SEQUENCES GENERATED BY ITERATION(1)

BY

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1. **Introduction.** A real valued function g of a real variable will be said to belong to the class $H(a_1, k)$ if there exists an $x_0>0$ such that

$$0 < g(x) < x, \qquad \text{for } 0 < x \leqq x_0,$$

and

$$g(x) = a_1x + x^{k+1}h(x), \qquad 0 \leqq x \leqq x_0,$$

where $0 \leqq a_1 \leqq 1$, k is a positive number, and h is continuous, and $|h(x)| < M$ for $0 \leqq x \leqq x_0$. A sequence $\{x_n\}$ is said to be *generated by* g if $0<x_1<x_0$ and $x_{n+1}=g(x_n)$, $n \geqq 1$.

The purpose of this article is to investigate the rapidity of convergence of sequences generated by functions of the class H and of certain subclasses. We first dispose of the cases $0<a_1<1$ (Theorem 2.1) and $a_1=0$ (Theorem 2.2) and then analyze the case $a_1=1$ in detail. In so doing we obtain not only the dominant term of x_n (Theorem 3.1 and Corollary 3.1), but, for sufficiently restricted g, also all terms that are not affected by the value of the initial element of the sequence (Theorem 5.1). We lead up to this result by considering the difference between sequences generated by different functions (Theorem 4.1 and Corollary 4.3) and sequences generated by the same function with different initial elements (Corollaries 4.1 and 4.2).

Some of the results presented here play a role in the solution of the functional equation of Schröder. Our Theorem 2.1 was certainly known to Koenigs [3], though with more restrictive assumptions on g. In a very recent investigation of Schröder's equation, Szekeres [5] proved, also with substantially stronger requirements on g than we are using, our Theorem 2.1, part of Theorem 2.2, Corollary 3.1, and Corollary 4.3 (for $r<k$ only). Corollary 3.1 was already proved by Polya and Szegö [4, p. 31]; their assumptions on g are somewhat stronger than ours but weaker than those of Szekeres.

A generalization of our Corollary 3.1 with weaker assumptions on g was proved by Karamata [2]. Finally De Bruijn devotes a chapter of a recent book [1] to iterated functions. He illustrates the methods that can be used by means of examples but gives no general theorems. Of interest is that, in a

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detailed discussion of the iteration of $g=\sin x$, he obtains the result of our Theorem 5.1 for this special case.

We now turn to a brief discussion of the ways in which sequences $\{x_n\}$, generated by a function g, can converge. We restrict ourselves to functions g which are continuous and have at least a first derivative, where needed. Then it is well known that a sequence $\{x_n\}$ can converge only if its generating function g has fixed points. A fixed point f is one for which $g(f)=f$. The limit of $\{x_n\}$ is one of these fixed points. Moreover, if we disregard the trivial case where $x_{n_0}=f$ and hence $x_n=f$ for all $n\geqq n_0$, the sequence $\{x_n\}$ can approach only one of those fixed points f for which $|g'(f)|\leqq 1$. The case $-1\leqq g(f)<0$ can be reduced to the case $0\leqq g'(f)\leqq 1$ by considering the sequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ separately. Both are generated by the function $G(x)=g(g(x))$ and one has $G'(f)=(g'(f))^2$. There is thus no loss of generality if we study only generating functions g near fixed points f for which $0\leqq g'(f)\leqq 1$. One can, by means of a simple transformation, arrive at a generating function g^* which satisfies $0<g^*(x)<x$, at least if $0<g'(f)<1$, and a sequence $\{x_n^*\}$ that is generated by g^* from some n_0 on. Assume that the same transformation has been applied in the other two cases. If $g'(0)=0$ (we have omitted the star again) it is possible that $g(x)=0$ for an infinite number of x tending to zero. In this case the condition $0<g(x)$ will not be satisfied and x_n may, in a quite irregular pattern, be sometimes positive, sometimes negative. Nevertheless $|x_n|$ approaches zero very rapidly, so that a further study of this case can be dispensed with. Considerably more complicated is the situation if $g'(0)=1$. Write $g(x)=x+x^{k+1}h(x)$. Then the sequence $\{x_n\}$ cannot converge to zero if either $x^kh(x)\geqq 0$ for all $|x|<\delta$, or $x_n>0$ for all $n>n_0$ and $x^kh(x)\geqq 0$ for all $0<x<\delta$, or $x_n<0$ for all $n>n_0$ and $x^kh(x)\geqq 0$ for all $-\delta<x<0$. Convergence to zero does take place, if there exist two non-negative quantities a and b of which at most one may be zero, so that $x^kh(x)<0$ for $-a<x<b$, $x\neq 0$, and if for some n_0 one has $-a<x_{n_0}<b$. These cases are subsumed under the condition $g(x)<x$. Uncovered remains the case where in every neighborhood of the origin the function h changes sign infinitely often. In this case not even the convergence to zero, and hence certainly not the rapidity of the convergence, depends solely on g.

