ON ABSOLUTELY CONVERGENT EXPONENTIAL SUMS([1])

BY

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Consider the following problem. Is it possible to represent zero by an absolutely convergent series of exponentials with bounded exponents:

$$\text{(1)} \qquad \sum_{1}^{\infty} a_n e^{\alpha_n z} \equiv 0, \qquad 0 < \sum |a_n| < \infty, \qquad |\alpha_n| < M \qquad (n \geqq 1)$$

where $\{a_n\}$ and $\{\alpha_n\}$ are complex numbers and the α_n are all distinct? Equation (1) is equivalent to

$$\text{(2)} \qquad \sum_{1}^{\infty} \frac{a_n}{z - \alpha_n} \equiv 0 \quad (|z| > M), \quad 0 < \sum |a_n| < \infty, \quad |\alpha_n| < M,$$

since (2) is the Borel-Laplace transform of (1) (see §1).

To our knowledge the solution to this problem was first given in 1921 by J. Wolff [21] who gave an example of such a representation of zero.

In this paper we consider the problem of characterizing those bounded complex sequences $\{\alpha_n\}$ for which coefficients $\{a_n\}$ exist such that (1) holds. In §2 a number of preliminary results are given. For example, (1) is impossible if the closure of the set $\{\alpha_n\}$ contains no interior and does not separate the plane, or if all the α_n lie in a convex domain and at least one, with a nonzero coefficient, lies on the boundary.

In §3 we obtain a characterization of those $\{\alpha_n\}$ for which (1) is possible, under the assumption that the α_n lie in the unit circle and have no interior limit points. The condition is that almost every boundary point $\exp(i\theta)$ be approachable nontangentially (i.e. inside of an angle) by a subsequence of $\{\alpha_n\}$. This is equivalent to

$$\sup_n |f(\alpha_n)| = \sup_{|z|<1} |f(z)|$$

for every bounded analytic f in the unit circle.

In §4 we study the set $E(G)$ of all those entire functions $h(z)$ admitting a representation

$$\text{(3)} \qquad h(z) = \sum a_n e^{\alpha_n z}, \qquad \sum |a_n| < \infty, \qquad \alpha_n \in G,$$

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where G is a Jordan domain. With a suitable norm $E(G)$ is a Banach space whose conjugate space is the space of all bounded analytic functions in G with the supremum norm. In case G is the unit circle we obtain a characterization of E as a quotient space of $L_1(0, 2\pi)$ $(E=L_1/H_1)$. This includes Wolff's result [21] that E contains all entire functions of exponential type less than 1. It also gives a representation of Fourier coefficients as solutions of infinite "Vandermonde" systems of equations.

In §5 we give some additional properties of the space E.

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1. **Background material.** We shall need a few standard results on entire functions of exponential type, and on bounded analytic functions in the unit circle. These are stated here, proofs may be found in Boas [3, Chapter 5] and Privaloff [15, Chapter 1].

Let $h(z)=\sum a_n z^n/n!$ be an entire function and let $M(r)$ denote the maximum of its modulus on a circle of radius r. The function h is said to be of exponential type 1 if

$$M(r) = O(e^{(1+\epsilon)r})$$

for every $\epsilon>0$.

The function $H(z)=\sum a_n/z^{n+1}$ is called the Borel transform (or Borel-Laplace transform) of h. The series for H converges for $|z|>1$, and vanishes at $z=\infty$. One may recover h as an integral

$$h(z) = \frac{1}{2\pi i}\int_{|w|=r} e^{zw}H(w)dw \tag{4}$$

for every $r>1$. The Borel transform establishes a one-to-one correspondence between the class of all entire functions of exponential type 1, and the class of all functions analytic for $|z|>1$ and vanishing at infinity.

Let H_∞ denote the class of all bounded analytic functions in the unit circle. It is a Banach space under the supremum norm. Each $f\in H_\infty$ has a radial limit

$$f(e^{it}) = \lim_{r\to 1^-} f(re^{it})$$

at almost every boundary point $p=\exp(it)$, and the resulting boundary function is a bounded measurable function whose essential supremum equals $\|f\|$. In addition, whenever the radial limit exists, the limit exists and has the same value inside of any angle with vertex at p. The radial limit can be zero only on a set of measure zero, unless $f\equiv 0$.

Let $\{\alpha_n\}$ be a sequence of points inside the unit circle. The necessary and sufficient condition that there exist an $f\in H_\infty$ vanishing at the α_n but nowhere else inside the unit circle is that $\sum(1-|\alpha_n|)<\infty$.

2. **Preliminary results.** Most of the results of this section apply to a more general representation of zero than is given in (1). Let μ be any finite, complex-valued Borel measure of compact support K in the plane. We say that μ represents zero if

$$\int e^{zw} d\mu(w) \equiv 0. \tag{1'}$$

Lemma 1. *Equation* (1′) *holds if and only if*

$$\int P(w) d\mu(w) = 0 \tag{5}$$

for all polynomials P.

Proof. Differentiating (1′) n times and evaluating at $z=0$ we obtain $\int w^n d\mu(w)=0$, and (5) follows. Conversely, (5) implies (1′) since $f_z(w)=e^{zw}$ may be uniformly approximated by polynomials on the support of μ.

Instead of using the exponential function in this lemma we could have used any entire function g that, together with all its derivatives, is different from zero at the origin. Equation (1′) then becomes: $\int g(zw)d\mu(w)\equiv 0$. This remark applies to several of the following lemmas.

Theorem 1. *If K contains no interior points and does not separate the plane, and if* (1′) *holds, then* $\mu=0$.

Proof. By a theorem of Lavrentieff (see Mergelyan, [13, Chapter 1, §4]) polynomials are uniformly dense in the space $C(K)$ of all continuous, complex-valued functions on K. Hence by Lemma 1, $\int f d\mu=0$ for all $f\in C(K)$, and so $\mu\equiv 0$.

Let G be a Jordan region (the interior of a simple closed Jordan curve). We shall say that a measure μ is "in" G if $\mu(E)=0$ for all sets E disjoint from G. This is not quite the same as saying that the support of μ is contained in G. (The support of μ is the smallest closed set K such that $\mu(E)=0$ for all E disjoint from K.) Thus in (1) μ is a sequence of mass points; μ is in the circle $|z|<M$ but the support of μ is $\{\alpha_n\}^-$ (the closure of the set $\{\alpha_n\}$).

Let $H_\infty(G)$ denote the set of all bounded analytic functions in G. If G is the interior of the unit circle we write simply H_∞ as in §1.

Lemma 2. *If μ is a measure in the Jordan region G and if* (1′) *holds then*

$$\int f(w) e^{zw} d\mu(w) \equiv 0 \tag{6}$$

for all $f\in H_\infty(G)$.

Proof. If f is a polynomial the result follows as in Lemma 1 by differentiating (1′). If f is the uniform limit of polynomials the result follows from a passage to the limit. Finally, if f is a bounded pointwise limit of functions for which (6) holds, then by the Lebesgue bounded convergence theorem, (6) will hold for f.

Now let $f\in H_\infty(G)$ and let ϕ be a Riemann mapping function taking the unit circle $(|\zeta| < 1)$ onto G. Then $g(\zeta) = f(\phi(\zeta)) \in H_\infty$. Let $g_r(\zeta) = g(r\zeta)$ $(0<r<1)$. Then g_r is analytic for $|\zeta| <1/r$, and in particular g_r is continuous for $|\zeta| \leqq 1$. It is known (see Bieberbach [2, p. 33]) that ϕ maps $|\zeta| \leqq 1$ homeomorphically onto G^-, and therefore $f_r(z) = g_r(\phi^{-1}(z))$ is analytic in G and continuous on G^-. Hence by a theorem of Walsh [19] f_r is the uniform limit of polynomials on G^-. The result now follows since f is the bounded pointwise limit of f_r as $r\rightarrow 1$.

