mu$, a_μ, b_μ integers such that, for $k \in (a_\mu, b_\mu)$, $g(k)$ satisfies the relation

$$1/\mu \leqq |g(k)| \leqq 2/\mu \pmod 1 \tag{3.2}$$

but $g(a_\mu - 1)$ and $g(b_\mu + 1)$ do not. Then $\max(|g(a_\mu)|, |g(b_\mu)|) \leqq 2/\mu \to 0$ as $\mu \to \infty$. Also

$$b_\mu - a_\mu \geqq \frac{|g(b_\mu) - g(a_\mu)|}{1/\mu^3} \geqq \frac{1/\mu - 2/\mu^3}{1/\mu^3} = \mu^2 - 2$$

so that $\lim_{\mu\to\infty} (b_\mu - a_\mu) \min(|g(a_\mu)|, |g(b_\mu)|) = \infty$, as required.

LEMMA 3.3. *A function* $g(n)$ *satisfies condition* (B) *for at least one sequence* $\{(a_\mu, b_\mu)\}$ *if there exists a sequence* $\{(c_\mu, d_\mu)\}$ $(d_\mu - c_\mu \to \infty)$, *such that, for* $n \to \infty$,

(i) $g(n)$ *does not converge* (mod 1) *to a rational number*, $n \in (c_\mu, d_\mu)$,

(ii) $\Delta g(n) \to 0$ (mod 1), $n \in (c_\mu, d_\mu)$.

Proof. From the hypothesis, $g(n)$, $n \in (c_\mu, d_\mu)$ has some irrational number α as a point of accumulation (mod 1). Thus, taking a subsequence of $\{(c_\mu, d_\mu)\}$ if necessary, there can be found an interval (a_μ, b_μ) in each (c_μ, d_μ) in such a manner that $b_\mu - a_\mu \to \infty$ and $|g(n) - \alpha| < 1/\mu$ (mod 1) for $n \in (a_\mu, b_\mu)$.

4. Applications. Let $a_n = |a_n| e^{2\pi i f(n)}$, $(n = 0, 1, \cdots)$, denote a bounded sequence of complex numbers satisfying

$$\lim_N \inf_k \sum_{n=k+1}^{k+N} |a_n| = \infty,$$

thus

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

is absolutely convergent if and only if $|z| < 1$. We have

THEOREM 3. *If for some integer* $r \geqq 1$, $\Delta^r f(n) \to \infty$, (*ultimately*) *in a monotone fashion, and* $\Delta^{r+1} f(n) \to 0$, *then* $F(z)$ *has* $|z| = 1$ *as its natural boundary.*

Proof. If $r = 1$ the stated result is contained in the corollary to Theorem 1. If $r \geqq 2$ it follows by combining Theorem 1′ and Lemma 3.2.

Suppose that $f(n)$ is a real generalized polynomial, that is, $f(n)$ can be represented as a finite sum of the form

$$f(n) = \sum_{i=0}^{m} C_i n^{\alpha_i},$$

the C_i and α_i denoting real constants, $C_i \neq 0$, $\alpha_i > \alpha_{i+1}$. If the leading term $C_0 n^{\alpha_0}$ has a *noninteger* exponent $\alpha_0 > 1$ then $f(n)$ satisfies the conditions of Theorem 3 (with $r = [\alpha_0]$). The same is true for the more general finite sums of the form

$$f(n) = \sum C_i n^{\alpha_i} (\log n)^{\beta_i} (\log \log n)^{\gamma_i}, \tag{4.1}$$

provided that $f(n)/n \to \infty$ and the *leading* term in $f(n)$ is not of the form $C_i n^{\alpha_i}$, α_i an integer $\geqq 2$; (actually, $f(0)$ and $f(1)$ are undefined). In dealing with the latter exceptional case, we shall assume $|a_n| = 1$; we have:

Theorem 4. *Suppose that the real function $f(n)$ can be represented as a finite sum of the form* (4.1). *Then*

$$G(z) = \sum_{n=0}^{\infty} e^{2\pi i f(n)} z^n$$

admits an analytic continuation across some arc of $|z| = 1$ if and only if $f(n)$ is of the form $f(n) = f_1(n) + f_2(n)$, where $f_1(n) = O(n)$, while $f_2(n)$ is a polynomial with integer exponents and rational coefficients.