After these preliminary considerations we now derive a few basic properties of sequences generated by functions g which are continuous on $0\leqq x\leqq x_0$ and satisfy

$$0 < g(x) < x$$

for $0<x\leqq x_0$. Clearly $x_{n+1}=g(x_n)>0$ for all $n\geqq 1$ and $x_{n+1}=g(x_n)<x_n$. The sequence $\{x_n\}$ therefore converges to a value L which must satisfy $0\leqq L<x_1$, and, in view of the continuity of g, $L=g(L)$. This is only possible if L is zero.

In what follows we shall encounter certain quantities that depend on g, x, h, or certain constants; we shall denote this dependence by writing $K=K(g, x_1, \cdots)$.

2. **The case** $g'(0)<1$. In this section we present two theorems showing, respectively, the rapidity with which $\{x_n\}$ tends to zero if $0<a_1<1$ and if $a_1=0$.

THEOREM 2.1. *Let* $g\in H(a_1, k)$, *where* $0<a_1<1$. *If the sequence* $\{x_n\}$ *is generated by* g, *then there exists a constant* $K_1(g, x_1)$ *such that*

$$\lim_{n\to\infty} \frac{x_n}{a_1^n} = K_1.$$

Proof. We can write

$$\frac{x_{n+1}}{x_n} = a_1 + x_n^k h(x_n).$$

Since the sequence $\{x_n\}$ decreases to zero, we can determine a p such that for $n\geqq p$

$$x_n^k M < \frac{1-a_1}{2}.$$

It follows that

$$\frac{x_{n+1}}{x_n} < \frac{1+a_1}{2} < 1.$$

This insures the convergence of the infinite product

$$\prod_{n=1}^{\infty}\left(1 + X_n^k \frac{h(x_n)}{a_1}\right). \tag{2.1}$$

If we now introduce $u_n=x_n/a_1^n$ we obtain

$$\frac{u_{n+1}}{u_n} = 1 + x_n^k \frac{h(x_n)}{a_1},$$

and hence

$$u_{n+1} = u_1 \prod_{m=1}^{n}\left(1 + x_m^k \frac{h(x_m)}{a_1}\right).$$

The proof of the theorem is then completed by setting $K_1(g, x_1)$ equal to u_1 times the value to which the product (2.1) converges.

THEOREM 2.2. *Let* $g\in H(a_1, k)$, *where* $a_1=0$. *If the sequence* $\{x_n\}$ *is generated by* g, *then there exists a constant* $K_2(g, x_1)$, *with* $0<K_2<1$, *such that*

$$x_n < K_2^{(k+1)^n}, \qquad \textit{for } n > n_0(g, x_1).$$

If in addition $\liminf_{x\to 0} h(x)>0$, *then there exists a* $K_3(g, x_1)$, *with* $0<K_3<1$, *such that*

$$\lim_{n\to\infty} x_n^{(k+1)^{-n}} = K_3.$$

Proof. For $a_1=0$ we have

$$\log x_{n+1} = (k+1)\log x_n + \log h(x_n).$$

We now introduce

$$v_n = (k+1)^{-n}\log x_n$$

and observe that, after division by $(k+1)^{n+1}$, our recursion relation can be expressed, in terms of the quantities v_n, as

$$\begin{aligned} v_{n+1} &= v_n + (k+1)^{-(n+1)}\log h(x_n) \\ &= \sum_{m=n_0}^{n} (k+1)^{-(m+1)}\log h(x_m) + v_{n_0}. \end{aligned} \tag{2.2}$$

If $\liminf h(x)>0$, then the series

$$\sum_{m=n_0}^{\infty} (k+1)^{-(m+1)}\log h(x_m) \tag{2.3}$$

converges to a value which can be called $\log K_3(g, x_1)-v_{n_0}$. It is then easily seen that

$$\lim_{n\to\infty} x_n^{(k+1)^{-n}} = K_3.$$

That $K_3<1$ will follow automatically once the remainder of the theorem has been established.

Now let us assume only that $0<h(x)<M$. Then $\log h(x_n)$ could approach $-\infty$ so that the series (2.3) might not converge. From (2.2) one can, however, derive the following inequality:

$$\begin{aligned} v_{n+1} &< \sum_{m=n_0}^{n} (k+1)^{-(m+1)}\log M + (k+1)^{-n_0}\log x_{n_0} \\ &= \log M(k+1)^{-(m_0+1)}\,\frac{1-(k+1)^{-n+n_0+1}}{1-(k+1)^{-1}} + (k+1)^{-(n_0+1)}\log x_{n_0}^{k+1}. \end{aligned}$$

If $\log M<0$, we choose n_0 so that $x_{n_0}<1$ and obtain

$$v_{n+1} < (k+1)^{-(n_0+1)}\log M < 0. \tag{2.4}$$

If $\log M\geqq 0$, the following inequality is valid

$$v_{n+1} < (k+1)^{-(n_0+1)}\log\left(M^{(1+k)/k}\, x_{n_0}^{k+1}\right). \tag{2.4$'$}$$

We can then choose n_0 large enough so that the expression on the right is negative. The proof of the theorem is then completed by setting $\log K_2(g, x_1)$ equal to the right-hand side of (2.4) or (2.4)′ depending on whether $\log M<0$ or $\log M\geqq 0$.