LEMMA 3. *If μ is a measure in the Jordan region G then* (1′) *holds if and only if*

$$\int f(w)d\mu(w) = 0 \tag{7}$$

for every $f\in H_\infty$.

This follows immediately from Lemmas 1 and 2. We do not know to what extent it holds for more general regions.

COROLLARY. *If μ is a measure in the Jordan region G for which* (1′) *holds and if ϕ is any analytic function mapping G into itself, then*

$$\int e^{z\phi(w)}d\mu(w) \equiv 0. \tag{8}$$

Proof. We define a new measure σ by: $\sigma(E) = \mu(\phi^{-1}(E))$ for all Borel sets E. One verifies easily that

$$\int f(w)d\sigma(w) = \int f(\phi(w))d\mu(w) \tag{9}$$

for all continuous f. Now let $f\in H_\infty(G)$. Then $f(\phi(w))\in H_\infty(G)$ and so the right side of (9) vanishes by Lemma 3. Thus $\int f d\sigma = 0$ for all $f\in H_\infty(G)$, and so by the other half of Lemma 3, (1′) holds for σ. But this is precisely equation (8) which was to be proven.

The last two results of this section deal with equation (1) rather than with (1′). The first result enables us to add one more term to a given representation of zero.

LEMMA 4. *Let* (1) *hold, except that instead of the condition $|\alpha_n| < M$ we assume that the exponents all lie in a Jordan region G and have no interior limit points. Let $\alpha\in G$. Then coefficients $\{b_n\}$ exist such that*

$$\exp(\alpha z) + \sum b_n \exp(\alpha_n z) \equiv 0, \qquad \sum |b_n| < \infty. \tag{10}$$

Proof. We may assume $\alpha \neq \alpha_n$ ($n=1, 2, \cdots$) for otherwise (10) is trivial. If we differentiate (1) k times and evaluate at $z=0$ we obtain

$$\sum_{n=1}^{\infty} a_n \alpha_n^k = 0 \qquad (k = 0, 1, 2, \cdots). \tag{11}$$

Let now $b_n' = a_n/(\alpha_n - \alpha)$, $b' = -\sum b_n'$, and define the functions $h(z)$, $\phi(z)$ by

$$h(z) = \sum b_n' \exp(\alpha_n z), \qquad \phi(z) = b' \exp(\alpha z) + h(z). \tag{12}$$

From (11) we see that ϕ and all its derivatives vanish at the origin and thus $\phi \equiv 0$. If $b' \neq 0$ we have (10) and the proof is complete.

Now suppose $b'=0$. From (12) we see that $h \equiv 0$. We repeat the process, letting $b_n'' = b_n'/(\alpha_n - \alpha)$, and $b'' = -\sum b_n''$. Just as before, if $b'' \neq 0$ the proof is complete. If $b''=0$ we repeat the process once more.

We claim that eventually there must be an integer p for which $b^{(p)} \neq 0$. For if not then

$$\sum \frac{a_n}{(\alpha_n - \alpha)^p} = 0 \qquad (p = 0, 1, 2, \cdots). \tag{13}$$

Let

$$g(z) = \sum \frac{a_n}{(\alpha_n - z)}, \qquad z \in G, z \notin \{\alpha_n\}.$$

From (13) g and all its derivatives must vanish at α, so $g \equiv 0$. But this implies that all coefficients vanish, in contradiction to (1). Indeed, if, say, $a_1 \neq 0$, then by choosing z sufficiently close to α_1, g could not vanish. This completes the proof.

This lemma can be somewhat modified. For example, the proof would still work if the α_n had at most a countable number of interior limit points, and if $\alpha \notin \{\alpha_n\}^-$. This modified lemma would serve equally well for Theorem 3. The proof of the next theorem will suggest additional ways in which the hypothesis of Lemma 4 could be modified; this proof is in Levin's book [11, Chapter 1, §20].

Theorem 2 (Levin). *Equation* (1) *is impossible if there is a compact convex set D containing all the exponents such that at least one exponent α_p, with a nonzero coefficient a_p, lies on the boundary of D.*

Proof. We pass to the Borel transform so that (1) becomes (2), valid for all D in the exterior of D. There is an index N such that $\sum_{N+1}^{\infty} |a_n| < 1/2$. Further, we take $p=1$ and $a_1=1$. Then from (2) we have

$$0 \geqq \frac{1}{|z-\alpha_1|} - \sum_{2}^{N} \frac{|a_n|}{|z-\alpha_n|} - \sum_{N+1}^{\infty} \frac{|a_n|}{|z-\alpha_n|}.$$

Let L be a line of support to D at α_1 and let L' be the ray perpendicular to L, starting at α_1 and going out away from D. If $z \in L'$ then $|z-\alpha_1| < |z-\alpha_n|$ $(n=2, 3, \cdots)$. Hence

$$0 > \frac{1}{2|z-\alpha_1|} - \sum_2^N \frac{|a_n|}{|z-\alpha_n|}.$$

But if $z \to \alpha_1$ on L' we have a contradiction and the theorem is proven.

3. **The principal results.** Let G be a bounded domain. By $E(G)$ we denote the family of all those entire functions $h(z)$ that admit a representation of the form

$$h(z) = \sum a_n \exp(\alpha_n z), \qquad \sum |a_n| < \infty, \qquad \alpha_n \in G. \tag{14}$$

In case G is the unit circle we shall write simply E.

Different domains may have the same sets $E(G)$. It will follow from Theorem 3 that if G_1 is the unit circle, G_2 is the unit circle with the unit interval $(0 \leqq x \leqq 1)$ deleted, and G_3 is any annulus $(r < |z| < 1)$, then $E(G_1) = E(G_2) = E(G_3)$.

Clearly $E(G)$ is a linear vector space over the complex numbers. We introduce a norm by the formula

$$\|h\| = \inf \sum |a_n| \tag{15}$$

taken over all representations (14) of h. We shall see later (§4) that if G is a Jordan domain then $E(G)$ is complete in this norm.

As usual, l_1 and l_∞ denote the Banach spaces of absolutely convergent sequences, and of bounded sequences, respectively.

THEOREM 3. *Let G be a Jordan region and let $S = \{\alpha_n\}$ $(n=1, 2, \cdots)$ be a sequence of distinct points in G with no interior limit points. Then the following three properties of S are equivalent.*

(i) *S represents zero, i.e., coefficients $\{a_n\} \in l_1$ exist, not all zero, such that* (1) *holds.*

(ii) *S represents all of $E(G)$, i.e., if $h \in E(G)$ and $\epsilon > 0$ are given, then coefficients $\{a_n\}$ exist such that*

$$h(z) = \sum a_n \exp(\alpha_n z), \qquad \sum |a_n| < \|h\| + \epsilon. \tag{16}$$

(iii) $\sup_n |f(\alpha_n)| = \|f\|$ *for all $f \in H_\infty(G)$.*

In case G is the unit circle, then these three properties are all equivalent to the following condition.

(iv) *Almost every boundary point $p = \exp(i\theta)$ may be approached nontangentially (i.e., inside of some angle with vertex at p) by points of S.*

Condition (iii) was suggested by P. C. Rosenbloom, and condition (iv) by G. Piranian.

Proof. The order of proof will be: (iv)→(iii)→(ii)→(i)→(iii)→(iv). We do not know of any direct way to go from (i) to (ii).

(iv)→(iii). Let $f\in H_\infty$ and let $\epsilon>0$ be given. There is a set of positive measure on the boundary of the unit circle such that $|f(e^{i\theta})|\geqq\|f\|-\epsilon$ for $\exp(i\theta)$ in this set. At almost all of these points $f(z)$ has a nontangential limit, and almost all of them are approachable nontangentially by points of S; therefore, $\sup|f(\alpha_n)|\geqq\|f\|-\epsilon$.