Proof. 1. *Necessity.* Let $f_2(n)$ denote the sum of all those terms in (4.1) which are of the form Cn^α where α denotes an integer $\geqq 2$ and C a rational number. Then $f_2(n)$ is periodic (mod 1), in fact, the least common multiple q of the denominator of its coefficients is a period (mod 1). Let $f_1(n) = f(n) - f_2(n)$ and suppose that $f_1(n)$ is of larger order than n, thus, the leading term Cn^α $(\log n)^\beta (\log \log n)^\gamma$ $(C \neq 0)$ in $f_1(n)$ is such that $\alpha > 1$ or $\alpha = 1$, $\beta > 0$, or $\alpha = 1$, $\beta = 0$, $\gamma > 0$. Moreover, if $\beta = \gamma = 0$ then either α is not an integer, or α is an integer and C is irrational. Let $r = [\alpha]$ unless α is an integer $\geqq 2$ and $\beta < 0$ or $\beta = 0$, $\gamma < 0$, in which case we put $r = \alpha - 1$; anyway, $r \geqq 1$. Moreover, $\Delta^r f_1 \to \infty$ unless the leading term in $f_1(n)$ is of the form Cn^r $(r \geqq 2)$ in which case $\Delta^r f_1$ converges to the irrational number $r!C$. Finally, $\Delta^{r+1} f_1 \to 0$. If $r \geqq 2$ then Lemma 3.3 and Theorem 2′ imply that $G(z)$ has $|z| = 1$ as its natural boundary. If $r = 1$ then $\Delta f_1 \to \infty$ and $\Delta^2 f_1 \to 0$, hence, $\Delta f_1(qn)$ is dense (mod 1). But $\Delta f_2(qn)$ is constant (mod 1), thus, $\Delta f(qn)$ is dense (mod 1). From Theorem 2, also in this case, $|z| = 1$ is a natural boundary for $G(z)$.

2. *Sufficiency.* Suppose that $f(n) = f_1(n) + f_2(n)$, where $f_2(n+q) = f_2(n)$ while $f_1(n) = Cn + o(n)$ (C a constant). Here, $f_1(n)$ is defined by a formula of the type (4.1), enabling us to extend $f_1(n)$ to a single-valued and analytic function $Cw + \phi(w)$ in some right half plane, where $\phi(w) = o(w)$. From a theorem due to Le Roy (cf. Lindelöf, [7, p. 109]),

$$H_p(u) = \sum_{m=0}^{\infty} e^{2\pi i\phi(p+mq)}u^m$$

has $u=1$ as its only singularity on its circle of convergence $|u|=1$. Moreover,

$$G(z) = \sum_{p=0}^{q-1}\sum_{m=0}^{\infty} e^{2\pi i f(p+mq)}z^{p+mq}$$

$$= \sum_{p=0}^{q-1} e^{2\pi i f_2(p)}(ze^{2\pi iC})^p H_p((ze^{2\pi iC})^q),$$

showing that only the finitely many points $e^{2\pi i(-C+s/q)}$, $(s=0, 1, \cdots, q-1)$ can be singular points of $G(z)$ on $|z|=1$. Also functions of the type $f(n) = n^{\lambda}\sin n^{1/2}$ $(\lambda>1/2)$ give rise to noncontinuable analytic functions as is shown by:

