SOME REMARKS ON ABSTRACT MACHINES

BY

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Introduction. In 1954 the mathematical entity called a (sequential) machine was found to be a valuable tool in designing sequential switching circuits [2; 8; 9]. Since then there has been considerable mathematical activity by mathematicians and nonmathematicians relating to the analysis and the synthesis of these machines. As was to be expected of a topic which arose because of an engineering need, most of these results have appeared in engineering and computing journals. Recently though, some of the papers have appeared in mathematical journals [3; 4; 5; 6; 10]. Also, much of the recent literature has dealt with questions almost exclusively of mathematical, as contrasted with engineering, interest [1; 3; 4; 5; 6; 10; 12]. The present paper is written in that spirit.

The Moore-Mealy (complete, sequential) machine is defined [8; 9] as a nonempty set K (of "states"), a nonempty set D (of "inputs"), a nonempty set F (of "outputs"), and two functions δ (the "next state" function), and λ (the "output" function), δ mapping $K\times D$ into K and λ mapping $K\times D$ into F. Then δ and λ are extended to sequences of inputs $I_1 \cdots I_k$ (written without commas) by

$$\delta(q_1, I_1 \cdots I_k) = q_{k+1}$$

and

$$\lambda(q_1, I_1 \cdots I_k) = \lambda(q_1, I_1)\lambda(\delta(q_1, I_1), I_2 \cdots I_k),$$

where $q_{j+1}=\delta(q_j, I_j)$ for $1\leqq j\leqq k$. The properties of machines studied usually involve sequences of inputs and sequences of outputs. The present paper arose by observing that many facts about machines, for example, those on submachines, could be phrased more elegantly by calling "sequences of inputs" and "sequences of outputs," "inputs" and "outputs" respectively. From this it was natural to consider both inputs and outputs as elements from abstract semi-groups, subject of course, to certain restrictions on the next state function and the output function under the product of inputs. This led to the concept of an abstract "quasi-machine." The term "quasi-machine" was used because certain desirable properties associated with the Moore-Mealy machines no longer held. By adding a technical condition on the output semi-group, namely the left cancellation law, the lost properties were restored. This led to the definition of an abstract "machine." While the

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emphasis and interest here is on machines, results have been stated wherever applicable, to the more general notion of quasi-machines.

The material is divided into seven sections, the first two and the last dealing with certain relations between machines, and the remaining four with properties about states, inputs, and outputs. The main results are in the last five sections.

In §1, some basic concepts of machines, such as "submachines," "strongly connected" machines, and "stable" sets are introduced.

§2 considers properties related to composite machines, i.e., machines obtained by using the outputs of one machine as inputs to another.

§3 deals with distinguishability and indistinguishability of states between machines. A sample result is the following: Let $\{S_b/b\in B\}$ be a family of denumerable state quasi-machines, at least one of the S_b having just a finite number of states. Suppose that for each sequence of inputs $\{I_i\}$ and each finite collection of machines $\{S_1, \cdots, S_n\}$ there exists a set of states $\{p_j/p_j\in S_j, j\leqq n\}$ such that the output of p_j from $I_1 \cdots I_i$ is the same for $j\leqq n$. Then there exists a set $\{p^b/p^b\in S_b, b=1, \cdots, n\}$ of states which are pairwise indistinguishable.

§4 discusses the topic of "essentially different" inputs. Two inputs I_1 and I_2 are said to be input-distinguishable if a state p can be found so that some two sequences of inputs, starting with I_1 and I_2 respectively and identical thereafter, yield different outputs. That is, the inputs I_1 and I_2 are essentially different when measured by the accompanying outputs. It is shown (Theorem 4.1), that each machine may be reduced to one which is input-distinguished (i.e., all inputs are pairwise input-distinguishable). A test for determining if a free machine is input-distinguished (Theorem 4.3) is given. For a free machine S, with a finite number of states and finite number of generating outputs, a "best" upper bound on the maximum number of elements in the input alphabet in order for S to be input-distinguished is given (Theorem 4.5).

§5 is concerned with a state being output complete, that is, a state where all the outputs are actually assumed. In particular, conditions are given which ensure a machine having an output complete state (Theorems 5.1 and 5.3).

§6 deals with the notion of a rational state, i.e., a state where each ultimately periodic sequence of inputs yields an ultimately periodic sequence of outputs. Intuitively, this can be interpreted as meaning that the machine does not yield "wild" sequences of outputs under "repetitive" sequences of inputs. Relations are given between the period of the input sequence and the period of the output sequence, in conjunction with the number of states in the machine (Theorem 6.3).

The final section is concerned with machines which have inverses. Roughly speaking, an inverse S^* of S undoes the action performed by S. Questions on existence, uniqueness, and "practical tests" for inverses are then answered.

The emphasis throughout is on questions of mathematics as compared to those of logic. As such, proofs and counterexamples have been given from an existential point of view. Thus the axiom of choice is used freely. It is recognized, however, that many of these same proofs and counterexamples can be made more constructive if desired. For example, when dealing with machines which have a denumerably infinite number of states, specific relations of outputs to inputs, instead of existential ones, can be given.

In conclusion, the author wishes to thank C. C. Chang for the many stimulating conversations which transpired during the writing of this paper.

1. **Submachines.** The basic notions relating to machines and submachines are now defined. Some elementary properties are then given.

Definition. A *quasi-machine* S is a 5-tuple $(K_S, W_S, Y_S, \delta_S, \lambda_S)$ satisfying the following properties.

(1) K_S is a nonempty set of "states."

(2) W_S (the set of "outputs") and Y_S (the set of "inputs") are nonempty semi-groups(1).

(3) δ_S (the "next state" function) is a mapping of $K_S \times Y_S$ into K_S such that $\delta_S(q, I_1I_2) = \delta_S[\delta_S(q, I_1), I_2]$ for each q in K_S and each I_1 and I_2 in Y_S.

(4) λ_S (the "output" function) is a mapping of $K_S \times Y_S$ into W_S such that $\lambda_S(q, I_1I_2) = \lambda_S(q, I_1)\lambda_S[\delta_S(q, I_1), I_2]$ for each q in K_S and each I_1 and I_2 in Y_S.

The subscript S on K_S, W_S, etc., is omitted when S is understood.

Unless stated to the contrary, in all examples used in the sequel, only generating sets(2) for W_S and Y_S are used.

Given K, W, and Y, it follows from (3) and (4) that a quasi-machine is uniquely determined when the two functions δ and λ are known for each input of a generating set of inputs and for each state.

On numerous occasions, as when discussing distinguishability of states, it is convenient to restrict the semi-group of outputs in a particular way.

Definition. A *machine* S is a quasi-machine in which the semi-group of outputs satisfies the left cancellation law(3).

For the purposes of this paper, the most important semi-groups which satisfy the left cancellation law are the free semi-groups(4). Thus, if W_S is a free semi-group, then S is a machine.

(1) A semi-group is a set of elements A and an operation "o" such that $a \circ b$ is in A and $(a \circ b) \circ c = a \circ (b \circ c)$ for every a, b, and c in A. When no confusion can arise, we write ab instead of $a \circ b$ and abc instead of $(a \circ b) \circ c$ or $a \circ (b \circ c)$ etc.

(2) A subset M of a semi-group S is said to *generate*, or be a *generating set* for S, if the closure of M under the semi-group operation is S.

(3) A semi-group S is said to satisfy the left cancellation law if, for any elements a, b, and c in S, $ab = ac$ implies that $b = c$.

(4) Let D be an abstract set and M the set of all finite sequences of elements of D. Endow M with the operation of concatenation. That is, if σ_1 and σ_2 are two sequences in M, let $\sigma_1\sigma_2$ be the sequence consisting of the elements of σ_1 followed by the elements of σ_2. Then M is a semi-group, called the free semi-group based on the alphabet D.

DEFINITION. Let S be a machine. Suppose that W_S and Y_S are both free semi-groups based on the output alphabet F and the input alphabet D. If for each state q in S and each element I of D there exists an element E of F so that $\lambda_S(q, I) = E$, then S is said to be a *free* machine.

Suppose that Y and W are two free semi-groups (based on the alphabets D and F respectively) and that K is an abstract set. For each I in D and q in K define $\delta(q, I)$ to be an element of K and $\lambda(q, I)$ to be an element of F. By extending the definition of δ and λ to all of Y, in the obvious manner, so as to satisfy (3) and (4) in the definition of a quasi-machine, a free machine is obtained. It is clear that the theory of free machines is really the theory of the Moore-Mealy machines([5]). In the sequel, free machines will be described by defining δ and λ only on D.

We now turn to the subject of subquasi-machines.

DEFINITION. A *subquasi-machine* T of a quasi-machine S is a quasi-machine such that $K_T \subseteq K_S$, $W_T = W_S$, $Y_T = Y_S$, δ_T is δ_S acting on $K_T \times Y_S$, and λ_T is λ_S acting on $K_T \times Y_S$.

A subquasi-machine has the same set of inputs as the original quasi-machine. The definition of a subquasi-machine T could be modified so as to allow Y_T and W_T to be subsemi-groups of Y_S and W_S respectively. Since this generalization has not proved fruitful for the topics considered here, it has not been used.

In dealing with subquasi-machines, it is frequently convenient to study certain subsets of K_S.

DEFINITION. A nonempty set of states A of a quasi-machine S is said to be *stable* if $\delta(q, I)$ is in A for every state q in A and every input I.

It immediately follows from the definitions that in order for a subset A of K_S to serve as $A = K_T$ for a subquasi-machine T of S, it is necessary and sufficient that A be stable. T is said to be the subquasi-machine associated with A, and A the stable set associated with T.

Let $\{A_c / c \in C\}$ be a family of stable sets. Since

$$\left\{\delta(q, I)/q \in \bigcup_c A_c\right\} \subseteq \bigcup_c \ \{\delta(q, I)/q \in A_c\} \subseteq \bigcup_c A_c$$

and

$$\left\{\delta(q, I)/q \in \bigcup_c A_c\right\} \subseteq \bigcup_c \ \{\delta(q, I)/q \in A_c\} \subseteq \bigcap_c A_c$$

for each input I, the set $\bigcup_c A_c$ is stable; and if $\bigcap_c A_c$ is nonempty, this set is also stable.

Let A be an arbitrary subset of K_S and let $F(A)$ be the family of all stable

([5]) The Moore-Mealy model [8; 9] consists of the input alphabet, output alphabet, states, and next state and output functions defined for each state and each input in the input alphabet.

sets which contain A. Since K_S is in $F(A)$, $F(A)$ is nonempty. Then the set B, defined by $B=\bigcap_{A_c \in F(A)} A_c$ is the smallest stable set containing A. B may also be described as the set containing A and all states of the form $\delta(q, I)$, for all q in A and all inputs I. B is said to be the stable set *generated* by A. If A consists of the single state q, then the subquasi-machine associated with the stable set generated by q is said to be a *principal* subquasi-machine, in particular, the principal subquasi-machine generated by q.

DEFINITION. A quasi-machine S is said to be *strongly connected*(6) if for every two states q_1 and q_2 in S there exists an input I so that $\delta(q_1, I)=q_2$.

THEOREM 1.1. *A subquasi-machine S is strongly connected if and only if the stable set A associated with S contains no proper subset which is stable, i.e., A is minimum stable.*

Proof. Suppose that S is a strongly connected quasi-machine and the associated stable set A is not minimum stable. Let B be a stable, proper subset of A, q an element of $A-B$ and q_1 in B. Since S is strongly connected, there exists an input I such that $\delta(q_1, I)=q$. Since B is stable q is in B, a contradiction. Hence A is minimum.

Now suppose that A is minimum stable. Let q_1 and q_2 be in A and let B be the stable set generated by q_1. Since A is minimum stable and B is stable, $B=A$. Thus q_2 is in B, i.e., $\delta(q_1, I)=q_2$ for some input I. Hence S is strongly connected.

COROLLARY 1. *A quasi-machine S contains no proper subquasi-machine if and only if S is strongly connected.*

COROLLARY 2. *For each state q in the finite state quasi-machine S, there exists a strongly connected subquasi-machine T and an input I such that $\delta(q, I)$ is in T.*

COROLLARY 3. *A necessary and sufficient condition that each state in a quasi-machine S be in a minimum stable set is that S be the sum*(7) *of strongly connected (sub)quasi-machines.*

Proof. In view of Theorem 1.1, only the necessity has to be shown. Therefore let each state in S be in a minimum stable set. Since the intersection of two stable sets is stable if it is nonempty, two minimum stable sets are either identical or disjoint. Let $\{H_\alpha/\alpha \in M\}$ be the collection of all minimum stable sets. For each α let S_α be the subquasi-machine associated with H_α. By Theo-

(6) This follows the terminology of Moore [9] who defined strongly connected in terms of sequences of inputs of the input alphabet of a free machine.

(7) Let $\{S(a)/a \in A\}$ be a family of quasi-machines, all with the same input semi-group Y and same output semi-group W respectively. Without loss of generality, it is assumed that $K_{S(a)} \cap K_{S(b)}$ is empty for each $a \neq b$ in A. Define K_S to be $\bigcup_{a \in A} K_{S(a)}$. Let $Y_S=Y$ and $W_S=W$. For q in K_S and I in Y, let $\lambda_S(q, I)=\lambda_{S(a)}(q, I)$ and $\delta_S(q, I)=\delta_{S(a)}(q, I)$, where q is in $K_{S(a)}$. Then S is a quasi-machine and is said to be the *sum* of the family $\{S(a)/a \in A\}$.

rem 1.1, each H_α is strongly connected. Since the $\{H_\alpha\}$ are a partition of K, S is the sum of strongly connected quasi-machines $\{S_\alpha/\alpha\in M\}$.

From Corollary 3 there follows

COROLLARY 4. *A necessary and sufficient condition that a quasi-machine S be the sum of strongly connected quasi-machines is that for each two states q_1 and q_2, if there exists an input I so that $\delta(q_1, I)=q_2$, then there exists an input J so that $\delta(q_2, J)=q_1$.*

COROLLARY 5. *Each permutation quasi-machine*(8) *with a finite number of states is the sum of strongly connected quasi-machines.*

Proof. By Corollary 3, it is sufficient to show that for each state q and each input I, there exists an input I_1 such that $\delta(q, II_1)=q$. If $\delta(q, I)=q$, it is sufficient to let $I_1=I$. Suppose that $\delta(q, I)=q_1\neq q=q_0$. For each integer $j>0$ define $q_j=\delta(q_{j-1}, I)$. Let k be the smallest integer such that $q_k=q_i$ for some $i<k$. The existence of k is guaranteed by S having just a finite number of states. Suppose that $q_k\neq q_0$. Then $q_k=\delta(q_{k-1}, I)=q_i=\delta(q_{i-1}, I)$ since $i>0$. As I is a permutation, thus one to one, $q_{k-1}=q_{i-1}$. This contradicts the minimality property of k. Therefore $q_k=q_0$. Let $I_1=II\cdots I(k-1$ times). Then $q=\delta(q, II_1)$. Q.E.D.

It is easily seen that Corollary 5 is no longer true if either the finiteness condition is removed or if one of the inputs does not affect a permutation.

In passing, it can readily be shown that each quasi-machine is the sum of indecomposable quasi-machines(9), the decomposition being unique to within rearrangement of the indecomposable quasi-machines.

2. **Composite quasi-machines.** The notion of a composite quasi-machine arises when the outputs of one quasi-machine are used as the inputs to another quasi-machine.

DEFINITION. Let $S_1, \cdots, S_n$ be n quasi-machines. For each $i\leqq n$ let Y_i be the input semi-group, W_i the output semi-group, δ^i the next state function, and λ^i the output function, of S_i. Suppose that $W_i\subseteq Y_{i+1}$ for $i\leqq n-1$. The *composite* quasi-machine $T=S_1\rightarrow\cdots\rightarrow S_n$ is defined as follows. The inputs I_j^T of T are the inputs I_j^1 of S_1 and the outputs E_k^T of T are the outputs E_k^n of S_n. The states q_j^T of T are the n-tuples $(q^1, \cdots, q^n)$, where each q^i is a state of S_i. δ^T and λ^T are defined by

$$\delta^T[(q^1, \cdots, q^n), I^T] = (q_0^1, \cdots, q_0^n) \quad\text{and}\quad \lambda^T[(q^1, \cdots, q^n), I^T] = E^n,$$

where $E^1 = \lambda^1(q^1, I^T)$, $E^i = \lambda^i(q^i, E^{i-1})$ for $i > 1$, $q_0^1 = \delta^1(q^1, I^T)$, and $q_0^i = \delta^i(q^i, E^{i-1})$ for $i>1$.

(8) A quasi-machine S is said to be a *permutation* quasi-machine if each input affects a permutation of K.

(9) A quasi-machine is said to be *indecomposable* if it is not the sum of at least two sub-quasi-machines.

Both $S_1 \to (S_2 \to S_3)$ and $(S_1 \to S_2) \to S_3$ are isomorphic[10] in the obvious manner, to $S_1 \to S_2 \to S_3$. Because of this, the parentheses are omitted. Similar remarks hold for four or more quasi-machines.

Clearly $S_1 \to \cdots \to S_n$ is a machine if and only if S_n is a machine.

If, for each $i \leqq n$, S_i is a free machine, then so is $S_1 \to \cdots \to S_n$. The converse is not true.

Definition. Let S and T be two quasi-machines. Then g is said to be a *box function* of (S, T) if (i) $W_S \subseteq Y_T$, and (ii) g maps K_S into K_T such that $g[\delta_S(q, I)] = \delta_T[g(q), \lambda_S(q, I)]$ for each state q and each input I of S.

Given the box functions g_i of (S_i, S_{i+1}), $1 \leqq i \leqq n-1$, a subquasi-machine $H(g_1, \cdots, g_{n-1})$ of $S_1 \to \cdots \to S_n$ is determined in a natural way. $H(g_1, \cdots, g_{n-1})$ is the quasi-machine obtained from the stable set of states

$$A = \left\{ (q^1, q^2, \cdots, q^n)/q^1 \in S_1 \text{ and } q^i = g_{i-1}(q^{i-1}) \text{ for } i > 1 \right\}.$$

To see that this set is stable let $p = (q^1, \cdots, q^n)$ be in A and let I be any input of $S_1 \to \cdots \to S_n$. Then $\delta(p, I) = (q_0^1, \cdots, q_0^n)$, where $E^1 = \lambda^1(q^1, I)$, $E^i = \lambda^i(q^i, E^{i-1})$ for $i > 1$, $q_0^1 = \delta^1(q^1, I)$, and $q_0^i = \delta^1(q^i, E^{i-1})$ for $i > 1$. Let $E^0 = I$. Then for each $i > 1$,

$$\begin{aligned} q_0^i = \delta^i(q^i, E^{i-1}) &= \delta^i[g_{i-1}(q^{i-1}), \lambda^{i-1}(q^{i-1}, E^{i-1})] \\ &= g_{i-1}[\delta^{i-1}(q^{i-1}, E^{i-1})], \text{ since } g_{i-1} \text{ is a box function of } (S_{i-1}, S_i), \\ &= g_{i-1}(q_0^{i-1}). \end{aligned}$$

Thus $(q_0^1, \cdots, q_0^n)$ is in A, i.e., A is stable.

To see if g is a box function it is sufficient to check g for all the inputs of a generating sets of inputs. More precisely

Lemma 2.1. *Let S and T be two quasi-machines and H a generating set for Y_S. If* (i) $W_S \subseteq Y_T$ *and* (ii) *g is a function mapping K_S into K_T such that $g[\delta_S(q, I_1)] = \delta_T[g(q), \lambda_S(q, I_1)]$ for each state q and each I_1 in H, then g is a box function of* (S, T).