(iii)→(ii). It will be sufficient to show that S represents all exponentials. In other words, we must show that if $\alpha_0\in G$ is given ($\alpha_0\neq\alpha_n$, $n=1, 2, \cdots$), and $\epsilon>0$ is given, then there exist coefficients $\{a_n\}$ such that

$$\exp(\alpha_0 z) = \sum_1^\infty a_n \exp(\alpha_n z), \qquad \sum_1^\infty |a_n| < 1+\epsilon. \tag{17}$$

Consider the mapping T: $H_\infty(G)\to l_\infty$ defined by $Tf=\{f(\alpha_n)\}_0^\infty$. By (iii) this is an isometric imbedding of $H_\infty(G)$ in l_∞. Let $B=T(H_\infty(G))$. Then B is a norm closed subspace of l_∞. Our first goal is to show that B is actually weak-star closed. See Banach [1, Chapitre VIII, and Annexe] and Hille-Phillips [9, §2.10] for the relevant definitions and theorems.

Since l_∞ is the conjugate space of a separable space, it is sufficient to show that every weak-star convergent sequence in B converges to an element of B [1, Théorème 5, Chapitre VIII]. Weak-star convergence of a sequence in l_∞ is equivalent to boundedness of the norms plus convergence in each fixed coordinate. Since B and $H_\infty(G)$ may be identified, a weak-star convergent sequence may be identified with a uniformly bounded sequence of functions $\{f_k\}\subset H_\infty(G)$ such that $\lim f_k(\alpha_n)$ $(k\to\infty)$ exists for each fixed n.

The functions $\{f_k\}$ form a normal family. There cannot be two subsequences converging to different limits f and g, for f and g would agree at the points α_n and then by (iii) $f-g$ would vanish identically. Therefore $g(z)=\lim f_k(z)$ exists, uniformly on compact subsets of G. Thus g is the weak-star limit of the sequence $\{f_k\}$, and therefore B is weak-star closed.

Consider now the element $p=(1, 0, 0, \cdots)\in l_\infty$. We show that $\operatorname{dist}(p, B)=1/2$. Indeed, the constant function $f\equiv 1/2$ has distance $1/2$ from p, and if any $f\in H_\infty(G)$ were at a smaller distance from p then we would have $|f(\alpha_0)|>\sup|f(\alpha_n)|$ $(n\geqq 1)$, contrary to (iii).

Now we require a theorem of Banach which states that if E is a Banach space, E^* its conjugate, B a weak-star closed subspace of E^*, $p\notin B$, $\epsilon>0$, then there exists an $x\in E$ such that: $x\perp B$, $(x, p)=1$, $\|x\|<(1/d)+\epsilon$, where $d=\operatorname{dist}(p, B)$ and (x, p) denotes the value of the linear functional p at the point x.

Applying this lemma to our present situation we obtain $\{a_n\}_0^\infty\in l_1$ such that:

I. $\sum_0^\infty a_n f(\alpha_n)=0$ for all $f\in H_\infty(G)$;

II. $a_0=1$;

III. $\sum_0^\infty |a_n|<2+\epsilon$.

By Lemma 3 we see that I is equivalent to $\sum a_n \exp(\alpha_n z) \equiv 0$. Now applying II and III we have (17), except for a change of sign in the coefficients. This completes the proof of (ii).

(ii)→(i). Let α_0 be any point in G different from all α_n. By (ii) there exist coefficients such that (17) holds. If we apply Lemma 2 to (17) with $f(w) = w - \alpha_0$ we obtain (i).

(i)→(iii). Assume that (i) holds but that (iii) does not. Then there is an $f \in H_\infty(G)$ and an α ($|\alpha| < 1$) for which $|f(\alpha)| > \sup |f(\alpha_n)|$. By Lemma 4 there exist coefficients such that

$$\exp(\alpha z) + \sum b_n \exp(\alpha_n z) \equiv 0, \qquad \sum |b_n| < \infty.$$

Without loss of generality we may assume that $f(\alpha) = 1$. Then by Lemma 3 for each positive integer p we have $1 + \sum b_n [f(\alpha_n)]^p = 0$. But this is impossible for large p.

(iii)→(iv). Let us assume that condition (iv) fails; we shall show that then condition (iii) also fails. Since (iv) is false, there is a set E of positive measure on the unit circumference $|z| = 1$, such that no point of E can be approached nontangentially by the α_n. This means that any angle with vertex at a point of E can contain only a finite number of the α_n. In particular this is true for a right angle, placed so that the radius to the point bisects the angle.

This implies that at each point $p = \exp(i\theta) \in E$ there is a right triangle $\triangle_\theta$, with the right angle vertex at the point p and the other two vertices inside the unit circle, having the radius to p as an axis of symmetry, and containing none of the α_n. There will be a number $b > 0$ and a closed subset $E_1 \subset E$ of positive measure such that at each point of E_1 the altitude of the triangle $\triangle_\theta$, measured from the vertex p, has length $\geqq b$.

Choose now a closed arc I, whose endpoints are in E_1, for which $|E_1 \cap I| > 0$ and $|I| < b$, where the vertical bars denote Lebesgue measure. Let G be the complement of E_1 with respect to the arc I; then G is the union of a set of open arcs $\{I_n\}$. (If $I \subset E_1$, then G is the null set.) Take one of these arcs, I_j, with endpoints $\exp(i\alpha)$ and $\exp(i\beta)$, and draw the two triangles $\triangle_\alpha$ and $\triangle_\beta$. Then one sees easily that the sides of these triangles cross over the interval I_j to form a little "triangle" T_j, one side of which is the arc I_j. Thus if t denotes a point of I_j, then any of the α_n sufficiently near to t must lie in T_j.

Let $k(\phi)$ be the characteristic function of the set G, and define $f(z)$ by

$$f(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} k(\phi)\,\frac{z + e^{i\phi}}{z - e^{i\phi}}\,d\phi\right\}. \tag{18}$$

Then

$$|f(z)| = \exp\left\{\frac{-1}{2\pi}\int_0^{2\pi} k(\phi)\,\frac{1 - r^2}{|z - e^{i\phi}|^2}\,d\phi\right\} \tag{19}$$

for $z=r\exp(i\theta)$, and so $|f(z)|\leqq 1$ for $|z|<1$.

It is a well known property of the Poisson integral that

$$\lim |f(re^{i\theta})| = 1 \tag{20}$$

at almost all points $\exp(i\theta)$ of E_1.

We shall now show that

$$|f(z)| \leqq e^{-1/2} \qquad (z\in T_j, j = 1, 2, \cdots). \tag{21}$$

Consider the arc I_j, with endpoints $\exp(i\alpha)$ and $\exp(i\beta)$ $(0\leqq\alpha\leqq\beta<2\pi)$. From (19) we have $|f(z)|=\prod|f_n(z)|$ where

$$f_n(z) = \exp\left\{\frac{1}{2\pi}\int_{I_n}\frac{z+e^{i\phi}}{z-e^{i\phi}}\,d\phi\right\}.$$

Since each $|f_n(z)|\leqq 1$ for $|z|<1$ this means that $|f(z)|\leqq|f_j(z)|$. So (21) will be proven if we can show that

$$\frac{1}{2\pi}\int_\alpha^\beta\frac{1-|z|^2}{|z-e^{i\phi}|^2}\,d\phi \geqq \frac{1}{2} \qquad (z\in T_j). \tag{22}$$

It is well known that this integral has a simple geometrical interpretation (see, for example, Nevanlinna [5, pp. 6–7]). Namely, one extends the line segment from $\exp(i\alpha)$ to z till it meets the boundary of the unit circle at a point w_1. Similarly, extend the line segment from $\exp(i\beta)$ to z till it meets the boundary in a point w_2. Then the integral is equal to the arc length from w_1 to w_2 (in the counterclockwise direction) divided by 2π. Using this, one sees that the minimum of the integral in (22) for $z\in T_j$ is attained at the interior vertex of T_j, and that this minimum value is $1/2+(\beta-\alpha)/2\pi$. This establishes (22) and therefore also (21).