THEOREM 5. *Let r denote a fixed positive integer and let $H(u)$ $(-\infty<u<\infty)$, $\phi(x)$ and $\psi(x)$ $(x>0)$ be real valued functions admitting $r+1$ continuous derivatives such that $H(u)$ is a nonconstant periodic function. Suppose further that $\phi(x)\to\infty$, $\psi(x)\to\infty$ when $x\to+\infty$,*

$$\lim_{x\to+\infty}(\phi')^r\psi = C, \qquad \lim_{x\to\infty}(\phi')^{r+1}\psi = 0, \tag{4.2}$$

where C denotes a positive constant, allowing $C=+\infty$, ($\phi'=\phi^{(1)}$, $g^{(k)}$ denoting the kth derivative of g). Only if $r=1$ and C is finite, assume that $CH^{(1)}(x)$ has a range of length $\geqq 1$. Finally, suppose that for $x\to\infty$,

$$\phi^{(j)} = o((\phi')^j),\ \psi^{(k)} = o((\phi')^k\psi)\ (j = 2, \cdots, r+1, k = 1, \cdots, r+1). \tag{4.3}$$

ASSERTION. *If, for n sufficiently large, $f(n)=\psi(n)H(\phi(n))$ then*

$$G(z) = \sum_{n=0}^{\infty} e^{2\pi i f(n)}z^n$$

has $|z|=1$ as its natural boundary.

EXAMPLE. Let $\phi(x) = Ax^{\alpha}(\log x)^{\beta}$, $\psi(x) = Bx^{\lambda}(\log x)^{\mu}$, $((A\neq 0,\ B\neq 0)$, $H(u)$ any nonconstant infinitely differentiable periodic function, e.g., $H(u) = \sin u)$, thus

$$f(n) = Bn^{\lambda}(\log n)^{\mu}H(An^{\alpha}(\log n)^{\beta})].$$

The above condition holds for a suitable choice of the positive integer r provided that: (i) $0\leqq\alpha\leqq 1$ and $\alpha+\lambda\geqq 1$; (ii) if $0<\alpha<1$ and $\alpha+\lambda=1$ then $\mu+\beta\geqq 0$; (iii) if $\alpha=1$ then $\lambda=0$, $\mu\geqq-\beta>0$; (iv) if $\alpha=0$ then $\beta>1$; if $\alpha=0$, $\lambda=1$ then $\mu+(\beta-1)\geqq 0$; (v) in the following cases $CH'(x)$ has a range $\geqq 1$, (C as specified); (v)$'$ $0<\alpha\leqq 1$, $\alpha+\lambda=1$, $\mu+\beta=0$; then $C=AB\alpha$; (v)$''$ $\alpha=0$, $\lambda=1$, $\mu+\beta=1$; then $C=AB\beta$.

Proof of Theorem 5. Let $f(x)=\psi(x)H(\phi(x))$. Then, for $x\to\infty$,

$$f^{(k)}(x) = \psi(x)\phi'(x)^k\{H^{(k)}(\phi) + o(1)\}, \tag{4.4}$$

$(k=1, \cdots, r+1)$, for, $f^{(k)}$ is the sum of the function $\psi(\phi')^kH^{(k)}(\phi)$ and, further, a finite number of terms of the form

$$a\psi^{(i_0)}(\phi')^{i_1}(\phi'')^{i_2}\cdots(\phi^{(k)})^{i_k}H^{(i_1+\cdots+i_k)}(\phi).$$

Here, a is a constant and the i_j denote non-negative integers, such that $i_0+i_1+2i_2+\cdots+ki_k=k$ and $i_1<k$, (thus, $i_0\geqq 1$ or $i_2\geqq 1, \cdots$, or $i_k\geqq 1$). Observing that $H^{(\nu)}(u)$ is bounded, $(\nu\leqq r+1)$, (4.3) yields (4.4).