Proof. Let I_1 and I_2 be in H. Then

$$\begin{aligned} g[\delta_S(q, I_1 I_2)] &= g[\delta_S(\delta_S(q, I_1), I_2)] \\ &= \delta_T(g[\delta_S(q, I_1)], \lambda_S[\delta_S(q, I_1), I_2]) \\ &= \delta_T(\delta_T[g(q), \lambda_S(q, I_1)], \lambda_S[\delta_S(q, I_1), I_2]) \\ &= \delta_T(g(q), \lambda_S(q, I_1)\lambda_S[\delta_S(q, I_1), I_2]) \\ &= \delta_T[g(q), \lambda_S(q, I_1 I_2)]. \end{aligned}$$

(10) Two quasi-machines S and T are said to be *isomorphic* if $Y_S = Y_T$, $W_S = W_T$, and there exists a one to one function f of K_S onto K_T so that for each input I and each state q in S, $f[\delta_S(q, I)] = \delta_T[f(q), I]$ and $\lambda_S(q, I) = \lambda_T[f(q), I]$. If S and T are not isomorphic to each other, then they are said to be nonisomorphic.

Continuing by induction we see that g and δ interchange[11] for any finite sequence of inputs $I_1 \cdots I_n$, each I_i in H. As H generates Y_S, g and δ interchange for all I in Y_S.

To each box function g of (S, T) there is associated a "box decomposition" of S in the following sense. A *box decomposition* of S is a decomposition of K_S[12] into a family $J=\{A_h/h \in H\}$ of classes such that whenever states p_1 and p_2 are in the same class of J and whenever I_1 and I_2 in Y_S are such that $\lambda_S(p_1, I_1)=\lambda_S(p_2, I_2)$, then $\delta_S(p_1, I_1)$ and $\delta_S(p_2, I_2)$ are in the same class of J. The box decomposition associated with the box function g is the family $J=\{A_q/q \in g(K_S)\}$, where $A_q=\{p/p \in K_S,\ g(p)=q\}$; that is, p_1 and p_2 are in the same class if and only if $g(p_1)=g(p_2)$. It is obvious that J as just described is a box decomposition.

As a partial converse to the above we have

THEOREM 2.1. *If S is a free machine, then for each box decomposition $J=\{A_h/h \in H\}$ of S there is at least one free machine T and one box function g of (S, T) such that the box decomposition of S associated with g is J.*

Proof. Let $K_T=H$ and $W_S=Y_T=W_T$. Let q be in K_S. Then q is in exactly one class of J, say A_h. Define $g(q)$ to be h. Now let I be an element of the output alphabet F of S and let h be an element of K_T. If there exists a state q_0 in A_h and an input E_0 in the input alphabet D of S such that $\lambda_S(q_0, E_0)=I$, then let $\delta_T(h, I)=h^*$, where $g[\delta_S(q_0, E_0)]=h^*$, i.e., $\delta_S(q_0, E_0)$ is in A_{h^*}. If no such state q_0 in A_h and input E_0 exist, define $\delta_T(h, I)$ to be h. Since J is a box decomposition, $\delta_T(h, I)$ is uniquely defined. For each h in H and each I in F, let $\delta_T(h, I)=I$. Since Y_T and W_T are free semi-groups (both based on the alphabet F) with elements of F mapped on elements of F under λ_T, T is a free machine. By construction, g and δ interchange on F. By Lemma 2.1, g is a box function. It is obvious that the box decomposition associated with g is J. Q.E.D.

It is easy to find examples showing that Theorem 2.1 is no longer true if the condition that S be a free machine is removed.

If there exists a box function g of (S, T) then certain properties in one of the two quasi-machines, S and T, imply certain properties in the other. Several results of this nature are listed in the following theorem, the proof of which, being straightforward, is omitted.

THEOREM 2.2. (a) *If g is an onto box function of (S, T) and if S is strongly connected, then T is strongly connected.*

(b) *If g is an onto box function of (S, T) and if q in K_S is connected to each*

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(11) If $g[\delta_S(q, I)]=\delta_T[g(q), \lambda_S(q, I)]$, then g and δ are said to *interchange* (for I).

(12) By *decomposition* of a set P is meant a family of nonempty disjoint sets whose set union is P.

state in K_S *(i.e., for each* p *in* K_S *there exists* I_p *so that* $\delta_S(q, I_p) = p$*), then* $g(q)$ *is connected to each state in* K_T.

(c) *If* g *is a one to one box function of* (S, T)*, if* $Y_T = W_S$ *and* S *is output complete*(13)*, and if* T *is strongly connected; then* S *is strongly connected.*

(d) *If* g *is a box function of* (S, T)*, if* $W_S = Y_T$*, if* A *is a stable set of* S*, and if* S *is output complete at each state* q *in* A*; then* $g(A)$ *is a stable set of* T.

3. **Distinguishability of states.** In this section we shall be dealing with the notion of "distinguishability" and "indistinguishability" between states in a set of quasi-machines. We shall assume that each quasi-machine in a set of quasi-machines has the same semi-group Y of inputs and the same semi-group W of outputs.

DEFINITION. State q in a quasi-machine S is said to be *distinguishable* from state $\bar{q}$ in a quasi-machine T if there exists an input I such that $\lambda_S(q, I) \neq \lambda_T(\bar{q}, I)$. If q and $\bar{q}$ are not distinguishable, they are said to be *indistinguishable.* A quasi-machine S is said to be *distinguished* if each two distinct states in S are distinguishable.

The following result, in the case of free machines, is implied in [9].

LEMMA 3.1. *If state* q *in machine* S *is indistinguishable from state* $\bar{q}$ *in machine* T*, then for every input* E*,* $\delta_S(q, E)$ *is indistinguishable from* $\delta_T(\bar{q}, E)$.

Proof. Let I be an input. Since q and $\bar{q}$ are indistinguishable, $\lambda_S(q, E) = \lambda_T(\bar{q}, E)$ and

$$\lambda_S(q, E)\lambda_S[\delta_S(q, E), I] = \lambda_S(q, EI) = \lambda_T(\bar{q}, EI) = \lambda_T(\bar{q}, E)\lambda_T[\delta_T(\bar{q}, E), I].$$

Since S and T are machines, W satisfies the left cancellation law. Thus $\lambda_S[\delta_S(q, E), I] = \lambda_T[\delta_T(\bar{q}, E), I]$. Consequently $\delta_S(q, E)$ is indistinguishable from $\delta_T(\bar{q}, E)$.

Lemma 3.1 is no longer true if the word "machine" is replaced by "quasi-machine."

EXAMPLE 3.1. Let $K_S = \{q_1, q_2\}$. Let Y_S and W_S be generated by I_1 and I_2, subject to the relations $I_1I_2 = I_1I_1$ and $I_2I_1 = I_2I_2$. Let $\delta_S(q_k, I_i) = q_k$, $\lambda_S(q_1, I_i) = I_i$, and $\lambda_S(q_2, I_i) = I_j$, where $i, j, k = 1, 2, i \neq j$. Let $K_{\bar{S}} = K_S$, $\delta_{\bar{S}}(q_i, I_k) = q_j$, $\lambda_{\bar{S}}(q_1, I_i) = I_i$, and $\lambda_{\bar{S}}(q_2, I_i) = I_j$, where again $i, j, k = 1, 2, i \neq j$. Then q_1 in S is indistinguishable from q_1 in $\bar{S}$. This follows from the fact that the output from q_1 in S and $\bar{S}$ under any sequence of elements of Y_S, beginning with I_i is $I_i \cdots I_i$, $i = 1, 2$. However $q_1 = \delta_S(q_1, I_1)$ in S is distinguishable from $q_2 = \delta_T(q_1, I_1)$ in T.

DEFINITION. Two quasi-machines S and T are said to be *equivalent* if for each state q_s in S there exists a state q_t in T which is indistinguishable from q_s,

(13) See §5 for the definition of output complete.

and conversely, for each state q_t in T there exists q_s in S which is indistinguishable from q_t.

It is a well known (but unpublished) result that to each free machine S there corresponds a unique (up to isomorphism), distinguished machine which is equivalent to S. Using Lemma 3.1, this result is now extended to machines.

THEOREM 3.1. *To each machine S there corresponds a unique (up to isomorphism), distinguished machine which is equivalent to S.*

Proof. Let $Y_T = Y_S$ and $W_T = W_S$. For states q_1 and q_2 in S, write q_1Rq_2 if q_1 and q_2 are indistinguishable. Then R is an equivalence relation decomposing K_S into a set of equivalence classes K_T. Denote by $[q]$ the equivalence class containing q. Let $\lambda_T([q], I) = \lambda_S(q, I)$ and $\delta_T([q], I) = [\delta_S(q, I)]$ where q is an element of $[q]$. Since the elements of $[q]$ are indistinguishable, λ_T is uniquely defined. By Lemma 3.1, δ_T is uniquely defined. In view of Lemma 3.1, λ_T and δ_T satisfy properties (3) and (4) of a quasi-machine since λ_S and δ_S satisfy these properties. Thus T is a machine. It is obvious that T is equivalent to S.

Now suppose that S^* and T are two distinguished machines, each equivalent to S. For q^* a state in S^*, let q be a state in S which is indistinguishable from q^*, and $\bar{q} = f(q^*)$ the unique state in $\bar{S}$ which is indistinguishable from q, thus from q^*. Since S^* and T are distinguished, f is a one to one mapping of K_{S^*} onto K_T. Using Lemma 3.1, it is seen that S^* and T are isomorphic. Q.E.D.

Theorem 3.1 cannot be extended to hold for quasi-machines. For example, the two quasi-machines given in Example 3.1 are distinguished, equivalent to each other, but nonisomorphic. Because of this, whenever the notion of equivalence appears, it is usually necessary to assume that the quasi-machines involved are actually machines.

Theorem 3.1 cannot be extended to hold for quasi-machines even if the uniqueness condition is dropped. In other words, it is not true that to each quasi-machine there corresponds at least one distinguished, equivalent machine. This is a fundamental difference between machines and quasi-machines.

EXAMPLE 3.2. Let W be the semi-group generated by the set of nine elements $\{O_i/i \leqq 9\}$ subject to the three relations

(1) $O_1O_4 = O_8O_3$, $O_2O_5 = O_9O_3$, and $O_3O_6 = O_3O_7$.

Let Y be the semi-group generated by the two elements I_1 and I_2, subject to the relation $I_1I_1 = I_2I_2$.

Let $K_S = \{q_1, \bar{q}_1, q_2, q_3, q_4, q_5, q_6, q_7\}$. The output function λ_S is defined by $\lambda_S(q_6, I_2) = O_1$, $\lambda_S(q_7, I_2) = O_2$, $\lambda_S(q_1, I_1) = \lambda_S(q_1, I_2) = \lambda_S(\bar{q}_1, I_1) = \lambda_S(\bar{q}_1, I_2) = O_3$, $\lambda_S(q_4, I_1) = \lambda_S(q_4, I_2) = O_4$, $\lambda_S(q_5, I_1) = \lambda_S(q_5, I_2) = O_5$, $\lambda_S(q_2, I_1) = \lambda_S(q_2, I_2) = O_6$, $\lambda_S(q_3, I_1) = \lambda_S(q_3, I_2) = O_7$, $\lambda_S(q_6, I_1) = O_8$, and $\lambda_S(q_7, I_1) = O_9$. The next state function δ_S is defined by $\delta_S(q_6, I_1) = q_1$, $\delta_S(q_7, I_1) = \bar{q}_1$, $\delta_S(q_1, I_1) = \delta_S(q_1, I_2)$

$= \delta_S(q_2, I_1) = \delta_S(q_2, I_2) = \delta_S(q_4, I_1) = \delta_S(q_4, I_2) = q_2$, $\delta_S(\bar{q}_1, I_1) = \delta_S(\bar{q}_1, I_2) = \delta_S(q_3, I_1) = \delta_S(q_3, I_2) = \delta_S(q_5, I_1) = \delta_S(q_5, I_2) = q_3$, $\delta_S(q_6, I_2) = q_4$, and $\delta_S(q_7, I_2) = q_5$. It is readily verified that S is a quasi-machine.

Suppose that T is a distinguished quasi-machine which is equivalent to S. Since S has seven distinguishable states (only q_1 and $\bar{q}_1$ are indistinguishable), the states of T may be labelled q_1 to q_7, with q_i in T indistinguishable from q_i in S, for $i \leqq 7$. Thus $\lambda_T(q_i, I) = \lambda_S(q_i, I)$ for all q_i in K_T and all I in Y. Consider the next state function δ_T. First note that

$$\text{if } O_xO_y = O_xE, \text{ where } x, y, \leqq 9, x \neq 3, \text{ then } O_y = E. \tag{2}$$

[Thus left cancellation occurs in certain cases.] Now

$$\begin{aligned} O_6O_6 &= \lambda_T(q_2, I_1I_1) \\ &= \lambda_T(q_2, I_1)\lambda_T[\delta_T(q_2, I_1), I_1] \\ &= O_6E, \end{aligned}$$

whence $E = O_6$. Since q_2 is the only state q_x such that $\lambda_T(q_x, I_1) = O_6$, $\delta_T(q_2, I_1) = q_2$. Using the same procedure it is seen that δ_T agrees with δ_S for (q_2, I), (q_3, I), (q_4, I), (q_5, I), (q_6, I_2), and (q_7, I_2), where $I = I_1$ or I_2. Furthermore, $\delta_T(q_6, I_1) = \delta_T(q_7, I_1) = q_1$ since q_1 and $\bar{q}_1$ in S are combined to form q_1 in T. Consider $\delta_T(q_1, I_1)$. It is easily seen that $\delta_T(q_1, I_1)$ is either q_2 or q_3, say q_3. Then

$$\begin{aligned} \delta_T(q_6, I_1I_1) &= \delta_T[\delta_T(q_6, I_1), I_1] \\ &= \delta_T(q_1, I_1) = q_3. \end{aligned}$$

Since $I_1I_1 = I_2I_2$,

$$\begin{aligned} \delta_T(q_6, I_1I_1) &= \delta_T(q_6, I_2I_2) \\ &= \delta_T[\delta_T(q_6, I_2), I_2] \\ &= \delta_T(q_4, I_2) = q_2. \end{aligned}$$

This is a contradiction. Another contradiction, using q_7, arises if it is assumed that $\delta_T(q_1, I_1)$ is q_2. Hence T cannot be a quasi-machine.

Therefore S is a quasi-machine for which there is no equivalent, distinguished, quasi-machine.

In the proof of Theorem 9 of [9] the following result, applied to a free machine, is used.(14)

LEMMA 3.2. *If S and T are two nonisomorphic, strongly connected, distinguished machines, then each state q of S is distinguishable from each state $\bar{q}$ of T.*

The proof of Lemma 3.2 follows immediately from Lemma 3.1 and the hypothesis.

(14) Communicated to the author by Moore.

NOTATION. For a nonempty set H let $J(H)(J_f(H))$ be the set of all strongly connected (finite state) distinguished machines such that $K \subseteq H$.

The question arises as to when a state is distinguishable from each state of each machine in $J(H)(J_f(H))$. The answer is given by

THEOREM 3.2. (a) *Let S be a distinguished machine with $K_S \subseteq H$. A state q_1 of machine S is distinguishable from each state of each machine in $J(H)$ if and only if q_1 is contained in no strongly connected submachine of S.*

(b) *Let S be a distinguished, finite state machine with $K_S \subseteq H$. A state q_1 of S is distinguishable from each state of each machine in $J_f(H)$ if and only if q_1 is contained in no strongly connected submachine of S.*

Proof. (a) The necessity being obvious, only the sufficiency shall be shown. Therefore suppose that q_1 is contained in no strongly connected submachine of S and that q_1 is indistinguishable from state $\bar{q}_1$ of machine T in $J(H)$. Since q_1 is contained in no strongly connected submachine of S, there exists state q_2 in S and an input E_0 such that $\delta_S(q_1, E_0) = q_2$ but for no input E is $\delta_S(q_2, E) = q_1$. Let $\bar{q}_2 = \delta_T(\bar{q}_1, E_0)$. By Lemma 3.1, q_2 and $\bar{q}_2$ are indistinguishable. Since T is strongly connected, there exists E_1 such that $\delta_T(\bar{q}_2, E_1) = \bar{q}_1$. Let $q_3 = \delta_S(q_2, E_1)$. Then q_3 and $\bar{q}_1$, thus q_3 and q_1, are indistinguishable. Since S is distinguished $q_1 = q_3$. Thus $q_1 = \delta_S(q_2, E_1) = q_1$ contradicting the selection of q_2. Hence the result.

(b) An analogous argument yields the proof of (b).

Via personal correspondence, Moore has conjectured the validity of the next result for the case when S is a free machine.

THEOREM 3.3. *Let L be a set of distinguished, finite state machines, each $K \subseteq H$, containing $J_f(H)$. Furthermore, suppose that for every two nonisomorphic machines S and T in L, every state of S is distinguishable from every state of T. Then $L = J_f(H)$.*

Proof. Suppose that S is a machine in L which is not strongly connected. By hypothesis, K_S is finite. Since each finite stable set contains a minimum stable set, by Theorem 1.1 S contains a strongly connected, thus proper, submachine T. By hypothesis, T is in L. Then S and T are two nonisomorphic machines, but not all the states of S are distinguishable from all the states of T, a contradiction. Thus no such machine S is in L, i.e., $L = J_f(H)$.

If the machines in L are permitted to have an infinite number of states, then Theorem 3.3, with $J_f(H)$ replaced by $J(H)$, is no longer true. For let S be any distinguished machine, with $K_S \subseteq H$, containing no strongly connected submachine and let $L = J(H) \cup \{S\}$. By Lemmas 3.1 and 3.2, for two nonisomorphic machines T_1 and T_2 in L, each state of T_1 is distinguishable from each state of T_2.

NOTATION. Given two quasi-machines S and T, by $S \leqq T$ is meant that for each state q in S and each input E, there exists state $\bar{q}$ in T (depending on q and E) such that $\lambda_S(q, E)=\lambda_T(\bar{q}, E)$.

EXAMPLE 3.3. An example is now given of an infinite state distinguished machine S and a finite state distinguished machine T having the property that $S \leqq T$ and $T \leqq S$. Let I_1 and I_2 generate the inputs. For machine S let $\delta_S(q_n, I_1)=q_{n-1}$ for $n>1$, $\delta_S(q_1, I_1)=q_1$, $\delta_S(q_n, I_2)=q_1$ and $\lambda_S(q_n, I_1)=0$ for all n, $\lambda_S(q_n, I_2)=0$ for $n \neq 2$, and $\lambda_S(q_2, I_2)=1$. For machine T, let $\delta_T(q_2, I_2)=\delta_T(q_1, I_1)=\delta_T(q_1, I_2)=q_1$, $\delta_T(q_2, I_1)=q_2$, $\lambda_T(q_2, I_2)=1$, $\lambda_T(q_1, I_1)=\lambda_T(q_1, I_2)=\lambda_T(q_2, I_1)=0$.

Observe that q_1 in S is indistinguishable from q_1 in T. It is shown below that such states always exist when $T \leqq S$, S has only a finite number of states, and T has a denumerable number of states.

