BOUNDS FOR DETERMINANTS WITH POSITIVE DIAGONALS

BY

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1. **Introduction.** In this paper upper and lower bounds are found for the determinant of a real, $n \times n$ matrix $A=(a_{ij})$, with positive diagonal elements satisfying

$$a_{ii} \geqq \sum_{j \neq i} |a_{ij}|, \qquad i = 1, 2, \cdots, n, \tag{1}$$

and a lower bound is found for determinants whose elements satisfy

$$a_{ii} \geqq nA_i^+ - \sum_{j \neq i} a_{ij}, \qquad i = 1, 2, \cdots, n, \tag{2}$$

where

$$A_i^+ = \frac{1}{2}\left(\max_{j \neq i} a_{ij} + \left|\max_{j \neq i} a_{ij}\right|\right).$$

G. B. Price [6], A. Ostrowski [4] and [5], J. L. Brenner [1] and [2], and H. Schneider [7] have given lower and upper bounds for the absolute value of determinants satisfying more general conditions than (1). However, the following theorem, proved in §2, is not implied by any of the above results:

THEOREM 1. *If $A=(a_{ij})$ has elements satisfying* (1), *it is possible to define L_i and R_i such that*

$$\begin{aligned} a_{ii} &= L_i + R_i, \\ L_i &\geqq \sum_{j<i} |a_{ij}|, \\ R_i &\geqq \sum_{j>i} |a_{ij}|. \end{aligned} \qquad (i = 1, \cdots, n). \tag{3}$$

Then, for any choice of L_i and R_i satisfying (3),

$$\sum_{k=0}^{n}\left(\prod_{i=1}^{k} L_i \prod_{i=k+1}^{n} R_i\right) \leqq \det A \leqq \sum_{k=0}^{n} \prod_{i=1}^{k-1} (L_i + 2R_i) L_k \prod_{i=k+1}^{n} R_i, \tag{4}$$

where an empty product is defined to be 1.

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Ostrowski [5] has shown that if A satisfies (2) then $\det A \geqq 0$. In §3 we prove

THEOREM 2. *If A satisfies* (2), *then*

$$\det A \geqq \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} - nA_i^+\right).$$

In proving this we prove a result which may be used to improve any bound depending only on the nondiagonal elements in the row, when the diagonal elements are positive.

2. **Proof of Theorem 1.** To prove Theorem 1 we need the following bound given by Price [6]:

If (1) holds, then

$$\prod_{i=1}^{n}(a_{ii} - r_i) \leqq \det A \leqq \prod_{i=1}^{n}(a_{ii} + r_i) \tag{5}$$

where $r_i = \sum_{j>i} |a_{ij}|$.

We proceed by induction on n. Let D_n represent $\det A$, when A is of order n, and suppose the elements of A satisfy (3).

1. For $n=2$,

$$D_2 = \begin{vmatrix} L_1 + R_1 & a_{12} \\ a_{21} & L_2 + R_2 \end{vmatrix}.$$

Expanding D_2 by the diagonal elements,

$$D_2 = \begin{vmatrix} L_1 & a_{12} \\ 0 & L_2 \end{vmatrix} + \begin{vmatrix} L_1 & 0 \\ 0 & R_2 \end{vmatrix} + \begin{vmatrix} R_1 & 0 \\ a_{21} & R_2 \end{vmatrix} + \begin{vmatrix} R_1 & a_{12} \\ a_{21} & L_2 \end{vmatrix}.$$

Therefore

$$L_1L_2 + L_1R_2 + R_1R_2 \leqq D_2 \leqq R_1R_2 + L_1R_2 + (L_1 + 2R_1)L_2,$$

since

$$0 \leqq \begin{vmatrix} R_1 & a_{12} \\ a_{21} & L_2 \end{vmatrix} \leqq (R_1 + a_{12})L_2 < 2R_1L_2,$$

by (3) and (5).

2. Assume that for any matrix of order $n-1$ with elements satisfying (3),

$$\sum_{k=0}^{n-1}\left(\prod_{i=1}^{k} L_i \prod_{i=k+1}^{n-1} R_i\right) \leqq D_{n-1} \leqq \sum_{k=0}^{n-1}\left(\prod_{i=1}^{k-1}(L_i + 2R_i)L_k \prod_{i=k+1}^{n-1} R_i\right). \tag{6}$$

If $D_n = \det A$, where $A = (a_{ij})$, $i, j = 1, \cdots, n$, and the elements a_{ij} satisfy (3), partition D_n as follows:

$$D_n = \begin{vmatrix} A_1 & a_2 \\ a_3 & L_n + R_n \end{vmatrix}$$

where $A_1 = (a_{ij})$, $i, j = 1, \cdots, n-1$; a_2 is the column vector with components a_{in}, $i = 1, \cdots, n-1$; a_3 is the row vector with components a_{nj}, $j = 1, \cdots, n-1$; and, as in (3),

$$L_n + R_n = a_{nn},$$
$$L_n \geqq \sum_{j=1}^{n-1} |a_{nj}|,$$
$$R_n \geqq 0.$$