Now let $t=\exp(i\theta)$ be a point of E_1, interior to the arc I, at which the relation (20) holds. For simplicity we assume $t=1$. Then there is an $\epsilon>0$ such that if $\mathrm{Re}(\alpha_n)>1-\epsilon$, then α_n is in one of the "triangles" T_j. By (21), $|f(\alpha_n)|\leqq\exp(-1/2)$ at all such points α_n. Let $g(z)=f(z)e^z$. Then $g\in H_\infty$ and $\|g\|=e$. But

$$|g(\alpha_n)| \leqq e^{1/2} \qquad \text{for } \mathrm{Re}\,(\alpha_n) > 1-\epsilon,$$
$$|g(\alpha_n)| \leqq e^{1-\epsilon} \qquad \text{for } \mathrm{Re}\,(\alpha_n) \leqq 1-\epsilon,$$

and so $\sup|g(\alpha_n)|<\|g\|$. In other words, condition (iii) is not satisfied.

This completes the proof of Theorem 3.

REMARK 1. The assumption that the α_n have no interior limit points in G was only needed for the implication (i)→(iii) where Lemma 4 was applied. As noted after Lemma 4, this assumption can be weakened slightly.

REMARK 2. Let $\{\beta_n\}$ be a sequence of points inside the unit circle for which $\sum(1-|\beta_n|)<\infty$. Then almost no boundary points are approachable nontangentially by subsequences of $\{\beta_n\}$.

Indeed, let B be the set of boundary points that can be approached nontangentially. Let $f \in H_\infty$ $(f \not\equiv 0)$ vanish on $\{\beta_n\}$. Then f has radial limits almost everywhere on B, and these limits must all be zero as the radial limit is the same as the limit inside of any angle. But the limit cannot be zero on a set of positive measure, and therefore $|B| = 0$.

The converse to this remark is false. In fact, one can give examples of sequences $\{\beta_n\}$ such that almost no boundary points are approachable nontangentially, and such that if C is any sector of the unit circle then

$$\sum_{\beta_n \in C} (1 - |\beta_n|) = \infty.$$

REMARK 3. If $\{\alpha_n\}$ satisfies condition (iv) of Theorem 3 and $\{\alpha_{n_j}\}$ is a subsequence such that

$$\sum (1 - |\alpha_{n_j}|) < \infty \tag{23}$$

then $\{\alpha_n\} \backslash \{\alpha_{n_j}\}$ satisfies condition (iv) (the symbol $\backslash$ denotes set-theoretic difference).

This follows from Remark 2.

REMARK 4. Let $\{\alpha_n\}$ satisfy condition (iv). Then there are coefficients $\{a_n\}$, *all of which are different from zero*, such that (1) holds. Further, if h is any entire function admitting a representation (14) then the coefficients $\{a_n\}$ may be chosen all different from zero.

Indeed, by (ii) there are coefficients $\{a_{i1}\}$ $(i = 2, 3, \cdots)$ such that

$$0 = \frac{1}{2} \exp(\alpha_1 z) + \sum_{i=2}^{\infty} a_{i1} \exp(\alpha_i z), \qquad \sum_2^{\infty} |a_{i1}| < 1.$$

Choose b_2 such that $|b_2| < 1/4$ and $b_2 \neq -a_{21}$. Then applying (ii) again we find coefficients $\{a_{i2}\}$ $(i = 3, 4, \cdots)$ such that

$$0 = b_2 \exp(\alpha_2 z) + \sum_{i=3}^{\infty} a_{i2} \exp(\alpha_i z), \qquad \sum_3^{\infty} |a_{i2}| < \frac{1}{2} \cdot$$

Choose b_3 such that $|b_3| < 1/8$ and $b_3 \neq -(a_{31} + a_{32})$. Then there are coefficients $\{a_{i3}\}$ $(i = 4, 5, \cdots)$ such that

$$0 = b_3 \exp(\alpha_3 z) + \sum_{i=4}^{\infty} a_{i3} \exp(\alpha_i z), \qquad \sum_4^{\infty} |a_{i3}| < \frac{1}{4} \cdot$$

This process may be continued. If we add up all the resulting equations we have a representation of zero in which no coefficients vanish.

Let now h admit a representation (14) and let $\{b_n\}$ all different from zero be chosen such that $\sum b_n \exp(\alpha_n z) \equiv 0$. Then

$$h(z) = \sum a_n \exp(\alpha_n z) + \lambda \sum b_n \exp(\alpha_n z)$$

for all complex λ. It only remains to choose λ such that

$$a_n + \lambda b_n \neq 0 \qquad (n = 1, 2, \cdots).$$

REMARK 5. Let $\{\alpha_n\}$ satisfy condition (iv) and let $\{\alpha_{n_j}\}$ be a subsequence for which (23) holds. Let $h(z)$ be any entire function admitting a representation (14), and let $\{b_j\} \in l_1$ be given. Then one can find coefficients $\{a_n\} \in l_1$ such that $h(z) = \sum a_n \exp(\alpha_n z)$ and $a_{n_j} = b_j$. In other words, certain infinite subsets of coefficients can be prescribed arbitrarily in l_1.

Indeed, let $\phi(z) = \sum b_j \exp(\alpha_{n_j} z)$. By Remark 2 both $\phi(z)$ and $h(z)$ can be represented in terms of the exponents $\{\alpha_n\} \setminus \{\alpha_{n_j}\}$:

$$\phi(z) = \sum c_n \exp(\alpha_n z) \qquad (c_{n_j} = 0, j = 1, 2, \cdots),$$
$$h(z) = \sum d_n \exp(\alpha_n z) \qquad (d_{n_j} = 0, j = 1, 2, \cdots).$$

Then

$$h(z) = \sum d_n \exp(\alpha_n z) - \sum c_n \exp(\alpha_n z) + \sum b_j \exp(\alpha_{n_j} z),$$

so we may choose $a_n = (d_n - c_n)$ for $n \notin \{n_j\}$, and $a_{n_j} = b_j$.

REMARK 6. It is easy to give examples of sequences $\{\alpha_n\}$ satisfying condition (iv). For example, let $r_n \to 1$ $(r_n < 1)$, let $c > 0$ be given, and let $\phi(n)$ be any integer-valued function for which

$$\phi(n) > \frac{c}{1 - r_n} \qquad (n = 1, 2, \cdots).$$

The sequence $\{\alpha_n\}$ is now formed by taking $\phi(n)$ points, equally spaced, on the circle $|z| = r_n$ $(n = 1, 2, \cdots)$. It can be verified that every boundary point p can be approached inside of an angle of opening 2β, placed symmetrically about the radius to p, for all β such that $\tan \beta > \pi/c$.

From this it follows by conformal mapping that in any simply connected domain G (with at least two boundary points) sequences $\{\alpha_n\}$ exist that satisfy (iii), and have no interior limit points.

REMARK 7. It might be asked whether one couldn't require something more than absolute convergence of the coefficients in (1). For example, is it possible to have a representation of zero in which $\sum |a_n|^\delta < \infty$ for every $\delta > 0$?

In this connection Denjoy has shown that if D is any closed set in the extended plane, and U is an open set containing D, and ϕ is any function analytic on U (and vanishing at $z = \infty$ if D is unbounded), and $\epsilon > 0$ is given, then there exist sequences $\{a_n\}$, $\{\alpha_n\}$, with $\alpha_n \notin D$ $(n = 1, 2, \cdots)$ such that

$$\phi(z) = \sum \frac{a_n}{\alpha_n - z} \qquad (z \in D)$$

and

$$|a_n| < k \exp(-n^{1/2-\epsilon}).$$

See [5, p. 3] and [6].

REMARK 8. The sets $\{\alpha_n\}$ that satisfy condition (iv) seem to play a role analogous to the boundary. Thus one is led to look for analogues of theorems about the boundary. Consider, for example, the following theorem: if f is analytic for $|z|<1$ and continuous for $|z|\leqq 1$ and if f is schlicht on the boundary, then f is schlicht inside.