From (4.4) and (4.2),

$$\Delta^{r+1}f(n) = f^{(r+1)}(n+\theta)\to 0 \text{ if } n\to\infty, \qquad (0<\theta<r+1). \tag{4.5}$$

Thus,

$$\Delta^r f(n) = f^{(r)}(n+\theta') = f^{(r)}(n) + o(1), \tag{4.6}$$

$(0<\theta'<r)$. Further, let L denote the range of the nonconstant periodic function $H^{(r)}(u)$, $L>0$. From $\phi(x)\to\infty$, $\phi'(x)\to 0$, (cf. (4.2)), it follows that

$$\limsup H^{(r)}(\phi(n)) - \liminf H^{(r)}(\phi(n)) = L.$$

Hence, from (4.6), (4.4) and (4.2),

$$\limsup \Delta^r f(n) - \liminf \Delta^r f(n) = CL > 0, \tag{4.7}$$

showing that, for $n\to\infty$, $\Delta^r f(n)$ does not converge. In view of (4.5), applying Lemma 3.3 and Theorem 2′, the stated assertion follows if $r\geqq 2$. On the other hand, if $r=1$ then $1\leqq CL\leqq\infty$, hence, from (4.5) and (4.7), $\Delta^2 f(n)\to 0$ while $\Delta f(n)$ is dense (mod 1). Now, the assertion follows from the corollary to Theorem 1.

Many more applications of the results in §3 could be obtained. We prefer to indicate instead another class of applications of the fundamental Theorem 1. It implies that $F(z)$ has $|z|=1$ as its natural boundary if, for *each* positive integer N, there exists a sequence $\{(a_\mu, b_\mu)\}$ on which the system $\Delta^j f(n)$, $(j=1, \cdots, N)$ is uniformly distributed (mod 1). This in turn holds true if, for each N, the system $f(n+j)$, $(j=0, 1, \cdots, N)$ is uniformly distributed (mod 1) on $\{(0, \mu), \mu=1, 2, \cdots\}$.

If $g(n)$ is a fixed function and $f(n)=\xi g(n)$, the latter is true for almost all values ξ if $\Delta g(n)\geqq 1$ and

$$\lim_{n\to\infty} \Delta g(n+1)/\Delta g(n) = \rho,$$

exists, where either ρ is a transcendental number or $\rho=0$ or $\rho=\infty$, cf. L. E. Grosh [4, p. 85]. For instance, if $\theta>1$ is a fixed transcendental number then, for almost all ξ, the function

$$F(z) = \sum_{n=0}^{\infty} a_n e^{i\xi\theta^n} z^n$$

has $|z| = 1$ as its natural boundary, (the a_n satisfying the conditions mentioned at the beginning of this section).

5. **Application to generalized hypergeometric series.** This last application of Theorem 1 is a somewhat surprising one. It is shown that under certain conditions the function given for $|z|$ small by a generalized hypergeometric series is noncontinuable beyond its circle of convergence. Let

$$h(w) = \prod_{k=1}^{m} \{\Gamma(c_k w + d_k)\}^{\delta_k}, \tag{5.1}$$

where $\delta_k = \pm 1$, c_k and d_k are complex constants with $\operatorname{Re}(c_k) \geqq 0$. Consider the generalized hypergeometric series,

$$H(z) = \sum{}' h(n) z^n, \tag{5.2}$$

where the summation is taken over all positive integers n except those at which $h(w)$ has a pole. Let

$$K = -\sum_{k=1}^{m} \delta_k c_k. \tag{5.3}$$

From a suitable extension of Stirling's formula for $\Gamma(w)$, (to the entire region $\{w = u + iv : u \geqq 1 \text{ or } |v| \geqq 1\}$ by means of the identity $\Gamma(w)\Gamma(1-w) = \pi/\sin \pi w$), it can be seen that for $\operatorname{Re}(w) \geqq u_0$, u_0 sufficiently large,

$$h(w) = C(w) e^{\sigma w} w^{\tau} w^{-Kw} (1 + o(1)) \tag{5.4}$$

where $|\arg w| < \pi/2$,

$$\sigma = \sum_{k=1}^{m} c_k \delta_k (\log c_k - 1),$$

$$\tau = \sum_{k=1}^{m} \delta_k (d_k - 1/2),$$

($|\arg c_k| \leqq \pi/2$), and $C(w)$ is bounded and bounded away from zero.