Then we can write D_n as the sum of two determinants, i.e.,

$$D_n = \Delta + R_n \det A_1 \tag{7}$$

where

$$\Delta = \begin{vmatrix} A_1 & a_2 \\ a_3 & L_n \end{vmatrix}.$$

But the elements of Δ satisfy (1), hence, by (3) and (5),

$$\Delta \geqq \prod_{i=1}^{n} (a_{ii} - r_i) \geqq \prod_{i=1}^{n} (a_{ii} - R_i) = \prod_{i=1}^{n} L_i, \tag{8}$$

and

$$\Delta \leqq L_n \prod_{i=1}^{n-1} (a_{ii} + r_i) \leqq L_n \prod_{i=1}^{n-1} (L_i + 2R_i).$$

Also, by the inductive assumption, since A_1 is of order $n-1$, and, by (3),

$$R_i \geqq \sum_{j=i+1}^{n} |a_{ij}| \geqq \sum_{j=i+1}^{n-1} |a_{ij}|,$$

we have, using (6), (7) and (8),

$$D_n \geqq \prod_{i=1}^{n} L_i + R_n \sum_{k=0}^{n-1} \left(\prod_{i=1}^{k} L_i \prod_{i=k+1}^{n-1} R_i \right) = \sum_{k=0}^{n} \left(\prod_{i=1}^{k} L_i \prod_{i=k+1}^{n} R_i \right),$$

and

$$D_n \leqq L_n \prod_{i=1}^{n-1} (L_i + 2R_i) + R_n \sum_{k=0}^{n-1} \left(\prod_{i=1}^{k-1} (L_i + 2R_i) L_k \prod_{i=k+1}^{n-1} R_i \right)$$
$$= \sum_{k=0}^{n} \left(\prod_{i=1}^{k-1} (L_i + 2R_i) L_k \prod_{i=k+1}^{n} R_i \right).$$

3. **Proof of Theorem** 2. Suppose we have a lower bound for the determinant of A which holds when the diagonal elements are individually bounded from below. That is, suppose

$$a_{ii} \geqq c_i(A) \geqq 0, \qquad i = 1, 2, \cdots n, \tag{9}$$

implies

$$\det A \geqq m(A) \geqq 0,$$

where $c_i(A)$ is a function of the nondiagonal elements in the ith row, with $c_i(A)=0$ if $a_{ij}=0$ for all $j\neq i$, and $m(A)$ is a function of all the nondiagonal elements of A.

Let $D=\operatorname{diag}(d_1, d_2, \cdots, d_n)$ with $d_i\geqq 0$. We shall show that

$$\det(A + D) \geqq m(A) + \prod_{i=1}^{n} d_i. \tag{10}$$

It is this result which we will use to improve Ostrowski's lower bound for determinants satisfying (2).

Since $d_{ii}\geqq c_i(D)=0$ for all $i=1, 2, \cdots, n$, (10) is a direct consequence of the following theorem which we proved in [3]:

If two matrices A and B of the same order satisfy the same row hypothesis (i.e., (9)) which is, in turn, sufficient to prove $\det A\geqq 0$ *and* $\det B\geqq 0$, *then* $\det(A+B)\geqq \det A+\det B$.

A determinant whose elements satisfy (2) does not necessarily have a dominant diagonal unless $A_i^+=0$ $(i=1, \cdots, n)$. Ostrowski's result shows, however, that such determinants are never negative, and, as a corollary he proved that, if λ is any root of A,

$$\lambda \geqq \mu,$$

where

$$\mu = \min_i \left(\sum_{j=1}^{n} a_{ij} - nA_i^+ \right).$$

Since the determinant is the product of the roots, this would imply

$$\det A \geqq \mu^n.$$

This bound is, in general, less than that given by Theorem 2.

If A satisfies (2), let

$$b_{ij} = a_{ij}, \qquad j \neq i,$$
$$b_{ii} = nA_i^+ - \sum_{j\neq i} a_{ij}.$$

Then $B_i^+=A_i^+$, and B satisfies (2). Thus $\det B\geqq 0$.

Let

$$d_i = a_{ii} - b_{ii} = \sum_{j=1}^{n} a_{ij} - nA_i^+, \qquad i = 1, 2, \cdots, n,$$

and set $D = \text{diag}(d_1, d_2, \cdots, d_n)$. Since (2) implies $a_{ii} \geqq b_{ii}$ we have $d_i \geqq 0$ for each $i = 1, 2, \cdots, n$. Then (10) implies

$$\det A = \det (B + D) \geqq \prod_{i=1}^{n} d_i$$

which proves Theorem 2.

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