An analogue might be: let $\{\alpha_n\}$ satisfy condition (iv) and let $f\in H_\infty$ be such that to every $\delta>0$ there corresponds an $\epsilon>0$ such that $|\alpha_n-\alpha_m|\geqq\delta$ implies $|f(\alpha_n)-f(\alpha_m)|\geqq\epsilon$; then f is schlicht in $|z|<1$. (It would not be enough to require merely that f be one-to-one on $\{\alpha_n\}$; for example, if no two of the α_n are diametrically opposed then z^2 is one-to-one on $\{\alpha_n\}$.) We do not know if this conjecture is true, nor do we know if this property implies condition (iv).

Consider now the problem of determining the values of an analytic function inside the circle by means of its values on or near the boundary. For example, if f is analytic for $|z|<1$ and continuous for $|z|\leqq 1$ then the Cauchy integral

$$f(z) = \frac{1}{2\pi i}\int_{|w|=1} f(w)\,\frac{1}{w-z}\,dw$$

gives the solution to the problem. This may be written

$$f(z) = \int f(w)\,d\mu_z(w) \tag{24}$$

where the measure μ_z depends only on the point z and not on the function f. Another possible choice for the measures μ_z is

$$d\mu_z(w) = \left(\frac{1}{w-z} + \frac{\bar{z}}{1-\bar{z}w}\right)dw \tag{25}$$

(Macintyre, Rogosinski [12, p. 304]). These measures are all of total variation 1 on $|w|=1$, independently of z ($|z|<1$). (The absolute value of the expression in parentheses is the Poisson kernel.)

Theorem 3 enables us to obtain a formula analogous to (24), with the integral replaced by a series. Again let $S=\{\alpha_n\}$ be a sequence of distinct points inside the unit circle with no interior limit points.

THEOREM 4. *The sequence $S=\{\alpha_n\}$ has the property* (v) *defined below if and only if it has property* (i) *of Theorem* 3.

(v) *For each point z inside the unit circle and each $\epsilon>0$ there exist coefficients $\{a_n(z;\epsilon)\}$ for which*

$$f(z) = \sum a_n(z;\epsilon)f(\alpha_n), \qquad \sum |a_n(z;\epsilon)| < 1+\epsilon, \tag{26}$$

for all $f\in H_\infty$.

Proof. Let (v) hold, and choose a point $z_0 \notin S$. From (26) and Lemma 3 we obtain a representation of zero using z_0 and $\{\alpha_n\}$. Using Lemma 2 we can drop out the z_0 term, thus obtaining (1).

Conversely, let (1) hold, and assume $z_0 \notin S$ (since otherwise (26) is trivial). Then (26) follows from (17) and Lemma 3.

Unfortunately, Theorem 4 does not give any explicit method for determining the coefficients $\{a_n(z; \epsilon)\}$. If one seeks to eliminate the ϵ in (v), then one must eliminate the ϵ in (17). We do not know whether this can be done or not. However, J. Wolff [21] has shown that if z_0 is given, then there exist numbers $\{\alpha_n\}$ with $z_0 \notin \{\alpha_n\}$, and there exist coefficients $\{a_n\}$ such that (26) holds with $\epsilon = 0$. In his example the α_n have many interior limit points. See the remarks following Theorem 8 of §4 for a further discussion.

We now mention two applications of Theorem 3 that were pointed out to us by John Wermer. Let $\{\alpha_n\}$ be a bounded sequence of distinct complex numbers. We shall say that $\{\alpha_n\}$ "represents zero" if coefficients $\{a_n\}$ can be found, not all zero, such that (1) holds.

Let T be a bounded normal operator on Hilbert space H. Let the eigenvectors of T span H and let $\alpha_1, \alpha_2, \cdots$ be the distinct eigenvalues of T. Wermer [20] has shown that the following statements are equivalent.

(a) Every invariant subspace of T is spanned by the eigenvectors it contains.

(b) Every invariant subspace of T contains an eigenvector.

(c) Every subspace invariant under T is also invariant under T^*.

(d) $\{\alpha_n\}$ does not represent zero.

J. E. Scroggs [17] has a number of further results in this direction.

Tord Hall [7] has considered the following problem. Let $\{A_n\}$, $\{\alpha_n\}$ be two sequences of complex numbers with the α_n all distinct and different from zero. It is further assumed that $A_n \geqq 1$ $(n = 1, 2, \cdots)$ and that

$$\lim_n |\alpha_n|^k / A_n = 0 \qquad (k \geqq 0). \tag{27}$$

Let Q denote the set of all polynomials such that $|p(\alpha_n)| \leqq A_n$ $(n = 1, 2, \cdots)$, and let $M = \sup |p(0)|$ $(p \in Q)$.

Hall proves that $M < \infty$ if and only if there exist coefficients $\{a_n\}$ such that

$$\sum |a_n| \leqq 1, \qquad \sum_n \frac{a_n}{A_n} (\alpha_n)^k = 0 \ (k > 0), \qquad \sum \frac{\alpha_n}{A_n} \neq 0. \tag{28}$$

Condition (27) with $k = 0$ implies $A_n \to \infty$. If the set $\{\alpha_n\}$ is bounded, then condition (28) is equivalent to

$$\sum \frac{a_n}{A_n} \exp(\alpha_n z) \equiv \text{const.} \ (\neq 0), \qquad \sum |a_n| \leqq 1.$$

Thus if $\{\alpha_n\}$ cannot represent zero, then $M=\infty$, no matter how the $\{A_n\}$ are chosen.

4. **Functions representable by exponentials.** In this section we study the space $E(G)$ where G is a Jordan region (see (14)).

THEOREM 5. *Let G be a Jordan region. Then $E(G)$ is a Banach space in the norm* (15), *and the conjugate space is $H_\infty(G)$ with the supremum norm.*

Proof. By Theorem 3 and Remark 6 there exists a sequence $\{\alpha_n\}$ in G such that every function in $E(G)$ can be represented using only the exponents $\{\alpha_n\}$, and the norm is not increased. Let N denote the subset of l_1 consisting of those sequences $\{a_n\}$ for which $\sum a_n \exp(\alpha_n z)\equiv 0$. N is a closed linear subspace of l_1 and $E(G)$ is isomorphic to the quotient space l_1/N. Hence $E(G)$ is a Banach space.

The conjugate space of l_1/N is $N^\perp$, the set of all elements in l_∞ that are orthogonal to N. Let B be the set of those bounded sequences that come from bounded analytic functions: $\{f(\alpha_n)\}$, $f\in H_\infty(G)$. One half of Lemma 3 tells us that $B\subset N^\perp$, while the other half tells us that B is weak-star dense in $N^\perp$ (since $N^{\perp\perp}=N$). The proof of (iii)$\to$(ii) in Theorem 3 showed that B is weak-star closed. Hence $B=N^\perp$. Since $\|f\|=\sup|f(\alpha_n)|$, the conjugate space is isomorphic to $H_\infty(G)$, which completes the proof.

Let $h=\sum a_n \exp(\alpha_n z)\in E(G)$, and let $f\in H_\infty$. Then (h, f), the value of the linear functional f at the point h, is given by

$$(h,f) = \sum a_n f(\alpha_n). \tag{29}$$

If G is the unit circle this can be rewritten in terms of the Taylor series of f and h. Let $h=\sum A_n z^n/n!$ and $f=\sum B_n z^n$. It can be shown that

$$(h,f) = \lim_{r\to 1^-} \sum A_n B_n r^n \tag{30}$$

(first consider the case when the Taylor series for f contains but a single term).

Let now G be the unit circle. By H_1 we denote as usual the Hardy space of functions $f(z)$, analytic in the unit circle and satisfying the condition

$$\|f\|_1 = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \infty. \tag{31}$$

H_1 is a Banach space in this norm (whose conjugate space is not H_∞!). The radial limits $f(e^{it})=\lim f(re^{it})$ $(r\to 1^-)$ exist for almost all t, and $f(e^{it})\in L_1(0, 2\pi)$. Further, $\|f(e^{it})\|_{L_1}=\|f\|_{H_1}$, so H_1 is isometrically imbedded as a subspace of L_1.