THEOREM 3.4. *Let k be a positive integer and for each positive integer $j \leqq k$ let A_j be a denumerable set of states of the quasi-machine S_j and let H_j be the stable subset of S_j generated by A_j. Suppose that for every sequence $\{I_i\}$ of inputs there exists a set of states $\{q^j/q^j \text{ in } A_j,\ j \leqq k\}$ such that $\lambda_j(q^j, I_1 \cdots I_i)=\lambda_m(q^m, I_1 \cdots I_i)$ for all j, $m \leqq k$ and all i. Then there exists a set $\{p^j/p^j \text{ in } H_j, j \leqq k\}$ of states which are pairwise indistinguishable.*

Proof. Since each A_i is denumerable, the set of all k-tuples $\{(q^1 \cdots q^j \cdots q^k)/q^j \in A_j\}$ is denumerable and may be relabelled as $\{(p_i^1 \cdots p_i^k)/i=1, 2, \cdots\}$. Suppose that the conclusion of the theorem is false. Then there exists an input F_1 and integers $s(1)$ and $t(1)$ so that $\lambda_{s(1)}(p_1^{s(1)}, F_1) \neq \lambda_{t(1)}(p_1^{t(1)}, F_1)$, and for each integer $i>1$ there exists an input F_i and integers $s(i)$ and $t(i)$ so that $\lambda_{s(i)}(\delta_{s(i)}(p_i^{s(i)}, F_1 \cdots F_{i-1}), F_i) \neq \lambda_{t(i)}(\delta_{t(i)}(p_i^{t(i)}, F_1 \cdots F_{i-1}), F_i)$. This is so since $\delta_j(p_i^j, F_1 \cdots F_{i-1})$ is in H_j. Applying the hypothesis to the sequence of inputs $\{F_i\}$, there exists an integer, say n, such that $\lambda_j(p_n^j, F_1 \cdots F_i)=\lambda_m(p_n^m, F_1 \cdots F_i)$ for all i, thus $\lambda_j(p_n^j, F_1 \cdots F_n)=\lambda_m(p_n^m, F_1 \cdots F_n)$ for j, $m \leqq k$. If $n>1$, then $\lambda_j(\delta_j(p_n^j, F_1 \cdots F_{n-1}), F_n)=\lambda_m(\delta_m(p_n^m, F_1 \cdots F_{n-1}), F_n)$, contradicting the manner in which F_n was selected. If $n=1$, then $\lambda_j(p_1^j, F_1)=\lambda_m(p_1^m, F_1)$ for j, $m \leqq k$, another contradiction. Hence the result.

REMARKS. (1) What has actually been proved in Theorem 3.4 is the slightly stronger result that, under the given hypothesis, there exists a set of states $\{p^j/p^j \text{ in } A_j,\ j \leqq k\}$ and an input E such that the states in $\{\delta_j(p^j, E)/j \leqq k\}$ are pairwise indistinguishable.

(2) It is easy to find examples showing that if either the denumerability condition or the sequence $\{I_i\}$ assumption is removed, then Theorem 3.4 is no longer true.

COROLLARY 1. *Let k be a positive integer and for each positive integer $j \leqq k$ let A_j be a finite set of states of the quasi-machine S_j. Let H_j be the stable subset of S_j generated by A_j. Suppose that for every input E there exists a set of states*

$\{q^j/q^j$ *in* $A_j, j \leqq k\}$ *such that* $\lambda_j(q^j, E) = \lambda_m(q^m, E)$ *for all* $j, m \leqq k$. *Then there exists a set* $\{p^j/p^j$ *in* $H_j, j \leqq k\}$ *of states which are pairwise indistinguishable.*

Proof. Let $\{I_i\}$ be any sequence of inputs. Two cases arise.

(a) Suppose that $\{I_i\}$ is a finite sequence, with I_t the last I_i. Then there exists $\{q^j/q^j \in A_j, j \leqq k\}$ such that $\lambda_j(q^j, I_1 \cdots I_t) = \lambda_m(q^m, I_1 \cdots I_t)$ for all $j, m \leqq k$. Thus $\lambda(q^j, I_1 \cdots I_i) = \lambda_m(q^j, I_1 \cdots I_i)$ for all $j, m \leqq k$ and all $i \leqq t$.

(b) Suppose that $\{I_i\}$ is an infinite sequence. For each integer i there exists $\gamma_i = \{q_i^j/q_i^j$ in $A_j, j \leqq k\}$ such that $\lambda_j(q_i^j, I_1 \cdots I_i) = \lambda_j(q_i^m, I_1 \cdots I_i)$ for all $j, m \leqq k$. Since the sets A_i are finite there are only a finite number of distinct elements in the sequence $\{\gamma_i\}$. Hence one of them, say $\gamma_r = \{q_r^j/j\}$ occurs an infinite number of times in the sequence $\{\gamma_i\}$. It follows that $\lambda_j(q_r^j, I_1 \cdots I_s) = \lambda_m(q_r^m, I_1 \cdots I_s)$ for all $j, m \leqq k$ and all s.

Hence the hypothesis of Theorem 3.4 is satisfied in either case whence the conclusion.

Suppose that S_1 and S_2 are two quasi-machines such that S_2 has a finite number of states and $S_1 \leqq S_2$. Letting A_1 consist of just one state (any one) of S_1 and $A_2 = K_{S_2}$, there follows from Corollary 1

COROLLARY 2. *If* $S_1 \leqq S_2$ *and* S_2 *has a finite number of states, then there exist states* q_1 *of* S_1 *and* q_2 *of* S_2 *which are indistinguishable.*

It is not difficult to see that Corollary 2 is no longer true if S_2 is allowed to have an infinite number of states.

A generalization of Theorem 3.4 to an infinite family of quasi-machines is as follows:

THEOREM 3.5. *Let* $\{S_b/b \in B\}$ *be a family of quasi-machines, and for each* b *in* B *let* A_b *be a denumerable family of states of* S_b. *Let* H_b *be the stable subset of* S_b *generated by* A_b *and let one of the* H_b, *say* H_{b_0}, *contain just a finite number,* n, *of states. Suppose that for each sequence of inputs* $\{I_i\}$ *and each* n *or fewer of the* A_b, *say* $D^1, \cdots, D^r$, *there exists a set* $\{p/p \in A_{b_0}\} \cup \{d^j/d^j \in D^j, j \leqq r\}$ *of* $r+1$ *states such that* $\lambda_j(d^j, I_1 \cdots I_i) = \lambda_{b_0}(p, I_1 \cdots I_i)$ *for all* $j \leqq r$ *and all* i. *Then there exists a set* $\{p^b/p^b \in H_b, b \in B\}$ *of states which are pairwise indistinguishable.*

Proof. Let the n states of H_{b_0} be $\{q_i/i \leqq n\}$. For each q_i let G_i be the family of all sets H_b which contain a state indistinguishable from q_i. Assume that the conclusion of the theorem is false. Then for each $i \leqq n$ there exists an H_b, say $H_{\sigma(i)}$, such that each state in $H_{\sigma(i)}$ is distinguishable from q_i, i.e., $H_{\sigma(i)}$ is not in G_i. Now the $n+1$ or fewer sets $\{H_{b_0}, A_{\sigma(1)}, \cdots, A_{\sigma(n)}\}$ satisfies the hypothesis of Theorem 3.4. Hence there exists a set $\{q_s, p_i/q_s$ in H_{b_0}, p_i in $H_{\sigma(i)}, i \leqq n\}$ of states which are pairwise indistinguishable. Since q_s and p_s are indistinguishable, $H_{\sigma(s)}$ is in G_s. But this is a contradiction. Thus the theorem is true.

In an analogous manner, using Corollary 1 of Theorem 3.4 instead of Theorem 3.4 itself, it is seen that

THEOREM 3.6. *Let $\{S_b/b\in B\}$ be a family of quasi-machines, and for each b in B let A_b be a finite family of states of S_b. Let H_b be the stable subset of S_b generated by A_b and let one of the H_b, say H_{b_0}, contain just a finite number, n, of states. Suppose that for each input E and each n or fewer of the A_b, say $D^1, \cdots, D^r$, there exists a set $\{p/p\in A_{b_0}\}\cup\{d^j/d^j\in D^j, j\leqq r\}$ of $r+1$ states such that $\lambda_j(d^j, E)=\lambda_{b_0}(p, E)$ for all $j\leqq r$. Then there exists a set $\{p^b/p^b\in H_b, b\in B\}$ of states which are pairwise indistinguishable.*

REMARKS. (1) The conclusion of Theorem 3.5 is no longer true if every H_b is allowed to have an infinite number of states. In fact, it is not even true if one replaces the finiteness condition with the hypothesis that for each sequence of inputs $\{I_i\}$ there exists a set $\{q^b/q^b \text{ in } A_b, b\in B\}$ such that $\lambda_{b_1}(q^{b_1}, I_1\cdots I_i)=\lambda_{b_2}(q^{b_2}, I_1\cdots I_i)$ for all b_1 and b_2 in B and all i. This is shown by the following example.

EXAMPLE 3.4. For each positive integer m let R_m consist of all sequences, each term of which is either 1 or 2, of length m. Let $\{q_0^1\}\cup\bigcup_{m=1}^{\infty}\{q_\sigma^1/\sigma\in R_m\}$ be the set of states of S_1. For each integer $i>1$ let

$$\{{}_jp_\sigma^i/\sigma\in R_{i-1}; j=1, 2, \cdots, i-1\}\cup\bigcup_{m=i-1}^{\infty}\{q_\sigma^i/\sigma\in R_m\}$$

be the set of states of S_i. Clearly the number of each machine is denumerable. Let the inputs be generated by I^1 and I^2. For each machine S_i, the functions δ_i and λ_i are defined as follows. For $\sigma=a_1a_2\cdots a_{i-1}$ and $j<i-1$ let $\delta_i({}_jp_\sigma^i, I^1)={}_{j+1}p_\sigma^i$ and $\delta_i({}_jp_\sigma^i, I^2)={}_jp_\sigma^i$ if $a_j=1$, $\delta_i({}_jp_\sigma^i, I^1)={}_jp_\sigma^i$ and $\delta_i({}_jp_\sigma^i, I^2)={}_{j+1}p_\sigma^i$ if $a_j=2$, $\lambda_i({}_jp_\sigma^i, I^1)=a_1a_2\cdots a_{j-1}1$, and $\lambda_i({}_jp_\sigma^i, I^2)=a_1a_2\cdots a_{j-1}2$. When $\sigma=a_1a_2\cdots a_{i-1}$ let $\delta_i({}_{i-1}p_\sigma^i, I^1)=q_\sigma^i$ and $\delta_i({}_{i-1}p_\sigma^i, I^2)={}_{i-1}p_\sigma^i$ if $a_{i-1}=1$, $\delta_i({}_{i-1}p_\sigma^i, I^1)={}_{i-1}p_\sigma^i$ and $\delta_i({}_{i-1}p_\sigma^i, I^2)=q_\sigma^i$ if $a_{i-1}=2$, $\lambda_i({}_{i-1}p_\sigma^i, I^1)=a_1\cdots a_{i-2}1$, and $\lambda_i({}_{i-1}p_\sigma^i, I^2)=a_1\cdots a_{i-2}2$. Finally, let $\delta_i(q_\sigma^i, I^1)=q_{\sigma 1}^i$, $\delta_i(q_\sigma^i, I^2)=q_\sigma^i 2$, $\lambda_i(q_\sigma^i, I^1)=\sigma 1$, and $\lambda_i(q_\sigma^i, I^2)=\sigma 2$.

To show that the hypothesis pertaining to the sequence $\{I_i\}$ of inputs holds, it is obviously sufficient to show that it holds when the sequence is infinite and each I_i is either I^1 or I^2. Accordingly, let $\{I_i\}$ be any infinite sequence of inputs, each I_i being either I^1 or I^2. Consider the set consisting of q_0^1 and all ${}_1p_{\sigma(i)}^{i+1}$, where $\sigma(i)=a_1\cdots a_i$, with $a_j=1$ if $I_j=I^1$ and $a_j=2$ if $I_j=I^2$. The output from each of these states under $I_1\cdots I_m$, for all m, is the same, namely $\sigma(1), \sigma(2), \cdots, \sigma(m)$. Thus the hypothesis is satisfied.

To see that there is no set $\{q_i/q_i \text{ in } S_i\}$ of states which are pairwise indistinguishable, first note that S_1 (as well as each of the other machines) is distinguished. Now assume that $\{q_i/q_i \text{ in } S_i\}$ is a set of pairwise indistinquishable states. Obviously $q_1\neq q_0^1$. Suppose that $q_1=q_\sigma^1$, where $\sigma=a_1\cdots a_m$. Since q_τ^{m+2} is indistinguishable from q_τ^1 for each $\tau=b_1\cdots b_mb_{m+1}\cdots$, q_{m+2}

must be one of the ${}_jp_\gamma^{m+2}$, where $j \leqq m+1$ and $\gamma = c_1 \cdots c_{m+1}$. Hence there exists an input, either I^1 or I^2, say I, such that $\delta_{m+2}({}_jp_\gamma, I) = {}_jp_\gamma$. Then $\lambda_{m+2}({}_jp_\gamma, II) = EE$ for some output E, whereas $\lambda_1(q_0^1, II) = \sigma_1\sigma_2$ with $\sigma_1 \neq \sigma_2$. Thus q_1 and q_{m+2} are distinguishable.

(2) The above example also shows that the condition that one of the H_b be finite cannot be relaxed to the extent that one of the A_b be finite. [Let $H_b = K_{S_1}$ and $A_b = \{q_0^1\}$.]

(3) The machines S_i constructed in Example 3.4 possess the property of having an infinite number of generating outputs. The machines may be modified so as to have but a finite number of generating outputs. A brief sketch doing this is now given.

EXAMPLE 3.5. Let B be the set of all irrational numbers s, $0 < x < 1$. In binary form let each x in B be $x = (.a_0a_1 \cdots a_i \cdots)$, each $a_i = 0$ or 1. For x and y in B write xRy if $2^ix - 2^jy$ is a rational number for some $i, j = 0, \pm 1, \pm 2, \cdots$. Clearly R is an equivalence relation, decomposing B into an infinite number of equivalence classes $\{B_a\}$. Let $\{B_i/i = 1, 2, \cdots\}$ be a denumerably infinite number of these equivalence classes. For each i let $x_i = (.a_0^ia_1^i \cdots a_n^i \cdots)$ be an element of B_i. Since x_i is irrational, its sequence of a_n^i cannot be ultimately periodic.

To define the machine $\bar{S}_1$ let the states of $\bar{S}_1$ be $\{\bar{q}_0^1\} \cup \bigcup_{m=1}^{\infty} \{\bar{q}_\sigma^1/\sigma \in R_m\}$, R_m being as in Example 3.4. The elements of $K_{\bar{S}_1}$ are arranged into a sequence $\{p_i\}$, with $\bar{q}_0^1$ the first element, followed by the elements of $\{\bar{q}_\sigma^1/\sigma \in R_1\}$ arranged in any order; and in general, the elements of $\{\bar{q}_\sigma^1/\sigma \in R_i\}$ are to be followed by the elements of $\{\bar{q}_\sigma^1/\sigma \in R_{i+1}\}$ arranged in any order. Let $E_1 = I^1$ and for each $i > 1$ let $E_i = E_{i-1}I^1$. Associate a_0^1 with p_1 and a_i^1 with $\delta_1(p_1, E_i)$. Continuing by induction, suppose that each a_j^i, $i \leqq k$, $j = 1, 2, \cdots$ is associated with a state $\bar{q}_\sigma^1$. Let $\bar{p}_{k+1}$ be the first state in $\{p_i\}$ which is not associated with any a_j^i. Associate a_0^{k+1} with $\bar{p}_{k+1}$ and a_i^{k+1} with $\delta_1(\bar{p}_{k+1}, E_i)$ for $i = 1, 2, \cdots$. It is easily seen that this association establishes a one to one, onto, correspondence, p_i with b_i, between the set of p_i and the set of a_j^i. Define $\lambda_1(\bar{q}_\sigma^1, I) = b_i$ if and only if $\delta_1(q_\sigma^1, I) = p_i$, $I = I^1$ or I^2.

The machines $\bar{S}_i$, $i > 1$, are now constructed in the obvious manner from $\bar{S}_1$ as S_i was constructed from S_1. It is readily seen that (1) for each sequence of inputs $\{I_i\}$ there exists a set $\{\bar{q}_j/\bar{q}_j \text{ in } \bar{S}_j\}$ such that $\lambda_j(\bar{q}_j, I_1 \cdots I_i) = \lambda_k(\bar{q}_k, I_1 \cdots I_i)$ for all j, k, and i; (2) there is no set $\{\bar{q}_i/\bar{q}_i \text{ in } \bar{S}_i\}$ of states which are pairwise indistinguishable; and (3) the outputs are generated by two outputs, 0 and 1.

4. **Input-distinguishability.** In the previous section we discussed the discrimination between *states* by considering the outputs obtained from identical *inputs*. In this section we discuss the discrimination between *inputs* by considering the outputs obtained from identical *states*.

DEFINITION. Two inputs I_1 and I_2 of a quasi-machine S are said to be *input-indistinguishable* if $\lambda(q, I_1) = \lambda(q, I_2)$ and $\lambda(q, I_1I) = \lambda(q, I_2I)$ for each state q and each input I. Otherwise I_1 and I_2 are said to be *input-distinguishable*. A quasi-machine is said to be *input-distinguished* if each two distinct inputs are input-distinguishable.

In case S is a machine with n (finite) states and M is a generating set for Y, I_1 and I_2 are input-indistinguishable if and only if for each state q and each sequence $A_1, \cdots, A_{n-1}$ of inputs from M, $\lambda(q, I_1)=\lambda(q, I_2)$ and $\lambda(q, I_1A_1 \cdots A_{n-1})=\lambda(q, I_2A_1 \cdots A_{n-1})$. This follows from the following result, which is an obvious modification of Theorem 6 of [9]. "Two states q_1 and q_2 in a machine with n (finite) states and generating set M for Y, are distinguishable if and only if there exists a sequence $A_1, \cdots, A_{n-1}$ of inputs from M so that $\lambda(q_1, A_1 \cdots A_{n-1}) \neq \lambda(q_2, A_1 \cdots A_{n-1})$."

The property of being input-distinguished does not carry over to subquasi-machines. Thus S may be input-distinguished and contain a subquasi-machine which is not input-distinguished. If S is not input-distinguished, then no subquasi-machine is input-distinguished.

Suppose that S, T, and $S \to T$ are quasi-machines. If both S and T are input-distinguished, then so is $S \to T$. To see this, let I_1 and I_2 be any two inputs of $S \to T$, thus of S. Since S is input-distinguished, there exists a state p of S so that $E_1=\lambda_S(p, I_1) \neq \lambda_S(p, I_2)=E_2$. Since T is input-distinguished, there exists a state q in T so that

$$\lambda_{S\to T}[(p, q), I] = \lambda_T(q, E_1) \neq \lambda_T(q, E_2) = \lambda_{S\to T}[(p, q), I_2].$$

Thus $S \to T$ is input-distinguished. It is not difficult to find examples showing that $S \to T$ is not input-distinguished if either S or T is not input-distinguished. On the other hand, if $S \to T$ is input-distinguished, then it is easily seen that S is also input-distinguished.

Suppose that for each state q, $\lambda(q, I_1)=\lambda(q, I_2)$ and $\delta(q, I_1)$ and $\delta(q, I_2)$ are indistinguishable states. Then

$$\begin{aligned} \lambda(q, I_1I) &= \lambda(q, I_1)\lambda[\delta(q, I_1), I] \\ &= \lambda(q, I_2)\lambda[\delta(q, I_2), I] \\ &= \lambda(q, I_2I), \end{aligned}$$

so that I_1 and I_2 are input-indistinguishable. The converse is not true, that is, if I_1 and I_2 are input-indistinguishable, it is not necessarily true that $\delta(q, I_1)$ and $\delta(q, I_2)$ are indistinguishable states.

EXAMPLE 4.1. Let $K_S=\{q_1, q_2, q_3\}$, Y be the free semi-group generated by I_1 and I_2, and W the semi-group generated by O_1 and O_2 subject to the relations (i) $O_1O_2=O_1O_1$, and (ii) $O_2O_1=O_2O_2$. Let $\delta(q_1, I_1)=q_2$, $\delta(q_1, I_2)=q_3$, $\delta(q_2, I)=q_2$, and $\delta(q_3, I)=q_3$ for any I. Let $\lambda(q_1, I)=O_1$, $\lambda(q_2, I)=O_1$, and $\lambda(q_2, I)=O_2$ for $I=I_1$ or I_2. Then I_1 and I_2 are input-indistinguishable but $\delta(q_1, I_1)$ and $\delta(q_1, I_2)$ are distinguishable states.