3. **The dominant term of x_n for $g'(0)=1$.** From now on we shall be concerned with a detailed investigation of the case $g'(0)=1$. As a first result we have

THEOREM 3.1. *Let $g\in H(a_1, k)$, where $a_1=1$. Then $B_1=\liminf_{x\to 0+} -h(x)\geqq 0$, $B_2=\limsup_{x\to 0+} -h(x)\leqq M$. Let $\{x_n\}$ be a sequence generated by g, and let $\epsilon>0$ be given; then there exists an $N(\epsilon, g, x_1)$ so that*

$$x_n > [(B_2+\epsilon)kn]^{-1/k}, \qquad n > N.$$

If in addition $B_1>0$ and ϵ is chosen less than B_1, then the following inequality also holds:

$$x_n < [(B_1-\epsilon)kn]^{-1/k}, \qquad n > N'(\epsilon, g, x_1).$$

Proof. The requirements $g(x)=x+x^{k+1}h(x)$, $g(x)<x$, and $|h(x)|<M$ together insure that $0\leqq -h(x)<M$, for $0\leqq x\leqq x_0$. This proves the assertions about the lim sup $-h(x)$ and lim inf $-h(x)$.

We first consider the case $k=1$, employing a method suggested by O. Perron. We set

$$-h(x_n) = d_n$$

and then have

$$x_{n+1} = x_n(1-x_nd_n).$$

It follows that

$$\frac{1}{x_{n+1}} = \frac{1}{x_n}\cdot\frac{1}{1-x_nd_n}.$$

Let ϵ be given and set it equal to 3η. We now choose an $n_1(g, x_1, \epsilon)$ such that for $n\geqq n_1$ the following three inequalities hold:

$$x_nd_n<1, \qquad \sum_{m=2}^{\infty} d_n^m x_n^{m-1}<\eta, \qquad B_1-\eta<d_n<B_2+\eta.$$

For $n\geqq n_1$ we then have

$$\begin{aligned}\frac{1}{x_{n+1}} &= \frac{1}{x_n}+d_n+\sum_{m=2}^{\infty} d_n^m x_n^{m-1} \\ &< \frac{1}{x_n}+B_2+2\eta.\end{aligned} \tag{3.1}$$

Hence

$$x_{n_1+m} > \frac{1}{m(B_2 + 2\eta) + x_{n_1}^{-1}}$$

and, for $n \geqq n_1$,

$$x_n > \frac{1}{n[(1 - n_1/n)(B_2 + 2\eta) + (nx_{n_1})^{-1}]}$$

$$> \frac{1}{n[B_2 + 2\eta + (nx_{n_1})^{-1}]}.$$

By imposing a further restriction on n, namely $(nx_{n_1})^{-1} < \eta$, we arrive at

$$x_n > \frac{1}{n(B_2 + \epsilon)}.$$

From equation (3.1) it follows that, for $n \geqq n_1$

$$\frac{1}{x_{n+1}} > \frac{1}{x_n} + B_1 - \eta,$$

and hence, provided $B_1 - \epsilon > 0$, that

$$x_n < \frac{1}{n[(1 - n_1/n)(B_1 - \eta) + (nx_{n_1})^{-1}]}.$$

Choosing $N' > n_1$ and such that for $n > N'(1 - n_1/n)(B_1 - \eta) > B_1 - 3\eta$ we obtain the desired inequality

$$x_n < \frac{1}{n(B_1 - \epsilon)}, \qquad \text{for } n > N'.$$

For the case $k \neq 1$ we introduce $w_n = x_n^k$. Now

$$x_{n+1} = g(x_n) = x_n(1 + x_n^k h(x_n)).$$

Hence

$$w_{n+1} = G(w_n) = [g(w_n^{1/k})]^k = w_n(1 + w_n h(w_n^{1/k}))^k$$
$$= w_n(1 + w_n H(w_n)).$$

Here $G(w) \in H(1, 1)$ for $0 \leqq w \leqq w_0 = x_0^k$, and it is $\liminf_{w \to 0+} -H(w) = kB_1$, $\limsup_{w \to 0+} -H(w) = kB_2$. With this observation the case $k \neq 1$ has been reduced to the case $k = 1$ and the proof of Theorem 3.1 is complete.

The result below follows immediately.