Let F denote the Banach space L_1/H_1. F can be interpreted as the space of those entire functions $h(z)$ that admit a representation

$$h(z) = \frac{1}{2\pi i}\int_{|w|=1} e^{zw}\phi(w)\,dw \qquad (\phi\in L_1(0, 2\pi)). \tag{32}$$

(We shall say that ϕ represents h.)

Indeed, if $\int e^{zw}\phi(w)dw\equiv 0$ then the Fourier coefficients of ϕ vanish on one side and therefore $\phi\in H_1$ [15, Chapter 2]. Conversely, if $\phi\in H_1$ then $\int e^{zw}\phi(w)dw\equiv 0$ by the Cauchy integral theorem.

The norm in F is the quotient space norm:

$$\|h\|_F = \inf \|\phi\|_{L_1} \tag{33}$$

taken over all ϕ that represent h.

The conjugate space of F is the annihilator of H_1 in L_∞. If $f\in L_\infty$, $\phi\in L_1$ we denote the value of the functional f at the point ϕ by (ϕ, f), defined by

$$(\phi, f) = \frac{1}{2\pi i}\int_{|w|=1} f(w)\phi(w)dw. \tag{34}$$

If $\phi\in H_1$ then it is analytic inside the unit circle. If f is the boundary function of a function in H_∞ then $(\phi, f)=0$ by the Cauchy integral theorem. Conversely, if $(\phi, f)=0$ for all $\phi\in H_1$ then the Fourier coefficients of f vanish on one side and therefore $f\in H_\infty$.

Thus the conjugate space of F is H_∞ with the supremum norm. If $h(z) = \sum A_n z^n/n!\in F$ and $f(z)=\sum B_n z^n\in H_\infty$ then

$$(h, f) = \lim_{r\to 1-} \sum A_n B_n r^n. \tag{35}$$

THEOREM 6. *The two function spaces E and F with the norms* (15) *and* (33) *are identical.*

Proof. We first show that F contains all pure exponentials $e^{\alpha z}(|\alpha|<1)$, and that $\|e^{\alpha z}\|_F\leqq 1$. Indeed, let

$$K_\alpha(w) = \frac{1}{w-\alpha} + \frac{\bar{\alpha}}{1-\bar{\alpha}w} \tag{36}$$

(see (25)). Then $e^{\alpha z}$ is represented by $K_\alpha(w)$ and $\|K_\alpha\|=1$, which is the result.

It follows that $E\subset F$ since by definition E consists of all absolutely convergent series of exponentials. Also, E and F have the same conjugate space: H_∞ in the supremum norm. If $h\in E$ and $f\in H_\infty$ then the value of the linear functional f at the point h, (h, f), is the same whether we regard h as an element of E or as an element of F (see (30) and (35)).

Therefore $\|h\|_E=\|h\|_F$ for all $h\in E$, since $\|h\|=\sup|(h, f)|$, taken over all $f\in H_\infty$ of norm one. So E is a closed subspace of F. But it cannot be a proper subspace, for then there would be a linear functional vanishing on E but not vanishing identically, which is impossible.

The first corollary is due to Wolff [21].

COROLLARY 1. *E contains all entire functions of exponential type less than one.*

Proof. For such a function the Borel transform is analytic part way inside the unit circle, and in the Borel transform representation of h (see (4)) we may take $r=1$. $F(w)$ is analytic on $|w|=1$, and thus is in $L_1(0, 2\pi)$.

In fact E contains all those entire functions whose Borel transforms are in H_1 in the exterior of the unit circle.

DEFINITION. A sequence $\{A_n\}$ $(n=0, 1, \cdots)$ will be said to be a Fourier sequence on one side if numbers $A_{-1}, A_{-2}, \cdots$ exist such that $\{A_n\}$ $(-\infty<n<\infty)$ is the sequence of complex Fourier coefficients of some integrable function on $(0, 2\pi)$.

COROLLARY 2. *A necessary and sufficient condition that $\{A_k\}$ $(k=0, 1, \cdots)$ be a Fourier sequence on one side is that there exist two sequences of complex numbers, $\{a_n\}$, $\{\alpha_n\}$ $(\sum|a_n|<\infty, |\alpha_n|<1$ $(n=1, 2, \cdots))$ such that*

$$\sum_{n=1}^{\infty} a_n(\alpha_n)^k = A_k \qquad (k = 0, 1, 2, \cdots). \tag{37}$$

Proof. Let $\{a_n\}$, $\{\alpha_n\}$ be given and let $h(z)=\sum a_n \exp(\alpha_n z)$. By Theorem 6 there is a $\phi\in L_1$ such that $h(z)=(1/2\pi i)\int e^{zw}\phi(w)dw$. Differentiating these two representations k-times and evaluating at $z=0$ we obtain (37), where the A_k are the Fourier coefficients on one side of ϕ.

Conversely, given $\phi\in L_1$ we define h by (32). By Theorem 4, $h\in E$ and so h also has a representation (14). Just as in the preceding paragraph we obtain (37). In other words, the coefficients of any given $\phi\in L_1$ have a representation (37).

REMARK 9. If μ is a measure concentrated entirely on the unit circumference $|w|=1$, and μ is not purely absolutely continuous, then $\int e^{zw}d\mu(w)$ is not in E.

Indeed, if it were in E then by Theorem 4 there would be a purely absolutely continuous measure σ on $|w|=1$ such that $\int e^{zw}d(\mu-\sigma)\equiv 0$. But by a theorem of F. and M. Riesz [15, Chapter 2, §5] this would imply that $\mu-\sigma$ were absolutely continuous, which would be a contradiction.

H. S. Shapiro has pointed out to us that the work of Macintyre-Rogosinski [12], Havinson [8], and Rogosinski-Shapiro [16] provides further information about the space E and enables us to compute the norms of elements in E in simple cases. (There is considerable duplication in the last two papers, the last one having been written in ignorance of Havinson's work.)

Theorem A of [16] applied to E yields the following result.

THEOREM 7. *For each $h\in E$ $(h\neq 0)$ there exists a unique $\phi\in L_1$ and a unique $f\in H_\infty$ such that ϕ represents h (see (32)) and $\|h\|_E=\|\phi\|_{L_1}=(h, f)$.*

Thus the infimum in (33) is uniquely attained.

COROLLARY. *Let $h\in E$. Then $\|h'\|\leqq\|h\|$, and there is a unique $h_1\in E$ such that $h_1'=h$ and $\|h_1\|=\|h\|$.*

Proof. The first statement is obvious since $\sum |a_n\alpha_n| < \sum |a_n|$. For the second statement, let $\phi \in L_1$ provide the minimal representation of h as in Theorem 7. Then $\phi(w)/w = \phi_1(w)$ represents h_1, and $\|\phi_1\| = \|\phi\|$. There could not be a second such primitive, say h_2, represented minimally by some ϕ_2. For then $w\phi_2$ would represent $h_2' = h$, but the minimal representation of h is unique.

REMARK 10. If $h \in E$ then $|h(0)| \leqq \|h\|$. Indeed, $|h(0)| = |\sum a_n| \leqq \sum |a_n|$.

We now give the norms of several elements of E, and of one linear operator on E.

1. $\|e^{\alpha z}\| = 1$ $(|\alpha| < 1)$.
2. $\|z^n/n!\| = 1$ $(n = 0, 1, 2, \cdots)$.
3. $\|ze^{\alpha z}\| = 1/(1 - |\alpha|^2)$.
4. $\|e^{\alpha z} - 1\| = 2|\alpha|/(1 + (1 - |\alpha|^2)^{1/2})$ $(|\alpha| < 1)$.
5. $\|e^{bz} - e^{az}\| = 2|c|/(1 + (1 - |c|^2)^{1/2})$ $(c = (b - a)/(1 - \bar{a}b))$.
6. $\|D_0^{-1}\| = 2$, where D_0^{-1} is the integration operator $(D_0^{-1}h)(z) = \int_0^z h(s)ds$.