The converse is true in case S is a machine, namely

LEMMA 4.1. *If S is a machine and I_1 and I_2 are input-indistinguishable, then for each state q, $\delta(q, I_1)$ and $\delta(q, I_2)$ are indistinguishable states. Hence if S is distinguished, then $\delta(q, I_1)=\delta(q, I_2)$ for each state q.*

Proof. Since $\lambda(q, I_1I)=\lambda(q, I_2I)$ we get

$$\begin{aligned}\lambda(q, I_1)\lambda[\delta(q, I_2), I] &= \lambda(q, I_1I)\\ &= \lambda(q, I_2I)\\ &= \lambda(q, I_2)\lambda[\delta(q, I_2), I].\end{aligned}$$

As $\lambda(q, I_1)=\lambda(q, I_2)$ and S is a machine, $\lambda[\delta(q, I_1), I]=\lambda[\delta(q, I_2), I]$ for each input I. Thus $\delta(q, I_1)$ and $\delta(q, I_2)$ are indistinguishable states.

COROLLARY 1. *If S is a machine, if I_1 and I_2 are input-indistinguishable, and if q_1 and q_2 are indistinguishable states; then $\delta(q_1, I_1)$ and $\delta(q_2, I_2)$ are indistinguishable states.*

COROLLARY 2. *Let S be a machine. If I_1 and I_2 are input-indistinguishable and if I_3 and I_4 are input-indistinguishable, then so are I_1I_3 and I_2I_4.*

Proof. Since I_1 and I_2 are input-indistinguishable, $\lambda(q, I_1)=\lambda(q, I_2)$. By Lemma 4.1, $\delta(q, I_1)$ and $\delta(q, I_2)$ are indistinguishable states. Thus

$$\begin{aligned}\lambda[\delta(q, I_1), I_3] &= \lambda[\delta(q, I_2), I_3]\\ &= \lambda[\delta(q, I_2), I_4],\end{aligned}$$

since I_3 and I_4 are input-indistinguishable. Then

$$\begin{aligned}\lambda(q, I_1I_3) &= \lambda(q, I_1)\lambda[\delta(q, I_1), I_3]\\ &= \lambda(q, I_2)\lambda[\delta(q, I_2), I_4]\\ &= \lambda(q, I_2I_4).\end{aligned}$$

Since $\delta(q, I_1)$ and $\delta(q, I_2)$ are indistinguishable states and S is a machine, $\delta(q, I_1I_3)=\delta[\delta(q, I_1), I_3]$ and $\delta[\delta(q, I_2), I_4]=\delta(q, I_2I_4)$ are indistinguishable states by Corollary 1. Then

$$\begin{aligned}\lambda(q, I_1I_3I) &= \lambda(q, I_1I_3)\lambda[\delta(q, I_1I_3), I]\\ &= \lambda(q, I_2I_4)\lambda[\delta(q, I_2I_4), I]\\ &= \lambda(q, I_2I_4I).\end{aligned}$$

Hence I_1I_3 and I_2I_4 are input-indistinguishable.

REMARKS. (1) If S is a machine, and if both I_1 and I_2, and I_1I_3 and I_2I_4 are input-indistinguishable, it is not necessarily true that I_3 and I_4 are input-indistinguishable.

EXAMPLE 4.2. S is to be the free machine defined as follows. Let $K_S = \{q_1, q_2\}$ and let $\{I_1, I_2, I_3, I_4\}$ and $\{E_1, E_2\}$ be the input and output alphabets respectively. Let $\delta(q_1, I_i)=q_2$ and $\lambda(q_1, I_i)=E_1$ for $i=1, 2, 4$, and let $\delta(q_2, I_j)=q_2$ and $\lambda(q_2, I_j)=E_1$ for $j=1, 2, 3, 4$. Let $\delta(q_1, I_3)=q_1$ and $\lambda(q_1, I_3)=E_2$. Then I_1 and I_2, and I_1I_3 and I_2I_4, are input-indistinguishable. But I_3 and I_4 are not input-indistinguishable since $\lambda(q_1, I_3)\neq\lambda(q_1, I_4)$.

(2) Observe in Example 4.2 that for no state q is either $\delta(q, I_1)=q_1$ or $\delta(q, I_2)=q_1$. This is no accident due to the particular example chosen. For let S be a machine such that I_1 and I_2, and I_1I_3 and I_2I_4, are input-indistinguishable. If, for each state q, there exists a state $\bar{q}$ so that either $\delta(\bar{q}, I_1)=q$ or $\delta(\bar{q}, I_2)=q$, then I_3 and I_4 are input-indistinguishable. For let q be any state of S and $\bar{q}$ such that, say, $q=\delta(\bar{q}, I_1)$. To see that I_3 and I_4 are input-indistinguishable, it is sufficient to show that (a) $\lambda(q, I_3)=\lambda(q, I_4)$, and (b) $\lambda(q, I_3I)=\lambda(q, I_4I)$ for all inputs I. Since I_1I_3 and I_2I_4 are input-indistinguishable, $\lambda(\bar{q}, I_1I_3)=\lambda(\bar{q}, I_2I_4)$. Thus

$$\lambda(\bar{q}, I_1)\lambda[\delta(\bar{q}, I_1), I_3] = \lambda(\bar{q}, I_2)\lambda[\delta(\bar{q}, I_2), I_4].$$

As I_1 and I_2 are input-indistinguishable, $\lambda(\bar{q}, I_1)=\lambda(\bar{q}, I_2)$. Since S is a machine, W_S satisfies the left cancellation law. Thus $\lambda[\delta(\bar{q}, I_1), I_3]=\lambda[\delta(\bar{q}, I_2), I_4]$. By Corollary 1, $\delta(\bar{q}, I_1)$ and $\delta(\bar{q}, I_2)$ are indistinguishable states. Thus $\lambda(q, I_3)=\lambda(q, I_4)$. As to (b), repeating the above procedure, from $\lambda(\bar{q}, I_1I_3I)=\lambda(\bar{q}, I_2I_4I)$ there follows $\lambda[\delta(\bar{q}, I_1), I_3I]=\lambda[\delta(\bar{q}, I_2), I_4I]$, whence $\lambda(q, I_3I)=\lambda(q, I_4I)$. Therefore (a) and (b) are satisfied, so that I_3 and I_4 are input-indistinguishable.

(3) If S is a machine and if both I_3 and I_4, and I_1I_3 and I_2I_4, are input-indistinguishable, it is not necessarily true, as simple examples show, that I_1 and I_2 are input-indistinguishable. Reasonable conditions on S which will guarantee that I_1 and I_2 are input-indistinguishable are not known.

The counterpart to Theorem 3.1 is

THEOREM 4.1. *To each machine S there corresponds an input-distinguished machine T with the following properties*:

(1) *$K_S=K_T$ and $W_S=W_T$.*

(2) *There exists a homomorphism k of Y_S onto Y_T such that $\lambda_S(q, I)=\lambda_T[q, k(I)]$ for all inputs I in Y_S and each state q.*

(3) *If S is distinguished, then any input-distinguished machine T^* satisfying conclusions* (1) *and* (2) *is widely isomorphic*[15] *to T.*

Proof. For each pair of inputs I_1 and I_2 of S write $I_1\equiv I_2$ if I_1 and I_2 are input-indistinguishable. Clearly "$\equiv$" is an equivalence relation on K_S. Denote the equivalence class containing I by I^-. Define $I_1^-I_2^-$ to be $[I_1I_2]^-$, where I_1 and I_2 are arbitrarily elements of I_1^- and I_2^- respectively. In view of

[15] Two quasi-machines S and T are said to be *widely isomorphic* if there exists an isomorphism h of Y_S onto Y_T, an isomorphism g of W_S onto W_T, and a one to one mapping f of K_S onto K_T such that $f[\delta_S(q, I)]=\delta_T[f(q), h(I)]$ and $g[\lambda_S(q, I)]=\lambda_T[f(q), h(I)]$ for each state q and each input I. The term "isomorphic" is reserved for the case when $Y_S=Y_T$, $W_S=W_T$, and g and h are the identity mappings.

Corollary 2, $I_1^- I_2^-$ is independent of the I_1 in I_1^- and I_2 in I_2^- chosen. Clearly the set of I^- form a semi-group, which we denote by Y_T.

Define $\lambda_T(q, I^-)$ to be $\lambda_S(q, I)$ for some arbitrary I in I^-. Since the inputs I and I^- are input-indistinguishable, $\lambda_S(q, I)$ is independent of the I selected from I^-. For each state q, let q^- be the set of states of K_S which are indistinguishable from q, and let p_q be a definite element of q^-. Define $\delta_T(q, I^-)$ to be p_{q^*}, where I is in I^- and $q^* = \delta_S(q, I)$. By Lemma 4.1, $\delta_T(q, I^-)$ is independent of the I selected from I^-.

To prove that T is a quasi-machine (thus machine) it is sufficient to show that conditions (3) and (4) in the definition of a quasi-machine hold. As to (3), $\delta_T(q, I_1^- I_2^-) = \delta_T(q, [I_1 I_2]^-)$ is an element of $\delta_S(q, I_1 I_2)^-$. Also, $\delta_T(q, I_1^-)$ is in $\delta_S(q, I_1)^-$. Thus $\delta_T[\delta_T(q, I_1^-), I_2^-]$ is in $\delta_S(\delta_S(q, I_1), I_2)^-$. Since $\delta_S[\delta_S(q, I_1), I_2] = \delta_S(q, I_1 I_2)$ and the element p_q of q^- is unique, $\delta_T(q, I_1^- I_2^-) = \delta_T[\delta_T(q, I_1^-), I_2^-]$. As to (4),

$$\begin{aligned} \lambda_T(q, I_1^- I_2^-) &= \lambda_T(q, [I_1 I_2]^-) \\ &= \lambda_S(q, I_1 I_2) \\ &= \lambda_S(q, I_1)\lambda_S[\delta_S(q, I_1), I_2] \\ &= \lambda_T(q, I_1^-)\lambda_S[\delta_T(q, I_1^-), I_2], \text{ since } \delta_S(q, I_1) \text{ and} \\ &\quad\ \delta_T(q, I_1^-) \text{ are indistinguishable states,} \\ &= \lambda_T(q, I_1^-)\lambda_T[\delta_T(q, I_1^-), I_2^-]. \end{aligned}$$

Define the function k by $k(I) = I^-$. T is obviously input-distinguished and k satisfies condition (2) of the conclusion.

Now suppose that S is a distinguished machine and that T^* is an input-distinguished machine satisfying (1) and (2) of the theorem, by the function k^*. Let q in K_T correspond with q in K_{T^*} and E in W_T with E in W_{T^*}. Let I_1 and E_1 be any two elements of I_1^- and consider $k^*(I_1)$ and $k^*(E_1)$. Since k^* is a homomorphism of Y_S onto Y_{T^*}, for each state q,

$$\begin{aligned} \lambda_{T^*}[q, k^*(I_1)] &= \lambda_S(q, I_1) \\ &= \lambda_S(q, E_1) \\ &= \lambda_{T^*}[q, k^*(E_1)]; \end{aligned}$$

and, for each I in Y_S,

$$\begin{aligned} \lambda_{T^*}[q, k^*(I_1)k^*(I)] &= \lambda_S(q, I_1 I) \\ &= \lambda_S(q, E_1 I) \\ &= \lambda_{T^*}[q, k^*(E_1)k^*(I)]. \end{aligned}$$

Since T^* is input-distinguished, $k^*(I_1) = k^*(E_1)$. Defining $h(I_1^-) = k^*(I_1)$, we see that h is a uniquely defined function of Y_T onto Y_{T^*}. Suppose that $I_1^- \neq I_2^-$. Then I_1 and I_2 are input-distinguishable by definition of I_1^-. If, for some q, $\lambda_S(q, I_1) \neq \lambda_S(q, I_2)$, then $\lambda_{T^*}[q, h(I_1^-)] \neq \lambda_{T^*}[q, h(I_2^-)]$. If for some q

and some $I, \lambda_S(q, I_1 I) \neq \lambda_S(q, I_2 I)$, then $\lambda_{T^*}[q, h(I_1^-)h(I^-)] \neq \lambda_{T^*}[q, h(I_2^-)h(I^-)]$. Thus h is one to one. Since k^* is a homomorphism of Y_S onto Y_{T^*},

$$h(I_1^- I_2^-) = h([I_1 I_2]^-) = k^*(I_1 I_2) = k^*(I_1)k^*(I_2) = h(I_1^-)h(I_2^-).$$

Thus h is an isomorphism of Y_T onto Y_{T^*}.

To show that T and T^* are widely isomorphic, it is sufficient to show that $q_1 = \delta_T(q, I_1^-) = \delta_{T^*}[q, h(I_1^-)] = q_2$ for each q and each I_1^-. Then $\lambda_T(q, I_1^-) = \lambda_{T^*}[q, h(I_1^-)]$ and

$$\begin{aligned} \lambda_T(q, I_1^-)\lambda_T[\delta(q, I_1^-), I^-] &= \lambda_T(q, I_1^- I^-) \\ &= \lambda_{T^*}(q, h(I_1^-)h(I^-)) \\ &= \lambda_{T^*}[q, h(I_1^-)]\lambda_{T^*}[\delta_{T^*}(q, h(I_1^-)), h(I^-)]. \end{aligned}$$

Hence $\lambda_T[\delta_T(q, I_1^-), I^-] = \lambda_{T^*}[\delta_{T^*}(q, h(I_1^-)), h(I^-)]$ for each I^-. Then

$$\lambda_S(q_1, I) = \lambda_T(q_1, I^-) = \lambda_{T^*}(q_2, h(I^-)) = \lambda_T(q_2, I^-) = \lambda_S(q_2, I).$$

Since S is distinguished, $q_1 = q_2$. Q.E.D.

In general, it is not true that a free machine S in Theorem 4.1 gives rise to a free machine T. In order for a free machine S to yield a free machine T it is necessary and sufficient that whenever $I_1 \cdots I_n \equiv I_1^* \cdots I_m^*$, where each I_i and I_i^* are in the input alphabet, that $I_1 \equiv I_1^*$, thus that $I_i \equiv I_i^*$ for each i.

By an obvious modification of Theorem 4.1, the following result about free machines is seen to hold.

THEOREM 4.2. *To each free machine S there corresponds a free machine T with the following properties*:

(1) $K_S = K_T$ *and* $W_S = W_T$.

(2) *There exists a homomorphism k of Y_S onto Y_T such that* $\lambda_S(q, I) = \lambda_T[q, k(I)]$ *for all inputs I in Y_S and each state q.*

(3) *Each two elements of the input alphabet D_T of Y_T are input-distinguished.*

(4) *If S is distinguished, then any free machine T^* satisfying* (1), (2), *and* (3) *is widely isomorphic to T.*

For a free machine, the question arises of determining a number k having the property that if all inputs of length k or less are pairwise input-distinguishable, then the machine is input-distinguished. One such value for k is now given.

THEOREM 4.3[16]. *Let S be a free machine with n (finite) states. Let* $k = \sum_{r_1+r_2=w;\, w=2}^{2n} (n!/(n-r_1)!)(n!/(n-r_2)!)$, *where $r_1 = r_2$ if w is even and*

[16] The idea in Theorem 4.3 of using the $2n$-tuples of states to prove the existence of a bound k on the length of inputs to determine whether or not S is input-distinguished is due to C. C. Chang. He attained a value of n^{2n}. The lowering of k by considering permutations of the distinguishable states is due to the author.

$r_1=r_2-1$ if w is odd. If every two distinct sequences $I_1 \cdots I_k$ and $E_1 \cdots E_k$ of elements from the input alphabet are input-distinguishable, then S is input-distinguished.

Proof. The following symbolism is used in the argument. The letters a and α (with or without a subscript or superscript) are $2n$-tuples of, respectively, states and outputs of S. Thus $a=(p_1, \cdots, p_{2n})$ and $\alpha=(O_1, \cdots, O_{2n})$, where each p_i is a state and each O_i an output. Let $b(a)=(p_1, p_3, \cdots, p_{2n-1})$, $c(a) = (p_2, p_4, \cdots, p_{2n})$, $b(\alpha) = (O_1, O_3, \cdots, O_{2n-1})$, and $c(\alpha) =(O_2, O_4, \cdots, O_{2n})$. Let $n_1(a)$ and $n_2(a)$ be the number of distinct states in $b(a)$ and $c(a)$ respectively. Given two inputs I and E let

$$\delta(a, I, E) = (\delta(p_1, I), \delta(p_2, E), \delta(p_3, I), \delta(p_4, E), \cdots, \delta(p_{2n}, E))$$

and

$$\lambda(a, I, E) = (\lambda(p_1, I), \lambda(p_2, E), \lambda(p_3, I), \lambda(p_4, E), \cdots, \lambda(p_{2n}, E)).$$

Since S is a free machine, two sequences of inputs from the input alphabet D of different lengths are input-distinguishable. It will now be shown that two distinct sequences of elements from D of the same length are input-distinguishable, thus proving the theorem. To this end, let m be the smallest integer such that two distinct sequences $I_1 \cdots I_m$ and $E_1 \cdots E_m$ of elements from D are input-indistinguishable. By hypothesis, $k<m$. It will now be shown that

(*) there exist two distinct sequences of elements from D, of the same length $<m$, which are input-indistinguishable.

This will affect a contradiction, thus proving the theorem. Accordingly, let

$$\overset{1}{a} = (q_1, q_1, q_2, q_2, \cdots, q_n, q_n) = (\overset{1}{p}_1, \cdots, \overset{1}{p}_{2n}),$$

where $q_1, q_2, \cdots, q_n$ are the n states of S. For each positive integer $i \leqq m$, let

$$\overset{i+1}{a} = \delta(\overset{i}{a}, I_i, E_i) = (\overset{i+1}{p_1}, \cdots, \overset{i+1}{p_{2n}})$$

and

$$\overset{i}{\alpha} = \lambda(\overset{i}{a}, I_i, E_i) = (\overset{i}{O}_1, \cdots, \overset{i}{O}_{2n}).$$

To prove that (*) holds it is sufficient to show that either (*) holds or else

(**) there exist two values of i, say s and t, such that $a^s=a^t$.

For suppose that (**) holds. Let $\bar{I} = I_1 \cdots I_{s-1}I_t \cdots I_m$ and $\bar{E}=E_1 \cdots E_{s-1}E_t \cdots E_m$. It is to be understood that $\bar{I}=I_t \cdots I_m$ and $\bar{E}=E_t \cdots E_m$ if $s=1$. Two cases arise.

(a) Suppose that $\bar{I}$ and $\bar{E}$ are distinct sequences. Since I and E are input-indistinguishable,

$$b[\lambda(a^1, I, E)] = c[\lambda(a^1, I, E)] = (O_1, O_3, \cdots, O_{2n-1})$$

and

$$b[\lambda(\overset{1}{a}, IA, EA)] = c[\lambda(\overset{1}{a}, IA, EA)] = (\overset{A}{O_1}, \overset{A}{O_3}, \cdots, \overset{A}{O_{2n-1}}),$$

where A is an arbitrary input (not necessarily in D), $O_i = O_i^1 O_i^2 \cdots O_i^m$, $O_i^A = O_i U_i^A$, and $b[\lambda(a^{m+1}, A, A)] = (U_1^A, U_3^A, \cdots, U_{2n-1}^A)$. From the fact that S is a free machine and D is the input alphabet, it follows that $O_{2i}^j = O_{2i-1}^j$ for each $j \leqq m$ and each $i \leqq n$. Since $a^s = a^t$,

$$b[\lambda(a^1, \bar{I}, \bar{E})] = (\bar{O}_1, \bar{O}_3, \cdots, \bar{O}_{2n-1}) = c[\lambda(a^1, \bar{I}, \bar{E})]$$

and

$$b[\lambda(a^1, \bar{I}A, \bar{E}A)] = (\bar{O}_1^A, \bar{O}_3^A, \cdots, \bar{O}_{2n-1}^A) = c[\lambda(\overset{1}{a}, \bar{I}A, \bar{E}A)],$$

where $\bar{O}_i = O_i^1 \cdots O_i^{s-1} O_i^t \cdots O_i^m$ and $\bar{O}_i^A = \bar{O}_i U_i^A$. Thus I and E satisfy (*).