COROLLARY 3.1. *Let* $g \in H(1, k)$ *and in addition let* $\lim_{x \to 0+} h(x) = a_{k+1} \neq 0$. *If* $\{x_n\}$ *is a sequence generated by* g, *then*

$$x_n = (-a_{k+1}kn)^{-1/k} + O(n^{-1/k}).$$

4. **Comparison of different sequences.** From now on, it will be convenient to use the following abbreviations:

$$\text{(4.1)} \qquad \begin{aligned} &\text{(i)} \quad \delta = 1 + 1/k, \\ &\text{(ii)} \quad \alpha_k = (-a_{k+1}k)^{-1/k}. \end{aligned}$$

To obtain sharper results about sequences $\{x_n\}$ we impose further restrictions on g. Let $g \in H(1, k)$ and in addition assume

$$g(x) = x + a_{k+1}x^{k+1} + R(x), \qquad |R(x)| = O(x^{k+1+r}), \quad r > 0,\ 0 \leqq x \leqq x_0.$$

We then say $g \in F(k, r)$. If $R'(x)$ exists for $0 < x < x_0$ and satisfies the condition $|R'(x)| = O(x^{k+\eta})$ for some $\eta > 0$, we say $g \in F'(k, r)$.

In the remainder of this article, the following lemma plays a fundamental role.

LEMMA 4.1. *Let* $\{\gamma_n\}$ *be a sequence of non-negative numbers satisfying the inequalities*

$$\gamma_{n+1} < \gamma_n\left(1 - \frac{\delta - \epsilon_n}{n}\right) + dn^{-(\delta+m/k)}, \qquad n \geqq 1,$$

where m *and* k *are positive constants, and* $\delta = 1 + 1/k$. *Then*

$$\begin{aligned} \gamma_n &= O(n^{-(m+1)/k}), && \text{if } m < k \text{ and } \epsilon_n = o(1) \\ \gamma_n &= O(n^{-\delta}\log n), && \text{if } m = k \text{ and } \epsilon_n = O(n^{-\eta}),\ \eta > 0, \\ \gamma_n &= O(n^{-\delta}), && \text{if } m > k \text{ and } \epsilon_n = O(n^{-\eta}),\ \eta > 0. \end{aligned}$$

Proof. We begin with a few preliminary observations. For sufficiently large n, provided $\epsilon_n = o(1)$, one has

$$\begin{aligned} \left(1 - \frac{\delta - \epsilon_n}{n}\right) &< \left(1 - \frac{\delta}{n}\right)\left(1 + \frac{2\epsilon_n}{n}\right) < \left(1 + \frac{\delta}{n}\right)\left(1 + \frac{2\epsilon}{n}\right) \\ &< \left(1 + \frac{1}{n}\right)^{-\delta}\left(1 + \frac{1}{n}\right)^{2\epsilon} = \left(1 + \frac{1}{n}\right)^{2\epsilon-\delta}. \end{aligned}$$

Hence

$$\prod_{v=n_0}^{n}\left(1 - \frac{\delta - \epsilon_v}{v}\right) < \left(\frac{n+1}{n_0}\right)^{2\epsilon-\delta}.$$

If $\epsilon_n = O(n^{-\eta})$, then $\prod(1 + 2\epsilon_n/n)$ increases monotonely to a limiting value P, so that in that case

$$\prod_{v=n_0}^{n}\left(1-\frac{\delta-\epsilon_v}{v}\right)<P\left(\frac{n+1}{n_0}\right)^{-\delta}, \qquad \epsilon_n = O(n^{-\eta}).$$

Next one sees easily that from

$$s_{n+1} < s_n r_n + d_n \qquad \text{for all } n \geqq n_0$$

follows

$$s_{n+1} < s_{n_0}\prod_{v=n_0}^{n} r_v + \sum_{\mu=n_0}^{n} d_\mu \prod_{v=\mu+1}^{n} r_v.$$

For γ_{n+1} one thus obtains in the case $m<k$

$$\begin{aligned}\gamma_{n+1} &< \gamma_{n_0}\left(\frac{n+1}{n_0}\right)^{2\epsilon-\delta} + d\sum_{\mu=n_0}^{n}\mu^{-(\delta+m/k)}\left(\frac{n+1}{\mu+1}\right)^{2\epsilon-\delta}\\ &< (n+1)^{-(\delta-2\epsilon)}\left[\gamma_{n_0}n_0^{\delta-2\epsilon} + d'\sum_{\mu=n_0}^{n}\mu^{-(m/k+2\epsilon)}\right].\end{aligned}$$

Here $d'=2^{\delta-2\epsilon}d$. Now

$$\sum_{\mu=n_0}^{n}\mu^{-(m/k+2)} < \int_{n_0-1}^{n} x^{-(m/k+2\epsilon)}dx.$$

It follows that, if ϵ has been chosen small enough so that $m/k+2\epsilon<1$,

$$\begin{aligned}\gamma_{n+1} &< (n+1)^{-(\delta-2\epsilon)}[O((n+1)^{1-(m/k+2\epsilon)})]\\ &= O((n+1)^{-(m+1)/k}).\end{aligned}$$

If $m\geqq k$ and hence by assumption $\epsilon_n=O(n^{-\eta})$, then

$$\begin{aligned}\gamma_{n+1} &< \gamma_{n_0}\left(\frac{n_0}{n+1}\right)^{\delta}P + d\sum_{\mu=n_0}^{n}\mu^{-(\delta+m/k)}\left(\frac{n+1}{\mu+1}\right)^{-\delta}P\\ &< (n+1)^{-\delta}P\left[\gamma_{n_0}n_0^{\delta} + d'\sum_{\mu=n_0}^{n}\mu^{-m/k}\right]\end{aligned}$$

and the assertions of the lemma are easily verified.