Proof. 1. This is obvious from Remark 10 and the definition of norm.

2. $z^n/n!$ can be represented by $1/w^{n+1}$, and hence $\|z^n/n!\| \leqq 1$. By differentiating n times we get the reverse inequality.

3. Since $\|h\| = \sup |(h, f)|$ $(\|f\| \leqq 1)$, the assertion to be proven is equivalent to: $\sup |f'(\alpha)| = 1/(1 - |\alpha|^2)$. This result is given on page 304 of [12].

4. $\|e^{\alpha z} - 1\| = \sup |f(\alpha) - f(0)|$ taken over all f in the unit ball of H_∞. By Theorem 16 of [16] the extremal f has the form: $f(z) = (z - a)/(1 - \bar{a}z)$. Therefore,

$$|f(\alpha) - f(0)| = \frac{|\alpha|(1 - |a|^2)}{|1 - \bar{a}\alpha|}.$$

It follows that the extremal f must have $a = r\alpha/|\alpha|$ for some $0 < r < 1$. Therefore,

$$|f(\alpha) - f(0)| = \frac{|\alpha|(1 - r^2)}{1 - r|\alpha|} = \phi(r).$$

We wish to maximize $\phi(r)$. Putting the derivative equal to zero and solving for r we obtain: $r|\alpha| = 1 - (1 - |\alpha|^2)^{1/2}$. (The plus sign is excluded since r must lie in the unit interval.) Substituting this value for r into the previous equation and simplifying we obtain the result.

5. $\sup |f(b) - f(a)| = \sup |f(\phi(b)) - f(\phi(a))|$ for any bilinear map ϕ of the circle onto itself. Choose $\phi = (z - a)/(1 - \bar{a}z)$, which reduces this case to the previous one: $\|e^{bz} - e^{az}\| = \|e^{cz} - 1\|$ for $c = (b - a)/(1 - \bar{a}b)$.

6. Let $h \in E$ be given and let h_1 be the minimal primitive (see the corollary to Theorem 7). Then $(D_0^{-1}h)(z) = h_1(z) - h_1(0)$, and therefore $\|D_0^{-1}h\| \leqq 2\|h\|$. On the other hand, $\|D_0^{-1}e^{\alpha z}\| = (1/\alpha)\|e^{\alpha z} - 1\| \to 2$ as $\alpha \to 1$, from 5.

We now consider the "positive" elements in E. Let E_+ be the set of all those $h \in E$ for which

$$h(0) = \|h\|. \tag{38}$$

Clearly E_+ is a positive cone, i.e., it is closed under addition and under multiplication by non-negative scalars. Also, if $h_1, \cdots, h_n \in E_+$ then $\|h_1 + \cdots + h_n\| = \|h_1\| + \cdots + \|h_n\|$.

THEOREM 8. *$h \in E_+$ if and only if h can be represented by a $\phi \in L_1$ for which $w\phi(w) \geqq 0$ almost everywhere on $|w| = 1$. In this case ϕ provides the unique minimal representation of h.*

Proof. Let $h \in E_+$ and let ϕ be the unique extremal kernel in L_1 representing h. Then

$$\begin{aligned} h(0) &= \frac{1}{2\pi i}\int_{|w|=1} \phi(w)dw = \frac{1}{2\pi}\int_0^{2\pi} \phi(w)dw\theta \leqq \frac{1}{2\pi}\int_0^{2\pi} |\phi(w)|\, d\theta \\ &= \|\phi\| = \|h\| = h(0), \qquad (w = e^{i\theta}) \end{aligned}$$

and so $w\phi(w) \geqq 0$ almost everywhere.

Conversely, if $w\phi(w) \geqq 0$ almost everywhere then

$$\|h\| \geqq |h(0)| = \frac{1}{2\pi}\int \phi(w)wd\theta = \frac{1}{2\pi}\int |\phi(w)|\, d\theta \geqq \|h\|,$$

the last inequality following from the definition of norm (33), and so $h \in E_+$ which completes the proof.

Suppose h can be represented by an exponential sum with non-negative coefficients

$$h(z) = \sum a_n \exp(\alpha_n z), \qquad a_n \geqq 0 \ (n = 1, 2, \cdots). \tag{39}$$

Then one sees easily that $h \in E_+$. We do not know whether the converse is true. However, the representation (39) is never unique when it does exist.

Indeed, Wolff [21] proved that if a set of disjoint circles is removed from the unit circle so that only a set of measure zero remains, then

$$1 = \sum r_n^2 \exp(\alpha_n z)$$

where r_n denotes the radius of the nth removed circle and α_n is its center. By applying a bilinear transformation to this example (see the corollary to Lemma 3) we may represent any given exponential $e^{\alpha z}(|\alpha| < 1)$ using the same coefficients and different exponents. It follows from this that if a function has a representation (39) then it has uncountably many such representations.

REMARK 11. If $h \in E$ has a minimizing sequence of exponents and coefficients, i.e. if there exist $\{a_n\}$, $\{\alpha_n\}$ such that $h(z) = \sum a_n \exp(\alpha_n z)$ and $\sum |a_n| = \|h\|$, then signum $a_n = $ const., and $|h(0)| = \|h\|$.

Indeed, let $f\in H_\infty$, $\|f\|=1$, be such that $(h, f)=\|h\|$. Then: $\|h\|=(h, f) = \sum a_n f(\alpha_n) \leqq \sum |a_n f(\alpha_n)| \leqq \sum |a_n| = \|h\|$. Hence $|f(\alpha_n)|=1$, and so by the maximum modulus theorem $f=$const. Therefore, signum $a_n=$const., and $|h(0)| = \sum |a_n|$.

We now consider a different representation of the space $E(G)$. Let G be an arbitrary Jordan domain. Let $M(G)$ be the set of all those entire functions $h(z)$ that can be represented in the form

$$h(z) = \int e^{zw} d\mu(w) \tag{40}$$

where μ is a Borel measure in G. We define a norm in $M(G)$ by

$$\|h\| = \inf \operatorname{Var}(\mu) \tag{41}$$

taken over all representations (40) of h.

THEOREM 9. *The two spaces* $E(G)$ *and* $M(G)$ *with the norms* (15) *and* (41) *are identical.*

Proof. Clearly $E(G)\subset M(G)$ and $\|h\|_M \leqq \|h\|_E$ for all $h\in E(G)$.

To prove the converse, let $h\in M(G)$. We first consider the case where h can be represented in (40) by a measure μ whose support is entirely inside G. Since the support (the smallest closed set containing all the mass) is a closed set, it is at a positive distance from the boundary of G. Hence the Borel transform of h is analytic on the boundary of G, and therefore by a result of Wolff [21], $h\in E(G)$.

To estimate the norm of h, let $\{\mu_n\}$ be a sequence of measures in G, each consisting of a finite number of mass points, such that: $\operatorname{Var}(\mu_n)=\operatorname{Var}(\mu)$, and $\int f d\mu_n \to \int f d\mu$ for all $f\in H_\infty(G)$. Let $h_n(z)=\int e^{zw} d\mu_n(w)$. Then h_n is a finite exponential sum and therefore $h_n\in E(G)$. Also, $(h_n, f)\to(h, f)$ for each $f\in H_\infty(G)$. Hence, $\|h\|_E \leqq \lim\inf \|h_n\|_E \leqq \operatorname{Var}(\mu)$. (If E were known to be weakly sequentially complete then Wolff's result would not be needed.)