(b) Suppose that $\bar{I}$ and $\bar{E}$ are not distinct. Since I and E are distinct, $I_s \cdots I_{t-1}$ and $E_s \cdots E_{t-1}$ are distinct. Then $M = I_1 \cdots I_{t-1}$ and $N = E_1 \cdots E_{t-1}$ are distinct sequences of elements from D. Clearly $b[\lambda(a^1, M, N)] = c[\lambda(a^1, M, N)]$. Since $\bar{I} = \bar{E}$, $I_1 \cdots I_{s-1} = E_1 \cdots E_{s-1}$. Now $b(a^1) = c(a^1)$. Thus $b(a^s) = c(a^s)$. Since $a^s = a^t$, it follows that $b(a^t) = c(a^t)$. Thus

$$b[\lambda(a^1, MA, NA)] = c[\lambda(a^1, MA, NA)],$$

where A is an arbitrary input (not necessarily in D). Thus M and N satisfy (*).

We now demonstrate that either (*) or (**) holds. Consider the sequence $\{a^i\}_{i \leqq m}$. Two elements a^i and a^j of the sequence are distinct if and only if either $b(a^i)$ and $b(a^j)$ or $c(a^i)$ and $c(a^j)$ are distinct. Clearly

(***) if two states p_d^i and p_e^i are identical, where d and e have the same parity, then p_d^{i+1} and p_e^{i+1} are identical.

Thus $n_1(a^i)$ and $n_2(a^i)$ are nonincreasing, strictly positive, functions of i. For $2 \leqq j \leqq 2n$, let $\alpha(j) = (n!/(n-r_1)!)(n!/(n-r_2)!)$, where $r_1 = r_2 = j/2$ if j is even and $r_1 = r_2 - 1 = (j-1)/2$ if j is odd. Then $k = \sum_{j=2}^{2n} \alpha(j)$. Given j, it follows from (***) that there exists at most one pair (n_1, n_2) of numbers, with $n_1 + n_2 = j$, such that $n_1(a^i) = n_1$ and $n_2(a^i) = n_2$ for some a^i. For $2 \leqq j \leqq 2n$, let $\theta(j)$ be the number of distinct integers i such that $n_1(a^i) + n_2(a^i) = j$. Clearly $m = \sum_{j=2}^{2n} \theta(j)$. Since $k < m$, there exists at least one integer j so that $\alpha(j) < \theta(j)$. Let w be the largest such integer j. Let (n_1^w, n_2^w) be the unique pair associated with w. Let $r_1^w = r_2^w = w/2$ if w is even, and $r_1^w = r_2^w - 1 = (w-1)/2$ if w is odd. Without loss of generality we may assume that $n_1^w \leqq n_2^w$ since otherwise we may interchange the role of n_1^w and n_2^w in what follows. If $n_1^w = n_2^w$ or $n_1^w = n_2^w - 1$, then $(n_1^w, n_2^w) = (r_1^w, r_2^w)$. If $n_1^w + n_2^w$ is even and $n_1^w < n_2^w$, then $n_1^w < r_1^w = r_2^w < n_2^w$. If $n_1^w + n_2^w$ is odd and $n_1^w < n_2^w - 1$, then $n_1^w < r_1^w = r_2^w - 1 < n_2^w - 1$. In both cases,

$$\frac{n!}{(n-n_1^w)!}\frac{n!}{(n-n_2^w)!}$$

$$\begin{aligned}
&= [n(n-1)\cdots(n-n_1^w+1)]\,[n(n-1)\cdots(n-n_2^w+1)] \\
&= [n(n-1)\cdots(n-n_1^w+1)]^2[(n-n_1^w)\cdots(n-n_2^w+1)] \\
&\leqq [n(n-1)\cdots(n-n_1^w+1)]^2[(n-n_1^w)\cdots(n-r_1^w+1)] \\
&\qquad \cdot[(n-n_1^w)\cdots(n-r_2^w+1)], \text{ since } n_1^w+n_2^w=r_1^w+r_2^w, \\
&= \frac{n!}{(n-r_1^w)!}\frac{n!}{(n-r_2^w)!} \\
&= \alpha(w).
\end{aligned}$$

Due to (***) and elementary permutation theory, the maximum number of distinct a^i for which $n_1(a^i)=n_1^w$ and $n_2(a^i)=n_2^w$ is $(n!/(n-n_1^w)!)(n!/(n-n_2^w)!)$. Since

$$\frac{n!}{(n-n_1^w)!}\frac{n!}{(n-n_2^w)!}\leqq \alpha(w)<\theta(w),$$

there exist two values of i, say s and t, such that $a^s=a^t$, i.e., (**) holds. Thus the theorem is proved.

It is known that the value for k given in Theorem 4.3 is not the best possible. For example, $\alpha(2n)$ can be lowered from $(n!)(n!)$ to $n!$ by observing that the $2n$-tuples can be replaced by the unordered n-tuples of pairs $[(p_1^t, p_2^t), \cdots, (p_{2n-1}^t, p_{2n}^t)]$. In fact, at any stage, the $2n$-tuples can be replaced by the two unordered n-tuples $[(p_1^t, p_2^t), \cdots, (p_{2n-1}^t, p_{2n}^t)]$ and $[(p_2^t, p_1^t), \cdots, (p_{2n}^t, p_{2n-1}^t)]$. It is not known if the value of k obtained by this refinement is the smallest possible.

Another bound for k, which in certain circumstances is better than that given in Theorem 4.3, is now given.

THEOREM 4.4. *Let S be a distinguished, free machine. Let $K_S=\{q_1, \cdots, q_r\}$. For each i, let n_i be the number of states in the principal submachine generated by q_i. Let $k=(n_1\cdots n_r)^2$. If every two sequences $I_1\cdots I_k$ and $E_1\cdots E_k$ of elements from the input alphabet are input-distinguishable, then S is input-distinguished.*

The proof is an obvious modification of Theorem 4.3.

The line of reasoning given in Theorems 4.3 and 4.4 is now applied to yield the following lemma.

LEMMA 4.2. *Let m and n be positive integers. Let S be a free machine with n states and m elements in the output alphabet F. Suppose that C is a subset of the* *input alphabet and that C contains r elements, where* $(m^k n)^n < r^k$. *Then there exists two inputs* $A_1 A_2 \cdots A_k$ *and* $B_1 B_2 \cdots B_k$, *each* A_i *and* B_i *in C, which are input-indistinguishable.*

Proof. Let $K = \{p_i / i \leqq n\}$. To each input $I = I_1 \cdots I_k$, each I_i in C, associate the n-tuple of triples

$$\tau_I = ((p_1, q_1, E_1), (p_2, q_2, E_2), \cdots, (p_n, q_n, E_n)),$$

where, for each $i \leqq n$, $q_i = \delta(p_i, I)$ and $E_i = \lambda(p_i, I)$. It is clear that there are at most $(m^k n)^n$ distinct such n-tuples since there are m elements in F and n in K. As there are $r^k > (m^k n)^n$ such inputs I, two of the associated n-tuples, say τ_I and τ_{I^*} are identical. Then I and I^* are input-indistinguishable. This is so since $\lambda(p, I) = \lambda(p, I^*)$ for each state p in S; and $\lambda(p, IA) = \lambda(p, I^*A)$ for each state p in S as $\delta(p, I) = \delta(p, I^*)$. Q.E.D.

Using the above lemma we are now able to prove a result which places bounds on the number of states, number of elements in the input alphabet, and number of elements in the output alphabet of a free machine S in order for S to be input-distinguished.

THEOREM 4.5. *Let n, r, and m be given positive integers. Let S be a free machine with n states, r elements in the input alphabet, and m elements in the output alphabet. If* $m^n < r$, *then S is not input-distinguished. If* $r \leqq m^n$, *then there exists a free, input-distinguished, machine with n states, r elements in the input alphabet, and m elements in the output alphabet.*

Proof. Let S be a free machine with $m^n < r$. Since $1 < r/m^n$, for some integer k, $n^n < r^k/m^{nk}$, i.e., $(m^k n)^n < r^k$. By Lemma 4.2, there exists two sequences $A_1 \cdots A_k$ and $B_1 \cdots B_k$ of elements from the input alphabet which are input-indistinguishable. Consequently, S is not input-distinguished.

Now suppose that $r \leqq m^n$. Let K be a set of n elements, say $K = \{p_i / i \leqq n\}$, and F a set of m elements, say $F = \{U_i / i \leqq m\}$. Let T be a set of r distinct n-tuples of the form

$$\tau = ((p_1, p_1, E_1), (p_2, p_2, E_2), \cdots, (p_n, p_n, E_n)),$$

where each E_i is in F. T exists since $r \leqq m^n$. To each τ in T associate a symbol I_τ and let $D = \{I_\tau / \tau \in T\}$. For each $i \leqq n$, define $\delta(p_i, I_\tau) = p_i$ and $\lambda(p_i, I_\tau) = E_i$, where (p_i, p_i, E_i) is the ith coordinate in τ. This defines a free machine V with n states, r elements in the input alphabet, and m elements in the output alphabet.

Let $A_1 \cdots A_s$ and $B_1 \cdots B_s$ be two distinct sequences of elements from D. Suppose that A_t is the first A_i such that $A_t \neq B_t$, but $A_i = B_i$ for $i < t$. Then $A_t = I_\sigma$ and $B_t = I_\tau$, with $\sigma \neq \tau$. For some integer, say j, the jth coordinate (p_j, p_j, E_j) of σ differs from the jth coordinate (p_j, p_j, E_j^*) of τ. Then $E_j \neq E_j^*$. From this it follows that $\lambda(p_j, A_1 \cdots A_s) \neq \lambda(p_j, B_1 \cdots B_s)$. Therefore V is input-distinguished.

The next result, in conjunction with Theorem 4.2, asserts that when considering free machines with a finite number of elements in the output alphabet, only a finite number of elements in the input alphabet is "actually needed."

THEOREM 4.6. *Let m and n be two positive integers. Let S be a free machine with n states and m elements in the output alphabet. Then every set of $(mn)^n+1$ elements in the input alphabet contains two which are input-indistinguishable. Furthermore, if $m\geqq 2$, there is a distinguished, free machine with n states, m elements in the output alphabet, and $(mn)^n$ pairwise input-distinguishable elements in the input alphabet.*

Proof. The first part of the theorem follows from Lemma 4.2 with $k=1$.

As to the second part, suppose that n and $m\geqq 2$ are given positive integers. Let K be a set of n elements, say $K=\{p_i/i\leqq n\}$, and F a set of m elements, say $F=\{U_i/i\leqq m\}$. Let T be the set of all $(mn)^n$ distinct n-tuples of the form

$$\tau = ((p_1, q_1, E_1), (p_2, q_2, E_2), \cdots, (p_n, q_n, E_n)),$$

where for each $i\leqq n$, q_i is in K and E_i in F. To each τ in T associate a symbol I_τ and let $D=\{I_\tau/\tau\in T\}$. Define $\delta(p_i, I_\tau)=q_i$ and $\lambda(p_i, I_\tau)=E_i$, where (p_i, q_i, E_i) is the ith coordinate in τ. This defines a free machine V with n states and $(mn)^n$ and m elements in the input and output alphabets respectively.

Finally, let I_σ and I_τ be two distinct inputs in D. Let i be the smallest integer such that the ith coordinate (p_i, q_i, E_i) in σ is not identical with the ith coordinate $(p_i, \bar{q}_i, A_i)$ in τ. Two cases arise.

(i) If $E_i\neq A_i$, then $\lambda(p_i, I_\sigma)\neq\lambda(p_i, I_\tau)$.

(ii) If $E_i=A_i$, then $q_i\neq\bar{q}_i$. As shown above, there exists I_μ in D so that $\lambda(q_i, I_\mu)\neq\lambda(\bar{q}_i, I_\mu)$. Thus $\lambda(p_i, I_\sigma I_\mu)\neq\lambda(p_i, I_\tau I_\mu)$. In either case we are forced to conclude that I_σ and I_τ are input-distinguishable. Thus each two elements of D are input-distinguishable. Q.E.D.

It is noted in passing that the analogue to Theorem 4.6 for the case where the number of inputs and the number of outputs are given, is not true. Namely, it is not true that a distinguished, input-distinguished, free machine, with n inputs and m outputs, has a finite number of states, depending solely on the integers m and n. For the free machine $\bar{S}_1$ in Example 3.5 is a distinguished, input-distinguished, machine, witu two elements in the input and two elements in the output alphabet respectively, and an infinite number of states.

5. **Output complete states.** We now consider states which yield all possible outputs.

DEFINITION. A state q of the quasi-machine S is said to be *output complete* if for each output E there exists an input I so that $\lambda(p, I)=E$. If each state of S is output complete, then S is said to be *output complete.*

If A is a generating set for W and if for each state q in S and each element E in A there exists an input I so that $\lambda(q, I)=E$, then S is output complete.

Given positive integers m, n, and r, there are only a finite number of non-isomorphic free machines having n states, m elements in the input alphabet, and r elements in the output alphabet. Thus there are only a finite number of distinguishable states in these machines. For each of these states p, either p is output complete or else there is a smallest integer $k(p)$ and a sequence $E_1^p, \cdots, E_{k(p)}^p$ of generating outputs with the property that $\lambda(p, I_1 \cdots I_{k(p)}) = E_1^p \cdots E_{k(p)}^p$ is false for every sequence $I_1, \cdots, I_{k(p)}$ of generating inputs. Let $k(m, n, r)$ be the largest of the numbers $k(p)$. The number $k(m, n, r)$ then has the following property. Given any free machine S of n states, with m and r elements in the input and output alphabets respectively, a state p in S is output complete if and only if, for each sequence $E_1, \cdots, E_{k(m,n,r)}$ of generating outputs, there exists a sequence $I_1, \cdots, I_{k(m,n,r)}$ of generating inputs such that $\lambda(p, I_1 \cdots I_{k(m,n,r)}) = E_1 \cdots E_{k(m,n,r)}$. There is no known explicit formula for $k(m, n, r)$ (in terms of m, n, and r, of course). In fact, there is no known upper bound for $k(m, n, r)$.

If $S \rightarrow T$ is a quasi-machine, and if p in S and q in T are both output complete states; then, as is easily seen, (p, q) is output complete in $S \rightarrow T$. Thus, if $S \rightarrow T$ exists, and if both S and T are output complete, then $S \rightarrow T$ is output complete. The obvious generalization to $S_1 \rightarrow \cdots \rightarrow S_n$ holds.

If T_1 and T_2 are subquasi-machines of S_1 and S_2 respectively and if $S_1 \rightarrow S_2$ exists, then $T_1 \rightarrow T_2$ is a subquasi-machine of $S_1 \rightarrow S_2$. The obvious converse does not hold. What is true, however, is the following easily proved result. "Suppose that S_1 is output complete, $S_1 \rightarrow S_2$ exists, and $W_{S_1} = Y_{S_1}$. If A is a subquasi-machine of $S_1 \rightarrow S_2$, then $F_1 = \{p/(p, q) \in K_A\}$ and $F_2 = \{q/(p, q) \in K_A\}$ are stable subsets of S_1 and S_2 respectively."

A property of quasi-machines which shall be of interest to us in connection with output complete states is now given.

Definition. A quasi-machine S is said to have *property* Q if whenever $\lambda(q, I) = E_1E_2$, with E_1 and E_2 in W, there exists I_1 and I_2 so that $I = I_1I_2$, $\lambda(q, I_1) = E_1$, and $\lambda[\delta(q, I_1), I_2] = E_2$.

A simple induction argument shows that if S has property Q and if $\lambda(q_1, I) = E_1 \cdots E_r$, with each E_i in W, then there exists $I_1, I_2, \cdots, I_r$ such that $\lambda(q_i, I_i) = E_i$, where $q_i = \delta(q_{i-1}, I_{i-1})$ for $2 \leqq i$.

If S is a machine, then in order for S to have property Q it is sufficient that whenever $\lambda(q, I) = E_1E_2$, with E_1 and E_2 in W, I_1 and I_2 in Y can be found so that $I = I_1I_2$ and $\lambda(q, I_1) = E_1$. This is so since

$$\begin{aligned}\lambda(q, I_1I_2) &= \lambda(q, I_1)\lambda[\delta(q, I_1), I_2]\\ &= E_1\lambda[\delta(q, I_1), I_2] = E_1E_2,\end{aligned}$$

whence $\lambda[\delta(q, I_1), I_2] = E_2$.

It is obvious that each free machine has property Q. Also, if S and T both have property Q, then so does $S \rightarrow T$.

Suppose that S has property Q. Furthermore, suppose that p is an output complete state, and I_0 an input with the following property. For any input I, if $\lambda(p, I)=\lambda(p, I_0)$ then $q=\delta(p, I_0)=\delta(p, I)$. Then q is also output complete. For let E be any output and let $E_0=\lambda(p, I_0)$. As p is output complete, there exists an input I such that $\lambda(p, I)=E_0E$. Since S has property Q, $I=I_1I_2$ such that $\lambda(p, I_1)=E_0$ and $\lambda[\delta(p, I_1), I_2]=E$. In view of the hypothesis on S and the fact that $\lambda(p, I_0)=\lambda(p, I_1)$, it occurs that $q=\delta(p, I_0)=\delta(p, I_1)$. Thus $\lambda(q, I_2)=E$, so that q is output complete.

Suppose that S is a free machine with the input alphabet D and the output alphabet F both having the same number (finite) of elements. Then (1) p being an output complete state, and (2) λ being one to one at p, are equivalent statements; and each implies that the principal submachine T_p generated by p is output complete. The equivalence of (1) and (2) is obvious since D and F have the same finite number of elements. If (1), thus (2) holds, then for any two inputs I and I_0, whenever $\lambda(p, I)=\lambda(p, I_0)$, $I=I_0$, so that $\delta(p, I)=\delta(p, I_0)$. Hence $\delta(p, I)$ is output complete, i.e., T_p is output complete.

The following auxiliary concept is now introduced([17]).

Definition. A subquasi-machine T of S is said to have *property* P if the following condition holds. For each state q in T, if E is an output such that $\lambda(q, I)=E$ is false for every input I, if $\bar{q}$ in S and I can be found so that $\delta(\bar{q}, I)=q$; then $\lambda(\bar{q}, H)=OE$ is false for all inputs H, where $O=\lambda(\bar{q}, I)$.

Lemma 5.1. *Let S be a quasi-machine, p a state which is output complete, and T the principal subquasi-machine generated by p. If T has property* P, *then T is output complete.*

Proof. It is sufficient to show that for each input A, $q=\delta(p, A)$ is output complete. Therefore suppose the contrary, that is, suppose that E is an output such that $\lambda(q, I)=E$ is false for every input I. Let $O=\lambda(p, A)$. Since T has property P, $\lambda(p, H)=OE$ is false for every input H. However, this contradicts p being output complete. Hence there exists an input I so that $\lambda(q, I)=E$, i.e., q is output complete.

Corollary 1. *Let S have property* Q. *Suppose that for each state q in S and each two inputs I_1 and I_2, $\delta(q, I_1)=\delta(q, I_2)$ whenever $\lambda(q, I_1)=\lambda(q, I_2)$. Then S has property* P.

Proof. Suppose that for a given state q, a given output E, and all inputs I, $\lambda(q, I)=E$ is false. Let $\bar{q}$ and A be such that $\delta(\bar{q}, A)=q$ and let $\lambda(\bar{q}, A)=O$. Suppose that H exists satisfying $\lambda(\bar{q}, H)=OE$. Since S has property Q, $H=H_1H_2$ such that $\lambda(\bar{q}, H_1)=O$ and $\lambda[\delta(\bar{q}, H_1), H_2]=E$. As $\lambda(\bar{q}, H_1)=\lambda(\bar{q}, A)$, it results that $\delta(\bar{q}, H_1)=\delta(\bar{q}, A)=q$. Thus $\lambda(q, H_2)=E$ which contradicts the selection of q and E. Hence S has property P.

([17]) The author is indebted to C. C. Chang for isolating and pointing out that property **P** would suffice in subsequent applications, instead of a more restrictive condition originally given by the author.

From Corollary 1 there immediately follows

COROLLARY 2. *Let S be a free machine such that for each state q, λ is one to one on the input alphabet. Then S has property* P.