THEOREM 4.1. *Let $g_1\in F'(k, r_1)$ and $g_2\in F(k, r_2)$ and let*

$$|g_1(x) - g_2(x)| = O(x^{k+1+m}),$$

where $m>0$. Let the sequences $\{x_n\}$ and $\{s_n\}$ be generated by g_1 and g_2, respectively. Then

$$|x_n - s_n| = \begin{cases} O(n^{-(m+1)/k}), & \text{if } m<k,\\ O(n^{-\delta}\log n), & \text{if } m=k,\\ O(n^{-\delta}), & \text{if } m>k.\end{cases}$$

Proof. We have

$$x_{n+1} - s_{n+1} = g_1(x_n) - g_2(s_n) = g_1(x_n) - g_1(s_n) + g_1(s_n) - g_2(s_n)$$
$$= (x_n - s_n)g_1'(\xi_n) + g_1(s_n) - g_2(s_n).$$

Hence, if we set $\gamma_n = |x_n - s_n|$ we obtain

$$\gamma_{n+1} \leqq \gamma_n \left| 1 - a_{k+1}(k+1)\xi_n^k + O(\xi_n^{k+\eta}) \right| + O(s_n^{k+1+m}).$$

Now ξ_n lies between x_n and s_n so that x_n, s_n, and ξ_n are all of the form

$$(\alpha_k + \epsilon_n)n^{-1/k}, \quad \text{where} \quad \epsilon_n = o(1).$$

One thus arrives at

$$\gamma_{n+1} \leqq \gamma_n \left(1 - \frac{\delta - \epsilon_n'}{n}\right) + dn^{-\delta+m/k}, \tag{4.2}$$

where $\epsilon_n' = o(1)$. If $m < k$ the assertion of the theorem then follows directly from Lemma 4.1.

To prove the theorem for $m \geqq k$ we first derive some preliminary results. It is easily verified that the function

$$g_0(x) = x(1 - a_{k+1}kx^k)^{-1/k}$$

satisfies $g_0 \in F'(k, k)$ and that it generates the sequence $\{\alpha_k n^{-1/k}\}$ if α_k is chosen as initial element. Now the two functions g_1 and g_2 satisfy

$$|g_i - g_0| = O(x^{k+1+\min(r_i,k)}), \qquad i = 1, 2.$$

Thus, if we set $0 < k\eta_1 < \min(r_1, r_2, k)$, we can use the part of the theorem already proved to show that x_n and s_n, and hence also ξ_n, are of the form $a_k n^{-1/k} + O(n^{-(1/k+\eta_1)})$. We are then able to conclude that the numbers ϵ_n' in formula (4.2) are of order $n^{-\min(\eta,\eta_1)}$ and therefore can apply Lemma 4.1 for $m \geqq k$ to complete the proof of Theorem 4.1.

COROLLARY 4.1. *If $g \in F'(k, r)$ and $\{x_n\}$ and $\{x_n'\}$ are two sequences generated by g starting with different initial elements x_1 and x_1', then*

$$|x_n - x_n'| = O(n^{-\delta}).$$

Proof. We apply Theorem 4.1 with $g_1 = g_2 = g$, and thus can choose m arbitrary large.

It is interesting to note that $|x_n - x_n'|$ is independent of r. That the same is probably not true for $g \in F(k, r)$, we shall see in the next corollary. In §5 we shall discuss an example to show that this result cannot be improved.

COROLLARY 4.2. *If $g \in F(k, r)$ and $\{x_n\}$ and $\{x_n'\}$ are two sequences generated by g starting with different initial elements x_1 and x_1', then*

$$|x_n - x_n'| = \begin{cases} O(n^{-(r+1)/k}), & \text{if } r < k, \\ O(n^{-\delta}\log n), & \text{if } r = k, \\ O(n^{-\delta}), & \text{if } r > k. \end{cases}$$

Proof. We introduce $g^* = x + a_{k+1}x^{k+1}$. By applying Theorem 4.1 twice, once to compare $\{x_n\}$ with a sequence generated by g^*, then to compare this sequence with $\{x_n'\}$, we arrive at the above result, if we note that $g^* \in F'(g, \infty)$ and that

$$|g^* - g| = O(x^{k+1+r}).$$

These corollaries do not answer the question of how much the terms of a sequence $\{x_n\}$ may differ from their dominant terms $\alpha_k n^{-1/k}$. The answer is provided below.

COROLLARY 4.3. *If $g \in F(k, r)$ and $\{x_n\}$ is generated by g, then*

$$|x_n - \alpha_k n^{-1/k}| = \begin{cases} O(n^{-(r+1)/k}), & \text{if } r < k, \\ O(n^{-\delta}\log n), & \text{if } r \geqq k. \end{cases}$$

Proof. Let the function g_1 of Theorem 4.1 be g_0 (defined in the proof of that theorem) and $g_2 = g$. Then

$$|g_0 - g| = O(x^{k+1+\min k, r}).$$

It is clear from the outline of the proof that, with the methods at our disposal, this result cannot be improved by replacing F by F'. In the next section we shall show that, unless the class of functions is much more severely restricted, no better estimate can be obtained.