From (5.4), if $\operatorname{Re}(K) > 0$, then $H(z)$ is an entire function; if $\operatorname{Re}(K) < 0$, the series converges only for $z = 0$; if $\operatorname{Re}(K) = 0$, the series has radius of convergence $|e^{-\sigma}|$. We shall be interested only in the last case.

THEOREM 6. *If $K = 0$, then $H(z)$ has only one singularity on its circle of convergence; moreover, it can be continued to a single-valued analytic function in the complement of the half-ray $ue^{-\sigma}$, $(1 \leqq u \leqq \infty)$. If $K = i|K| \neq 0$, then $H(z)$ has its circle of convergence as a natural boundary.*

Proof. If $K=0$, then, for $\operatorname{Re}(w) \geqq u_0$ (u_0 sufficiently large), the function $e^{-\sigma w}h(w)$ is analytic and satisfies

$$e^{-\sigma w}h(w) = O(w^{\tau}).$$

Thus, by a theorem of Le Roy, cf. Lindelöf [7, p. 109], the function given for $|z| < 1$ by

$$L(z) = \sum_{n=0}^{\infty}{}' e^{-\sigma n}h(n)z^n = H(ze^{-\sigma})$$

is single-valued and analytic in the complement of the half-ray $[1, \infty]$ on the positive real axis, proving the stated assertion for $K=0$.

If $K=i|K| \neq 0$, then, from (5.4),

$$e^{\sigma w}w^{-Kw}\{h(w)\}^{-1} = w^{-\tau}(1 + o(1)),$$

for $\operatorname{Re}(w) \geqq u_0$, u_0 sufficiently large. Again by Le Roy's theorem,

$$L_1(z) = \sum{}' \frac{e^{\sigma n}n^{-i|K|n}}{h(n)} z^n$$

has $z=1$ as its only singularity on its circle of convergence $|z|=1$. But by Theorem 1, cf. (2.8),

$$G(z) = \sum{}' e^{-i|K|n \log n}z^n$$

has $|z|=1$ as its natural boundary. By the Hadamard Multiplication Theorem, every singularity of $G(z)$ on $|z|=1$, that is, each point on the unit circle is a product of a singularity of $L_1(z)$ on $|z|=1$ (i.e. $z=1$) and a singularity of $H(ze^{-\sigma})$. Thus $H(ze^{-\sigma})$ has $|z|=1$ as its natural boundary, so that $|z|=|e^{-\sigma}|$ is the natural boundary for $H(z)$.

References

1. R. Cooper, *The behaviour of certain series associated with limiting cases of elliptic theta functions*, Proc. London Math. Soc. (2) vol. 27 (1928) pp. 410–426.
2. J. G. van der Corput, *Diophantische Ungleichungen*. I. *Zur Gleichverteilung Modulo Eins*, Acta Math. vol. 56 (1931) pp. 373–456.
3. E. Fabry, *Sur les points singuliers d'une fonction donnée par son développement de Taylor*, Ann. Sci. Ecole Norm. Sup. (3) vol. 13 (1896) pp. 367–399.
4. L. E. Grosh, Thesis, Purdue University, 1954.
5. E. Le Roy, *Sur les séries divergentes et les fonctions définies par un développement de Taylor*, Ann. Fac. Sci. Univ. Toulouse (2) vol. 2 (1930) pp. 317–430.
6. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloquium Publications, vol. 26, New York, 1940.
7. E. Lindelöf, *Le calcul des résidus*, Paris, 1905.
8. M. Riesz, *Über Potenzreihen*, Ark. Mat. Astr. och Fys. vol. 11 (1916) pp. 1–16.

PURDUE UNIVERSITY,
LAFAYETTE, INDIANA