THEOREM 5.1. *Let S be a quasi-machine with a denumerable number of states and having property* P. *If, for each sequence $\{E_n\}$ of outputs, there exists a state q and a sequence $\{I_n\}$ of inputs such that $\lambda(q, I_1 \cdots I_n) = E_1 \cdots E_n$ for each integer n; then there exists a subquasi-machine which is output complete. Thus, under these conditions, if S is strongly connected, then S is output complete.*

Proof. In view of Lemma 5.1 it is sufficient to show the existence of a state p which is output complete. Therefore, suppose that there is no state p which is output complete. Let the distinct states of S be $q_0, q_1, \cdots$. There exists an output E_1 such that $\lambda(q_0, I) = E_1$ is false for each input I. Let $q_{\alpha(1)}$ be the first q such that $\lambda(q, A_1) = E_1$ for some input A_1. In view of the hypothesis, $q_{\alpha(1)}$ exists. Obviously $\alpha(1) > 0$. Let $p_2 = \delta(q_{\alpha(1)}, A_1)$. Since p_2 is not output complete, there exists an output E_2 such that $\lambda(p_2, I) = E_2$ is false for every input I. By property P there is no input I such that $\lambda(q_{\alpha(1)}, I) = E_1E_2$.

Now suppose that for $2 \leqq i \leqq n$, E_i and $q_{\alpha(i-1)}$ are defined satisfying the following three properties:

(1) $\alpha(i-1) \geqq i-1$ and $\alpha(j-1) > \alpha(j-2)$ for $3 \leqq j \leqq n$;

(2) $\lambda(q_{\alpha(i-1)}, I) = E_1 \cdots E_i$ is false for every input I; and

(3) $q_{\alpha(i-1)}$ is the first q for which there exists an input I such that $\lambda(q, I) = E_1 \cdots E_{i-1}$.

Define $q_{\alpha(n)}$ to be the first q for which there exists an input, call it A_n, satisfying $\lambda(q, A_n) = E_1 \cdots E_n$. By the hypothesis pertaining to sequences of outputs, the element $q_{\alpha(n)}$ exists. From (2) and (3), $\alpha(n) > \alpha(n-1) \geqq n-1$. Let $p_{n+1} = \delta(q_{\alpha(n)}, A_n)$. Since p_{n+1} is not output complete, there exists an output E_{n+1} such that $\lambda(p_{n+1}, I) = E_{n+1}$ is false for every input I. By property P, $\lambda(q_{\alpha(n)}, I) = E_1 \cdots E_{n+1}$ is false for every input I. Thus (1)–(3) are true for $n+1$. By induction, (1)–(3) are true for every integer n.

By construction $\{E_n\}$ is a sequence of outputs. By hypothesis there exists a sequence of inputs $\{I_n\}$ and a state q of S such that $\lambda(q, I_1 \cdots I_n) = E_1 \cdots E_n$ for each n. Being a state of S, q is one of the q_i, say q_m. Then $\lambda(q_m, I_1 \cdots I_{m+1}) = E_1 \cdots E_{m+1}$. But $q_{\alpha(m+1)}$ is the first element such that $\lambda(q, I) = E_1 \cdots E_{m+1}$ for some input I; and $\alpha(m+1) > m$. Thus $\lambda(q_m, I_1 \cdots I_{m+1}) = E_1 \cdots E_{m+1}$ is false. From this contradiction it follows that the assumption of no state being output complete is false.

COROLLARY. *Let $\{S(i)/i = 1, 2, \cdots\}$ be a denumerable collection of denumerable state quasi-machines, all having the same input semi-group and same output semi-group respectively. Suppose that each quasi-machine $S(i)$ has property* P. *If for each sequence $\{E_n\}$ of outputs, there exists $S(i)$, a state q^i in $S(i)$, and a sequence $\{I_n\}$ of inputs such that $\lambda_{S(i)}(q^i, I_1 \cdots I_n) = E_1 \cdots E_n$ for each* *integer n; then there exists a subquasi-machine of one of the $S(i)$ which is output complete.*

The above corollary follows from the fact that the sum of the $S(i)$ satisfies the hypothesis of Theorem 5.1, whence the conclusion.

If the quasi-machine in Theorem 5.1 has just a finite number, say k, of states, then the proof shows that after at most k steps, no $q_{\alpha(k)}$ will exist. The hypothesis pertaining to sequences $\{E_n\}$ of outputs and sequences $\{I_n\}$ of inputs may be modified by only requiring that for each output E there exists a state q and input I so that $\lambda(q, I)=E$. In conjunction with the Corollary to Theorem 5.1, we thus get

THEOREM 5.2. *Let $\{S(i)/i=1, 2, \cdots, n\}$ be a finite collection of finite state quasi-machines, all having the same input semi-group and same output semi-group respectively. Suppose that each $S(i)$ has property* P. *If, for each output E there exists a quasi-machine $S(i)$, a state q^i in $S(i)$, and an input I such that $\lambda_{S(i)}(q^i, I)=E$; then there exists a subquasi-machine of one of the $S(i)$ which is output complete.*

Theorem 5.2 is no longer true if "finite collection of finite state" is replaced by "infinite collection of finite state."

Turning to a "practical" condition for determining when a state in a free machine is output complete we first prove

LEMMA 5.2. *Let S have property* Q. *Let T be a subquasi-machine, and M a generating set for W, with the following property*:

(P′) *For each state q in T, if an element E of M is such that $\lambda(q, I)=E$ is false for every input I, if state $\bar{q}$ in S and input I can be found so that $\delta(\bar{q}, I)=q$; then $\lambda(\bar{q}, H)=OE$ is false for all inputs H, where $O=\lambda(\bar{q}, I)$. Then T has property* P.

Proof. Let q be in T and $E=E_1E_2 \cdots E_k$, each E_i being in M, such that $\lambda(q, I)=E$ is false for every output I. Let p in S and A an input be such that $\delta(p, A)=q$. Suppose that there exists an input H so that $\lambda(p, H)=OE$, where $O=\lambda(p, A)$. By property P′, $k>1$. Two possibilities exist.

(1) Suppose that there is an input I so that $\lambda(q, I)=E_1$. Let t be the largest integer such that an input, call it $\bar{I}$, can be found so that $\lambda(q, \bar{I}) =E_1 \cdots E_t$. By assumption, $t<k$. Let $q_1=\delta(q, \bar{I})$. In view of the maximality property of t, for no input I does $\lambda(q_1, I)=E_{t+1}$. Since S has property Q, $H=H_1H_2$ or $H=H_1H_2H_3$, according as $t=k-1$ or $t<k-1$, such that $\lambda(p, H_1) =OE_1 \cdots E_t$ and $\lambda(p, H_1H_2)=OE_1E_2 \cdots E_{t+1}$. This contradicts T having property P′ since $\delta(p, A\bar{I})=q_1$ and $\lambda(p, A\bar{I})=OE_1E_2 \cdots E_t$.

(2) Suppose that for no input I does $\lambda(q, I)=E_1$. Since S has property Q, $H=H_1H_2H_3$, such that $\lambda(p, H_1)=O$ and $\lambda(p, H_1H_2)=OE_1$. This contradicts T having property P′.

In either case we are led to a contradiction. Thus, for no input H is $\lambda(p, H) = OE$, where $O = \lambda(p, A)$, i.e., T has property P.

THEOREM 5.3. *Let S be a free machine and let p be a state satisfying the following conditions:*

(1) *For each sequence $E_1, \cdots, E_r$, of length n or less, of elements of the output alphabet F, there exists a sequence $I_1, \cdots, I_r$ of elements of the input alphabet D such that $\lambda(p, I_1 \cdots I_r) = E_1 \cdots E_r$.*

(2) *The principal submachine T_p generated by p has n states and satisfies property* P′.

Then T_p is output complete. Thus, under these conditions, if S is strongly connected, then S is output complete.

Proof. By Lemma 5.2, T_p has property P.

The conclusion of the theorem is obviously true when $n=1$. Therefore suppose that $n>1$. Let P_j be the statement that for any sequence $E_1, \cdots, E_j$ of elements of F, of length j, there exists a sequence $I_1, \cdots, I_j$ of elements of D such that $\lambda(p, I_1 \cdots I_j) = E_1 \cdots E_j$. By hypothesis, P_j is true for each $j \leqq n$. Suppose that P_j is true for all $j \leqq k$, where $n \leqq k$. Let $E_1, \cdots, E_{k+1}$ be any sequence of elements of F, of length $k+1$, and let $\bar{q} = \delta(p, I_1 \cdots I_k)$, where $I_1, \cdots, I_k$ is some sequence of inputs such that $\lambda(p, I_1 \cdots I_k) = E_1 \cdots E_k$. Suppose that $\lambda(q, I) = E_{k+1}$ is false for every I in D.

(a) If $\bar{q} = p$, then from the hypothesis, $\lambda(\bar{q}, I) = E_1$ for some I in D. Thus $q \neq p$.

(b) Suppose that $\bar{q} \neq p$. It is readily seen that if a free machine has n states, if $\bar{q} \neq p$, and if there exists an input I so that $\delta(p, I) = \bar{q}$, then there exists a sequence $\bar{I}_1 \cdots \bar{I}_r$, $r \leqq n-1$, of elements of D such that $\delta(p, \bar{I}_1 \cdots \bar{I}_r) = \bar{q}$. Let $\lambda(p, \bar{I}_1 \cdots \bar{I}_r) = \bar{E}_1 \cdots \bar{E}_r$. By (2) there is no input I such that $\lambda(p, I) = \bar{E}_1 \cdots \bar{E}_r E_{k+1}$. This contradicts (1) since $\bar{E}_1 \cdots \bar{E}_r E_{k+1}$ is of length $\leqq n$.

Hence, both (a) and (b) lead to contradictions. Thus there exists an element I_{k+1} in D such that $\lambda(\bar{q}, I_{k+1}) = E_{k+1}$, whence

$$\lambda(p, I_1 \cdots I_k I_{k+1}) = \lambda(p, I_1 \cdots I_k)\lambda(\bar{q}, I_{k+1}) = E_1 \cdots E_k E_{k+1}.$$

Consequently P_{k+1} is true, so that, by induction, p is output complete. The theorem then follows from Lemma 5.1.

The theorem is no longer true if either condition (2) is removed, or if n is replaced by $n-1$ condition (1).

Using the previous result we obtain:

COROLLARY 1. *Let S be a free machine having property* P, *and let $K_S = \{q_i/i=1, \cdots, n\}$. For each i let m_i be the number of states in the principal submachine generated by $\bar{q}_i$, the notation being that $m_n \leqq m_i$. If for each sequence $A_1, \cdots, A_k$ of elements of the output alphabet F, $k \leqq 1 + \sum_1^{n-1} (m_i - 1)$, there* *exists a state q and an input I so that $\lambda(q, I)=A_1A_2 \cdots A_k$, then there exists a submachine of S which is output complete.*

Proof. Suppose that the conclusion is false. Then there exists a state q_0 and an output E_1 of F so that $\lambda(q_0, I)=E_1$ is false for every input I. Label the remaining states $q_1, q_2, \cdots, q_{n-1}$. Repeating the proof of Theorem 5.1 for each $i \geqq 1$, we obtain states $q_{\alpha(i)}$ and p_{i+1}, and output E_{i+1}, so that $\lambda(p_{i+1}, I) =E_{i+1}$ is false for every input I. Now p_{i+1} is in the principal submachine generated by $q_{\alpha(i)}$. By hypothesis and Theorem 5.3, such an output E_{i+1} can be found so that $E_{i+1}=A_1^{i+1}A_2^{i+1} \cdots A_{s(i+1)}^{i+1}$, where $s(i+1) \leqq m_{\alpha(i)}-1$ and each A_j^{i+1} is in F. The process in Theorem 5.1 must terminate with E_{r+1}, where $r \leqq n-1$. Then $E_1E_2 \cdots E_{r+1}$ is of length at most $1+\sum_1^{n-1}(m_i-1)$ elements of F, and for no pair (q, I) does $\lambda(q, I)=E_1 \cdots E_{r+1}$. Hence a state p, thus a submachine, exists which is output complete.

Since $m_i \leqq n$ always holds, there follows

COROLLARY 2. *Let S be a free machine having n (finite) states and satisfying property* P. *If for each sequence $A_1, \cdots, A_k$ of elements of the output alphabet F, $k \leqq (n-1)^2+1$, there exists a state q and an input I so that $\lambda(q, I) =A_1A_2 \cdots A_k$; then there exists a submachine which is output complete.*

It is not known if $(n-1)^2+1$ is the best possible bound.

In passing we prove a result on a state being sequence output complete, a state q being defined as *sequence output complete* if for each sequence $\{E_i\}$ of outputs there exists a sequence $\{I_i\}$ of inputs so that $\lambda(q, I_1 \cdots I_i) =E_1 \cdots E_i$ for all i.

An application of the following known result [7, p. 81] will be used in Theorem 5.4 below.

"Let $\{D_i\}$ be an infinite sequence of finite, nonempty, pairwise disjoint sets. Let G be a graph with the following two properties:

(1) The nodes of G are the elements in the set $\bigcup_1^\infty D_i$.

(2) Each element of D_{n+1} is joined with at least one element of D_n.

Then there exists an infinite sequence $\{P_i\}$ of nodes, P_i in D_i, such that each P_iP_{i+1} is an edge of G."

THEOREM 5.4. *Let S have property* Q *and let q_1 be an output complete state of S. Suppose that for each output E and each state p in the principal subquasi-machine generated by q_1, the set*

$$L(p, E) = \{\delta(p, I)/\lambda(p, I) = E \text{ for some input } I\}$$

is finite (possibly empty). Then q_1 is sequence output complete.

Proof. Let $\{E_i\}$ be any sequence of outputs. Let $C_1=\{q_1\}$ and

$$C_2 = \{\delta(q_1, I)/\lambda(q_1, I) = E_1 \text{ for some input } I\}.$$

By hypothesis, $C_2=L(q_1, E_1)$ is finite and nonempty. We now define C_j recursively. Suppose that for each $2<j\leqq s$, the finite, nonempty set C_j is defined so that

$$C_j = \{\delta(q, I)/ \text{ for some } q \text{ in } C_{j-1} \text{ and some input } I, \lambda(q, I) = E_{j-1}\}.$$

Let

$$C_{s+1} = \{\delta(q, I)/\text{for some } q \text{ in } C_s \text{ and some input } I, \lambda(q, I) = E_s\}.$$

As q_1 is output complete, there exists an input H so that $\lambda(q_1, H)=E_1E_2 \cdots E_s$. Since S has property Q, inputs $H_1, H_2, \cdots, H_s$ can be found so that $H=H_1 \cdots H_s$ and $\lambda(q_i, H_i)=E_i$ for $i\leqq s$, where $q_{i+1}=\delta(q_i, H_i)$. Thus each q_i is in C_i, so that q_{s+1} is in C_{s+1}, i.e., C_{s+1} is nonempty. Now $C_{s+1}=\bigcup_{q\in C_s} L(q, E_s)$. Since C_s and each $L(q, E_s)$ is finite, C_{s+1} is finite.

For each integer s and each element q in C_s associate the symbol $\bar{q}^s$. Let $D_s=\{\bar{q}^s/q \text{ in } C_s\}$. Clearly the D_s are pairwise disjoint. Let G be the abstract graph defined as follows. The nodes of G are the set $\bigcup_1^\infty D_s$. An edge $e(\bar{q}^s, \bar{q}_*^{s+1}, A)$ is in G if q is in C_s, $q_*=\delta(q, A)$, and $\lambda(q, A)=E_s$. Applying the result cited above, there exists an infinite sequence $\{e_n\}$ of edges, with $e_n=(\bar{q}_n^n, \bar{q}_{n+1}^{n+1}, I_n)$. For $n\geqq 1$, $q_{n+1}=\delta(q_n, I_n)$ and $\lambda(q_n, I_n)=E_n$. Thus $\lambda(q_1, I_1 \cdots I_i)=E_1 \cdots E_i$ for all i, i.e., q_1 is sequence output complete.

COROLLARY 1. *Let S be a finite state quasi-machine having property* Q. *Then each state which is output complete is sequence output complete.*

COROLLARY 2. *Let S be a free machine with a finite number of elements in the input alphabet. Then each state in S which is output complete is also sequence output complete.*

Theorem 5.4 is no longer true if the condition that for each output E and each state p in the principal subquasi-machine generated by q_1, the set $L(p, E)$ is finite is removed.

EXAMPLE 5.1. Let S be the free machine S defined as follows. Let $K=\{p_0\}\cup\{q_{i,j}/i, j\geqq 1\}$. Let Y and W be generated by the input alphabet $D=\{I^n/n\geqq 1\}$ and output alphabet $F=\{0, 1\}$ respectively. Since W is denumerably infinite, $W=\{\tau_n/n\geqq 1\}$, with $\tau_1=0$ and $\tau_2=1$. Define $\delta(p_0, I^1)=q_{1,1}$, $\delta(p_0, I^2)=q_{2,1}$, $\lambda(p_0, I^1)=0$, and $\lambda(p_0, I^2)=1$. For I in D, $k=1, 2$, and $i\geqq 1$, let $\delta(q_{k,i}, I)=q_{k,i+1}$ and $\lambda(q_{k,i}, I)=0$. For $n\geqq 3$ suppose that τ_n is a sequence of $m(n)$ elements of F, say $\tau_n=U_1^nU_2^n \cdots U_{m(n)}^n$. Let $\delta(p_0, I^n)=q_{n,1}$, $\lambda(p_0, I^n)=U_1^n$, and $\lambda(q_{n,i}, I^n)=U_{i+1}^n$ for $i\leqq m(n)-1$. For I in D and $i\geqq 1$, let $\delta(q_{n,i}, I)=q_{n,i+1}$, and $\lambda(q_{n,i}, I)=0$ for those $(q_{n,i}, I)$ not already defined. Clearly p_0 is output complete For each n let $E_n=1$. There is no sequence of inputs $\{I_n\}$ so that $\lambda(p_0, I_1 \cdots I_n)=E_1 \cdots E_n$ for all n. Hence p_0 is not sequence output complete.

6. **Rational states.** It is common knowledge that in a finite state Moore-Mealy machine ultimately periodic sequences[18] of inputs applied to a state yield ultimately periodic sequence of outputs. We now consider this property for states of a quasi-machine.

DEFINITION. A state q of a quasi-machine S is said to be *rational* if for each ultimately periodic sequence $\{I_n\}$ of inputs, $\lambda(q, \{I_n\})$[19] is an ultimately periodic sequence of outputs.

It is easily seen that if p is a rational state of a machine and q is indistinguishable from p, then q is rational. Also, if $S \rightarrow T$ exists, if p is a rational state of S and q a rational state of T; then (p, q) is a rational state of $S \rightarrow T$.

Given a state q and an ultimately periodic sequence $\{I_n\}$ of inputs, in checking to see if q is rational, it is sufficient to assume that $\{I_n\}$ is an infinite sequence. This observation will tacitly be used in the sequel.

When checking to determine if a state is rational, we need only consider those ultimately periodic infinite sequences consisting of elements from a generating set for Y. Specifically we have the following result whose proof, being quite straightforward (and messy), is omitted.

THEOREM 6.1. *Let S be a quasi-machine, M a generating subset of Y, and q a given state of S. If $\lambda(q, \{I_n\})$ is ultimately periodic for every ultimately periodic infinite sequence $\{I_n\}$ of elements of M, then q is a rational state.*

The next result gives a sufficient condition for a state to be rational.