5. **Other terms of x_n not depending on the initial element of the sequence.** By considering a more restricted class of functions than has been investigated so far, we are able to make much more precise statements about the size of the elements x_n. The results we obtain are not unexpected in the light of Theorem 4.1. However, Theorem 4.1 alone is not strong enough to prove them.

THEOREM 5.1. *Let the function $g \in H(1, k)$ and satisfy the additional requirement*

$$g(x) = x + \sum_{v=1}^{k+1} a_{k+v}\, x^{k+v} + R(x), \qquad 0 \leqq x \leqq x_0,$$

where k is a positive integer, $a_{k+1} < 0$, and $|R(x)| = O(x^{2k+1+\epsilon})$ for some $\epsilon > 0$. If $\{x_n\}$ is a sequence generated by g, then there exist real valued constants $c_1, \cdots, c_{k+1}$, where $c_1 = \alpha_k$ and $c_v = c_v(a_{k+1}, \cdots, a_{k+v})$, such that

$$x_n - \sum_{v=1}^{k} c_v n^{-v/k} - c_{k+1} n^{-\delta}\log n = O(n^{-\delta}).$$

Proof. We define

$$G(x) = x + \sum_{v=1}^{k+1} a_{k+v} x^{k+v}.$$

Then $g = G + R$. Let

$$p_n = \sum_{v=1}^{k} c_v n^{-v/k} + c_{k+1} n^{-\delta} \log n.$$

The main part of the proof consists in showing that we can determine the c_v in such a way that

$$p_{n+1} - G(p_n) = O(n^{-(2k+1+\eta)}), \qquad \eta > 0.$$

Assume this has been done and that we obtained $c_1 = \alpha_k$. Now let $\{z_n\}$ be a sequence generated by G and set $z_n = p_n + \gamma_n$. For γ_n we obtain the relation

$$z_{n+1} = p_{n+1} + \gamma_{n+1} = G(p_n + \gamma_n).$$

Hence

$$\begin{aligned} \gamma_{n+1} &= G(p_n + \gamma_n) - G(p_n) + G(p_n) - p_{n+1} \\ &= \gamma_n G'(\xi_n) + O(n^{-(2k+1+\eta)}). \end{aligned}$$

Here ξ_n lies between p_n and z_n. By Corollary 4.3

$$z_n = \alpha_k n^{-1/k} + O(n^{-2/k}).$$

Since p_n is also of this form, the same must be true for ξ_n. We can therefore write

$$\gamma_{n+1} = \gamma_n \left(1 - \frac{\delta - \epsilon_n}{n}\right) + O(n^{-(2k+1+\eta)}),$$

where $\epsilon = O(n^{-1/k})$. An application of Lemma 4.1 then leads to the conclusion $\gamma_n = O(n^{-\delta})$. Finally we employ Theorem 4.1 to compare sequences generated by g and G. Since $|g - G| = O(x^{2k+1+\epsilon})$ we obtain

$$x_n = p_n + O(n^{-\delta}).$$

Here $\{x_n\}$ is any sequence generated by g.

We now turn to the evaluation of $p_{n+1} - G(p_n)$. It is convenient to introduce

$$y = n^{-1/k}.$$

Then

$$(n+1)^{-1/k} = (1 + y^{-k})^{-1/k} = y - \frac{1}{k} y^{k+1} + \frac{k+1}{2k^2} y^{2k+1} + y^{3k+1}\phi_1,$$

where ϕ_1 is a convergent power series in y for $|y|<1$. The coefficients of ϕ_1 depend only on k. Hence ϕ_1 is bounded for $|y|<1/2$.

We can now write p_{n+1} in terms of y as follows

$$\begin{aligned} p_{n+1} &= \sum_{v=1}^{k} c_v\left(y - \frac{1}{k}y^{k+1} + \frac{k+1}{2k^2}y^{2k+1} + \phi_1 y^{3k+1}\right)^v \\ &\quad + c_{k+1}\log(n+1)\left(y - \frac{1}{k}y^{k+1} + \cdots\right)^{k+1} \\ &= \sum_{v=1}^{k} c_v y^v - \sum_{v=1}^{k} c_v \frac{v}{k} y^{k+v} + c_1\frac{k+1}{2k^2}y^{2k+1} + c_{k+1}\log(n+1)y^{k+1} \\ &\quad - \delta c_{k+1}\log(n+1)y^{2k+1} + y^{2k+2}\phi_2 + y^{2k+2}\log(n+1)\phi_3. \end{aligned}$$

Here ϕ_2 and ϕ_3 are polynomials in y and ϕ_1. Their coefficients depend only on k and the c_v. Hence the two polynomials are bounded for $|y|<1/2$ and fixed c_v.