Now let $h\in M(G)$ be arbitrary. Let σ be any measure in G that represents h. Since G is an open set, there are compact sets $C_1\subset C_2 \subset \cdots$ whose union is G. Let σ_1 be the restriction of σ to C_1, and for $n>1$ let σ_n be the restriction of σ to $C_n \backslash C_{n-1}$. Then $\sum \operatorname{Var}(\sigma_n)=\operatorname{Var}(\sigma)$.

Let $h_n(z)=\int e^{zw} d\sigma_n(w)$. Then $\|h_n\|_M \leqq \operatorname{Var}(\sigma_n)$ and therefore $h=\sum h_n$, the series converging in $M(G)$, and also converging uniformly on compact subsets of the plane. By our previous result, $h_n\in E(G)$ and $\|h_n\|_E \leqq \operatorname{Var}(\sigma_n)$. Hence the series converges in $E(G)$. Therefore $h\in E(G)$ and $\|h\|_E \leqq \operatorname{Var}(\sigma)$. Consequently $\|h\|_E \leqq \inf \operatorname{Var}(\sigma) = \|h\|_M$. This completes the proof.

5. **Additional properties of the space** E. In this section we work entirely in the unit circle and accordingly we write E instead of $E(G)$. We do not know whether E has a Schauder basis or not, but we shall show that the Taylor series is not such a basis.

THEOREM 10. *There is an $h \in E$ whose Taylor series does not converge in the norm of E.*

Proof. Let T_n be the projection operator sending each $h \in E$ onto the nth partial sum of its Taylor series. If the Taylor series always converged the T_n would be uniformly bounded. Since $\|e^{\alpha z}\| = 1$ for $|\alpha| < 1$ we have:

$$\|T_n\| \geqq \|T_n e^{\alpha z}\| = \left\| \sum_0^n \alpha^k z^k / k! \right\| = \sup \left| \sum_0^n B_k \alpha^k \right|$$

where the supremum is over all $f(z) = \sum B_k z^k \in H_\infty$ such that $\|f\| \leqq 1$. The last equality is merely the statement that $\|h\| = \sup |(h, f)|$.

Since α is arbitrary inside the unit circle we have

$$\|T_n\| \geqq \sup \left| \sum_0^n B_k \right|,$$

and by a result of Landau [10, pp. 26–27] this is asymptotic to $(1/\pi) \log n$, which completes the proof.

COROLLARY. *There are functions $h(z) = \sum A_n z^n / n! \in E$ and $f(z) = \sum B_n z^n \in H_\infty$ for which the series $\sum A_n B_n$ is not convergent.*

Proof. It is known (Day [4, p. 69]) that a weak Schauder basis is a strong basis. Hence the Taylor series is not weakly convergent for all $h \in E$. QED.

The series is of course always Abel summable (see (30)).

PROBLEM (TAYLOR [18, p. 33]). If $\{A_n\}$ has the property that $\lim \sum A_n B_n r^n$ $(r \to 1^-)$ exists for all $f(z) = \sum B_n z^n \in H_\infty$, does it follow that $h(z) = \sum A_n z^n / n!$ is in E? This would follow if it were known that E is weakly sequentially complete.

The space H_∞ may be considered as a space of linear operators on E. Indeed, if $f \in H_\infty$ and $h(z) = \sum a_n \exp(\alpha_n z) \in E$ then we define the operator $T_f \colon E \to E$ by:

$$(T_f h)(z) = \sum a_n f(\alpha_n) \exp(\alpha_n z). \tag{42}$$

T_f may also be considered as an infinite order differential operator: if $f(z) = \sum B_n z^n$ and $D = d/dz$ then formally $T_f = \sum B_n D^n$.

In analogy to the problem stated above we may ask: if h belongs to the domain of all the operators T_f $(f \in H_\infty)$, must $h \in E$?

The adjoint operator $T_f^* \colon H_\infty \to H_\infty$ is simply multiplication by f.

$$(T_f^* g)(z) = f(z) g(z). \tag{43}$$

REMARK 12. If $f(e^{i\theta})$ is essentially bounded away from zero (i.e. if $|f(e^{i\theta})| \geqq c > 0$ almost everywhere), then T_f is onto.

Indeed, if h is given by (32) let $h_1(z)$ be represented by $\phi(w)/f(w)$. Then $T_f h_1 = h$.

REMARK 13. If f is analytic for $|z| \leqq 1$ and has no zeros for $|z| < 1$, then T_f is one-to-one.

Indeed, let $h \in E$ and assume $T_f h = 0$. For any $\epsilon > 0$, h maybe expressed in terms of its Borel transform (4) with $r = 1 + \epsilon$. Then

$$(T_f h)(z) = \frac{1}{2\pi i} \int_{|w|=1+\epsilon} e^{zw} f(w) H(w) dw \equiv 0, \tag{44}$$

where ϵ is chosen so that f is analytic on $|w| = 1 + \epsilon$. From (44) it then follows that $f(w)H(w)$ ($|w| = 1 + \epsilon$) are the boundary values of a function analytic for $|w| < 1 + \epsilon$. Since H is analytic for $|w| > 1$, and f has no zeros for $|w| < 1$, it follows that H can be at worst a rational function with poles only on $|w| = 1$. But then h would have the form $\sum P_k(z) \exp(w_k z)$, where the P_k are polynomials, and $|w_k| = 1$, and such a function is not in E (for example, if $H(w) = 1/(w-1)$ then $h(z) = e^z$). Thus H has no poles at all, and therefore $H = 0$.

BIBLIOGRAPHY

1. S. Banach, *Théorie des opérations linéaires*, Warszawa-Lwow, 1932.

2. L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, Leipzig-Berlin, 1927.

3. R. P. Boas, *Entire functions*, New York, Academic Press, 1954.

4. M. M. Day, *Normed linear spaces*, Ergebnisse der Mathematik, vol. 21, Berlin, Springer-Verlag, 1958.

5. A. Denjoy, *Sur les singularités des séries de fractions rationnelles*, Rend. Circ. Mat. Palermo vol. 50 (1926) pp. 1–95.

6. ———, *Sur les séries de fractions rationnelles*, Bull. Soc. Math. France vol. 52 (1924) pp. 418–434.

7. Tord Hall, *On polynomials bounded at an infinity of points*, Thesis, Uppsala, Appelbergs Boktryckeri, 1950.

8. S. Ya. Havinson, *On some extremal problems of the theory of analytic functions*, Moscov. Gos. Univ. Uč. Zap. vol. 148 (1951), Matematika vol. 4, pp. 133–143 (Russian).

9. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, 1957.

10. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 2d ed., 1929.

11. B. Ya. Levin, *Distribution of the roots of entire functions*, Moscow, 1956 (Russian).

12. A. J. Macintyre and W. W. Rogosinski, *Extremum problems in the theory of analytic functions*, Acta Math. vol. 82 (1950) pp. 275–325.

13. S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, Uspehi Mat. Nauk vol. 7, no. 2 (1952) pp. 31–122; Amer. Math. Soc. Translation, no. 101.

14. R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936.

15. I. I. Privaloff, *Boundary properties of analytic functions*, Moscow-Leningrad, 1950 (Russian); Deutscher Verlag der Wissenschaften, Berlin, 1956 (German translation).

16. W. W. Rogosinski and H. S. Shapiro, *On certain extremum problems for analytic functions*, Acta Math. vol. 90 (1953) pp. 287–318.

17. J. E. Scroggs, *Invariant subspaces of a normal operator*, Duke Math. J. vol. 26 (1959) pp. 95–112.

18. A. E. Taylor, *Banach spaces of functions analytic in the unit circle* II, Studia Math. vol. 12 (1951) pp. 25–50.

19. J. L. Walsh, *Über die Entwicklung einer analytischen Funktion nach Polynomen*, Math. Ann. vol. 96 (1926) pp. 430–436.

20. J. Wermer, *On invariant subspaces of normal operators*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 270–277.

21. J. Wolff, *Sur les séries* $\sum A_k/(z-\alpha_k)$, C. R. Acad. Sci. Paris vol. 173 (1921) pp. 1057–1058, 1327–1328.

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