THEOREM 6.2. *Let q be a given state with the property that for every ultimately periodic infinite sequence $\{I_i\}$ and every positive integer r, the corresponding sequence of states $\{q_i\}$ contains an element, say p_r, which occurs at least r times. Then q is rational.*

Proof. Let q be the given state and $\{I_n\}$ an ultimately periodic infinite sequence of inputs. Then for some integers m and m_0, $I_{n+m}=I_n$ for all $n \geqq m_0$. By hypothesis, $\{q_i\}$ contains an element p_r, where $r=(m_0+m)(m_0+m-1)$, which occurs r times. Let the subscripts i of q_i for which $p_r=q_i$ be $a(1)$, $a(2), \cdots, a(r)$. Since $\{I_i\}$ is ultimately periodic and $I_{n+m}=I_n$ for all $n \geqq m_0$, there are at most m_0+m-1 different I_i in $\{I_i\}$. Hence there exists a finite sequence of strictly increasing integers $s_1, s_2, \cdots, s_{m_0+m}$ so that $(q_{a(s_i)}, I_{a(s_i)}) =(q_{a(s_j)}, I_{a(s_j)})$. Note that $a(s_i) \geqq m_0$ for $i \geqq m_0$. Thus there are at least $m+1$ $a(s_i)$ with $a(s_i) \geqq m_0$. Consequently there exist two integers t_1 and t_2 in

[18] A sequence $\{I_i\}$ of elements of an abstract set A is said to be *ultimately periodic* if either the sequence is finite or else the sequence is infinite and there exist integers m and m_0 so that $I_{n+m}=I_n$ for all $n \geqq m_0$. m is said to be a *period* of the sequence $\{I_i\}$.

[19] Let S be a quasi-machine and $\{I_n\}$ a sequence of elements of Y. By $\lambda(q, \{I_n\})$ is meant the sequence $\{\lambda(q_i, I_i)\}$, where $q_1=q$ and for $i \geqq 2$, $q_i=\delta(q_{i-1}, I_{i-1})$. The sequence $\{q_n\}$ is called the *corresponding sequence* of $\{I_n\}$.

$\{s_i\}$, $m_0 \leqq a(t_1) < a(t_2)$, so that $a(t_2) - a(t_1)$ is divisible by m, i.e., $a(t_2) - a(t_1) = um$. It is now readily seen that

$$(q_{a(t_1)+i}, I_{a(t_1)+i}) = (q_{a(t_1)+um+i}, I_{a(t_1)+um+i})$$

for all $i \geqq 0$. Thus $\lambda(q_j, I_j) = \lambda(q_{j+um}, I_{j+um})$ for all $j \leqq a(t_1)$. Therefore $\lambda(q_j, \{I_j\})$ is an ultimately periodic sequence so that q is a rational state.

COROLLARY 1. *Let q be a given state with the property that for each ultimately periodic sequence $\{I_i\}$, the corresponding sequence of states $\{q_i\}$ contains just a finite number of different elements. Then q is rational.*

This follows from the fact that if $\{q_i\}$, the corresponding sequence of the ultimately periodic infinite sequence $\{I_i\}$, contains just a finite number of different elements, one of them occurs infinitely often.

COROLLARY 2. *A quasi-machine with the property that every principal subquasi-machine has only a finite number of states has each state rational. In particular, a quasi-machine with a finite number of states has each state rational.*

REMARKS. (1) The hypothesis of Theorem 6.2 may be weakened to merely require an element p_r, where $r = (m_0 + m)(m_0 + m - 1)$, m_0 and m being associated with the ultimately periodic infinite sequence $\{I_i\}$ as denoted in the proof.

(2) The hypothesis of Theorem 6.2 cannot be weakened to merely require that $\{q_i\}$ contain an element which occurs twice.

(3) There exist distinguished, denumerable state, free machines, each state rational, with the property that for each ultimately periodic infinite sequence $\{I_i\}$ and each state q, the elements of the associated sequence $\{q_i\}$ are all different. The example we have in mind is rather complicated and so is omitted.

Suppose that q is a rational state of the quasi-machine S, S having an infinite number of states. There need be no relation between the period of the ultimately periodic infinite input sequence $\{I_n\}$ and the period of the resulting output sequence $\{E_n\}$. That is, for a given rational state q, one might find a set of ultimately periodic infinite input sequences $\{I_n^j\}$ of period m, with the resulting output sequences $\{E_n^j\}$ of period m_j, such that $\{m_j/j\}$ is an unbounded set. The last statement is no longer true if the principal subquasi-machine generated by q is finite, as the next theorm indicates.

THEOREM 6.3. *Let p be a state in the quasi-machine S and suppose that the principal subquasi-machine T_p generated by p has at most n (finite) states. Let $\{I_j\}$ be an ultimately periodic input sequence, with m_0 and m two positive integers such that $I_{j+m} = I_j$ for all $j \geqq m_0$. Then there exist positive integers m_1 and m_2, with* (i) *$m_1 \leqq m_0 + (n-1)m$,* (ii) *$m_2 \leqq mn$, and* (iii) *$m_1 + m_2 \leqq m_0 + mn$, such that $E_{k+m_2} = E_k$ for all $k \geqq m_1$, where $\lambda(p, \{I_j\}) = \{E_j\}$. Furthermore, given* *positive integers* m_0, m, *and* n, *for each of the inequalities* (i), (ii), *and* (iii), *there exists a case where the equality sign holds.*

Proof. To see the first part let p, T_p, m_0, m, $\{I_j\}$, and $\{E_j\}$ be as in the hypothesis. Consider the finite sequence $\{(p_i, I_i)\}m_0 \leqq i \leqq m_0+mn$, where $p_1=p$ and $p_i=\delta(p_{i-1}, I_{i-1})$ for $i>1$. Since T_p has at most n states and there are exactly $mn+1$ terms in the finite sequence, one of the p_i occurs as part of a term at least $m+1$ times. Let the indices at which this p_i occurs as part of a term in the finite sequence be $a(1), a(2), \cdots, a(r)$, $r \geqq m+1$. Then two of the indices, say $a(u)$ and $a(v)$ differ by an integral multiple of m, i.e., $a(v)=a(u)+wm$, w a positive integer. As the sequence $\{I_j\}$ becomes periodic, of period m, at the m_0 term, $I_{a(u)}=I_{a(v)}$. Thus $(p_{a(u)}, I_{a(u)})=(p_{a(v)}, I_{a(v)})$, and as is easily seen, for $k \geqq a(u)=m_1$ and $m_2=wm$, $E_{k+m_2}=E_k$. This implies that $m_1+m_2 \leqq m_0+mn$. Since $a(v)$ and $a(u)$ are both elements of the set $\{i/m_0 \leqq i \leqq m_0+mn\}$,

$$m_2 = wm = a(v) - a(u) \leqq m_0 + mn - m_0 = mn.$$

Now the smallest value that w can assume is 1 and the largest that $a(v)$ can assume is m_0+mn. Thus

$$a(u) \leqq m_0 + mn - m = m_0 + (n-1)m.$$

The last statement of the theorem will now be demonstrated by two examples. Accordingly, let m_0, m and n be given positive integers.

(I) Let S be the free machine with the n states $q_1, \cdots, q_n$ which is defined as follows. The set $\{A_i/i \leqq m+1\}$ is the input alphabet and $\{B_i/i \leqq n+1\}$ is the output alphabet. For $i \leqq n$ and $j<m$, let $\delta(q_i, A_j)=q_i$. If $j=m$ or $m+1$, let $\delta(q_i, A_j)=q_{i+1}$ when $i<n$ and $\delta(q_i, A_j)=q_1$ when $i=n$. For $i \leqq n$, let $\lambda(p_i, I_j)=B_i$ when $j \leqq m$ and $\lambda(p_i, I_j)=B_{n+1}$ when $j=m+1$.

Let r be the non-negative integer $<n$ for which $m_0-1=yn+r$, y being an integer; and let $p=q_{n-r+1}$ if $r \geqq 1$ and $p=q_1$ if $r=0$. Let $\{I_j\}$ be the infinite sequence whose first m_0-1 terms are A_{m+1}, the remaining terms being the periodic sequence whose periodic part is $A_1, A_2, \cdots, A_m$. Then $I_{j+m}=I_m$ for $j \geqq m_0$. Let $\lambda(p, \{I_j\})=\{E_j\}$. As is easily seen, $\{E_j\}$ is the sequence whose first m_0-1 terms are B_{n+1}, the remaining terms being the periodic sequence whose periodic part is m consecutive B_1's, followed by m consecutive B_2's, $\cdots$, followed by m consecutive B_n's. Then $m_1=m_0$, $m_2=mn$, and $m_1+m_2=m_0+mn$, that is, the equality sign holds for both (ii) and (iii).

(II) Let S be the free machine which is defined as in (I) with the following changes: $\delta(q_n, I_m)=q_n$ and $\lambda(q_i, I_j)=E_{m+1}$ for $i<n$ and $j \leqq m$. Let $\{I_j\}$, r, and p be as in (I). Let $\lambda(p, \{I_j\})=\{E_j\}$. Then $\{E_j\}$ is the sequence whose first $m_0-1+m(n-1)$ terms are B_{n+1}, the remaining terms being B_n. Then $m_1=m_0+(n-1)m$ and $m_2=1$. Thus the equality sign holds for (i). Q.E.D.

7. **Inverses.** We now consider the situation of one machine "undoing" the work of another.

DEFINITION. A *semi-inverse* S^* of a quasi-machine S is a quasi-machine such that (1) $W_S = Y_{S^*}$; (2) $W_{S^*} = Y_S$; (3) for each state q in S there exists a state $g(q)$ in S^* such that $\lambda_{S^*}[g(q), \lambda_S(q, I)] = I$ for all inputs I of S; and (4) for each state q^* in S^* there exists a state $h(q^*)$ in S such that $\lambda_S[h(q^*), \lambda_{S^*}(q^*, E)] = E$ for all inputs E of S^*.

From the symmetry of the definition, it is clear that S is a semi-inverse of S^* whenever S^* is a semi-inverse of S. That is, S and S^* are semi-inverses of each other.

LEMMA 7.1. *Let S and S^* be semi-inverses of each other. Then the following statements are true*:

(a) *$hg(q)$ is indistinguishable from q and $gh(q^*)$ is indistinguishable from q^*.*

(b) *If S^* is distinguished, then $gh(q^*) = q^*$, g maps K_S onto K_{S^*}, and h is a one to one function of K_{S^*} into K_S. If S is also distinguished, then g and h are inverse functions.*

(c) *$\lambda_S[q, \lambda_{S^*}(g(q), E)] = E$ and $\lambda_{S^*}[q^*, \lambda_S(h(q^*), I)] = I$ for all inputs I in S and E in S^*.*

(d) *For fixed q, λ_S is a one to one function of Y_S onto W_S.*

(e) *If S is a machine, then so is S^*.*

Proof. (a) Due to symmetry it is sufficient to show that q and $hg(q)$ are indistinguishable. For any input I of S,

$$\lambda_S(q, I) = \lambda_S(hg(q), \lambda_{S^*}[g(q), \lambda_S(q, I)]) = \lambda_S(hg(q), I).$$

Since I is arbitrary, q and $hg(q)$ are indistinguishable.

(b) Suppose that S^* is distinguished. By (a), $gh(q^*)$ is indistinguishable from q^*, thus $gh(q^*) = q^*$. Since $g[h(q^*)] = q^*$, g is onto. If $h(q^*) = h(q_1^*)$, then $q^* = gh(q^*) = gh(q_1^*) = q_1^*$. Thus h is one to one. If S is also distinguished, then g and h are inverse functions by virtue of $gh(q^*) = q^*$ and $hg(q) = q$.

(c) Obviously it is sufficient to show that $\lambda_S[q, \lambda_{S^*}(g(q), E)] = E$. Since q and $hg(q)$ are indistinguishable,

$$\lambda_S[q, \lambda_{S^*}(g(q), E)] = \lambda_S[hg(q), \lambda_{S^*}(g(q), E)] = E.$$

(d) That λ_S is one to one follows from $\lambda_{S^*}[g(q), \lambda_S(q, I)] = I$. By (c), for E in W_S, $\lambda_S[q, \lambda_{S^*}(g(q), E)] = E$. Hence λ_S is onto.

(e) Suppose that S is a machine and that $I_1 I_2 = I_1 I_3$, where I_1, I_2, and I_3 are in Y_S. Let q be a fixed state of S. Then

$$\begin{aligned} \lambda_S(q, I_1)\lambda_S[\delta_S(q, I_1), I_2] &= \lambda_S(q, I_1 I_2) = \lambda_S(q, I_1 I_3) \\ &= \lambda_S(q, I_1)\lambda_S[\delta_S(q, I_1), I_3]. \end{aligned}$$

Since S is a machine, $\lambda_S[\delta_S(q, I_1), I_2] = \lambda_S[\delta_S(q, I_1), I_3]$. As S has a semi-inverse, λ_S is one to one. Thus $I_2 = I_3$, i.e., S^* is a machine.

LEMMA 7.2. *Let p in S and q in T be two indistinguishable states. If S^* and T^* are semi-inverses of S and T respectively, then $g_S(p)$ and $g_T(q)$ are indistinguishable states.*

Proof. For arbitrary E in W_S, consider $\lambda_{S^*}[g_S(p), E]$ and $\lambda_{T^*}[g_T(q), E]$. Since p and q are indistinguishable and λ_S is one to one onto, there exists I in Y so that $\lambda_S(p, I) = E = \lambda_T(q, I)$. Then $\lambda_{S^*}[g_S(p), \lambda_S(p, I)] = \lambda_{S^*}[g_S(p), E] = I = \lambda_{T^*}[g_T(q), \lambda_T(q, I)] = \lambda_{T^*}[g_T(q), E]$. This proves the lemma.

Lemma 7.2 is now applied in the following result:

THEOREM 7.1. *If S^* is a semi-inverse of the quasi-machine S, then $\overline{S}$ is a semi-inverse of S if and only if $\overline{S}$ is equivalent to S^*.*

Proof. Let S and S^* be semi-inverses of each other under g and h.

Suppose that $\overline{S}$ is equivalent to S^*. For each state $\bar{q}$ in $\overline{S}$ let $d(\bar{q})$ be a state in S^* which is equivalent to $\bar{q}$; and for each state q^* in S^* let $f(q^*)$ be a state in $\overline{S}$ which is equivalent to q^*. For q in S let $\bar{g}(q) = fg(q)$ and for $\bar{q}$ in $\overline{S}$ let $\bar{h}(\bar{q}) = hd(\bar{q})$. Then

$$\begin{aligned}\lambda_{\bar{S}}[\bar{g}(q), \lambda_S(q, I)] &= \lambda_{\bar{S}}[fg(q), \lambda_S(q, I)] \\ &= \lambda_{S^*}[g(q), \lambda_S(q, I)], \text{ as } g(q) \text{ and } fg(q) \text{ are indistinguishable,} \\ &= I.\end{aligned}$$

Also

$$\begin{aligned}\lambda_S[\bar{h}(\bar{q}), \lambda_{S}(\bar{q}, E)] &= \lambda_S[hd(\bar{g}), \lambda_{\bar{S}}(\bar{q}, E)] \\ &= \lambda_S[hd(\bar{q}), \lambda_{S^*}(d(\bar{q}), E)], \text{ as } \bar{q} \text{ and } d(\bar{q}) \text{ are indistinguishable,} \\ &= E.\end{aligned}$$

Thus S and $\overline{S}$ are semi-inverses of each other.

Now suppose that S and $\overline{S}$ are semi-inverses of each other under $\bar{g}$ and $\bar{h}$. Let q^* be an arbitrary state of S^*. Then $h(q^*)$ is in S. By Lemma 7.2, $gh(q^*)$ and $\bar{g}h(q^*)$ are indistinguishable states. By Lemma 7.1(a), q^* and $gh(q^*)$, thus q^* and $\bar{g}h(q^*)$ are indistinguishable. In a similar fashion it is seen that to each state $\bar{q}$ in $\overline{S}$, there corresponds $g\bar{h}(\bar{q})$ in S^* indistinguishable from $\bar{q}$. Thus S and $\overline{S}$ are equivalent.

In order to obtain "structure" results, i.e., results involving the next state function, the following notion is introduced.

DEFINITION. S^* is said to be an *inverse*[20] of S if $(i)S^*$ is a semi-inverse of S, and (ii) $g[\delta_S(q, I)] = \delta_{S^*}[g(q), \lambda_S^{-}(q, I)]$ for each q and I in S.

THEOREM 7.2. *If S^* is a distinguished machine which is a semi-inverse of S, then S^* is an inverse of S.*

Proof. It is necessary to show that $g[\delta_S(q, I)] = \delta_{S^*}[g(q), \lambda_S(q, I)]$ for each

([20]) Inverses were first introduced in [11] for special machines.

state q and each input I of S. For all inputs I_1 of S,

$$\begin{aligned} II_1 &= \lambda_{S^*}[g(q), \lambda_S(q, II_1)] \\ &= \lambda_{S^*}[g(q), \lambda_S(q, I)\lambda_S(\delta_S(q, I), I_1)] \\ &= \lambda_{S^*}[g(q), \lambda_S(q, I)]\lambda_{S^*}(q_1^*, \lambda_S[\delta_S(q, I), I_1]) \\ &= I\lambda_{S^*}(q_1^*, \lambda_S[\delta_S(q, I), I_1]), \end{aligned}$$

where $q_1^* = \delta_{S^*}[g(q), \lambda_S(q, I)]$. Since S^* is a machine, the left cancellation law holds for Y_S. Thus

$$\lambda_{S^*}(g[\delta_S(q, I)], \lambda_S[\delta_S(q, I), I_1]) = I_1 = \lambda_{S^*}[q_1^*, \lambda_S(\delta_S(q, I), I_1)].$$

As S^* is distinguished and I_1, thus $\lambda_S[\delta_S(q, I), I_1]$ is arbitrary,

$$g[\delta_S(q, I)] = q_1^* = \delta_{S^*}[g(q), \lambda_S(q, I)]. \qquad \text{Q.E.D.}$$

The theorem is no longer valid if S^* is not distinguished.

EXAMPLE 7.1. Let S and S^* be the free machines defined as follows. Let $K_S = \{q_1\}$. Let I generate both W_S and Y_S. Let $K_{S^*} = \{q_1^*, q_2^*\}$, and $\delta_S(q_i^*, I) = q^*$, where $i, j = 1, 2, i \neq j$. Then S^* and S are semi-inverses but S^* is not an inverse of S.

The above example also shows that being an inverse is not invariant under equivalence. For S is an inverse of S, S^* is equivalent to S, but S^* is not an inverse of S. What is true, however, is the following result, which is a consequence of Theorem 7.2.