For $G(p_n)$ we obtain

$$G(p_n) = \sum_{v=1}^{k} c_v y^v + c_{k+1}\log n y^{k+1} + \sum_{v=1}^{k+1} a_{k+v}\left(\sum_{v=1}^{k} c_v y^v + c_{k+1}\log n y^{k+1}\right)^{k+v}.$$

Taking the difference we arrive at

$$\begin{aligned} p_{n+1} - G(p_n) &= \sum_{v=1}^{k} d_v y^{k+v} + c_{k+1}y^{k+1}(\log(n+1) - \log n) \\ &\quad + c_1\frac{k+1}{2k^2}y^{2k+1} - \delta c_{k+1}y^{2k+1}\log(n+1) - a_{k+1}c_1^{k+1}(k+1)y^{2k+1}\log n \\ &\quad + \psi_{k+1}y^{2k+1} + y^{2k+2}(\phi_2 + \log(n+1)\phi_3 + \log n\,\phi_4 + \phi_5). \end{aligned}$$

Here ϕ_4 is a polynomial in y and ϕ_5 is a polynomial in $y\log n$ and y. The coefficients of both polynomials depend only on k, c_v, a_{k+v}. The two polynomials are therefore bounded for $|y|<1/2$.

For the coefficients d_v we have the equations

$$\begin{aligned} -d_1 &= c_1/k + a_{k+1}c_1^{k+1}, \\ -d_v &= c_v(v/k - \delta) + \psi_v(c_1, \cdots, c_{v-1}; a_{k+1}, \cdots, a_{k+v}), \qquad v = 2, \cdots, k. \end{aligned}$$

The system of equations $d_v=0$ can be solved for the c_v. For c_1 we obtain, by discarding the possible solution $c_1=0$,

$$c_1 = (-a_{k+1}k)^{-1/k} = \alpha_k.$$

Since $v/k-\delta\neq 0$ for all $v=2, \cdots, k$ the other equations can be solved successively. The solutions are unique, real valued, and of the form

$$c_v(a_{k+1}, \cdots, a_{k+v}).$$

Substituting the values thus obtained into $p_{n+1}-G(p_n)$, and observing that

$$a_{k+1}c_1^{k+1}(k+1) = -\delta$$

and

$$\log(n+1) - \log n = \log(1+y^k) = y^k - \frac{y^{2k}}{2} + \cdots$$

we arrive at

$$p_{n+1} - G(p_n) = c_{k+1}y^{2k+1} + c_1\frac{k+1}{2k^2}y^{2k+1} + \psi_{k+1}y^{2k+1} + O(y^{2k+1+\epsilon}),$$

where $0<\epsilon<1$. By setting $c_{k+1}=-c_1(k+1)/2k^2-\psi_{k+1}$ we then have the desired estimate for $p_{n+1}-G(p_n)$, and the proof of the theorem is complete.

The result just obtained can be extended to functions not possessing quite as much regularity as those just considered. We have the following result.

COROLLARY 5.1. *Let the function $g\in H(1, k)$ and satisfy*

$$g(x) = x + \sum_{v=1}^{r} a_{k+v}x^{k+v} + R(x), \qquad 0 \leqq x \leqq x_0,$$

where k and $r\leqq k$ are positive integers, $a_{k+v}<0$ and $|R(x)|=O(x^{k+r+\epsilon})$ for some $0<\epsilon\leqq 1$. Then there exist real valued constants $c_1, \cdots, c_r$, where $c_1=\alpha_k$ and $c_v=c_v(a_{k+1}, \cdots, a_{k+v})$ such that

$$x_n - \sum_{v=1}^{r} c_v n^{-v/k} = \begin{cases} O(n^{-(r+\epsilon)/k}), & \text{if } r+\epsilon < k+1, \\ O(n^{-\delta}\log n) & \text{if } r+\epsilon = k+1. \end{cases}$$

Proof. We make use of a sequence generated by G, as obtained in the proof of Theorem 5.1, and compare it with a sequence generated by the function g of the corollary. In this case

$$|g - G| = O(x^{k+r+\epsilon})$$

so that an application of Theorem 4.1 establishes the corollary.

It is of interest to know to what extent sequences generated by different functions can agree. To discuss this problem we make the following definition. Let

$$g = x + \sum_{v=1}^{r} a_{k+v}x^{k+v} + O(x^{k+r+\epsilon})$$

and let $\{x_n\}$ be generated by g. Then we call the sum

$$p_n = \begin{cases} \sum_{v=1}^{r} c_v n^{-v/k}, & \text{if } r \leqq k, \\ \sum_{v=1}^{k} c_v n^{-v/k} + c_{k+1} n^{-\delta} \log n, & \text{if } r > k, \end{cases}$$

the *principal part of* x_n, provided

$$|x_n - p_n| = \begin{cases} O(n^{-(r+\epsilon)/k}), & r + \epsilon < k + 1, \\ O(n^{-\delta} \log n), & r + \epsilon = k + 1, \\ O(n^{-\delta}), & r + \epsilon > k + 1. \end{cases}$$