COROLLARY. *If S^* and S are semi-inverses of each other, if T^* is a distinguished machine which is equivalent to S^*, and if T is a distinguished machine which is equivalent to S; then T^* and T are inverses of each other.*

When S and its inverse S^* are both distinguished quasi-machines, then the relation between S and S^* may be reversed, namely:

THEOREM 7.3. *If S and S^* are both distinguished quasi-machines, and if S^* is an inverse of S, then S is an inverse of S^*.*

Proof. It is necessary to show that $h[\delta_{S^*}(q^*, E)] = \delta_S[h(q^*), \lambda_{S^*}(q^*, E)]$. Let $q = h(q^*)$ and $I = \lambda_{S^*}(q^*, E)$. Then $g(q) = q^*$ since S^* is distinguished, and $E = \lambda_S(h(q^*), I)$. Since S^* is an inverse of S, $g[\delta_S(q, I)] = \delta_{S^*}[g(q), \lambda_S(q, I)]$. Thus $hg[\delta_S(q, I)] = h(\delta_{S^*}[g(q), \lambda_S(q, I)])$. Since S is distinguished,

$$hg[\delta_S(q, I)] = \delta_S(q, I) = \delta_S[h(q^*), \lambda_{S^*}(q^*, E)] = h[\delta_{S^*}(q^*, E)].$$

Turning to "uniqueness" of inverses we have

THEOREM 7.4. *If S^* and $\overline{S}$ are both distinguished quasi-machines which are inverses of S, then S^* and $\overline{S}$ are isomorphic to each other.*

Proof. Let $\bar{g}$ and $\bar{h}$ be the functions relating S and $\overline{S}$. Since S^* is distinguished, $\bar{g}h$ is a one to one function of K_{S^*} onto $K_{\bar{S}}$. Let $\bar{q}$ be in $K_{\bar{S}}$ and $q^*=g\bar{h}(\bar{q})$. Then $\bar{g}\bar{h}(\bar{q})=\bar{q}$ and $h(q^*)=hg[\bar{h}(\bar{q})]$ is indistinguishable from $\bar{h}(\bar{q})$. Since $\bar{S}$ is distinguished, by Lemma 7.1(e) $\bar{q}=\bar{g}\bar{h}(\bar{q})=\bar{g}h(q^*)$. Thus $\bar{g}h$ maps K_{S^*} one to one on $K_{\bar{S}}$. Next

$$\begin{aligned} E = \lambda_S[h(q^*), \lambda_{S^*}(q^*, E)] &= \lambda_S[\bar{h}\bar{g}h(q^*), \lambda_{\bar{S}}(\bar{g}h(q^*), E)] \\ &= \lambda_S[h(q^*), \lambda_{\bar{S}}(\bar{g}h(q^*), E)], \end{aligned}$$

the last equality occurring since $\bar{h}\bar{g}h(q^*)$ is indistinguishable from $h(q^*)$. Since λ_S is one to one, $\lambda_{S^*}(q^*, E)=\lambda_{\bar{S}}(\bar{g}h(q^*), E)$. Thus the outputs from q^* and $\bar{g}h(q^*)$ are identical. To complete the proof it is necessary to show that the next state functions correspond under $\bar{g}h$, i.e., $\delta_{\bar{S}}[\bar{g}h(q^*), E]=\bar{g}h[\delta_{S^*}(q^*, E)]$. Let $q=h(q^*)$ and $I=\lambda_{S^*}(q^*, E)$. Then

$$\begin{aligned} \delta_{\bar{S}}[\bar{g}h(q^*), E] &= \delta_{\bar{S}}[\bar{g}(q), \lambda_S(q, I)] \\ &= \bar{g}[\delta_S(q, I)], \text{ since } \bar{S} \text{ is an inverse of } S. \end{aligned}$$

Since $\delta_S(q, I)$ and $hg[\delta_S(q, I)]$ are indistinguishable and $\bar{S}$ is distinguished,

$$\begin{aligned} \bar{g}[\delta_S(q, I)] &= \bar{g}hg[\delta_S(q, I)] \\ &= \bar{g}h[\delta_{S^*}(g(q), \lambda_S(q, I))], \text{ since } S^* \text{ is an inverse of } S, \\ &= \bar{g}h[\delta_{S^*}(gh(q^*), E] \\ &= \bar{g}h[\delta_{S^*}(q^*, E)], \text{ since } S^* \text{ is distinguished.} \end{aligned}$$

Thus $\delta_{\bar{S}}[\bar{g}h(q^*), E]=\bar{g}h[\delta_{S^*}(q^*, E)]$, whence the theorem.

The theorem is not true if inverse is replaced by semi-inverse. For let S and $\bar{S}$ be as in Example 3.1. Then S and $\bar{S}$ are two nonisomorphic semi-inverses of S.

A characterization of those quasi-machines which have an inverse is now given.

THEOREM 7.5. *A necessary and sufficient condition that a quasi-machine S possess an inverse S^* is that for each state q, λ_S be a one to one function of Y_S onto W_S.*

Proof. The necessity being obvious, only the sufficiency shall be shown. Therefore assume that λ_S is a one to one function for each state q. To each q in K_S associate in a one to one manner, a symbol $q^*=g(q)$. Denote by K_{S^*} the set of all such q^*. Let h be the inverse function of g, let $W_{S^*}=Y_S$, and let $Y_{S^*}=W_S$. For each q^* in K_{S^*} and each E in Y_{S^*}, define $\lambda_{S^*}(q^*, E)$ to be I and $\delta_{S^*}(q^*, E)$ to be $g(\delta_S[h(q^*), \lambda_{S^*}(q^*, E)])$, where $\lambda_S[h(q^*), I]=E$. In view of the assumption on λ_S, $\delta_{S^*}(q^*, E)$ is uniquely defined.

It is now to be shown that S^* is an abstract quasi-machine. This is to be done by verifying properties (3) and (4) in the definition of a quasi-machine.

From the method of construction, it is obvious that if S^* is a quasi-machine, then it is an inverse of S.

Let q^* be in K_{S^*} and E_1 any element of Y_{S^*}. Let $q=h(q^*)$ and $\lambda_S(q, I_1) = E_1$. Clearly I_1 is uniquely defined. For any E_2 in Y_{S^*} let I_2 be such that $E_2=\lambda_S[\delta_S(q, I_1), I_2]$. Since S is a quasi-machine

$$\begin{aligned} E_1E_2 &= \lambda_S(q, I_1)\lambda_S[\delta_S(q, I_1), I_2] \\ &= \lambda_S(q, I_1I_2). \end{aligned}$$

Thus

$$\lambda_{S^*}(q^*, E_1E_2), = I_1I_2.$$

(a) Now

$$\begin{aligned} \delta_{S^*}[q^*, E_1E_2] &= g(\delta_S h(q^*), \lambda_{S^*}(q^*, E_1E_2)]) \\ &= g[\delta_S(q, I_1I_2)] \\ &= g(\delta_S[\delta_S(q, I_1), I_2]); \end{aligned}$$

and

$$\begin{aligned} \delta_{S^*}[\delta_{S^*}(q^*, E_1), E_2] &= \delta_{S^*}[g(\delta_S[h(q^*), \lambda_{S^*}(q^*, E_1)]), E_2] \\ &= \delta_{S^*}[g(\delta_S[q, I_1]), E_2] \\ &= g(\delta_S[hg[\delta_S(q, I_1)], \lambda_{S^*}(g[\delta_S(q, I_1)], E_2)]) \\ &= g(\delta_S[\delta_S(q, I_1), I_2]). \end{aligned}$$

Thus $\delta_{S^*}[q^*, E_1E_2]=\delta_{S^*}[\delta_{S^*}(q^*, E_1), E_2]$, that is, property (3) in the definition of a quasi-machine holds.

(b) $\lambda_{S^*}[q^*, E_1E_2]=I_1I_2$ and

$$\begin{aligned} \lambda_{S^*}(q^*, E_1)\lambda_{S^*}[\delta_{S^*}(q^*, E_1), E_2] &= I_1\lambda_{S^*}[g(\delta_S[h(q^*), \lambda_{S^*}(q^*, E_1)]), E_2] \\ &= I_1\lambda_{S^*}[g(\delta_S[q, I_1]), E_2] = I_1I_2. \end{aligned}$$

Thus $\lambda_{S^*}[q^*, E_1E_2]=\lambda_{S^*}(q^*, E_1)\lambda_{S^*}[\delta_{S^*}(q^*, E_1), E_2]$, that is, property (4) in the definition of a quasi-machine holds.

This completes the proof.

In certain cases, the property of a quasi-machine S having an inverse may be deduced from the behavior of λ_S on a generating set of inputs. Specifically we have

COROLLARY 1. *Let A be a generating set for Y_S and B a generating set for W_S such that each E in W_S is the product, in a unique way, of elements in B. If for each state q, λ_S is a one to one function of A onto B, then S has an inverse S^*.*

Proof. Let q be a state of S and E an element of W_S. Let $E=E_1E_2 \cdots E_n$, where each E_i is in B. Define q_1 to be q and let I_1 in A be such that $\lambda_S(q_1, I_1) = E_1$. I_1 exists since λ_S maps A onto B. Let $q_2=\delta_S(q_1, I_1)$. For each $i \leqq j < n$

suppose that q_i and I_i are defined so that $\lambda_S(q_i, I_i)=E_i$. Let $q_{j+1}=\delta_S(q_j, I_j)$ and let I_{j+1} be such that $\lambda_S(q_{j+1}, I_{j+1})=E_{j+1}$. In this way I_i becomes defined for each $i\leqq n$. Let $I=I_1 \cdots I_n$. There is no difficulty in seeing that $\lambda_S(q, I)=E$. Thus λ_S maps Y_S onto W_S. Now suppose that $\bar{I}$ is such that $\lambda_S(q, \bar{I})=E$. Let $\bar{I}=\bar{I}_1 \cdots \bar{I}_m$, each $\bar{I}_i$ in A. Define $\bar{q}_1$ to be q. Then $E=\bar{E}_1 \cdots \bar{E}_m$, where $\bar{E}_1=\lambda_S(\bar{q}_1, \bar{I}_1)$ and for $i\geqq 2$, $\bar{E}_i=\lambda_S(\bar{q}_i, \bar{I}_i)$, with $\bar{q}_i=\delta_S(\bar{q}_{i-1}, \bar{I}_{i-1})$. In view of the uniqueness property, $m=n$ and $\bar{E}_i=E_i$ for each E_i. Since λ_S maps A one to one onto B, $I_i=\bar{I}_i$ for each i. Thus $I=\bar{I}$, that is λ_S maps Y_S one to one onto W_S. Thus, from Theorem 7.5, S has an inverse S^*. Q.E.D.

Corollary 2. *Let S be a free machine. If for each state q, λ_S maps the input alphabet one to one onto the output alphabet, then S has an inverse S^*.*

In order to check that two quasi-machines S and S^* are inverses of each other, it is sufficient to verify the basic laws over a set of generating inputs and generating outputs. More precisely we have

Theorem 7.6. *Let S and S^* be two quasi-machines with $Y_S=W_{S^*}$ and $W_S=Y_{S^*}$. Let g be a one to one function of K_S into K_{S^*} and h a one to one function of K_{S^*} into K_S. Suppose that A is a generating set of inputs of Y_S and B a generating set of ouputs of W_S such that $g[\delta_S(q, I)]=\delta_{S^*}[g(q), \lambda_S(q, I)]$, $h[\delta_{S^*}(q^*, E)]=\delta_S[h(q^*), \lambda_{S^*}(q^*, E)]$, $\lambda_{S^*}[g(q), \lambda_S(q, I)]=I$, and $\lambda_S[h(q), \lambda_{S^*}(q^*, E)]=E$ for each I in A, E in B, q in K_S, and q^* in K_{S^*}. Then S and S^* are inverses of each other.*

Proof. We first show that

$$g[\delta_S(q, I)] = \delta_{S^*}[g(q), \lambda_S(q, I)] \tag{*}$$

for each input I in Y_S. Denote by C_n the set of all finite sequences $I_1 \cdots I_j$, $j\leqq n$ of inputs I_i in A. Since A is a generating set for Y_S, $Y_S=\bigcup_1^\infty C_n$. Thus it is sufficient to show that (*) holds for each I in C_n, $n=1, 2, \cdots$. By assumption (*) is true for I in C_1. Suppose that (*) is true for I in C_k, $k\leqq n$. For I in $C_{n+1}-C_n$, $I=UI_1$, where U is in C_n and I_1 in C_1. Then

$$\begin{aligned}
g[\delta_S(q,I)] &= g[\delta_S(q, UI_1)] \\
&= g[\delta_S(\delta_S(q, U), I_1)] \\
&= \delta_{S^*}[g(\delta_S(q, U)), \lambda_S(\delta_S(q, U), I_1)], \text{ since (*) holds for } C_1, \\
&= \delta_{S^*}[\delta_{S^*}\{g(q), \lambda_S(q, U)\}, \lambda_S(\delta_S(q, U), I_1)], \text{ since (*) holds for } C_n, \\
&= \delta_{S^*}[g(q), \lambda_S(q, U)\lambda_S(\delta_S(q, U), I_1)] \\
&= \delta_{S^*}[g(q), \lambda_S(q, UI_1)] \\
&= \delta_{S^*}[g(q), \lambda_S(q, I)].
\end{aligned}$$

Thus (*) holds for I in C_{n+1}, so that, by induction, (*) holds for each C_n.

Suppose that $\lambda_{S^*}[g(q), \lambda_S(q, U)]\neq U$ for some U in some C_m. Let n be the smallest integer such that there exists $I_1 \cdots I_n$ for which $\lambda_{S^*}[g(q), \lambda_S(q, I_1 \cdots I_n)] \neq I_1 \cdots I_n$. Then $n \geqq 2$. Let $I = I_1 I_2 \cdots I_{n-1}$. Hence $\lambda_{S^*}[g(q), \lambda_S(q, I)] = I$. Now

$$\begin{aligned}
\lambda_{S^*}&[g(q), \lambda_S(q, I_1 \cdots I_n)] \\
&= \lambda_{S^*}[g(q), \lambda_S(q, II_n)] \\
&= \lambda_{S^*}[g(q), \lambda_S(q, I)\lambda_S(\delta_S(q, I), I_n)] \\
&= \lambda_{S^*}[g(q), \lambda_S(q, I)]\lambda_{S^*}\{\delta_{S^*}[g(q), \lambda_S(q, I)], \lambda_S[\delta_S(q, I), I_n]\} \\
&= I\lambda_{S^*}\{g(\delta_S(q, I)), \lambda_S[\delta_S(q, I), I_n]\} \\
&= II_n = I_1 \cdots I_n.
\end{aligned}$$

This is a contradiction. Therefore $\lambda_{S^*}[g(q), \lambda_S(q, U)] = U$ for all U in Y_S.

In a similar fashion it can be shown that

$$h[\delta_{S^*}(q^*, E)] = \delta_S[h(q^*), \lambda_{S^*}(q^*, E)]$$

and

$$\lambda_S[h(q^*), \lambda_{S^*}(q^*, E)] = E \text{ for each } E \text{ in } Y_{S^*}$$

Consequently, all the laws for S and S^* to be inverses of each other are satisfied.

One application of Theorem 7.6 occurs when S and S^* are free machines. Then one only has to check the basic laws for the elements in the input alphabet D and the output alphabet F.

THEOREM 7.7. *Let S and S^* be two distinguished machines, each of n (finite) states. Let $Y_S = W_{S^*}$ and $Y_{S^*} = W_S$. Let g be a one to one function of K_S into K_{S^*} and h a one to one function of K_{S^*} into K_S. Suppose that A is a generating set of inputs of Y_S and B a generating set of inputs of Y_{S^*} such that λ_S maps A onto B, λ_{S^*} maps B onto A, $\lambda_{S^*}[g(q), \lambda_S(q, I)] = I$, and $\lambda_S[h(q), \lambda_{S^*}(q^*, E)] = E$ for each q in K_S, each q^* in K_{S^*}, each sequence $I_1 \cdots I_j$, $j \leqq n$ of elements I_i from A, and each sequence $E = E_1 \cdots E_k$, $k \leqq n$ of elements E_i from B. Then S and S^* are inverses of each other.*

Proof. In view of Theorem 7.6 it is sufficient to show that

$$g[\delta_S(q, I_1)] = \delta_{S^*}[g(q), \lambda_S(q, I_1)] \quad \text{and} \quad h[\delta_{S^*}(q^*, E_1)] = \delta_S[h(q^*), \lambda_{S^*}(q^*, E_1)]$$

for each I_1 in A and E_1 in B. Because of the symmetry of the situation, it is sufficient to show the former equality. Let q and I_1 be fixed.

Let $E_2 \cdots E_n$ be any sequence of $n-1$ elements from B. Since λ_S maps A onto B, there exists I_2 in A such that $\lambda_S[\delta_S(q, I_1), I_2] = E_2$. Similarly there exists I_3 in A such that $\lambda_S[\delta_S(\delta_S(q, I_1), I_2), I_3] = E_3$. Thus $\lambda_S[\delta_S(q, I_1), I_2 I_3] = E_2 E_3$. Continuing in the obvious manner we see that there exists $I_2, \cdots, I_n$ in A such that $\lambda_S[\delta_S(q, I_1), I_2 \cdots I_n] = E_2 \cdots E_n$.

Now let $I = I_2 \cdots I_n$ be any sequence of length $n-1$ of inputs of A.

Repeating the inequalities in Theorem 7.2 we get $I_1 I = I_1 \lambda_{S^*}(q_1^*, \lambda_S[\delta_S(q, I_1), I])$, where $q_1^* = \delta_{S^*}[g(q), \lambda_S(q, I_1)]$. By the left cancellation law and hypothesis

$$\lambda_{S^*}(q_1^*, \lambda_S[\delta_S(q, I_1), I]) = I = \lambda_{S^*}(g[\delta_S(q, I_1)], \lambda_S[\delta_S(q, I_1), I]).$$

Thus q_1^* and $g[\delta_S(q, I_1)]$ are indistinguishable by $\lambda_S[\delta_S(q, I_1), I]$. In the previous paragraph we showed that when I varies over all sequences of length $n-1$ of elements from A, $\lambda_S[\delta_S(q, I_1), I]$ varies over all sequences of length $n-1$ of elements from B. Now B is a generating set for all elements of Y_S. Repeating the proof given in Theorem 6 of [9] we see that in a machine (as defined here) with n states, two states are distinguishable if and only if they are distinguishable by some sequence of generating inputs of length $n-1$. Thus q_1^* and $g[\delta_S(q, I_1)]$ are indistinguishable. Since S^* is distinguished

$$\delta_{S^*}[g(q), \lambda_S(q, I_1)] = q_1^* = g[\delta_S(q, I_1)]. \qquad \text{Q.E.D.}$$

REMARKS. (1) The most important application of Theorem 7.7 is when S and S^* are free machines and A and B are input and output alphabets respectively of S.

(2) Theorem 7.7 is no longer true if the hypothesis on the length of the sequences $I_1 \cdots I_j$ is changed from n to $n-1$.

EXAMPLE 7.2. Let S and S^* be the free machines defined as follows: Let $n \geqq 3$, $K_S = K_{S^*} = \{q_i / i \leqq n\}$, $A = B = \{0, 1\}$, $g(q_i) = h(q_i) = q_i$, $\lambda_S(q_i, U) = U$ for $i \leqq n-1$, $\lambda_S(q_n, 0) = 1$, $\lambda_S(q_n, 1) = 0$, $\lambda_{S^*} = \lambda_S$, $\delta_S(q_i, U) = q_{i+1}$ for $2 \leqq i \leqq n-1$, $\delta_S(q_n, U) = q_2$, $\delta_S(q_1, 0) = q_3$, $\delta_S(q_1, 1) = q_2$, $\delta_S(q_i, U) = \delta_{S^*}(q_i, U)$ for $i > 1$, $\delta_{S^*}(q_1, 0) = q_3$, and $\delta_{S^*}(q_1, 1) = q_1$, where $U = 0, 1$. Then the hypothesis of Theorem 7.7 holds for sequences of inputs of A and of B of length $n-1$, but not of length n (consider the sequence of length n consisting of all 1's, starting at q_1). S and S^*, of course, are not inverses.

Turning briefly to properties preserved by inverse machines we have

THEOREM 7.8. *If S^* is a distinguished quasi-machine which is an inverse of S and if S is strongly connected, then S^* is strongly connected.*

Proof. Let q^* and q_1^* be any two states in S^*. Let $q = h(q^*)$ and $q_1 = h(q_1^*)$. Since S is strongly connected there exists an input I in S such that $\delta_S(q, I) = q_1$. As S^* is distinguished, $gh(q^*) = g(q) = q^*$ and $gh(q_1^*) = q_1^*$. Then

$$\begin{aligned} q_1^* &= g[\delta_S(q, I)] \\ &= \delta_{S^*}[g(q), \lambda_S(q, I)], \text{ since } S^* \text{ is an inverse of } S, \\ &= \delta_{S^*}[q^*, \lambda_S(q, I)]. \end{aligned}$$

Consequently S^* is strongly connected. Q.E.D.

Theorem 7.9 is a generalization of a result of [11]. The theorem is no longer true if S^* is not distinguished.

Another result, easily proved by the reader, is

THEOREM 7.9. *Let S^* be an inverse of the quasi-machine S. If A is a stable set of S, then $g(A)$ is stable and the quasi-machine associated with $g(A) \cup \bigcup_1^\infty (gh)^n[g(A)]$*(21) *is an inverse of the quasi-machine associated with $A \cup \bigcup_1^\infty (hg)^n(A)$. If S is distinguished and A is any set of states of S, then the subquasi-machine generated by $g(A)$ is an inverse of the subquasi-machine generated by A; furthermore, $g(A)$ is stable if and only if A is stable.*

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(21) By f^n is meant ff^{n-1}, with $f^1 = f$.