We are now able to state the theorem

THEOREM 5.2. *Let*

$$g_1(x) = x + \sum_{v=1}^{r} a_{k+v} x^{k+v} + O(x^{k+r+\epsilon_1}), \qquad 0 \leqq x \leqq x_0,$$

$$g_2(x) = x + \sum_{v=1}^{r} b_{k+v} x^{k+v} + O(x^{k+r+\epsilon_2}), \qquad 0 \leqq x \leqq x_0,$$

and $g_1 \in H(1, k)$, $g_2 \in H(1, k)$. *Then sequences generated by the two functions have the same principal part if and only if*

$$a_{k+v} = b_{k+v}, \qquad v = 1, \cdots, \min(r, k + 1).$$

Proof. If $a_{k+v} = b_{k+v}$, $v = 1, \cdots, \min(r, k+1)$ then the difference between the two functions is of order $x^{k+r+\epsilon}$, $0 < \epsilon < \min \epsilon_1, \epsilon_2$. It follows from Theorem 4.1 that the principal parts of sequences generated by the two functions are the same.

If the principal parts of the two sequences are the same, we note that the c_v are determined uniquely by the method employed in the proof of Theorem 5.1, and proceed by induction. It is certainly true that $a_{k+1} = b_{k+1}$, since

$$a_{k+1} = \frac{-1}{kc_1^k} = b_{k+1}.$$

Now assume the assertion is true up to $n-1 < \min r, k+1$, then $a_{k+v} = b_{k+v}$, $v \leqq n-1$ and hence

$$\begin{aligned} &\psi_n(c_1, \cdots, c_{n-1}; a_{k+1}, \cdots, a_{k+n-1}, a_{k+n}) \\ &\quad = \psi_n(c_1, \cdots, c_{n-1}; b_{k+1}, \cdots, b_{k+n-1}, b_{k+n}) \\ &\quad = \psi_n(c_1, \cdots, c_{n-1}; a_{k+1}, \cdots, a_{k+n-1}, b_{k+n}). \end{aligned}$$

The only way a_{k+n} enters into ψ_n (see proof of Theorem 5.1) is in the additive term $a_{k+n} c_1^{k+n}$. Hence

$$a_{k+n}c_1^{k+n} = b_{k+n}c_1^{k+n},$$

and it follows that $a_{k+n}=b_{k+n}$, since $c_1\neq 0$. This completes the proof of the theorem.

We are now able to throw some light on the questions raised in §4 as to whether some of the results obtained there could be improved. Consider the function (r an integer $\leqq k$)

$$\gamma(x) = g_0(x) + x^{k+r}h(x),$$

where $h(x)=0$ for $x=\alpha_k n^{-1/k}$, rises very rapidly to one near these points and remains at one until it has to dip down to zero again. Then one sequence generated by γ is $\{\alpha_k n^{-1/k}\}$ if the initial element x_1 is chosen to be α_k. However, for a suitably chosen x_1 we can obtain

$$x_n = \alpha_k n^{-1/k} + c_{k+r}n^{-r/k} + O(n^{-(r+1)/k}),$$

where $c_{k+r}\neq 0$ (this follows from Theorem 5.2). Now $\gamma\in F(k, r)$ and hence it follows that Corollary 4.2 cannot be improved. We also note that we can make h differentiable but that then at certain places h' would have to be of order x^k at least, which is not small enough to insure $g\in F'(k, r)$. The example also shows that Corollary 5.1 cannot be improved.

Another consequence of Theorem 5.2 is that a sequence $\{x_n\}$, generated by a function of the type considered in the theorem, can satisfy

$$|x_n - \alpha_k n^{-1/k}| = O(n^{-\delta})$$

only if it agrees with g_0 up to the term in x^{2k+1}, that is only if

$$g = x + a_{k+1}x^{k+1} + (k+1)a_{k+1}^2x^{2k+1} + \cdots.$$

This together with the previous example shows that Corollary 4.3 cannot be improved.

At least for functions satisfying the conditions of Theorem 5.1 it is now also quite clear that two sequences generated by the same function may differ by $O(n^{-\delta})$. For let $\{x_n\}$ be one such sequence and define $y_n=x_{n+m}$. The result follows from the relation

$$\frac{1}{n^{v/k}} - \frac{1}{(n+m)^{v/k}} = O(n^{-(1+v/k)}).$$

We conclude the article with two final remarks: Even if the function g has a Taylor's expansion going beyond the term x^{2k+1}, this additional information is useless as far as the determination of the terms of a sequence generated by g is concerned. More precisely, it is useless if we are only concerned with that part of x_n which does not depend on x_1. If we had tried to add a term $c_{k+2}n^{-\delta}$ to p_n in Theorem 5.1, we would not have been able to

determine it since in the determining equation the coefficient of c_{k+2} would have been $\delta-\delta$. There is a gap in the result of Corollary 5.1 if $g\in F'(k, 1)$. In that case there should be additional terms depending on g only and of order greater than $O(n^{-\delta})$. These terms probably depend on R in addition to $a_{k+1}, \cdots, a_{k+r}$, but we are at present unable to say what they are.

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